Noncommutative Geometry and the Quantum Hall Effect

Bachelor's Thesis

submitted by

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Statement in Lieu of an Oath

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

Ich versichere hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Saarbrücken, April 1, 2019

Mirko Stappert

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1. Introduction

This thesis presents a proof of the quantization of the quantum Hall effect due to Bellissard, van Elst and Schulz-Baldes in their article [2] from 1994.
The Hall effect was first described by its name giver Edwin Hall in 1879. Hall Sensors, devices based on this effect, are used all over the world daily. In more recent times, its quantum version, the quantum Hall effect is still very relevant. The Nobel prizes in 1985 and 1998 were dedicated to the discovery of the "integer" and "fractional" quantum Hall effect respectively. Both of these only occur in the regime of high magnetic fields and extremely low temperatures, but exhibit peculiar robustness and allow highly precise measurements. While there are phenomenological explanations of the fractional quantum Hall effect, a completely satisfying theory is yet to come. We will therefore focus our attention on the integer quantum Hall effect, for which a complete explanation exists. Our presentation relies on the language of noncommutative geometry, which will be introduced in the first two thirds of this thesis. The Hall effect itself will be discussed afterwards.
Noncommutative geometry is a relatively new field of mathematics with connections to (algebraic) topology, differential/algebraic geometry, (homological) algebra, functional analysis, operator algebras, physics and many more. It uses concepts of each of these areas, which makes it really interesting, but also notoriously hard to learn. We attempt to explain some basic notions of noncommutative geometry, that are most relevant to the quantum Hall effect.
In Chapter 2 we explain what noncommutative geometry is supposed to mean, how one comes up with the definition of a "noncommutative" space and how to work with them. In the three chapters after that we introduce three of the main players of noncommutative geometry:

- Chapter 3 deals with the Schatten ideals and the Dixmier trace, a noncommutative analogue of an integral.
- Chapter 4 introduces K-theory, a way to keep track of noncommutative vector bundles and the first ingredient of the index pairing.
- In Chapter 5 we define three related cohomology theories: Hochschild, cyclic and periodic cohomology which are the noncommutative analogue of de Rham cohomology. Afterwards, we construct a pairing between K-theory and cohomology.

Chapter 6 deals with Fredholm modules, an interesting subject on its own, but for us it will mainly be relevant because of the Chern character: A way to construct cohomology classes from Fredholm modules that helps us to refine the pairing of Chapter 5.
When we have these mathematical tools at our disposal, we will bring them to use in Chapter 7, the central part of this thesis, to give an explanation of the quantum Hall effect based on noncommutative geometry, following [2].
For the mathematical part we mainly consulted [7], [9] and [11].

2. The Idea Behind Noncommutative Geometry

2.1 Gelfand Duality

The story of noncommutative geometry starts in the 1940s with Gelfand duality, which we will shortly outline in the following. We follow [4, Chapter II.2] and [11, Chapter 1.1]. Given a compact[1] space X, we consider the set of all continuous complex-valued functions on it: $C(X) \coloneqq \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}$. We can define addition, complex conjugation, multiplication and scalar multiplication pointwise to make $C(X)$ a commutative, complex *-algebra, which has an unit provided by the constant 1 function. The compactness of X allows us to introduce the supremum norm:

$$\|f\|_\infty \coloneqq \sup_{x \in X} |f(x)| = \max_{x \in X} |f(x)|$$

A pointwise convergent sequence of continuous functions on a compact space is already uniformly convergent, thus the limit is again continuous. Therefore $C(X)$ is complete with respect to this norm, meaning it is an unital Banach *-algebra. Since the norm fulfils the C*-condition

$$\|f^* f\|_\infty = \|f^*\|_\infty \|f\|_\infty \quad \forall f \in C(X),$$

we can finally conclude with the statement:

$C(X)$ is an unital commutative C*-algebra.

