SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
Mathematics Department
Prof. Dr. Roland Speicher
Prof. Dr. Moritz Weber
M.Sc. Luca Junk

Operator Algebras
Summer term 2022

Problem set 3
To be submitted by Monday, **May 2**, 2 pm.

Problem 9 (6 Points)**.** Prove in the following steps that the mapping $t \mapsto \mathrm{ev}_t$ with $\mathrm{ev}_t : C([0,1]) \to \mathbb{C}, f \mapsto f(t)$ defines a homeomorphism Ψ between $[0,1]$ (with the usual topology) and $\mathrm{Spec}(C([0,1]))$.

(a) Verify that ev_t is an element of $\mathrm{Spec}(C([0,1]))$ for every $t \in [0,1]$.

(b) Show that Ψ is injective.
Hint: Consider the function $t \mapsto |s-t|$ for any fixed $s \in [0,1]$.

(c) Show that $C([0,1])$ is the unique closed ideal I of $C([0,1])$ with the property that for every $t \in [0,1]$ there exists some $f \in I$ with $f(t) \neq 0$.
Hint: Conclude from the compactness of $[0,1]$ that I contains an invertible element.

(d) Show that the image of $[0,1]$ under Ψ is all of $\mathrm{Spec}(C([0,1]))$.
Hint: Using (c), find for every $\varphi \in \mathrm{Spec}(C([0,1]))$ some $t \in [0,1]$ with $\ker(\varphi) = \ker(\mathrm{ev}_t)$.

(e) Show that Ψ is continuous.

(f) Show that also the inverse of Ψ is continuous.
Hint: Use the Hausdorff property of $\mathrm{Spec}(C([0,1]))$.

Problem 10 (4 Points)**.** Let A be a unital C^*-algebra, let $x \in A$ be normal and $f \in C(\mathrm{sp}(x))$. We denote by $f(x)$ the value of f under the continuous functional calculus of x.

(a) Show that
$$\mathrm{sp}(f(x)) = f(\mathrm{sp}(x)).$$

(b) Let $g \in C(\mathrm{sp}(f(x)))$ and denote by $g(f(x))$ the value of g under the continuous functional calculus of $f(x)$. Show that
$$g(f(x)) = (g \circ f)(x).$$
Hint: Use the uniqueness property of the functional calculus.

Problem 11 (6 Points)**.** Let A be a unital C^*-algebra.

(a) Show that for every invertible $x \in A$ we have
$$\mathrm{sp}(x^{-1}) = \{\lambda^{-1} \,|\, \lambda \in \mathrm{sp}(x)\}.$$

(b) Show that the spectrum of a unitary $u \in A$ is contained in the unit circle of $\mathbb{C}$, i.e.

$$\mathrm{sp}(u) \subseteq \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}.$$

Hint: Show $\|u\| = 1$ and use (a).

(c) Show that every unitary $u \in A$ with $\mathrm{sp}(u) \neq \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$ can be written in *polar coordinates*, i.e. there exists a selfadjoint element $y \in A$ such that

$$u = e^{iy},$$

where the right hand side of this equation is to be interpreted using the continuous functional calculus.