8. The Stone-von Neumann uniqueness theorem

8.1. Uniqueness Theorem: Each representation of the Weyl relations is unitarily equivalent to a (at most countable) direct sum of Schrödinger representations.
In particular: Each irreducible representation of the Weyl relations is unitarily equivalent to the Schrödinger representation.

8.2. Remarks: 1) "Beweisansätze" (first ideas for proof) in Stone, Proc. Nat. Acad. 1930

2) Detailed rigorous proof by von Neumann, Math. Ann. 1931

3) Idea of proof: Find vacuum Ω in representation and reconstruct from this everything!

To find Ω, we need projection P onto Ω and P must be constructed from our operators U_t, V_s.

In the following we consider a fixed representation (U_t, V_s) (on $\mathcal{H}$) of the Weyl relations.

8.3. Notations: 1) For $s, t \in \mathbb{R}$ we put

$$W(s,t) := e^{-\frac{1}{2} i s t} U_s V_t = e^{+\frac{1}{2} i s t} V_t U_s$$

2) For $h \in L^1(\mathbb{R}^2)$, i.e.,

$$\iint |h(s,t)| \, ds dt < \infty,$$

we put

$$W_h := \int h(s,t) W(s,t) \, ds dt \in B(\mathcal{H}).$$

This is rigorously defined by

$$\langle f, W_h g \rangle = \int h(s,t) \langle f, W(s,t) g \rangle \, ds dt$$

$$\forall f, g \in \mathcal{H}$$

<u>8.4. Proposition:</u> 1) All $W(s,t)$ are unitary and we have for all $s_1, s_2, t_1, t_2 \in \mathbb{R}$:

$$W(s_1,t_1)W(s_2,t_2) = W(s_1+s_2, t_1+t_2) \cdot e^{\frac{1}{2}i(s_1t_2 - s_2t_1)};$$

in particular: $W(s,t)^* = W(-s,-t)$

$$W(0,0) = 1$$

2) The map $h \mapsto W_h$ is injective, i.e. $W_h \neq 0$ for $h \neq 0$

3) For $h_1, h_2 \in L^1(\mathbb{R}^2)$ we have

$$W_{h_1} \cdot W_{h_2} = W_h$$

where $h \in L^1(\mathbb{R}^2)$ is given by

$$h(s,t) = \int h_1(s-\tilde{s}, t-\tilde{t})\, h_2(\tilde{s},\tilde{t}) \cdot e^{1/2\, i(s\tilde{t} - \tilde{s}t)}\, d\tilde{s}\, d\tilde{t}$$

<u>Proof:</u> 1) easy calculation

2) Assume that $W_h = 0$

$\Rightarrow W(-x,-y)\, W_h\, W(x,y) = 0 \quad \forall x,y \in \mathbb{R}$

i.e. $0 = W(-x,-y) \int h(s,t) W(s,t)\, ds\, dt \cdot W(x,y)$

$$= \int h(s,t)\, e^{\frac{1}{2}i(-xt+ys)} \underbrace{W(s-x,t-y)\, W(x,y)}_{W(s,t)\, e^{\frac{1}{2}i[(s-x)y-(t-y)x]}}\, ds\, dt$$

$$= \int h(s,t)\, e^{i(sy-tx)}\, W(s,t)\, ds\, dt$$

thus $\forall x,y \in \mathbb{R}\ \forall f,g \in \mathcal{H}$

$$0 = \underbrace{\int e^{isy} e^{-itx}\, h(s,t) \langle f, W(s,t) g\rangle\, ds\, dt}_{\text{Fourier transform at } (y,-x) \text{ of function}}$$

$$(s,t) \longmapsto h(s,t) \langle f, W(s,t) g\rangle$$

injectivity of Fourier transform $\Longrightarrow$ $h(s,t) \langle f, W(s,t) g\rangle = 0$ a.e.

