Mathematical Aspects of Quantum Mechanics

mathematical physics → quantum mechanics
functional analysis
(of operators on
Hilbert spaces)

centers around problems of

CCR ≙ canonical commutation relations

$$pq - qp = [\underset{\substack{\uparrow \\ \text{position}}}{p}, \underset{\substack{\uparrow \\ \text{momentum}}}{q}] = -i\hbar \cdot 1$$ one degree of freedom

$$[p_i, q_j] = -i\hbar \cdot \delta_{ij}$$ several degrees (finite ⇔ infinite)

I usual math. setting of QM:
observable ≙ selfadjoint operators

I will recall basic theory of Hilbert spaces and bounded operators

i) there is no realization of CCR by bounded operators
→ need for unbounded operators

ii) unbounded operator

$A : D(A) \subset \mathcal{H} \to \mathcal{H}$

is not defined everywhere, but comes with its own domain (and has some kind of "continuity" properties there)

we have now to distinguish

symmetric (hermitean)	$\longleftrightarrow$	selfadjoint
$\langle Ax, y \rangle = \langle x, Ay \rangle$ $\forall x, y \in D(A)$	$\Longleftrightarrow$	$A = A^*$ i.e. $D(A) = D(A^*)$ (much stronger)
easy to check		hard to check
formal operators in physics are of this form		
not much to say		there are strong mathematical results - spectral theorem - Theorem of Stone

iii) spectral theorem

$A = \int \lambda \, dE(\lambda)$

$\uparrow$ possible values of observable A in measurement

iv) $\mathcal{U}(t)\,\mathcal{U}(s) = \mathcal{U}(t+s)$ + continuity $\overset{\text{Stone}}{\Longleftarrow}$ $\mathcal{U}(t) = e^{itH}$

math. description of time evolution in Schrödinger picture

$\psi_t = \mathcal{U}(t)\,\psi$

H selfadjoint

Hamilton operator

$$\frac{d}{dt}\,\mathcal{U}(t) = i\,H\,\mathcal{U}(t)$$

Schrödinger equation

v) symmetric $\rightsquigarrow$ selfadjoint extensions

"boundary conditions"

There are cases with ∞-many extensions

no extension

general theory of "defect indices" by von Neumann

II Realisations of CCR (finite degree)

$$pq - qp = -i\hbar \cdot 1 \rightsquigarrow \quad \mathcal{U}(t) = e^{itp}$$
$$V(t) = e^{isq}$$

$$\mathcal{U}(t)\, V(s) = e^{its}\, V(s)\, \mathcal{U}(t)$$

CCR $\quad \hat{=} \quad$ "Weyl relations"

Uniqueness Theorem of von Neumann:

each representation of Weyl relations is equivalent to $\bigoplus$ (Schrödinger representation of CCR)

i.e. two representations of CCR are related by unitary transformation of coordinate system

$\rightsquigarrow$ we can work with fixed Hilbert space

free theory $\xleftrightarrow{\text{unitary transformation}}$ theory with interaction

both "live" in the same Hilbert space with "universal" p, q

but: this is only true for finite degree of freedom

III infinite degree of freedom [quantum field theory]

different physical theories correspond to $\underbrace{\text{non-equivalent representations}}$ of CCR

↳ there are many!

→ two such theories cannot be connected via a unitary transformation (e.g. via perturbation theory)

⇝ problem of renormalization

there is not a preferred choice of representation of CCR, but has to be chosen according to considered problem

⇝ algebraic point of view

$C^*(CCR)$ $\quad$ C^*-algebra

encodes algebraic properties of CCR

physics → state on CCR $\xrightarrow{GNS}$ realization on Hilbert space