7. CCR and Weyl relations

7.1. Def.: Two s.a. operators P and Q satisfy the <u>canonical commutation relations</u> (CCR), if they are defined on a dense $\mathcal{D} \subset \mathcal{H}$ s.th.

$$P : \mathcal{D} \to \mathcal{D} \quad , \quad Q : \mathcal{D} \to \mathcal{D} \quad \text{and}$$

$$[P, Q] := PQ - QP = -i \cdot 1 \quad \text{on } \mathcal{D}$$

$$(\text{or in physics } = -i\hbar 1)$$

7.2. Remarks: 1) As we know from 3.2 there are no bounded realizations of CCR.

2) If we put

$$a := \frac{Q + iP}{\sqrt{2}} \quad , \quad a^* = \frac{Q - iP}{\sqrt{2}}$$

then

$$[a, a^*] = \frac{1}{2}\Big(-i \underbrace{[Q, P]}_{i} + i \underbrace{[P, Q]}_{-i}\Big) = 1$$

thus: $\text{CCR} \iff [a, a^*] = 1$

3) There is also a "fermionic" analogue,

CAR = canonical anti-commutation relations

$$\{b, b^*\} := b b^* + b^* b = 1$$

Those have bounded realizations:

$$b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

7.3. Example: Schrödinger representation

$\mathcal{H} = L^2(\mathbb{R})$

$\mathcal{D} = \mathcal{S}(\mathbb{R})$ functions of rapid decrease

$$= \left\{ f \in L^2(\mathbb{R}) \;\middle|\; \lim_{|t| \to \infty} t^n \frac{d f}{d t^m} = 0 \quad \forall\, m, n = 0, 1, 2 \ldots \right\}$$

$$(Q f)(t) = t\, f(t)$$

$$(P f)(t) = \frac{1}{i} f'(t)$$

then: $Q : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$
$\quad\;\; P : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$

and Q, P are essentially s.a. on $\mathcal{S}(\mathbb{R})$

and they satisfy CCR:

$$(PQ - QP)(f) = -if \quad \forall f \in \mathcal{S}(\mathbb{R})$$

<u>Question:</u> Is this, up to unitary equivalence, the only irreducible representation of CCR?

<u>7.4. Remarks:</u> 1) Note that with

$$a = \frac{Q + iP}{\sqrt{2}}, \quad a^* = \frac{Q - iP}{\sqrt{2}}$$

we find a vector $\Omega \in D$ for which $a\Omega = 0$

namely:

$$0 = a\Omega = \frac{1}{\sqrt{2}}(Q + iP)\Omega$$

$$= \frac{1}{\sqrt{2}}(t\,\Omega(t) + \Omega'(t))$$

$$\Leftrightarrow \Omega'(t) = -t\,\Omega(t)$$

$$\Leftrightarrow \Omega(t) = c \cdot e^{-1/2 t^2}$$

[$\Omega \mathrel{\hat{=}}$ vacuum $\mathrel{\hat{=}}$ ground state of harmonic oscillator]

2) If we have an irreducible representation of CCR with such a vacuum vector Ω, i.e.

$$[a, a^*] = 1 \quad \text{and} \quad \underset{(\|\Omega\|)}{a\,\Omega = 0},$$

then everything is uniquely determined

Put $e_0 := \Omega$ and $e_n := a^{*n}\,\Omega$

then:
$$a^* e_n = e_{n+1} \qquad (n \geq 0)$$
$$a\, e_n = n\, e_{n-1} \qquad (n \geq 1)$$
$$a\, e_0 = 0$$

[$a\,e_n = a\,a^{*n}\,\Omega = \underbrace{a\,a^*}_{1+a^*a}\,\underbrace{a^{*(n-1)}\,\Omega}_{e_{n-1}}$

$= e_{n-1} + a^*\underbrace{a\,e_{n-1}}_{(n-1)\,e_{n-2} \text{ (via induction)}}$

$= n\,e_{n-1}$]

By irreducibility, $\{e_n\}$ span a dense subset of $\mathcal{H}$.

All inner products are also determined, as (for $n \geq m$)

$$\langle e_n, e_m \rangle = \langle a^{*n}\Omega, a^{*m}\Omega \rangle$$

$$= \langle \Omega, a^{n-m} \underbrace{a^m \underbrace{a^{*m}\Omega}_{e_m}}_{m!\,\Omega} \rangle$$

$$= \begin{cases} 0 & n > m \\ m! & n = m \end{cases}$$

thus: $e_n \perp e_m \quad n \neq m$

$$\|e_n\| = \sqrt{n!}$$

usually the e_n are normalized

$$f_n := \frac{1}{\sqrt{n!}} e_n \qquad \Rightarrow \{f_n\} \text{ ONB}$$

3) So if we are looking for irreducible representations of $[a, a^*] = 1$, then

$$\left.\begin{array}{l}\text{equivalence to} \\ \text{Schrödinger} \\ \text{representation}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\text{existence of} \\ \text{a vector } \Omega \\ \text{with } a\Omega = 0\end{array}\right.$$

4) Since in the form $[P, Q] = -i\mathbb{1}$ we are facing subtle problems around the domain of unbounded operator (and there are pathological examples), we rewrite our problem for the unitary groups corresponding to P and Q.