Of course, a duality has to go in two ways and the surprising fact is that each commutative unital C*-algebra actually arises in this way. We can even explicitly describe the corresponding compact space: Let A be an unital commutative C*-algebra. We define the spectrum $\hat{A}$ of A as the set of all nonzero characters:

$$\hat{A} \coloneqq \{\varphi : A \to \mathbb{C} \mid \varphi \text{ is a *-algebra morphism, not identically } 0\}$$

Giving $\hat{A}$ the topology of pointwise convergence of functionals makes it a compact space by Banach-Alaoglu[2]. Thus we can again look at the continuous complex-valued functions on this space. The following theorem relates this to the C*-algebra we started with:

[1] We could also allow X to be locally compact and take the continuous functions vanishing at infinity $C_0(X)$ instead of $C(X)$. This will still yield a commutative C*-algebra, but we lose the unit if X is not compact.

[2] Note that *-homomorphisms between C*-algebras are automatically continuous and in our case have operator norm 1. Thus $\hat{A}$ is a subset of the (closed) unit ball in the dual of A. It is easy to see that $\hat{A}$ is closed and thus compact as a closed subset of a compact space

Theorem 2.1. *(Gelfand Naimark) Let A be an unital commutative C*-algebra and $\hat{A}$ its spectrum. Then the following map is an isometric *-isomorphism:*

$$\begin{aligned} \Phi : A &\to C(\hat{A}) \\ a &\mapsto (\varphi \mapsto \varphi(a)) \end{aligned}$$

Therefore we want to treat general (meaning not necessarily commutative) C*-algebras as noncommutative topological spaces. We can now give a very simplified description of noncommutative geometry:
Starting from a topological space X and Gelfand duality, noncommutative geometry tries to capture geometric notions such as connectedness, a manifold structure, smoothness, dimension, homology and others by purely algebraic properties of $C(X)$. After completing said task, we can extend this equivalent definition and decide whether noncommutative spaces (i.e. C-algebras) are connected or manifolds if they fulfil the corresponding algebraic condition(s) and also calculate their dimension and homology groups.*
Some examples of these algebraic reformulations are given in the following table:

property/construction of X	property/construction of $C_0(X)$
metrizable	separable
connected	contains no nontrivial projections[3]
compact	unital
compactification	unitisation
Stone-Cech compactification	multiplier algebra (maximal unitisation)
one-point compactification	$C_0(X) \oplus \mathbb{C}$ (minimal unitisation)
open subset	ideal
closed subset	quotient
Borel measure	positive functional

2.2 Vector Bundles and the Serre Swan Theorem

So far these are only topological notions. Since we want to do noncommutative geometry, we now present a second correspondence with a more geometrical flavour [11, Chapter 1.2]. If we are a given a compact Hausdorff space X, we can look at vector bundles over X. These are continuous families of vector spaces, parameterized by X, that also satisfy a local triviality condition. Motivated by our previous observation that we can look at the function algebra of the space instead of the space itself, one might wonder if there is a possibility to capture the notion of a vector bundle in a purely algebraic sense. And indeed there is!
Let (E, π) be a vector bundle over X, where π denotes the projection from E to X. We know from classical differential geometry that the space of all sections of a given vector bundle $\Gamma(X, E) := \{s : X \to E \mid s \text{ is continuous}, \pi \circ s = \mathrm{id}_X\}$ is an interesting object to study[4]. Since E is a vector bundle we can add two sections, which makes $\Gamma(X, E)$

[3] I.e. $p = p^2 = p^* \implies p = 0$ or $p = 1$.

[4] In the case that X is a smooth manifold and we take the tangent bundle TX as our vector bundle, we have that $\Gamma(X, TX)$ is the set of all vector fields on X.

an abelian group, but we do additionally have an action of $C(X)$ on $\Gamma(X, E)$:

$$\begin{aligned} C(X) \times \Gamma(X, E) &\to \Gamma(X, E) \\ (f, s) &\mapsto (X \ni p \mapsto f(p) \cdot s(p)) \end{aligned}$$

Hence $\Gamma(X, E)$ is a $C(X)$-module, for which one can show that it is always finitely generated and projective. The Serre-Swan theorem tells us that we can go in the opposite direction:

Theorem 2.2. *(Serre-Swan) Let A be a finitely generated, projective $C(X)$-module, then there exists a vector bundle $E(A)$ over X such that $\Gamma(X, E(A)) \cong A$. This construction is functorial and inverse to the functor of sections $\Gamma(X, -)$. Thus we have the following equivalence of categories:*

$$\{\textit{f.d. vector bundles over } X\} \underset{E}{\overset{\Gamma(X,-)}{\rightleftarrows}} \{\textit{f.g., projective } C(X)\textit{-modules}\}$$

So we are able to describe geometrical information of a space by purely algebraic means.

