6 Theorem of Stone

time evolution of quantum mechanical system is given by time evolution operators $(\mathcal{U}_t)_{t\geq 0}$

$$\psi \xrightarrow{\mathcal{U}_t} \psi_t = \mathcal{U}_t \psi$$

system at time 0   system at time t

where:

- $\|\psi_t\| = \|\psi\|$, i.e. $\mathcal{U}_t$ isometry

  if $\mathcal{U}_t$ invertible $\Rightarrow$ $\mathcal{U}_t$ unitary

  (closed system, reversible time evolution)

- $\mathcal{U}_0 = 1$
- $\mathcal{U}_t \mathcal{U}_s = \mathcal{U}_{t+s}$ (semigroup property)

Question: Can we write $\mathcal{U}_t$ as

$$\mathcal{U}_t = e^{-itH}$$

for some operator H, where

$\mathcal{U}_t$ unitary $\Leftrightarrow$ H selfadjoint then: $\frac{d}{dt} \mathcal{U}_t = -i H \mathcal{U}_t$

or $i \frac{d \mathcal{U}_t}{dt} \Psi = H \mathcal{U}_t \Psi$

$i \frac{\partial \Psi_t}{\partial t} = H \Psi_t$ Schrödinger equation

H is Hamilton operator of system (which governs time evolution)

<u>mathematical problem:</u>

unitary group $(\mathcal{U}_t)_{t\in\mathbb{R}} \overset{?}{\longleftrightarrow} \mathcal{U}_t = e^{iAt}$

A is then called the <u>generator</u> of $(\mathcal{U}_t)$

<u>6.1 Theorem:</u> Let $A \in \mathcal{B}(\mathcal{H})$ be s.a. and define

$$\mathcal{U}_t := e^{itA} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} A^n \qquad (t\in\mathbb{R})$$

Then we have:

i) $\mathcal{U}_t$ unitary $\forall t \in \mathbb{R}$

ii) $\mathcal{U}_0 = \mathbb{1}$

iii) $\mathcal{U}_t \mathcal{U}_s = \mathcal{U}_{t+s} \quad \forall\, t,s \in \mathbb{R}$

iv) $\lim_{t\to 0} \|\mathcal{U}_t - 1\| = 0$

Proof: Exercise! $\square$

thus: $A = A^* \in B(\mathcal{H}) \Longrightarrow$ norm continuous unitary group $(\mathcal{U}_t)$

$$\overset{?}{\Longleftarrow}$$

<u>6.2. Theorem</u>: Let $(\mathcal{U}_t)_{t\in\mathbb{R}}$ be a norm continuous unitary group on a Hilbert space $\mathcal{H}$, i.e.

- $\mathcal{U}_t$ unitary $\forall\, t \in \mathbb{R}$
- $\mathcal{U}_0 = 1$, $\mathcal{U}_t \mathcal{U}_s = \mathcal{U}_{t+s} \quad \forall\, t,s \in \mathbb{R}$
- $\lim_{t\to 0} \|\mathcal{U}_t - 1\| = 0$

Then there exists a (uniquely determined) selfadjoint operator $A \in B(\mathcal{H})$ s.th.

$$\mathcal{U}_t = e^{itA}$$

<u>Proof</u>: Idea: A should be given as

$iA = \frac{d}{dt}\mathcal{U}_t \Big|_{t=0}$, but existence of this limit is not clear a priori better way is

$$\int_0^t U_\tau \, d\tau = \frac{1}{iA} \left[U_\tau \right]_0^t =$$

$$= \frac{1}{iA} \left[U_t - 1 \right]$$

i.e.

$$iA = (U_t - 1) \cdot \left[\int_0^t U_t \, d\tau \right]^{-1}$$

$$= \frac{U_t - 1}{t} \cdot \Big[\underbrace{\frac{1}{t} \int_0^t U_t \, d\tau}_{=: X_t} \Big]^{-1}$$

thus:

$$iA = \frac{U_t - 1}{t} \cdot X_t^{-1} = \frac{U_s - 1}{s} \cdot X_s^{-1}$$

$$\Rightarrow \frac{U_t - 1}{t} \cdot X_s = \frac{U_s - 1}{s} \cdot X_t \quad (*)$$

