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Operator Algebras
Summer term 2022

Problem set 12
To be submitted by Monday, **July 4**, 2 pm.

Problem 41 (4 points)**.** (a) Let $M \subseteq B(H)$ be a von Neumann algebra that has a faithful tracial state $\tau : M \to \mathbb{C}$ which is moreover normal. Show that M is finite. Is this still the case if τ is not required to be normal?

(b) Let $M \subseteq B(H)$ be factor and let $\tau : M \to \mathbb{C}$ be a faithful tracial state. Consider any two projections $e, f \in M$. Show that $e \sim f$ if and only if $\tau(e) = \tau(f)$.

Problem 42 (4 points)**.** Let M be a factor of type II_1 and let $\tau : M \to \mathbb{C}$ be its unique faithful normal tracial state. Show that

$$\tau(\mathcal{P}(M)) = [0, 1].$$

Hint: Fix and $t \in [0, 1]$ and consider the set $S_t := \{p \in \mathcal{P}(M) \mid \tau(p) \leq t\}$. Verify that S_t is partially ordered and use Zorn's lemma to prove that S_t contains a maximal element p_t. Finally show that $\tau(p_t) = t$.

Problem 43 (4 points)**.** Let $M \subseteq B(H)$ be type II_1 factor with its unique faithful normal tracial state $\tau : M \to \mathbb{C}$. Suppose that M possesses a cyclic and separating vector $\Omega \in H$ such that $\tau(x) = \langle x\Omega, \Omega\rangle$ for all $x \in M$. Let $J : M\Omega \to M\Omega$ be defined by $J(x\Omega) = x^*\Omega$ for all $x \in M$. Prove the following statements:

(a) The antilinear operator $J : M\Omega \to M\Omega$ extends uniquely to an antilinear isometry $J : H \to H$ that satisfies $J^2 = 1$ and $\langle J\xi, \eta\rangle = \langle J\eta, \xi\rangle$ for all $\xi, \eta \in H$; we call J the *canonical conjugation operator on* H.

(b) For all $x, y \in M$ is holds true that $JxJ(y\Omega) = yx^*\Omega$.

(c) For every $x \in M'$, we have that $Jx\Omega = x^*\Omega$.

Deduce that $JMJ = M'$ and show that also M' is a type II_1 factor. How does the unique faithful tracial state on M' look like?
Hint: For proving $JMJ = M'$, switch the roles of M and M'. What does (c) tell us about this case?