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Operator Algebras
Summer term 2022

Problem set 2
To be submitted by Monday, **April 25**, 2022, 2 pm.

Recall that for any compact Hausdorff space X the $\mathbb{C}$-valued continuous functions on X form a Banach-$*$-algebra $C(X)$ if equipped with the supremum norm $\|\cdot\|_\infty$ and the pointwise-defined addition, multiplication and involution.

Problem 5 (4 points)**.** (a) For any compact Hausdorff space X prove that $C(X)$ is a commutative unital C^*-algebra.

(b) Show that for any two compact Hausdorff spaces X and Y and any continuous mapping $h : X \to Y$, the map $\alpha_h : C(Y) \to C(X)$ defined by $f \mapsto f \circ h$ is a $*$-homomorphism.

(c) Prove that in the situation of (b), if h is a homeomorphism, the map α_h is an isometric $*$-isomorphism.

If $\mathbb{D} := \{z \in \mathbb{C} \,|\, |z| < 1\}$ denotes the open unit disk and $\overline{\mathbb{D}} = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ its closure, then the set

$$A(\mathbb{D}) := \{f : \overline{\mathbb{D}} \to \mathbb{C} \,|\, f \text{ is continuous on } \overline{\mathbb{D}} \text{ and holomorphic on } \mathbb{D}\}$$

with pointwise addition and multiplication forms a unital Banach-$*$-algebra when equipped with the supremum-norm and the involution $f^*(z) := \overline{f(\overline{z})}$, but it is not a C^*-algebra. This can be seen as follows.

Problem 6 (4 points)**.** Show that the identity function on $\overline{\mathbb{D}}$ is a self-adjoint element of $A(\mathbb{D})$ which has spectrum $\overline{\mathbb{D}}$ with respect to $A(\mathbb{D})$.

Please turn the page.

Problem 7 (8 points)**.** Let A be a non-unital C^*-algebra. A *double centralizer* of A is a pair (L, R) of linear maps $L, R : A \to A$ satisfying $L(ab) = L(a)b$, $R(ab) = aR(b)$ and $aL(b) = R(a)b$ for all $a, b \in A$. The *multiplier algebra* $M(A)$ of A is defined as

$$M(A) \coloneqq \{(L, R) \text{ double centralizer of } A\}$$

with the operations

$$\begin{aligned}(L_1, R_1) + (L_2, R_2) &\coloneqq (L_1 + L_2, R_1 + R_2)\\ (L_1, R_1) * (L_2, R_2) &\coloneqq (L_1 L_2, R_2 R_1)\\ (L, R)^* &\coloneqq (R^*, L^*)\\ \lambda(L, R) &\coloneqq (\lambda L, \lambda R)\end{aligned}$$

for $\lambda \in \mathbb{C}$ and $(L, R), (L_1, R_1), (L_2, R_2) \in M(A)$. (Note, that for a linear map $T : A \to A$, the map $T^* : A \to A$ is defined as $T^*(x) \coloneqq (T(x^*))^*$.) We define a norm on $M(A)$ via

$$\|(L, R)\| \coloneqq \|L\| = \|R\|$$

for $(L, R) \in M(A)$.

(a) Show that $M(A)$ is a unital C^*-algebra and that the map

$$\begin{aligned}\varphi : A &\to M(A)\\ a &\mapsto (L_a, R_a)\end{aligned}$$

where $L_a(x) = ax$ and $R_a(x) = xa$ for $x \in A$ is an isometric $*$-homomorphism. Furthermore prove that $\varphi(A)$ is an ideal in $M(A)$. Thus every non-unital C^*-algebra can be embedded as an ideal in a unital C^*-algebra.

(b) Prove that $M(A)$ is the largest unitization of A: If B is a unital C^*-algebra and $A \subseteq B$ as an ideal, then there exists a $*$-homomorphism from B to $M(A)$ which extends the embedding $A \subseteq M(A)$.