9. Symmetric Fock space and second quantization

We want to generalize the Schrödinger representation of CCR (or Weyl relations) to infinite number of degrees

We try to generalize the "harmonic oscillator" version

$[a, a^*] = 1$, compare 7.4.

[recall that we could "create" the Hilbert space there from the vacuum Ω by iterated application of a^*]

9.1. Remark: So our Hilbert space should contain

- a vacuum Ω field mode
- one particle elements $f \in \mathcal{H}$
- n-particle elements

$$f_1 \otimes \ldots \otimes f_n \in \mathcal{H}^{\otimes n}$$

- combinations of different numbers of particles $\in \underbrace{\bigoplus_n \mathcal{H}^{\otimes n}}_{\text{full Fock space}}$

Since $a_i^* a_j^* = a_j^* a_i^*$, we have to consider the symmetrized bosonic Fock space, where

$$f \otimes g = g \otimes f$$

<u>9.2. Def.:</u> 1) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. The (Hilbert space) <u>tensor product</u> $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of

$$\mathcal{H}_1 \odot \mathcal{H}_2 := \left\{ \sum_{\text{finite}} x_i \otimes y_i \;\middle|\; \begin{array}{l} x_i \in \mathcal{H}_1 \\ y_i \in \mathcal{H}_2 \end{array} \right\}$$

with respect to inner product given by linear extension of

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle := \langle x_1, x_2 \rangle \cdot \langle y_1, y_2 \rangle$$

We write $\mathcal{H} \otimes \mathcal{H} = \mathcal{H}^{\otimes 2}$

By iteration this extends to a finite number n of factors

$$\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n \quad \text{and} \quad \mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \ldots \otimes \mathcal{H}}_{n \text{ times}}$$

2) Let $\mathcal{H}_i$ $(i \in \mathbb{N})$ be Hilbert spaces. The <u>direct sum</u> of the $\mathcal{H}_i$ is the Hilbert space

$$\bigoplus_{i \in \mathbb{N}} \mathcal{H}_i := \Big\{ (x_1, x_2, \ldots) \;\Big|\; x_i \in \mathcal{H}_i, \ \sum_{i \in \mathbb{N}} \|x_i\|^2 < \infty \Big\}$$

$$(x_1, x_2, \ldots) \longleftrightarrow x_1 \oplus x_2 \oplus \ldots$$

Note that the inner product is determined by the requirement that $\mathcal{H}_i \perp \mathcal{H}_j$ for $i \neq j$, when we identify

$$\mathcal{H}_i \ni x \longleftrightarrow (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ i}}{x}, 0, 0, \ldots)$$

<u>9.3. Remarks:</u> 1) If $\{e_i\}_{i=1}^{d}$ is an ONB of $\mathcal{H}$, with $d \in \mathbb{N} \cup \{\infty\}$, then

$\{ e_{i(1)} \otimes \ldots \otimes e_{i(n)} \mid 1 \leq i(1), \ldots, i(n) \leq d \}$ is

ONB of $\mathcal{H}^{\otimes n}$. In particular,

$$\dim \mathcal{H}^{\otimes n} = (\dim \mathcal{H})^n$$

2) Since $\mathcal{H}_i$ has a unique zero 0, we can embedd direct sums into each other

$$\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n \cong \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n \oplus 0$$
$$\subseteq \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n \oplus \mathcal{H}_{n+1}$$

and the infinite direct sum can be seen as completion of all finite ones

$$\bigoplus_{i \in \mathbb{N}} \mathcal{H}_i = \overline{\bigcup_{n \in \mathbb{N}} \bigoplus_{i=1}^{n} \mathcal{H}_i}$$

There is <u>no</u> canonical one 1 in a Hilbert space, hence tensor products of Hilbert spaces cannot be embedded into each other in a clear way and the notion of an infinite tensor product is more subtle.

