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Operator Algebras
Summer term 2022

Problem set 10
To be submitted by Monday, **June 20**, 2 pm.

Let H be a complex Hilbert space.

Problem 34 (4+4* points)**.** (a) Let $M_1 = L^\infty([0,1],\lambda)$ and $M_2 = L^\infty([7,13],\lambda)$ (where λ denotes the respective restrictions of the Lebesgue measure). Show that M_1 and M_2 are $*$-isomorphic.

(b) Now let $L^\infty(X_1,\mu_1)$ and $L^\infty(X_2,\mu_2)$ be two L^∞-spaces such that the underlying measures μ_1,μ_2 have no atoms (meaning that no singleton set $\{x\}$ has positive measure) and the corresponding L^2-spaces are separable. Show that $L^\infty(X_1,\mu_1)$ and $L^\infty(X_2,\mu_2)$ are $*$-isomorphic.

(c) Let $M_1 = L^\infty([0,1],\lambda)$ and let $M_3 = L^\infty([0,1],\nu)$ where

$$\nu = \frac{1}{2}(\lambda + \delta_0)$$

is a measure with an atom of mass $1/2$. Show that M_1 and M_3 are not $*$-isomorphic.

(d) **Bonus exercise:* Consider now discrete measures. Which properties (number of atoms, their locations, their masses, etc.) determine whether such two L^∞-spaces are $*$-isomorphic? Can you give a complete classification of abelian von Neumann algebras acting on separable Hilbert spaces?

Problem 35 (4 points)**.** Let $M \subseteq B(H)$ be a von Neumann algebra such that 1 is a finite projection. Prove the following statements:

(a) Every projection $e \in M$ is finite.

(b) If $e \sim f$ then $1 - e \sim 1 - f$ for projections $e, f \in M$.

(c) If $e \sim f$ then there exists a unitary $u \in M$ such that $f = ueu^*$.

(d) The statements (b) and (c) are false if M is a von Neumann algebra where 1 is not finite.

Please turn the page.

Problem 36 (4+4* points)**.** Let $M \subseteq B(H)$ be a von Neumann algebra and let $p \in M$ be a projection. Show the following:

(a) Both $pMp \subseteq B(pH)$ and $pM'p \subseteq B(pH)$ are von Neumann algebras.

(b) If M is a factor, then also pMp is a factor.
Hint: Proof by contradiction. First argue using functional calculus that there exists a projection f in the center of pMp which is not a scalar multiple of the identity. Then deduce that $fM(p-f) = 0$ - a contradiction.

(c) **Bonus exercise:* If M is a factor of type I, II or III, then pMp is a factor of the same type.