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Operator Algebras
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Problem set 14
This sheet is entirely optional.

Problem 47. For $\zeta \in S^1 \subseteq \mathbb{C}$ and $n \in \mathbb{N}$ we define:

$$a_n(\zeta) := \frac{1}{2n+1} \sum_{j=-n}^{n} \zeta^j$$

Let $\vartheta \in \mathbb{R}$ and put $\lambda := e^{2\pi i \vartheta} \in S^1$. Let $l \in \mathbb{Z}$.

(a) Show that $(a_n(\zeta))_{n\in\mathbb{N}}$ converges to 0 if $\zeta \neq 1$.
Hint: Show that $\sum_{j=-n}^{n} \zeta^j$ is bounded using geometric series.

(b) Let $\vartheta \notin \mathbb{Q}$. Show that $(a_n(\lambda^l))_{n\in\mathbb{N}}$ converges to 1 if $l = 0$ and to 0 otherwise.

(c) Let $\vartheta = \frac{p}{q} \in \mathbb{Q}$ with $\gcd(p,q) = 1$. Show that $(a_n(\lambda^l))_{n\in\mathbb{N}}$ converges to 1 if $l \in q\mathbb{Z}$ and to 0 otherwise. In particular, $(a_n(\lambda^l))_{n\in\mathbb{N}}$ converges to 1 for infinitely many powers λ^l, $l \in \mathbb{Z}$. Hence, in the rational case, the map $x \mapsto \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} u^j x u^{-j}$ is very different from φ_1.

Problem 48. Let $\vartheta = \frac{p}{q} \in \mathbb{Q}$.

(a) Find a representation $\pi : A_\vartheta \to M_q(\mathbb{C})$.

(b) Find unital C^*-algebras B and D as well as unital $*$-homomorphisms $\varphi : A_\vartheta \to B$ and $\psi : A_\vartheta \to D$ such that $\varphi(v^q) = 1$ and $\psi(v^q) \neq 1$.

(c) Conclude that A_ϑ is not simple.

(d) Show that there is a $*$-homomorphism $\sigma : C(S^1 \times S^1) \to C^*(u^q, v^q) \subseteq A_\vartheta$ which maps the generators $\tilde{u}$ and $\tilde{v}$ of $C(S^1 \times S^1)$ to u^q and v^q. (In fact, this is a $*$-isomorphism.)

(e) Convince yourself, that none of these statements is true for A_ϑ with $\vartheta \notin \mathbb{Q}$.

Please turn the page.

Problem 49. Let $S_1, \dots, S_n \in \mathcal{O}_n$ be the standard generators of the Cuntz algebra.

(a) Let $p_1, \dots, p_n \in A$ be projections in a C^*-algebra A such that $\sum_{i=1}^n p_i = 1$. Show that $p_i p_j = 0$ whenever $i \neq j$. From this deduce that $(S_i S_i^*)(S_j S_j^*) = 0$ for $i \neq j$ and thus $S_i^* S_j = \delta_{ij}$.
Hint: Use the fact that $\sum_{i, i\neq j} p_j p_i p_j = 0$ is a sum of positive elements.

(b) Let μ, ν be multi-indices. Show:

$$\text{If } |\mu| = |\nu|, \text{ then } S_\mu^* S_\nu = \delta_{\mu\nu}$$
$$\text{If } |\mu| < |\nu|, \text{then } S_\mu^* S_\nu = \begin{cases} S_{\nu'} & \text{if } \nu = \mu\nu' \\ 0 & \text{otherwise} \end{cases}$$
$$\text{If } |\mu| > |\nu|, \text{then } S_\mu^* S_\nu = \begin{cases} S_{\mu'} & \text{if } \mu = \nu\mu' \\ 0 & \text{otherwise} \end{cases}$$

Conclude that all monomials in $\mathcal{O}_n$ are of the form $S_\mu S_\nu^*$ for multi-indices μ, ν.

(c) Let $k \in \mathbb{N}$ and denote by $\mathcal{M}(k)$ the set of all multi-indices of length k. Show that $\sum_{\alpha \in \mathcal{M}(k)} S_\alpha S_\alpha^* = 1$.

(d) Let μ, ν be multi-indices with $|\mu| \neq |\nu|$ and $|\mu|, |\nu| \leq k$. Let $\alpha, \beta \in \mathcal{M}(k)$. Put $S_\gamma := S_1^{2k} S_2$. Show that $S_\gamma^* S_\alpha^* (S_\mu S_\nu^*) S_\beta S_\gamma = 0$.

Problem 50. For $n \in \mathbb{N}$ consider the extended Cuntz algebra

$$\mathcal{E}_n := C^*(S_1, \dots, S_n \mid S_i^* S_j = \delta_{ij}).$$

(a) Show that $p := 1 - \sum_{i=1}^n S_i S_i^* \in \mathcal{E}_n$ is a projection.

(b) Let μ, ν be multi-indices. Show that the elements $f_{\mu\nu} := S_\mu p S_\nu^*$ fulfill the relations $f_{\mu\nu}^* = f_{\nu\mu}$ and $f_{\mu\nu} f_{\mu'\nu'} = \delta_{\nu\mu'} f_{\mu\nu'}$.

(c) Show that there is a short exact sequence of the form

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{O}_n \longrightarrow 0$$

(H being a separable Hilbert space). How does this sequence look like for $n = 1$?

(d) Find a representation of $\mathcal{E}_n$ which is not a representation of $\mathcal{O}_n$.

One defines $\mathcal{O}_\infty := \mathcal{E}_\infty$ (why does the definition of $\mathcal{O}_n$ only make sense for $n < \infty$?). It can be shown that $\mathcal{O}_\infty$ is also purely infinite, which is false for $\mathcal{E}_n$ with $n < \infty$ (why?).