$$\Rightarrow h(s,t) \underbrace{W(s,t) g}_{\neq 0 \text{ for } g \neq 0} = 0 \quad \text{a.e. } \forall g$$

$$\Rightarrow h = 0 \quad \text{a.e.}$$

3) direct calculation

$\square$

8.5. Remark: We will claim now that

$$P := \frac{1}{2\pi} \int W(s,t)\, e^{-1/4(s^2+t^2)}\, ds\, dt$$

gives the projection onto the vaccum.
Let us check that this is right choice in the Schrödinger representation.
Recall that there the vacuum is given by

$$\Omega(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2} \qquad \text{(see 7.4)}$$

In the Schrödinger representation we have

$$(W(s,t) f)(x) = e^{\frac{1}{2} ist} e^{itx} f(x+s)$$

and thus

$$(Pf)(x) = \frac{1}{2\pi} \int e^{\frac{1}{2}ist} e^{itx} f(x+s)\, e^{-\frac{1}{4}(s^2+t^2)}\, ds\, dt$$

$$= \dots$$

$$= \frac{1}{\sqrt{\pi}} \left(\int f(\tilde{s})\, e^{-\frac{1}{2}\tilde{s}^2}\, d\tilde{s} \right) e^{-\frac{1}{2}x^2}$$

$$= \langle \Omega, f \rangle \Omega$$

so indeed ...

Proof of 8.1. : Put

$$P = \frac{1}{2\pi} \int W(s,t)\, e^{-1/4(s^2+t^2)}\, ds\,dt$$

$$= W_h$$

for $h(s,t) = \frac{1}{2\pi} e^{-1/4(s^2+t^2)}$

Claim: P is orthogonal projection, i.e.

i) $P^* = P$ and ii) $P^2 = P$

$$(i)\quad P^* = \frac{1}{2\pi} \int \underbrace{W(s,t)^*}_{W(-s,-t)}\, e^{-1/4(s^2+t^2)}\, ds\,dt$$

$$= P$$

For (ii) one calculates more generally

iii) $P\,W(x,y)\,P = e^{-1/4(x^2+y^2)}\, P \quad \forall\, x,y \in \mathbb{R}$

by using 8.4.

then: (ii) for $x = 0 = y$ we get

$$P^2 = P\,W(0,0)\,P = P$$

Since $h \neq 0$ we have $P = W_h \neq 0$,

thus $P\mathcal{H} \neq \{0\}$

Choose an ONB $(\Omega_n)_{n=1}^{N}$ of $P\mathcal{H}$

(where $N \in \mathbb{N}$ or $N = \infty$ countable)

and put for $n = 1, \dots, N$

$$\mathcal{H}_n := \overline{\operatorname{span}\{W(s,t)\Omega_n \mid s,t \in \mathbb{R}\}}$$

Then we have

a) each $\mathcal{H}_n$ invariant subspace for the $W(x,y)$

b) $\mathcal{H}_n \perp \mathcal{H}_m$ for $n \neq m$

c) $\bigoplus_{n=1}^{N} \mathcal{H}_n = \mathcal{H}$

d) in particular: if representation is irreducible $\Rightarrow N = 1$, $\mathcal{H}_1 = \mathcal{H}$

a) is clear, since

$$\underbrace{W(x,y)\,W(s,t)\,\Omega_n}_{e^{\cdots}\,W(x+s,\,y+t)\,\Omega_n \,\in\, \mathcal{H}_n}$$

b) $\langle W(x,y)\overset{P}{\Omega_n}, W(s,t)\overset{P}{\Omega_m}\rangle =$

$= \langle \Omega_n, \underbrace{P\, W(s,t)\, P}_{= \lambda \cdot P \text{ for } \lambda \in \mathbb{C}, \text{ by (iii)}} \Omega_m \rangle$