3. Schatten Ideals and Dixmier Trace

In this chapter we will outline the idea behind "noncommutative calculus", mainly following [12, Chapter 5] and [7, Chapter 4].
All the Hilbert spaces in this chapter will be assumed to be infinite-dimensional.

3.1 Compact Operators and Singular Values

To do calculus, we need an appropriate notion of an "infinitesimal element", which will be provided by compact operators. We will denote the set of all compact operators on a Hilbert space H by $\mathcal{K}(H) \subseteq B(H)$. Although the compactness of an operator is certainly a smallness condition in some sense, the label infinitesimal needs some explaining. To do this we recall the spectral theorem for selfadjoint compact operators on a Hilbert space (see for example [15, Chapter 16]):

Theorem 3.1. *Let H be a Hilbert space and T a compact, selfadjoint operator, then the following holds:*
(i) The spectrum of T consists only of its eigenvalues and 0. Each nonzero eigenvalue has only finite multiplicity.
(ii) The operator T has only countably many, mutually different eigenvalues $\lambda_1, \lambda_2, \dots$. They are all real numbers and any sequence consisting of them converges to 0.
(iii) There is an orthonormal basis $(v_n)_{n\in\mathbb{N}}$ such that T can be written in the following form[1]:

$$T = \sum_{n\in\mathbb{N}} \lambda_n \left|v_n\right\rangle \left\langle v_n\right| \tag{3.1.1}$$

For a generic (not necessarily selfadjoint) compact operator, we may pass to the absolute value of the polar decomposition $|T| = \sqrt{T^*T}$, which then is a compact selfadjoint operator, thus admitting an expansion in the form of (3.1.1)

$$|T| = \sum_{k\in\mathbb{N}} s_k(T) \left|v_k\right\rangle \left\langle v_k\right|, \quad \text{where } v_k \text{ is an orthonormal basis.}$$

The $s_k(T)$ are called **singular values** of T. In general infinitely many of them are nonzero, but they will always converge to zero from above[2]. Thus given any $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that for all $k \geq N$ we have $s_k < \varepsilon$. This gives some idea why compact operators might be called infinitesimal elements. To push this analogy a little bit further, the faster the singular values converge to 0, the more infinitesimal the corresponding operator should be. This idea leads to the following definition:

Definition 3.2. *Let $a > 0$:*
$T \in \mathcal{K}(H)$ is called ***an infinitesimal of order a****, if $s_n(T) = O(n^{-a})$ which means that there exists a constant $C > 0$ such that:*

$$s_n(T) \leq Cn^{-a} \quad \forall n \in \mathbb{N}$$

[1] We use bra-ket notation: $\left|v\right\rangle \left\langle w\right| : H \to H,\ x \mapsto \langle w, x\rangle v$.
[2] Since $|T|$ is positive, all eigenvalues are positive.

3.2 Schatten Ideals

Something related that is unfortunately not quite the right concept are the so-called **Schatten Ideals** $\mathcal{L}^p(H) \subseteq \mathcal{K}(H)$. They will be helpful to construct the noncommutative integral in the next section.

$$\mathcal{L}^p(H) := \{T \in \mathcal{K}(H) \mid (s_n(T))_{n\in\mathbb{N}} \in l^p(\mathbb{N})\}$$

We will also write $\mathcal{L}^p := \mathcal{L}^p(H)$ if we consider a generic Hilbert space or the one in question is clear from context.
In the cases $p = 1, 2$ we already know them:

$$T \in \mathcal{L}^1 \iff T \text{ is trace class}$$
$$T \in \mathcal{L}^2 \iff T \text{ is Hilbert-Schmidt}$$

