

# Mathematical Aspects of Quantum Mechanics [1]

mathematical physics  $\rightarrow$  quantum mechanics  
functional analysis  
(of operators on Hilbert spaces)

centers around problems of

CCR  $\hat{=}$  canonical commutation relations

$$pq - qp = \begin{matrix} \uparrow & \uparrow \\ p & q \end{matrix} = -i\hbar \cdot 1 \quad \begin{matrix} \text{one degree} \\ \text{of freedom} \end{matrix}$$

position      momentum

$$[p_i, q_j] = -i\hbar \cdot \delta_{ij} \quad \begin{matrix} \text{several degrees} \\ \text{(finite } \rightarrow \text{ infinite)} \end{matrix}$$

○ I usual math. setting of QM:

observable  $\hat{=}$  selfadjoint operators

I will recall basic theory of Hilbert spaces and bounded operators

i) there is no realization of CCR by bounded operators

$\rightarrow$  need for unbounded operators

ii) unbounded operator

$$A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H}$$

is not defined everywhere, but comes with its own domain

(and has some kind of "continuity" properties there)

we have now to distinguish

symmetric  $\longleftrightarrow$  selfadjoint  
(hermitean)

$$\langle Ax, y \rangle = \langle x, Ay \rangle \longleftrightarrow A = A^*$$

$$\forall x, y \in \mathcal{D}(A)$$

$$\text{i.e. } \mathcal{D}(A) = \mathcal{D}(A^*)$$

(much stronger)

hard to check

easy to check

formal operators in physics are of this form

not much to say

there are strong mathematical results

- spectral theorem

- Theorem of Stone

iii) spectral theorem

$$A = \int \lambda dE(\lambda)$$

$\hat{=}$  possible values of observable  $A$  in measurement

$$i\hbar) U(t)U(s) = U(t+s) \left. \begin{array}{l} \text{Stone} \\ \leftarrow \right\} \end{array} \right\} U(t) = e^{itH} \quad [3]$$

+ continuity

$H$  selfadjoint

$\uparrow$   
Hamilton operator

$\uparrow$   
math. description of  
time evolution in  
Schrödinger picture

$$\Psi_t = U(t)\Psi$$

$$\frac{d}{dt} U(t) = iH U(t)$$

Schrödinger equation

v) symmetric  $\rightsquigarrow$  selfadjoint extensions  
"boundary conditions"

There are cases with  $\infty$ -many extensions  
no extension

general theory of "defect indices"  
by von Neumann

## II Realizations of CCR (finite degree) 4

$$pq - qp = -i\hbar \cdot 1 \rightarrow U(t) = e^{itp}$$

$$V(t) = e^{isq}$$

$$U(t)V(s) = e^{its} V(s) U(t)$$

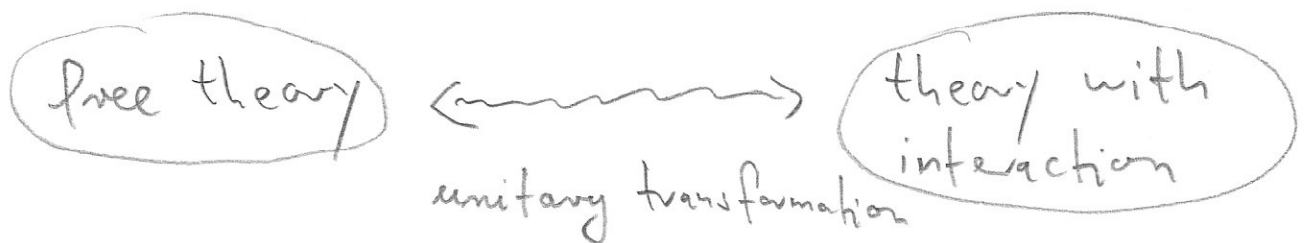
CCR  $\cong$  "Weyl relations"

Uniqueness Theorem of von Neumann:

each representation of Weyl relations is equivalent to  $\oplus$  (Schrödinger representation of CCR)

i.e. two representations of CCR are related by unitary transformation of coordinate system

$\rightarrow$  we can work with fixed Hilbert space



$\uparrow$   $\rightarrow$   
both "live" in the same Hilbert space  
with "universal"  $p, q$

but: this is only true for finite degree of freedom

### III infinite degree of freedom [quantum field theory] <sup>15</sup>

different physical theories correspond to non-equivalent representations of CCR

↳ there are many!

→ two such theories cannot be connected via a unitary transformation (e.g. via perturbation theory)

→ problem of renormalization

there is not a preferred choice of representation of CCR, but has to be chosen according to considered problem

→ algebraic point of view

$C^*(\text{CCR})$

$C^*$ -algebra

encodes algebraic properties of CCR

physics → state on CCR  $\xrightarrow{\text{GNS}}$  realization on Hilbert space