$= \lambda \underbrace{\langle \Omega_n, \Omega_m \rangle}_{= 0 \text{ for } n \neq m}$ since $P\Omega_m = \Omega_m$

c) Put $\mathcal{K} := \bigoplus_{n=1}^{N} \mathcal{H}_n$

Then $\{W(s,t)\}|_{\mathcal{K}^\perp}$ gives a representation of Weyl relations and the projection for this would be $P|_{\mathcal{K}^\perp}$, which cannot be zero, thus there is

$0 \neq f \in \mathcal{K}^\perp$ with $\underset{\underset{\mathcal{K}}{\cap}}{Pf} = f \in \mathcal{K}^\perp$

$\Rightarrow \mathcal{K}^\perp = \{0\}$

$\mathcal{K} = \mathcal{H}$

<u>thus:</u> each representation of Weyl relations

$\hat{=} \bigoplus$ (representation on $\mathcal{H}_n = \overline{\mathrm{span}}\{W(s,t)\Omega_n\}$)

$\nearrow$

it remains to see that each of them is $\hat{=}$ Schrödinger representation

So fix n and put $\Omega = \Omega_n$

For $s,t \in \mathbb{R}$ we put

$$f_{s,t} := W(s,t)\,\Omega$$

Then we have

$$W(x,y) f_{s,t} = \underbrace{W(x,y)\, W(s,t)}_{e^{i/2(xt-ys)}\, W(x+s,y+t)} \Omega$$

$$= e^{i/2(xt-ys)} f_{x+s,y+t}$$

and

$$\langle f_{x,y}, f_{s,t} \rangle = \langle W(x,y)\Omega, W(s,t)\Omega \rangle$$

$$= \langle \Omega, \underbrace{W(-x,-y)\, W(s,t)}_{e^{\frac{1}{2}i[-xs+yt]}\, W(s-x,t-y)} \Omega \rangle$$

$$= e^{\frac{1}{2}i[-xs+yt]}\, e^{-\frac{1}{4}((s-x)^2+(t-y)^2)}$$

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$$= e^{\frac{1}{2} i [-xs + yt]} e^{-\frac{1}{4}[(s-x)^2 + (t-y)^2]} \underbrace{\langle \Omega, \underbrace{P\Omega}_{\Omega} \rangle}_{= 1}$$

Thus all inner product and actions of $W(x,y)$ is uniquely determined:
If we have two such representations

$$\Omega, W(s,t), \mathcal{H} \text{ and } \Omega', W'(s,t), \mathcal{H}'$$

then the mapping

$$f_{s,t} = W(s,t)\Omega \mapsto f'_{s,t} = W'(s,t)\Omega'$$

extends to a unitary map $\mathcal{H} \to \mathcal{H}'$ which intertwines the action of $W(s,t)$ and of $W'(s,t)$.
Thus all irreducible representations are unitarily equivalent, in particular equivalent to the Schrödinger representation.

$\square$

8.6 Remark: 1) The theorem and the proof can be extended to a finite ~~number of~~ $n<\infty$ degrees of freedom.

CCR: $[P_i, Q_j] = -i\,\delta_{ij}$ $\qquad i,j = 1,\dots,n$

$$[Q_i, Q_j] = 0 = [P_i, P_j]$$

Weyl relations: $U_i(t) = e^{itP_i}$

$$V_j(s) = e^{isQ_j}$$

$$U_i(t)\,U_j(s) = U_j(s)\,U_i(t)$$

$$V_i(t)\,V_j(s) = V_j(s)\,V_i(t)$$

$$U_i(t)\,V_j(s) = e^{i\delta_{ij}st}\,V_j(s)\,U_i(t)$$

Schrödinger representation on $L^2(\mathbb{R}^n)$:

$Q_j \hat{=}$ multiplication with x_j

$P_i \hat{=}$ derivative with respect to x_i

Uniqueness Theorem: Each irreducible representation of Weyl relations is unitarily equivalent to Schrödinger representation

2) <u>But</u>: proof breaks down for $n = \infty$ [as there is no Gaussian measure on $\mathbb{R}^\infty$] and theorem does <u>not</u> hold anymore!