A lot of properties which are proven for trace class or Hilbert-Schmidt operators can actually be proven in this more general setting. We collect a few of them in the following lemma:

Lemma 3.3. *(i)* $\mathcal{L}^p \subseteq \mathcal{L}^q \subseteq \mathcal{K}$*, if* $p \leq q$*.*
(ii) $\mathcal{L}^p$ *is a Banach space for* $p > 1$*, where the norm is given by the* l^p*-norm of the corresponding sequence of singular values.*
(iii) $\mathcal{L}^p$ *is a two-sided ideal in* $B(H)$*, that is in general not closed in the norm topology.*
(iv) (Hölders inequality) $\mathcal{L}^{p_1}\mathcal{L}^{p_2}\dots\mathcal{L}^{p_n} \subseteq \mathcal{L}^p$*, if* $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{p}$*.*

3.3 The Dixmier Trace

Now that we have our infinitesimals, we want to define an integral. As we have remarked in the introduction the noncommutative analogue will be provided by a trace. Similar to classical calculus we want this trace to neglect infinitesimals of order $a > 1$. Our trace should therefore have the following two properties:

- The infinitesimals of order 1 are in the domain of the trace.
- The trace vanishes on infinitesimals of order bigger than 1.

The regular trace Tr with domain $\mathcal{L}^1$ fails to fulfil both them:

- An operator with singular value sequence $1/n$ is infinitesimal of order 1 but does not lie in $\mathcal{L}^1$.
- The diagonal operator $T = \operatorname{diag}(1, \frac{1}{4}, \frac{1}{9}, \dots)$ has singular value sequence $s_n(T) = 1/n^2$ and is therefore infinitesimal of order 2 but $\operatorname{Tr}(T) = \frac{\pi^2}{6} \neq 0$.

Nevertheless, we can modify the regular trace to rectify both problems. Regarding the first one we notice that in the case of an infinitesimal T of order 1 the divergence for the sum $\operatorname{Tr}(T) = \sum_{k\geq 1} s_k(T)$ will always be logarithmic. In a first step we could therefore look at the functional

$$T \mapsto \lim_{N\to\infty} \frac{1}{\ln(N)} \sum_{n=1}^{N} s_n(T) \tag{3.3.1}$$

This will almost be sufficient to solve our first problem (and coincidentally the second one as well), but let us first specify the domain of this functional:

Definition 3.4. *Let $p \geq 1$, we define*

$$\mathcal{L}^{p+} := \{T \in \mathcal{K} \mid \limsup_{N\to\infty} \frac{1}{\ln(N)} \sum_{n=1}^{N} s_n(T) < \infty\}.$$

They are again two-sided ideals and we have the following inclusions for $p \geq 1$

$$\mathcal{L}^{p} \subseteq \mathcal{L}^{p+} \subseteq \mathcal{L}^{p+\varepsilon} \quad \forall \varepsilon > 0.$$

At the moment we will only need $\mathcal{L}^{1+}$, which are exactly the infinitesimals of order 1. The spaces $\mathcal{L}^{p+}$ for $p > 1$ will appear again later.
We now want to define the Dixmier trace for elements of $\mathcal{L}^{1+}$. Unfortunately, the sequence in 3.3.1 is not necessarily convergent. We still want to define the Dixmier trace for all elements of $\mathcal{L}^{1+}$, so we need to find a way to extend 3.3.1 in the case of a non-convergent sequence. There is no canonical way to do this, but we get one possibility for each positive functional ω on $l^{\infty}(\mathbb{N})$, that coincides with the regular limit in the case of a convergent sequence and is also scale invariant in the following sense:

$$\omega((a_1, a_1, a_2, a_2, a_3, a_3, \dots)) = \omega((a_n)_{n\in\mathbb{N}}).$$

One example of a functional, satisfying the above conditions, is the Banach limit. Now fix such a functional ω. We can now finally come to the definition of the Dixmier trace, the noncommutative analogue of an integral.