Consider the Schrödinger representation as in 7.3.

$$(Qf)(x) = x \cdot f(x), \quad (Pf)(x) = \frac{1}{i} f'(x)$$

and consider corresponding

$$\mathcal{U}_t := e^{itP}, \quad V_t := e^{itQ}.$$

Those act as

$$(\mathcal{U}_t f)(x) = f(x+t)$$

$$(V_t f)(x) = e^{itx} f(x)$$

and thus

$$(\mathcal{U}_t \cdot V_s f)(x) = (V_s f)(x+t)$$

$$= e^{is(x+t)} f(x+t)$$

and

$$(V_s \mathcal{U}_t f)(x) = e^{isx}(\mathcal{U}_t f)(x)$$
$$= e^{isx} f(x+t)$$

thus

$$\mathcal{U}_t \cdot V_s = e^{ist} V_s \mathcal{U}_t$$

<u>7.5. Def.</u>: 1) A pair $(\mathcal{U}_t, V_s)_{s,t \in \mathbb{R}}$ of strongly continuous unitary groups $(\mathcal{U}_t)_{t\in\mathbb{R}}$ and $(V_s)_{s\in\mathbb{R}}$ (on the same Hilbert space $\mathcal{H}$) is a representation of the <u>Weyl relations</u> if we have

$$\mathcal{U}_t V_s = e^{its} V_s \mathcal{U}_t \qquad \forall s,t \in \mathbb{R}$$

2) The representation is <u>irreducible</u>, if there is no non-trivial sub-Hilbert space $\mathcal{K} \subset \mathcal{H}$ (i.e., $\{0\} \neq \mathcal{K} \neq \mathcal{H}$) for which

$$\left.\begin{array}{l} \mathcal{U}_t \mathcal{K} \subset \mathcal{K} \\ V_s \mathcal{K} \subset \mathcal{K} \end{array}\right. \quad \forall s,t \in \mathbb{R}$$

3) Two representations $(\mathcal{U}_t, V_s)$ and $(\mathcal{U}_t', V_s')$ (on $\mathcal{H}$ and $\mathcal{H}'$) of the Weyl relations are <u>unitarily equivalent</u>, if there is a unitary operator $W : \mathcal{H} \to \mathcal{H}'$ s.th.

$$\mathcal{U}_t = W^* \mathcal{U}_t' W \qquad \forall\, s,t \in \mathbb{R}$$
$$V_s = W^* V_s' W$$

$$\begin{tikzcd} \mathcal{H} \arrow[r, "W"] \arrow[d, "\mathcal{U}_t \; V_s"'] & \mathcal{H}' \arrow[d, "\mathcal{U}_t' \; V_s'"] \\ \mathcal{H} & \mathcal{H}' \arrow[l, "W^*"'] \end{tikzcd}$$

<u>7.6. Proposition:</u> The Schrödinger representation of the Weyl relations on $\mathcal{H} = L^2(\mathbb{R})$ by

$$(\mathcal{U}_t f)(x) = f(x+t)$$
$$(V_t f)(x) = e^{itx} f(x)$$

is irreducible.

<u>Proof</u>: Assume there is a non-trivial invariant subspace $\mathcal{K} \subset \mathcal{H}$

with $\mathcal{U}_t \mathcal{K} \subset \mathcal{K}$, $V_s \mathcal{K} \subset \mathcal{K}$ $\forall s,t$

Then we have

$0 \neq f \in \mathcal{K} \neq \{0\}$

$0 \neq g \in \mathcal{K}^\perp \neq \{0\}$

and thus

$$\langle g, \underbrace{V_t \underbrace{\mathcal{U}_s f}_{\in \mathcal{K}}}_{\in \mathcal{K}} \rangle = 0 \quad \forall s,t$$

i.e., $$\underbrace{\int \overline{g(x)}\, e^{itx} f(x+s)\, dx}_{\text{Fourier transform } \hat{h}_s(t)} = 0 \quad \forall s,t$$

of $h_s(x) := \overline{g(x)}\, f(x+s)$

thus: $\hat{h}_s = 0 \implies h_s = 0 \quad \forall s$

$\implies g = 0$ or $f = 0$ contradiction