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Operator Algebras
Summer term 2022

Problem set 6
To be submitted by Monday, **May 23**, 2 pm.

Recall from the lecture that an injective representation of a C^*-algebra is called *faithful*. A state φ on a C^*-algebra is called *faithful*, if $\varphi(x^*x) = 0$ already implies $x = 0$ for every element x.

Problem 19 (4 points)**.** Let A be a C^*-algebra, let φ be a state on A and let $(H_\varphi, \pi_\varphi, \xi_\varphi)$ be the GNS-representation of A corresponding to φ. Prove the following statements.

(a) For every closed ideal I in A we have $I \subseteq \ker(\pi_\varphi)$ if and only if $I \subseteq \ker(\varphi)$.

(b) If φ is a faithful state, then π_φ is a faithful representation.

Problem 20 (4 points)**.** Let $N \in \mathbb{N}$ be fixed. We denote by tr the normalized trace on $M_N(\mathbb{C})$, i.e. the linear functional $M_N(\mathbb{C}) \to \mathbb{C}$, $T = (T_{ij})_{i,j=1}^N \mapsto \frac{1}{N}\sum_{i=1}^N T_{ii}$. For a positive matrix $B \in M_N(\mathbb{C})$ we denote by τ_B the linear functional $M_N(\mathbb{C}) \to \mathbb{C}$, $T \mapsto \mathrm{tr}\,(BT)$.

(a) Show that tr is a faithful state on $M_N(\mathbb{C})$.

(b) Show that the mapping $B \mapsto \tau_B$ defines a bijection between the positive matrices in $M_N(\mathbb{C})$ and the positive linear functionals on $M_N(\mathbb{C})$.
Hint: The matrix units from Problem 4 constitute a basis of $M_N(\mathbb{C})$. Moreover, Problem 14 may be useful.

(c) Determine the preimage of the set of states on $M_N(\mathbb{C})$ under the mapping from (b).

Problem 21 (4 points)**.** We fix $N \in \mathbb{N}$. Consider $A = M_N(\mathbb{C})$ together with the normalized trace tr as in Problem 20.

(a) What does the GNS-construction yield for tr? Determine all components of $(H_{\mathrm{tr}}, \pi_{\mathrm{tr}}, \xi_{\mathrm{tr}})$. Is this representation faithful?

(b) Let $B = E_{11} = (\delta_{1,i}\delta_{1,j})_{i,j=1}^N \in M_N(\mathbb{C})$. Consider τ_B as in Problem 20. Determine the components of the GNS-representation of $M_N(\mathbb{C})$ with respect to τ_B. Is this representation faithful? Is τ_B faithful?

Please turn the page.

Problem 22 (4 points)**.** Let A be a separable C^*-algebra, i.e. there exists a countable dense subset $\{a_n \mid n \in \mathbb{N}\} \subseteq A$. For $n \in \mathbb{N}$ let φ_n be a state on A with $\varphi_n(a_n^* a_n) = \|a_n\|^2$ (Theorem 5.13 guarantees the existence of such a state). Let (H_n, π_n, ξ_n) be the corresponding GNS-construction and put $H := \bigoplus_{n\in\mathbb{N}} H_n$, $\pi := \bigoplus_{n\in\mathbb{N}} \pi_n$.

(a) Show that H_n is separable for each $n \in \mathbb{N}$ and decuce that H is separable.

(b) Given $a \in A$ and $0 < \varepsilon < \frac{1}{2}\|a\|$, let $n \in \mathbb{N}$ such that $\|a - a_n\| < \varepsilon$. Show that $\|\pi_n(a)\| > 0$.

(c) Deduce that $\pi : A \to B(H)$ is faithful.