Definition 3.5. *Let T be a positive operator in $\mathcal{L}^{1+}$, we define the **Dixmier trace***

$$\mathrm{Tr}_{\mathrm{Dix}}(T) = \omega((s_k(T))_{k\in\mathbb{N}}).$$

For general elements of $\mathcal{L}^{1+}$ we extend the definition by linearity[3].

One can show that, as the name suggests, $\mathrm{Tr}_{\mathrm{Dix}}$ is indeed a trace, meaning it is linear and $\mathrm{Tr}_{\mathrm{Dix}}(AB) = \mathrm{Tr}_{\mathrm{Dix}}(BA)$, for $A \in B(H)$ and $B \in \mathcal{L}^{1+}$ (see [7, Chapter 4.2]). Although the Dixmier trace is of great importance in noncommutative geometry[4], we will only need it in one step of our main proof. To finish this chapter we remark that for many operators, which arise in application, the Dixmier trace is independent of the chosen functional ω; such operators are then called **measurable**, see [9, Chapter 7.5]. This also partly justifies that we simply write $\mathrm{Tr}_{\mathrm{Dix}}$ and omit the dependence on ω.

[3]Note that we could in principle define Tr_ω by the same formula for nonpositive operators but this will not be linear since we would have $0 = \mathrm{Tr}_\omega(T - T) = \mathrm{Tr}_\omega(T) + \mathrm{Tr}_\omega(-T) = 2\,\mathrm{Tr}_\omega(T) \neq 0$.

[4]Especially due to its connection with the Wodzicki Residue, see for example [7, Chapter 4.2] and [12, Chapter 5].

4. K-Theory

There are two classical forms of K-theory: topological K-theory and algebraic K-theory. The former is applied to topological spaces, the latter to rings. Since a C*-algebra is a noncommutative topological space as well as a ring, we could try to apply/extend both to C*-algebras. At least for our purposes, they yield the same result so we decided to choose the topological approach. A comprehensive treatment of operator K-theory with a review of topological K-theory is given in [3].

4.1 Topological K-Theory

Vector bundles over a topological space are interesting and natural objects to study. If we want to get the maximal amount of information that vector bundles can tell us about a given space X we of course have to look at the set of all (isomorphism classes of) vector bundles over this space, denoted by $\mathrm{Vect}(X)$. We can add two vector bundles via the pointwise direct sum of vector spaces and the trivial bundle[1] serves as a neutral element for this addition. Thus we can endow $\mathrm{Vect}(X)$ with the structure of an abelian monoid. Since abelian groups are easier to deal with than abelian monoids, we would like to turn $\mathrm{Vect}(X)$ into an abelian group. Luckily it is possible to get a natural and functorial map from an arbitrary abelian monoid to an abelian group, called the Grothendieck group of said monoid, see [3, Chapter 1]. The Grothendieck group of $\mathrm{Vect}(X)$ is a known object and its properties are studied under the header (topological[2]) K-theory.
To transport this concept into the realm of noncommutative geometry, we therefore have to look at vector bundles over noncommutative spaces. By the Serre-Swan theorem this amounts to consider the projective, finitely generated modules over a C*-algebra.

4.2 Operator K-Theory

We want to reformulate this task in a way to make it more approachable from a C*-algebraic point. By definition, if P is a finitely generated, projective module over A, then there exists $n \in \mathbb{N}$ and an A-module Q such that $P \oplus Q = A^n$. Thus $M_n(A) \ni p : P \oplus Q \to P \oplus 0$ is a projection, meaning $p^2 = p^* = p$. Since projections are easier to work with, we will develop K-theory based on them instead of the equivalent approach via projective, finitely generated modules.

[1] I.e. the trivial vector space attached at each point.

[2] The earlier mentioned algebraic K-theory is defined as the Grothendieck group of the abelian monoid of isomorphism classes of projective, finitely generated modules over a given ring.

In classical K-theory we only work with equivalence classes of vector bundles, so we will also need an appropriate definition of equivalence for projections.

