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Operator Algebras
Summer term 2022

Problem set 1
To be submitted by Monday, **April 18**, 2022, 2 pm.

In the following, $0 \notin \mathbb{N}$.

Problem 1 (4 points)**.** For any set X an orthonormal basis of the complex Hilbert space $\ell^2(X)$ of square-summable $\mathbb{C}$-valued X-indexed families (with addition and multiplication with scalars from $\mathbb{C}$ explained pointwise) is given by $(e_x)_{x \in X}$, where for each $x \in X$, by definition, $e_x : X \to \mathbb{C}$, $x' \mapsto \delta_{x,x'}$. See also Example 1.35.

(a) In the case $X = \mathbb{N}$ there exists a unique bounded linear operator S on $\ell^2(\mathbb{N})$, the *unilateral shift*, with the property that $Se_n = e_{n+1}$ for each $n \in \mathbb{N}$.
  (i) Prove that the adjoint operator S^* of S satisfies $S^* e_n = e_{n-1}$ for all $n \in \mathbb{N}$ with $2 \leq n$ as well as $S^* e_1 = 0$.
  (ii) Deduce that S is an isometric but *not* a unitary operator on $\ell^2(\mathbb{N})$, i.e., that $S^* S = \mathrm{id}$ but $SS^* \neq \mathrm{id}$.

(b) For $X = \mathbb{Z}$ the *bilateral shift* is the unique bounded linear operator $\tilde{S}$ on $\ell^2(\mathbb{Z})$ with the property that $\tilde{S} e_n = e_{n+1}$ for each $n \in \mathbb{Z}$. Decide, with proof, whether $\tilde{S}$ is a unitary operator.

(c) Propose a reasonable analog of the bilateral shift $\tilde{S}$ in the case $X = \{1, \ldots, N\}$, where $N \in \mathbb{N}$.

Problem 2 (4 points)**.** Prove the following statements about the unilateral shift S from Problem 1 (a):

(a) The element $\lambda\,\mathrm{id} - S$ is invertible in $B(\ell^2(\mathbb{N}))$ for any $\lambda \in \mathbb{C}$ with $1 < |\lambda|$.
(b) The point spectrum of S over $\ell^2(\mathbb{N})$ is empty.
(c) Each $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ is an eigenvalue of S^* over $\ell^2(\mathbb{N})$.
(d) The spectrum of S in $B(\ell^2(\mathbb{N}))$ is given by $\{\lambda \in \mathbb{C} \,|\, |\lambda| \leq 1\}$.

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Given any complex Hilbert space H, any bounded linear operator V on H is called a *partial isometry* in $B(H)$ if and only if $VV^*V = V$, where V^* denotes the Hilbert adjoint of V.

Moreover, recall from Definition 1.33 that any bounded linear operator P on H is called a *projection* in $B(H)$ if and only if $PP = P$ and $P = P^*$.

Problem 3 (4 points)**.** Let V be any bounded linear operator on any complex Hilbert space H. Prove that the following statements about V are equivalent:

(1) V is a partial isometry in $B(H)$.

(2) V^*V is a projection in $B(H)$.

(3) VV^* is a projection in $B(H)$.

(4) There exists a Hilbert subspace (i.e., a closed $\mathbb{C}$-linear subspace) K of H such that $\langle Vx, Vy\rangle = \langle x, y\rangle$ for all $\{x, y\} \subseteq K$ and such that $Vz = 0$ for any $z \in K^\perp$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product of H and $(\cdot)^\perp$ orthogonal complementation with respect to $\langle\cdot,\cdot\rangle$.

Any C^*-algebra A is called *simple* if and only if it has no proper ideals, i.e., if $I = \{0\}$ or $I = A$ for any closed two-sided ideal I of A.

Problem 4 (4 points)**.** Let $N \in \mathbb{N}$ be arbitrary, let $M_N(\mathbb{C})$ denote the C^*-algebra of $N \times N$-matrices with complex coefficients (and with addition and $\mathbb{C}$-scalar multiplication explained entrywise and with matrix multiplication as product), and for any pair (i, j) of indices from $\{1, \ldots, N\}$ let $E_{i,j}$ be the element of $M_N(\mathbb{C})$ whose (i', j')-entry is given by $\delta_{i,i'}\delta_{j,j'}$ for any pair (i', j') of indices from $\{1, \ldots, N\}$, the *matrix (i, j)-unit*.

(a) For any closed two-sided ideal I of $M_N(\mathbb{C})$ prove the following:

  (i) For any pair (i_0, j_0) of indices from $\{1, \ldots, N\}$, if there exists an element $T = (t_{i,j})_{i,j=1}^N$ of I with $t_{i_0,j_0} \neq 0$, then I already contains E_{i_0,j_0}.
  Hint: Multiply T with appropriate matrix units to see this.

  (ii) If there exists a pair (i_0, j_0) of indices from $\{1, \ldots, N\}$ such that I contains E_{i_0,j_0}, then $E_{i,j}$ is an element of I for *any* pair (i, j) of indices from $\{1, \ldots, N\}$.

(b) Deduce that $M_N(\mathbb{C})$ is simple.
Hint: Note that $\sum_{i=1}^N E_{i,i} = \mathrm{id}$.