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Operator Algebras
Summer term 2022

Problem set 11
To be submitted by Monday, **June 27**, 2 pm.

Problem 37 (4 points)**.** Reflect on the details of the characterization of type I factors: Let $M \subseteq B(H)$ be a type I factor and let $e \in M$ be a minimal projection. By division with remainder, we may find a family $(e_i)_{i\in I}$ of pairwise orthogonal projections in M such that each e_i is Murray-von Neumann equivalent to e and $\sum_{i\in I} e_i = 1$. Let t_i be the corresponding partial isometries with $t_i^* t_i = e_i$ and $t_i t_i^* = e$. Put $H_1 := \ell^2(I)$ and $H_2 := eH$. Show that the isomorphism between H and $H_1 \hat{\otimes} H_2$ induces an isomorphism between $M \subseteq B(H)$ and $B(H_1) \otimes 1 \subseteq B(H_1 \hat{\otimes} H_2)$ where $x \in M$ is mapped to $x_1 \otimes 1$ and $x_1 \in B(\ell^2(I))$ is a matrix with coefficients $(\lambda_{ij})_{i,j\in I}$ satisfying $t_j^* x t_i = \lambda_{ij} t_j^* t_i$.

Problem 38 (4+4* points)**.** Consider the symmetric group S_3 (which has 6 elements). What is the dimension of the associated group von Neumann algebra $L(S_3)$? How else can $L(S_3)$ be written? Consider also $L(S_4)$.
**Bonus question:* What role do the irreducible representations of S_3 (respectively S_4) play?

Problem 39 (4 points)**.** Let $L(\mathbb{Z})$ be the left group von Neumann algebra of the discrete group $(\mathbb{Z}, +)$.

(a) Show that $L(\mathbb{Z})$ is an abelian von Neumann algebra.

(b) Prove that $L(\mathbb{Z})$ is $*$-isomorphic to $L^\infty(\mathbb{T}, m)$ where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ denotes the unit circle and m the arc length measure on $\mathbb{T}$. Furthermore, show that the tracial state $\tau : L(\mathbb{Z}) \to \mathbb{C}$, $x \mapsto \langle x\delta_0, \delta_0\rangle$ corresponds under this isomorphism to the linear functional on $L^\infty(\mathbb{T}, m)$ that is given by $f \mapsto \int_{\mathbb{T}} f(\zeta) dm(\zeta)$.

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Problem 40 (8 points)**.** Consider the chain of inclusions

$$M_2(\mathbb{C}) \hookrightarrow M_{2^2}(\mathbb{C}) \hookrightarrow M_{2^3}(\mathbb{C}) \hookrightarrow \ldots \hookrightarrow M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C}) \hookrightarrow \ldots$$

given by

$$\begin{aligned} \iota_n : M_{2^n}(\mathbb{C}) &\hookrightarrow M_{2^{n+1}}(\mathbb{C}) \\ x &\mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \end{aligned}$$

(a) Justify that the union $A := \bigcup_{n=1}^{\infty} M_{2^n}(\mathbb{C})$ is a complex unital $*$-algebra and show that there exists a (well-defined!) linear functional $\tau_0 : A \to \mathbb{C}$ such that $\tau_0(x) = \mathrm{tr}_{2^n}(x)$ holds for every $x \in M_{2^n}(\mathbb{C})$, where tr_{2^n} denotes the normalized trace on $M_{2^n}(\mathbb{C})$. Deduce that τ_0 is unital, positive, faithful and tracial.

(b) Denote by H the Hilbert space which is obtained by completion of A with respect to the inner product given by $\langle x, y \rangle = \tau_0(xy^*)$. Prove that each $y \in A$ induces a bounded linear operator on H, i.e. we can view A as a subalgebra of $B(H)$.

(c) Consider the von Neumann algebra $\mathcal{R} := A'' \subseteq B(H)$. Show that there exists a unique faithful normal tracial state τ on $\mathcal{R}$.

(d) Prove that $\mathcal{R} \subseteq B(H)$ is a factor of type II_1.
Hint: The center $Z(\mathcal{R}) = \mathcal{R} \cap \mathcal{R}'$ is generated by its positive elements. So as soon as we have shown that any positive $z \in Z(\mathcal{R})$ is a positive multiple of 1, it follows that $\mathcal{R}$ is a factor. For doing so, use the result obtained in (c).