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Operator Algebras
Summer term 2022

Problem set 9
To be submitted by Monday, **June 13**, 2 pm.

Let H be a complex Hilbert space.

Problem 31 (4 points)**.** Let $M \subseteq B(H)$ be an abelian von Neumann algebra and $0 \neq \xi \in H$ a cyclic vector for M. Let $A \subseteq M$ be a unital C^*-subalgebra which is dense in the strong operator topology. Let μ be a Radon measure on $X := \text{Spec}(A)$ and $\chi : A \to C(X)$ be the Gelfand isomorphism. Assume

$$\langle \chi^{-1}(f)\xi, \xi \rangle = \int_X f(x)\, d\mu(x), \qquad \forall f \in C(X)$$

(a) Show that $U_0 : C(X) \to H$, $U_0 f := \chi^{-1}(f)\xi$ satisfies $\|U_0 f\|_H = \|f\|_{L^2(X,\mu)}$
(b) Show that U_0 extends to a unitary $U : L^2(X,\mu) \to H$.
(c) Show that $U^*\chi^{-1}(f)U = M_f$ for all $f \in C(X) \subseteq L^\infty(X,\mu)$ (where $M_f \in B(L^2(X,\mu))$ is the multiplication operator with symbol f) and conclude that

$$A \cong U^*AU = \{M_f \mid f \in C(X)\} \cong C(X)$$

Problem 32 (4 points)**.** Suppose H is separable and let $M \subseteq B(H)$ be an abelian von Neumann algebra. Show that there exists a single selfadjoint operator $a \in M$ which generates M, i.e. $\{a\}'' = M$.

Hint: Choose a sequence of projections $(p_n)_{n\in\mathbb{N}}$ in M which is strongly dense in the set of all projections in M (why does such a sequence exist?) and consider the operator $\sum_{n=0}^{\infty} 3^{-n} p_n$. You may use without proof the fact that the unit ball of $B(H)$ is separable and metrizable in the strong operator topology.

Problem 33 (4 points)**.** Let $M \subseteq B(H)$ be a von Neumann algebra and let $(p_i)_{i\in I}$ be an arbitrary family of projections in M. The *supremum* of the family $(p_i)_{i\in I}$ is the projection onto the subspace

$$\overline{\text{span}}\left(\bigcup_{i\in I} p_i H\right) \subseteq H$$

and is denoted by $\bigvee_{i\in I} p_i$. Show that $\bigvee_{i\in I} p_i \in M$.