SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
Mathematics Department
Prof. Dr. Roland Speicher
Prof. Dr. Moritz Weber
M.Sc. Luca Junk

Operator Algebras
Summer term 2022

Problem set 4
To be submitted by Monday, **May 9**, 2 pm.

Problem 12 (6 points). Let A be a unital C^*-algebra and $x \in A$ be selfadjoint. Prove the following statements.

(a) The element x is invertible if and only if $0 \notin \text{sp}(x)$.
(b) If x is invertible, then x^{-1} is selfadjoint.
(c) If $\text{sp}(x) \subseteq (0, \infty)$, then $\text{sp}(x^{-1}) \subseteq (0, \infty)$.
(d) If $f, g \in C(\text{sp}(x))$, then $f(x)$ and $g(x)$ commute.
(e) We have $\text{sp}(x - 1) \subseteq [0, \infty)$ if and only if $\text{sp}(x) \subseteq [1, \infty)$.
(f) If $\text{sp}(x) \subseteq [1, \infty)$, then $\text{sp}(1 - x^{-1}) \subseteq [0, \infty)$.

Problem 13 (6 points). Let H be a complex hilbert space and $T \in B(H)$.

(a) Show that there is a unique decomposition $T = VP$ where $V \in B(H)$ is a partial isometry and $P \in B(H)$ is positive such that $\ker(V) = \ker(P) = \ker(T)$.
Hint: Take $P := |T| := \sqrt{T^*T}$ and $V := V_0 \oplus 0$ where $V_0 : \overline{\text{ran}(|T|)} \to \overline{\text{ran}(T)}$, $Px \mapsto Tx$. In particular, justify that these expressions are well-defined and make sense. For the uniqueness statement, first prove that P is unique.
(b) Show that V is unitary whenever T is invertible.
(c) How does the decomposition $T = VP$ look like in the case $H = \mathbb{C}$?

Problem 14 (4 points). Let H be a complex hilbert space and $T \in B(H)$. Prove the following statements.

(a) $T = 0$ if and only if $\langle Tx, x\rangle = 0$ for all $x \in H$.
(b) $T = T^*$ if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.
(c) $T \geq 0$ if and only if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.
Hint: Show that for every $\lambda < 0$, the operator $\lambda 1 - T$ is bounded from below, i.e. there exists a constant $c > 0$ such that $\|(\lambda 1 - T)x\| \geq c\|x\|$ for all $x \in H$.