SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
Mathematics Department
Prof. Dr. Roland Speicher
Prof. Dr. Moritz Weber
M.Sc. Luca Junk

Operator Algebras
Summer term 2022

Problem set 5
To be submitted by Monday, **May 16**, 2 pm.

Problem 15 (4 points)**.** Let A be a C^*-algebra and let $x, y \in A$. Show that

$$\mathrm{sp}(xy) \cup \{0\} = \mathrm{sp}(yx) \cup \{0\}$$

and provide an example where $\mathrm{sp}(xy)$ and $\mathrm{sp}(yx)$ do not coincide.

Problem 16 (4 points)**.**

(a) Show that the following holds in any commutative (not necessarily unital) C^*-algebra A: If $h \in A$ is positive and $x \in A$ is selfadjoint with $h \geq x$, then also $h \geq x_+$.

(b) Find a counterexample to the above statement in $A = M_2(\mathbb{C})$.

Problem 17 (4 points)**.** Let A be a (*not necessarily unital*) C^*-algebra, I a closed ideal in A and $a, b \in I$ positive with $\|a\| < 1$ and $\|b\| < 1$. Show that there exists a positive element $c \in I$ with $\|c\| < 1$ and $a \leq c$ as well as $b \leq c$.

One way to do this is to follow these steps:

(a) Prove that the expressions $a' := a(1-a)^{-1}$ and $b' := b(1-b)^{-1}$ make sense and that they define positive elements in I.

(b) Do the same with $c := c'(1+c')^{-1}$ for $c' := a' + b'$ and show that $\|c\| < 1$.

(c) Give meaning to the expression $1 - (1+x')^{-1}$ for $x = a, b, c$ and show that $1 - (1+x')^{-1} \leq 1 - (1+c')^{-1}$ for $x = a, b$.

(d) Give meaning to the expression $x'(1+x')^{-1}$ and show that $x = x'(1+x')^{-1}$ for $x = a, b$.

Problem 18 (4 points)**.**

(a) Let B, C, D be C^*-algebras and let $\phi : B \to C$, $\psi : C \to D$ be $*$-homomorphisms. Prove the following: If the sequence

$$0 \longrightarrow B \xrightarrow{\phi} C \xrightarrow{\psi} D \longrightarrow 0$$

is exact, then

(i) ϕ is injective,
(ii) ψ is surjective,
(iii) $\phi(B)$ is a closed ideal in C,
(iv) ψ induces a $*$-isomorphism $C/\phi(B) \to D$ via $c + \phi(B) \mapsto \psi(c)$.

(b) Let A be a C^*-algebra and let I be a closed ideal in A. Denote the inclusion map $I \to A$ by ι and the quotient map $A \to A/I$ by π. Show that

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0$$

defines a short exact sequence of C^*-algebras.