(*) is what we want if we define

$$X_t := \frac{1}{t} \int_0^t U_\tau \, d\tau$$

norm continuity of U_t gives

$\|X_t - 1\| \to 0$ for $t \to 0$

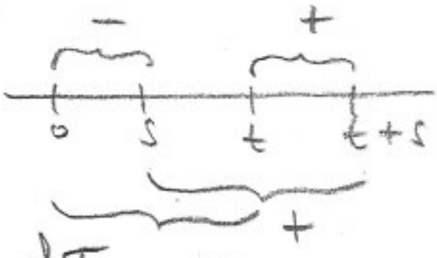
and thus

X_t^{-1} exists in $B(\mathcal{H})$ for t sufficiently small, say $t < t_0$

now show $(*)$ for $0 < s, t < t_0$

$$\frac{U_t - 1}{t} \cdot X_s = \frac{U_t - 1}{t} \, \frac{1}{s} \int_0^s U_\tau \, d\tau$$

$$= \frac{1}{st} \int_0^s \big(\underbrace{U_t U_\tau}_{U_{t+\tau}} - 1 \cdot U_\tau \big) \, d\tau$$



$$\overset{!}{=} \frac{1}{ts} \int_0^t \big(U_{s+\tau} - U_\tau \big) \, d\tau$$

$$= \frac{U_s - 1}{s} \, X_t$$

Now fix t and take $s \to 0$ (note $X_s \to 1$)

$$\frac{U_t - 1}{t} \cdot X_s = \frac{U_s - 1}{s} \cdot X_t$$

$$\Big\downarrow s \to 0$$

$$\frac{U_t - 1}{t} \cdot 1$$

$$\Rightarrow \frac{U_t - 1}{t} = \left(\lim_{s\to 0} \frac{U_s - 1}{s} \right) \cdot X_t$$

$$\Rightarrow \underbrace{\lim_{s\to 0} \frac{U_s - 1}{s}}_{=: iA} = \frac{U_t - 1}{t} \cdot X_t^{-1} \quad \text{for all } 0 < t < t_0$$

$$\Rightarrow iA\, X_t = \frac{U_t - 1}{t}$$

$$\Rightarrow U_t - 1 = iA\, t X_t = iA \int_0^t U_\tau \, d\tau$$

$$\Rightarrow U_t = 1 + iA \int_0^t U_\tau \, d\tau$$

$$= 1 + iA \int_0^t \left[1 + iA \int_0^\tau U_\sigma \, d\sigma \right] d\tau$$

$$= 1 + iAt + (iA)^2 \underbrace{\iint_{0 \le \sigma \le \tau \le t} U_\sigma \, d\sigma \, d\tau}_{\frac{t^2}{2} + \iiint U_\nu \, d\nu \, d\sigma \, d\tau}$$

$$\overset{\text{iteration}}{\Rightarrow} U_t = \sum_{n=0}^\infty \frac{(iAt)^n}{n!} = e^{itA}$$

check that $U_t^* = U_{-t}$ implies $A = A^*$ $\square$

thus: $A = A^* \in B(\mathcal{H}) \longleftrightarrow$ norm continuous $\mathcal{U}_t = e^{itA}$ unitary group $(\mathcal{U}_t)$

<u>but:</u> norm continuous time evolutions, or bounded Hamiltonians, are not quite realistic

$\rightsquigarrow$ generalization to unbounded s.a generators

<u>6.3. Theorem:</u> Let T be an unbounded s.a operator and define

$$\mathcal{U}_t := e^{itT} \quad (t \in \mathbb{R})$$

via functional calculus, i.e.

