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Operator Algebras
Summer term 2022

Problem set 8
To be submitted by Monday, **June 6**, 2 pm.

Let H be a complex Hilbert space.

Problem 27 (4 points)**.** (a) Show that if $h_1, h_2 \in C(\mathbb{R})$ are SOT-continuous and at least one of them is bounded, then also $h_1 \cdot h_2$ is SOT-continuous.

(b) Put $B(H)_1 := \{x \in B(H) \mid \|x\| \leq 1\}$. Show that if $(x_\lambda)_{\lambda\in\Lambda}$ is any net in $B(H)_1$ which converges strongly to some $x \in B(H)$, then necessarily $x \in B(H)_1$.

Problem 28 (4 points)**.** Show that the set $A_0 := \{f \in C_0(\mathbb{R}) \mid f \text{ is SOT-continuous}\}$ is closed in $C_0(\mathbb{R})$ with respect to the supremum norm. Moreover, show that the function f given by $f(z) = 1/(1+z^2)$ is an element of A_0.

Problem 29 (6 points)**.** Let $A \subseteq B(H)$ be a $*$-subalgebra. Prove the following: For any $x \in B := \overline{A}^{\mathrm{SOT}}$ we find a net $(x_\lambda)_{\lambda\in\Lambda}$ in A such that

$$x_\lambda \xrightarrow[\mathrm{SOT}]{\lambda} x \qquad \text{and} \qquad \sup_{\lambda\in\Lambda} \|x_\lambda\| \leq \|x\|$$

If x is selfadjoint, then $(x_\lambda)_{\lambda\in\Lambda}$ can be chosen to consist of selfadjoint operators.
If the Hilbert space H is separable, then there even exists a sequence $(x_n)_{n\in\mathbb{N}}$ with the above properties.

Hint: For proving the existence of the sequence $(x_n)_{n\in\mathbb{N}}$, take any sequence $(\xi_k)_{k\in\mathbb{N}}$ of vectors in H for which $\{\xi_n \mid n \in \mathbb{N}\}$ is dense in H and find $x_n \in A$ for each $n \in \mathbb{N}$ such that $\|x_n\| \leq \|x\|$ and $\|x_n\xi_k - x\xi_k\| < \frac{1}{n}$ holds for $k = 1, \dots, n$. Deduce that $(x_n)_{n\in\mathbb{N}}$ converges strongly to x.

Problem 30 (2 points)**.** A von Neumann algebra $M \subseteq B(H)$ is called *maximal abelian*, if it is abelian and for any other abelian von Neumann algebra $N \subseteq B(H)$ containing M, we have $M = N$.
Show that an abelian von Neumann algebra $M \subseteq B(H)$ is maximal abelian if and only if $M' = M$.