Definition 4.1. *Let p, q be projections in a C^*-algebra[3] A. Then we call p, q*
(i) ***Murray-von Neumann equivalent****, if there exists $u \in A$ such that $uu^* = p, u^*u = q$.*
(ii) ***homotopic****, if there exists a continuous path $\gamma : [0,1] \to A$ such that $\gamma(t)$ is a projection for all $t \in [0,1]$ and $\gamma(0) = p, \gamma(1) = q$.*

The first definition might seem strange, but it is quite easy to work with. As it turns out Murray-von Neumann equivalence coincides with the more intuitive notion of being homotopic, if we look at the projections inside of $M_2(A)$ instead of A.

Lemma 4.2. *Let p, q be projections in A, then we have*
$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ is homotopic to $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A) \iff p$ and q are Murray-von Neumann equivalent in A.

Since we need to look at modules of all dimensions, we also have to consider projections in $M_n(A)$ for arbitrary $n \in \mathbb{N}$. The correct setting for this is therefore the direct limit over all $M_n(A)$, where the map between $M_n(A)$ and $M_{n+1}(A)$ is simply given by the inclusion $p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. We denote this limit by $M_\infty(A) := \varinjlim M_n(A)$. This is a normed *-algebra, where the norm of an element is given by its norm in $M_n(A)$ for a sufficiently large $n \in \mathbb{N}$. Because of this we can look at the set of all equivalence classes of projections in $M_\infty(A)$, which we denote by $H(A)$. Due to Lemma 4.2, it does not matter, whether we choose Murray-von Neumann or homotopy as our equivalence relation.
In the same way as for $\mathrm{Vect}(X)$ we now want to endow $H(A)$ with the structure of an abelian monoid. We could define addition by $[p] + [q] = [p + q]$ but $p + q$ is only a projection if p and q are orthogonal. Fortunately we are working in the more "spacious" $M_\infty(A)$, so we can augment our projections by sufficiently many zeros to guarantee that they are orthogonal:

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right],$$

where the zeros stand for rectangular zero-matrices of size $n \times m$ and $m \times n$, if $p \in M_n(A),\ q \in M_m(A)$.
This addition is associative, commutative and has neutral element $[0]$, so we can apply the Grothendieck construction to $(H(A), +)$ and get a group.

Definition 4.3. *Let A be an unital *-algebra. The group $K_0(A)$ is defined to be the Grothendieck group of $(H(A), +)$.*

The subscript indicates the existence of higher K-groups, which is partially true. There is also K_1, but no $K_2, K_3, \dots$.
To define K_1 we will again look at equivalence classes of infinite matrices:
In the same way as for $M_\infty(A)$ we now consider the limit over n of the unitary groups

[3](i) can be defined more generally for *-algebras and (ii) for normed *-algebras

$U_n(A) = \{u \in M_n(A) \mid u \text{ is unitary}\}$, which we denote by $U^\infty(A) := \varinjlim U_n(A)$. This is again a group, so can look at the path-connected component of the identity $U_0^\infty(A)$, meaning all elements in $U^\infty(A)$, that can be connected to the identity by a continuous path. Since we have for any topological group that the path-connected component of the identity is a normal subgroup, we can make the following definition.

Definition 4.4. *Let A be an unital C*-algebra. We define $K_1(A) := U^\infty(A)/U_0^\infty(A)$*

Although $U^\infty(A)$ and $U_0^\infty(A)$ are nonabelian, the quotient $K_1(A)$ surprisingly is abelian. So far we have only described K_0 and K_1 on objects, but they are both functors, so we need to explain how they act on morphisms:
Let $\alpha : A \to B$ be an unital *-homomorphism between unital C*-algebras, this induces a map $H(\alpha)$ between $H(A)$ and $H(B)$ as follows: $H(\alpha)(p_{i,j}) = (\alpha(p_{i,j}))$, which is independent of the choice of representative p. The Grothendieck construction is functorial, so we get a map $K_0(\alpha) : K_0(A) \to K_0(B)$.
For K_1 we proceed in the same way: $K_1(\alpha)(u_{i,j}) := (\alpha(u_{i,j}))$, which again gives a well-defined map $K_1(\alpha) : K_1(A) \to K_1(B)$.
In a comprehensive presentation of K-theory we would have at least needed to mention continuity, half exactness, Bott periodicity and applications to the classification of C*-algebras. But for us, the main relevance of K-theory lies in the index pairing, which we will introduce in the next chapter.
We conclude this chapter by giving two basic examples, that we will need later on.

