SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
Mathematics Department
Prof. Dr. Roland Speicher
Prof. Dr. Moritz Weber
M.Sc. Luca Junk

Operator Algebras
Summer term 2022

Problem set 7
To be submitted by Monday, **May 30**, 2 pm.

Problem 23 (4 points)**.** Let H be a complex Hilbert space.

(a) Show that the involution $^* : B(H) \to B(H)$ is continuous in the norm topology and the weak operator topology, but not in the strong operator topology.

(b) Let $(x_\lambda)_{\lambda\in\Lambda}$, $(y_\lambda)_{\lambda\in\Lambda}$ be two nets in $B(H)$ (over the same index set Λ) with $x_\lambda \to x$ and $y_\lambda \to y$ in the strong operator topology. Show that if $(x_\lambda)_{\lambda\in\Lambda}$ is bounded, then $(x_\lambda y_\lambda)_{\lambda\in\Lambda}$ converges strongly to xy. (This fails to be true in general if $(x_\lambda)_{\lambda\in\Lambda}$ is not assumed to be bounded. A counterexample may be found in the book *A Hilbert Space Problem Book* by Paul Halmos.)

Problem 24 (4 points)**.** Let H be a complex Hilbert space and let $S, T \subseteq B(H)$ be subsets with $S \subseteq T$.

(a) Show that the commutant $S' \subseteq B(H)$ of S is a weakly closed unital subalgebra.

(b) Show that if $S = S^*$, then the commutant $S' \subseteq B(H)$ of S is a strongly closed unital $*$-subalgebra (and hence a von Neumann algebra).

(c) Show that $S \subseteq S''$ and $T' \subseteq S'$. Deduce that $S''' = S'$.

Problem 25 (4 points)**.** Let H be a complex Hilbert space.

(a) Consider a strongly continuous linear functional $\varphi : B(H) \to \mathbb{C}$. Show that there are $n \in \mathbb{N}$, vectors $\xi_1, \dots, \xi_n \in H$ and some $C > 0$ such that

$$|\varphi(x)| \leq C \left(\sum_{i=1}^{n} \|x\xi_i\|^2 \right)^{1/2} \qquad \text{for all } x \in B(H)$$

(b) Let $\varphi : B(H) \to \mathbb{C}$ be a linear functional. Prove that the following statements are equivalent:

(i) φ is continuous with respect to the weak operator topology.

(ii) φ is continuous with respect to the strong operator topology.

(iii) There are $n \in \mathbb{N}$ and vectors $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H$ such that

$$\varphi(x) = \sum_{i=1}^{n} \langle x\xi_i, \eta_i \rangle \qquad \text{for all } x \in B(H)$$

Please turn the page.

Hint: Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). For proving (ii) $\Rightarrow$ (iii), use (a) and consider the Hilbert space $K \subseteq H^n$ that is given as the closure of $K_0 := \{(x\xi_1, \ldots, x\xi_n) \mid x \in B(H)\}$; show that the assignment $(x\xi_1, \ldots, x\xi_n) \mapsto \varphi(x)$ is well-defined on K_0 and extends to a continuous linear functional $\psi : K \to \mathbb{C}$; finally apply the Riesz representation theorem to ψ.

Problem 26 (4 points)**.** Show that every element of a unital C^*-algebra can be written as a linear combination of at most four unitaries.

Hint: Consider first the case of selfadjoint elements of norm less or equal than one and use the continuous functional calculus.