9.4. Theorem: 1) Let $\sigma \in S_n$ be a permutation of $\{1, \dots, n\}$. Then we can extend

$$\mathcal{U}_\sigma \, x_1 \otimes \dots \otimes x_n = x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$$

uniquely to an operator

$$\mathcal{U}_\sigma : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$$

$\mathcal{U}_\sigma$ is unitary and we have

- $\mathcal{U}_{\sigma\tau} = \mathcal{U}_\sigma \mathcal{U}_\tau$ for all $\sigma, \tau \in S_n$
- $\mathcal{U}_{id} = 1$ for the identity permutation id
- $\mathcal{U}_\sigma^* = \mathcal{U}_{\sigma^{-1}}$

2) The operator

$$P_n := \frac{1}{n!} \sum_{\sigma \in S_n} \mathcal{U}_\sigma$$

is an orthogonal projection on $\mathcal{H}$ and projects onto the symmetric tensors, i.e.,

$$x \in P_n(\mathcal{H}^{\otimes n}) \iff \mathcal{U}_\sigma x = x \quad \forall \sigma \in S_n$$

<u>Proof</u>: 1) Note that

$$\sum_i x_1^{(i)} \otimes \dots \otimes x_n^{(i)} = 0 \Rightarrow \sum_i x_{\sigma(1)}^{(i)} \otimes \dots \otimes x_{\sigma(n)}^{(i)} = 0$$

and that, for an ONB $\{e_i\}_{i=1}^d$ of $\mathcal{H}$, U_σ maps the ONB $\{e_{i(1)} \otimes \dots \otimes e_{i(d)} \mid \dots\}$ of $\mathcal{H}^{\otimes n}$ bijectively onto itself, thus U_σ is unitary

2) $$P_n^* = \frac{1}{n!} \sum_{\sigma \in S_n} \underbrace{U_\sigma^*}_{U_{\sigma^{-1}}} = P_n$$

$$P_n^2 = \frac{1}{n!\,n!} \underbrace{\sum_{\sigma \in S_n} \sum_{\tau \in S_n} \underbrace{U_\sigma U_\tau}_{U_{\sigma\tau}}}_{= \sum_{\pi \in S_n} U_\pi}$$

$$= \frac{1}{n!\,n!}\, n! \sum_{\pi \in S_n} U_\pi = P_n$$

Suppose $U_\sigma x = x \;\; \forall \sigma \in S_n$

$$\Rightarrow P_n x = \frac{1}{n!} \sum_\sigma \underbrace{U_\sigma x}_{x} = x$$

Suppose $P_n x = x$

$$\Rightarrow U_\sigma x = \overbrace{U_\sigma P_n}^{P_n} x = P_n x = x \qquad \square$$

<u>9.5. Def.</u>: Let $\mathcal{H}$ be a Hilbert space.

The <u>symmetric</u> (or <u>bosonic</u>) <u>Fock space</u> over $\mathcal{H}$ is the Hilbert space

$$\mathcal{F}_+(\mathcal{H}) = \mathcal{F}_{sym}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \underbrace{P_n \mathcal{H}^{\otimes n}}_{\mathcal{H}_+^{\otimes n} = \mathcal{H}^{\odot}}$$

where $P_0 \mathcal{H}^{\otimes 0} = \mathcal{H}^{\otimes 0} = \mathbb{C}$.

We call

$\Omega := (1, 0, 0, \ldots) \in \mathcal{H}^{\otimes 0}$ vacuum

$\mathcal{H} \mathrel{\hat{=}}$ one-particle space

$\mathcal{H}_+^{\otimes n} = P_n \mathcal{H}^{\otimes n}$ n-particle space (symmetric)

<u>9.6. Example:</u> If $\mathcal{H} = L^2(\mathbb{R})$, then

$$\mathcal{H}^{\otimes n} \cong L^2(\mathbb{R}^n)$$

and

$$\mathcal{H}_+^{\otimes n} \cong \{ f \in L^2(\mathbb{R}^n) \mid f \text{ symmetric} \}$$

i.e., $f(t_1, \ldots, t_n) = f(t_{\sigma(1)}, \ldots, t_{\sigma(n)})$ $\forall \sigma \in S_n$

9.7. Remark: Let $T \in B(\mathcal{H})$, then $T^{\otimes n}$ on $\mathcal{H}^{\otimes n}$ is defined by linear extension of

$$T^{\otimes n} x_1 \otimes \dots \otimes x_n = (Tx_1) \otimes (Tx_2) \otimes \dots \otimes (Tx_n)$$

