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Operator Algebras
Summer term 2022

Problem set 13
To be submitted by Monday, **July 11**, 2 pm.

Problem 44 (4 points)**.** We consider the universal C^*-algebras

$$C^*(p, 1 \mid p \text{ is a projection, i.e. } p = p^2 = p^*)$$
$$C^*(s, 1 \mid s \text{ is a symmetry, i.e. } s^*s = ss^* = 1,\ s = s^*).$$

(Here we consider 1 as a generator with the relations $1 = 1^* = 1^2$ and $1x = x1 = x$ for every other generator x.)

(a) Show that these C^*-algebras are isomorphic by writing down an explicit isomorphism. To do so, find a symmetry $s' \in C^*(p, 1)$ and a projection $p' \in C^*(s, 1)$ and use the universal property twice. (It might help to solve (a) and (b) at the same time.)

(b) How does the spectrum of a projection and the spectrum of a symmetry look like? Since the C^*-algebras above are commutative, they are isomorphic to the algebra of continuous functions on the spectrum, i.e.

$$C^*(p, 1) \cong C(\mathrm{sp}(p)), \qquad C^*(s, 1) \cong C(\mathrm{sp}(s))$$

What are images of $\mathrm{id}_{\mathrm{sp}(p)}$ and $\mathrm{id}_{\mathrm{sp}(s)}$ under the isomorphism between $C^*(p, 1)$ and $C^*(s, 1)$?

Problem 45 (4 points)**.** Show that the following C^*-algebras are isomorphic.

- $C(\{1, \ldots, n\})$
- $\mathbb{C}^n = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$
- $C^*(p_1, \ldots, p_n, 1 \mid p_i \text{ are projections, } \sum_{i=1}^n p_i = 1)$
- $C^*(u, 1 \mid u^*u = uu^* = 1,\ u^n = 1)$

Please turn the page.

Problem 46 (8 points)**.** Let H be a complex Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and let $\tilde{H}$ be a complex Hilbert space with orthonormal basis $(\tilde{e}_n)_{n\in\mathbb{Z}}$. For $\lambda \in S^1 \subseteq \mathbb{C}$ we define shift and diagonal operators via

$$\begin{aligned} S : H &\to H, & \tilde{S} : \tilde{H} &\to \tilde{H}, & d(\lambda) : H &\to H, & \tilde{d}(\lambda) : \tilde{H} &\to \tilde{H} \\ e_n &\mapsto e_{n+1} & \tilde{e}_n &\mapsto \tilde{e}_{n+1} & e_n &\mapsto \lambda^n e_n & \tilde{e}_n &\mapsto \lambda^n \tilde{e}_n \end{aligned}$$

Prove the following assertions.

(a) S is an isometry (i.e. $S^*S = 1$), such that $1 - SS^*$ is the projection onto the one-dimensional subspace $\mathbb{C}e_1 \subseteq H$, while $\tilde{S}, d(\lambda), \tilde{d}(\lambda)$ are unitaries with

$$d(\lambda)^* = d(\bar{\lambda}), \quad d(\lambda)d(\lambda') = d(\lambda\lambda'), \quad \tilde{d}(\lambda)^* = \tilde{d}(\bar{\lambda}), \quad \tilde{d}(\lambda)\tilde{d}(\lambda') = \tilde{d}(\lambda\lambda').$$

(b) It holds that $d(\lambda)S = \lambda S d(\lambda)$ and $\tilde{d}(\lambda)\tilde{S} = \lambda\tilde{S}\tilde{d}(\lambda)$, and more generally

$$\tilde{d}(\lambda)^k \tilde{S}^l = \lambda^{kl} \tilde{S}^l \tilde{d}(\lambda)^k, \qquad k, l \in \mathbb{Z}.$$

Conclude that the set $\mathcal{S}$ of finite linear combinations of $\tilde{d}(\lambda)^k \tilde{S}^l$ is a dense $*$-subalgebra of $C^*(\tilde{S}, \tilde{d}(\lambda)) \subseteq B(\tilde{H})$.

(c) The maps

$$\begin{aligned} \beta_\lambda : B(H) &\to B(H), & \tilde{\beta}_\lambda : B(\tilde{H}) &\to B(\tilde{H}) \\ T &\mapsto d(\lambda)Td(\lambda)^*, & T &\mapsto \tilde{d}(\lambda)T\tilde{d}(\lambda)^* \end{aligned}$$

are $*$-isomorphisms with $\beta_\lambda(C^*(S)) = C^*(S)$ and $\tilde{\beta}_\lambda(C^*(\tilde{S})) = C^*(\tilde{S})$.

(d) Use (c) to show that $\mathrm{sp}(\tilde{S}) = S^1$ and $\mathrm{sp}(\sigma(S)) = S^1$, where $\sigma : B(H) \to B(H)/\mathcal{K}(H)$ is the quotient map.