$$T = \int \lambda \, dE(\lambda) \rightsquigarrow \mathcal{U}_t = \int e^{it\lambda} \, dE(\lambda).$$

Then we have:

i) $\mathcal{U}_t$ is unitary $\forall t \in \mathbb{R}$

ii) $\mathcal{U}_0 = 1$

iii) $\mathcal{U}_t \mathcal{U}_s = \mathcal{U}_{t+s} \quad \forall t, s \in \mathbb{R}$

iv) $\lim\limits_{t \to 0} \| \mathcal{U}_t x - x \| = 0 \quad \forall x \in \mathcal{H}$

<u>Proof:</u> Exercise $\square$

6.4. Def.: $(\mathcal{U}_t)_{t\in\mathbb{R}}$ with the properties (i) - (iv) is called a strongly continuous (one-parameter) unitary group.

6.5. Theorem of Stone: Let $(\mathcal{U}_t)_{t\in\mathbb{R}}$ be a strongly continuous unitary group. Then there exists a (uniquely determined) unbounded s.a. operator T s.th.

$$\mathcal{U}_t = e^{itT}$$

Proof: $T = \frac{1}{i} \frac{d}{dt} \mathcal{U}_t \Big|_{t=0}$

more precisely:

$$D(T) = \{ x \in \mathcal{H} \mid \underbrace{\lim_{t\to 0} \frac{\mathcal{U}_t x - x}{t}}_{=:\, Tx \text{ for } x\in D(T)} \text{ exists} \}$$

main problem: to see that $D(T)$ dense in $\mathcal{H}$

as before, put

$$X_t := \frac{1}{t} \int_0^t \mathcal{U}_\tau \, d\tau \qquad (t > 0)$$

then: $\| X_s y - y \| \to 0$ for $s \to 0$

and we still have (*), thus

$$\frac{U_t - 1}{t} \underbrace{X_s y}_{\xrightarrow{s \to 0} y} = \frac{U_s - 1}{s} \underbrace{X_t y}_{=: x}$$

$$\overset{s \searrow 0}{\Longrightarrow} \quad \frac{U_t - 1}{t} y = \lim_{s \to 0} \frac{U_s x - x}{s}$$

thus $\quad x = X_t y \in D(T) \quad \forall y \in \mathcal{H}, t > 0$

Since $\| X_t y - y \| \to 0$ for $t \to 0$,

we have: $\{ X_t y \mid y \in \mathcal{H}, t > 0 \}$ is

dense in $\mathcal{H}$

Now show that T is s.a.:

Consider $y \in D(T^*)$, i.e. there

is $z \in \mathcal{H}$ s.th. $\forall x \in D(T)$

$$\langle Tx, y \rangle = \langle x, z \rangle \qquad (z = T^* y)$$

$\shortparallel$

$$= \left\langle \frac{1}{i} \lim_{t \to 0} \frac{\mathcal{U}_t x - x}{t}, y \right\rangle$$

$$= i \lim_{t \to 0} \frac{1}{t} \left(\underbrace{\langle \mathcal{U}_t x, y \rangle}_{\langle x, \mathcal{U}_t^* y \rangle = \langle x, \mathcal{U}_{-t} y \rangle} - \langle x, y \rangle \right)$$

$$= -i \lim_{t \to 0} \frac{1}{-t} \langle x, \mathcal{U}_{-t} y - y \rangle$$

$$= \left\langle x, \frac{1}{i} \lim_{t \to 0} \frac{\mathcal{U}_{-t} y - y}{-t} \right\rangle$$

$$\overset{!}{=} \langle x, T^* y \rangle \qquad \forall x \in D(T) \underset{\text{dense}}{\subset} \mathcal{H}$$

$$\Rightarrow \lim_{t \to 0} \frac{\mathcal{U}_t y - y}{t} \text{ exists and } = i T^* y$$

thus: $y \in D(T)$ and $T^* y = T y$

$\Rightarrow T = T^*$

then easy to show that $\mathcal{U}_t = e^{itT}$

$\square$