Since

$$\mathcal{U}_\sigma T^{\otimes n} = T^{\otimes n} \mathcal{U}_\sigma \quad \forall \sigma \in S_n$$

we have for $x \in \mathcal{H}_+^{\otimes n}$ (i.e. $P_n x = x$):

$$T^{\otimes n} x = T^{\otimes n} \mathcal{U}_\sigma x \quad \forall \sigma \in S_n$$

$$= \mathcal{U}_\sigma T^{\otimes n} x \quad \text{—}\,''\,\text{—}$$

$$\Rightarrow P_n T^{\otimes n} x = T^{\otimes n} x$$

i.e. $T^{\otimes n}(\mathcal{H}_+^{\otimes n}) \subset \mathcal{H}_+^{\otimes n}$

and we can put

$$\Gamma_+^n(T) := T^{\otimes n}\big|_{\mathcal{H}_+^{\otimes n}}$$

Note:

- $\Gamma_+^n(1) = 1$
- $\Gamma_+^n(T)^* = \Gamma_+^n(T^*)$
- $\Gamma_+^n(ST) = \Gamma_+^n(S)\,\Gamma_+^n(T)$

but in general

$$\Gamma_+^n(S+T) \neq \Gamma_+^{(n)}(S) + \Gamma_+^{(n)}(T)$$

$$- \ \| \Gamma_+^n(T) \| = \| T \|^n \quad \leftarrow \text{Exercise}$$

Thus $\| \Gamma_+^n(T) \|$ stays bounded for all n, if and only if $\| T \| \leq 1$

<u>9.8. Def</u>: Let $T \in B(\mathcal{H})$ be a contraction, i.e. $\| T \| \leq 1$. Then the <u>(symmetric) second quantization</u> $\Gamma_+(T)$ of T is defined as

$$\Gamma_+(T) := \bigoplus_{n=0}^{\infty} \Gamma_+^n(T) \in B(\mathcal{F}_+(\mathcal{H}))$$

[Nelson: First quantization is a mystery, but second quantization is a functor.]

<u>9.9. Remark</u>: Let $U_t = e^{iHt}$ be a unitary group on $\mathcal{H}$. Then $\Gamma_+^n(U_t)$ is a unitary group on $\mathcal{H}_+^{\otimes n}$ with generator

$$d\Gamma_+^n(H) = H \otimes 1 \otimes \dots + \otimes 1 + 1 \otimes H \otimes \dots \otimes 1 + \dots$$
$$\dots + 1 \otimes 1 \otimes \dots \otimes H$$

thus

$$\Gamma_+^n(e^{iHt}) = e^{i\, d\Gamma_+^n(H)\cdot t}$$

and with

$$d\Gamma_+(H) = \bigoplus_{n=0}^{\infty} d\Gamma_+^n(H)$$

on the symmetric Fock space

$$\Gamma_+(e^{iHt}) = e^{i\, d\Gamma_+(H)\cdot t}.$$

$d\Gamma_+(H)$ is called <u>differential second quantization</u> of H.

<u>9.10. Notation:</u> For $x_1, \dots, x_n \in \mathcal{H}$ we put

$$x_1 \circ x_2 \circ \dots \circ x_n := \sqrt{n!}\; P_n(x_1 \otimes x_2 \otimes \dots \otimes x_n) \in \mathcal{H}_+^{\otimes n}$$

<u>9.11. Remarks:</u> Note

1) $x_{\sigma(1)} \circ \dots \circ x_{\sigma(n)} = x_1 \circ \dots \circ x_2 \quad \forall \sigma \in S_n$

2) If $\{e_i\}_{i\in I}$ is an ONB of $\mathcal{H}$, then

$\{e_{i(1)} \circ \dots \circ e_{i(n)} \mid i(1) \le i(2) \le \dots \le i(n)\}$

is a basis of $\mathcal{H}_+^{\otimes n}$ which is orthogonal, but <u>not</u> orthonormal

$$\langle e_{i(1)} \circ \ldots \circ e_{i(n)}, e_{j(1)} \circ \ldots \circ e_{j(n)} \rangle =$$

$$= n! \left\langle \frac{1}{n!} \sum_{\sigma \in S_n} e_{i(\sigma(1))} \otimes \ldots \otimes e_{i(\sigma(n))}, \right.$$

$$\left. \frac{1}{n!} \sum_{\pi \in S_n} e_{j(\pi(1))} \otimes \ldots \otimes e_{j(\pi(n))} \right\rangle$$

$$= \frac{1}{n!} \sum_{\sigma, \pi \in S_n} \langle e_{i(\sigma(1))}, e_{j(\pi(1))} \rangle \cdots$$

$$\cdots \langle e_{i(\sigma(n))}, e_{j(\pi(n))} \rangle$$

$$= \begin{cases} 0 & \text{if } (i(1), \ldots, i(n)) \neq (j(1), \ldots, j(n)) \\ \sum_{\tau \in S_n} \langle e_{i(1)}, e_{i(\tau(1))} \rangle \cdots \langle e_{i(n)}, e_{i(\tau(n))} \rangle & \text{if } (i(1), \ldots, i(n)) = (j(1), \ldots, j(n)) \end{cases}$$

e.g.: $\langle e_1 \circ e_2 \circ \ldots \circ e_n, e_1 \circ e_2 \circ \ldots \circ e_n \rangle = 1$

but

$$\langle \underbrace{e_1 \circ e_1 \circ \ldots \circ e_1}_{}, e_1 \circ e_1 \circ \ldots \circ e_1 \rangle = n!$$

note: those correspond to the e_n in 7.4

this notation has advantage that creation and annihilation operators have nice form

9.12. Def.: For $f \in \mathcal{H}$ we define the annihilation operator $A(f)$ and the creation operator $A^+(f)$ on the domain

$$D := \{ \bigoplus_{n=0}^{\infty} f^{(n)} \in \mathcal{F}_+(\mathcal{H}) \mid \sum_{n=0}^{\infty} n \|f^{(n)}\|^2 < \infty \}$$

by linear extension of

$$A(f)\,\Omega = 0$$

$$A(f)\, f_1 \circ \dots \circ f_n = \sum_{k=1}^{n} \langle f, f_k \rangle \cdot f_1 \circ \dots \circ f_{k-1} \circ f_{k+1} \circ \dots \circ f_n$$

and

$$A^+(f)\, f_1 \circ \dots \circ f_n = f \circ f_1 \circ \dots \circ f_n$$

9.13. Theorem: 1) $A(f)$ and $A^+(f)$ are closeable and adjoints of each other.

2) $A(f)$ and $A^+(f)$ leave D invariant

$$A(f) : D \to D$$

$$A^+(f) : D \to D$$

3) On $\mathcal{D}$ we have CCR: for all $f, g \in \mathcal{H}$ we have:

$$[A(f), A(g)] = 0$$

$$[A^*(f), A^*(g)] = 0$$

$$[A(f), A^*(g)] = \langle f, g \rangle \cdot \mathbb{1}$$

<u>9.14. Remarks:</u> Let $\{e_i\}_{i \in I}$ be an ONB of $\mathcal{H}$. Then we put

$$A_i := A(e_i), \quad A_i^* = A^*(e_i)$$

and

$$Q_i := \frac{A_i + A_i^*}{\sqrt{2}}, \quad P_i := \frac{A_i - A_i^*}{i \cdot \sqrt{2}}$$

The Q_i and P_i are then essentially selfadjoint.

We have then

$$[A_i, A_j] = 0$$

$$[A_i^*, A_j^*] = 0$$

$$[A_i, A_j^*] = \delta_{ij} \cdot \mathbb{1}$$

and

$$[Q_i, Q_j] = 0$$

$$[P_i, P_j] = 0$$

$$[P_i, Q_j] = -i\,\delta_{ij}$$

If we consider the corresponding unitary groups

$$U_k(t) := e^{iP_k t} \quad \text{and} \quad V_k(t) := e^{iQ_k t}$$

then they satisfy the Weyl relations

$$U_k(t)\,V_\ell(s) = V_\ell(s)\,U_k(t)\,e^{i\delta_{k\ell} st}$$

For $\dim \mathcal{H} = \infty = |I|$ we get thus a realization of the ∞-dimensional version of the Weyl relations.