Example 4.5. *(i) $A = \mathbb{C}$: Any element in $H(\mathbb{C})$ is represented by a projection in $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Since projections are equivalent if and only if the subspaces they project on have the same dimension, we get $H(A) \cong \mathbb{N} \cup \{0\}$ and hence $K_0(\mathbb{C}) \cong \mathbb{Z}$. Because $U_n(\mathbb{C})$ is path-connected for all $n \in \mathbb{N}$ we have that $K_1(\mathbb{C})$ is trivial.*
(ii) $A = C(S^1)$: $K_0(C(S^1)) \cong \mathbb{Z} = K_1(C(S^1))$, the proof uses some basic tools of K-theory, that we have not developed.

5. Cohomology and Index Pairing

In this chapter, we will introduce three notions of cohomology. They are the second constituent of the index pairing, that will be defined at the end of the chapter. We mainly follow [9, Chapter 8] and [11, Chapter 3,4].

5.1 Hochschild Cohomology

Let A be an algebra over $\mathbb{C}$. We define the chain groups to be $C_n(A) = A^{\otimes(n+1)}$ and the Hochschild boundary operator

$$\begin{aligned} b : C_n(A) &\to C_{n-1}(A) \\ b(a_0 \otimes a_1 \otimes \dots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \dots \otimes (a_i a_{i+1}) \otimes \dots \otimes a_n \\ &\quad + (-1)^n (a_n a_0) \otimes a_1 \otimes \dots \otimes a_{n-1} \end{aligned}$$

One can calculate that $b^2 = 0$, so we are able to define the homology of the thus formed Hochschild complex

$$\dots \xrightarrow{b} C_n(A) \xrightarrow{b} C_{n-1}(A) \xrightarrow{b} \dots \xrightarrow{b} C_1(A) \xrightarrow{b} C_0(A) = A \tag{5.1.1}$$

and denote the Homology groups by $HH_*(A)$.

Definition 5.1. *The **Hochschild Homology groups** are defined as follows:*

$$HH_n(A) \coloneqq \frac{\ker(b : C_n(A) \to C_{n-1}(A))}{\operatorname{im}(b : C_{n+1}(A) \to C_n(A))}$$

We will only need Hochschild cohomology, so we need to dualize the complex 5.1.1: Define the cochain groups $C^n(A) = \operatorname{Hom}(C_n(A), \mathbb{C})$ and differential:

$$\begin{aligned} b^* : C^n(A) &\to C^{n+1}(A) \\ (b^*(f))(a_0 \otimes a_1 \otimes \dots \otimes a_{n+1}) &= f(b(a_0 \otimes a_1 \otimes \dots \otimes a_{n+1})) \end{aligned}$$

This yields the cochain complex:

$$C^0(A) = A^* \xrightarrow{b^*} C^1(A) \xrightarrow{b^*} C^2(A) \xrightarrow{b^*} \dots \xrightarrow{b^*} C^n(A) \xrightarrow{b^*} \dots, \tag{5.1.2}$$

with cohomology groups $HH_*(A)$.

Definition 5.2. *The **Hochschild cohomology groups** are defined as follows:*

$$HH^n(A) \coloneqq \frac{\ker(b^* : C^n(A) \to C^{n+1}(A))}{\operatorname{im}(b^* : C^{n-1}(A) \to C^n(A))}$$

5.2 Cyclic Cohomology

We will now define a subcomplex of 5.1.2, with interesting cohomology. For this purpose we consider **cyclic cochains**. These are those cochains $f \in C^n(A)$ that fulfil: $f(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n f(a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})$. We denote the set of all cyclic cochains by $C^n_\lambda(A)(\subseteq C^n(A))$. As differential we simply take the restriction of b^* to $C^n_\lambda(A)$, which we will also denote by b^*. Of course, a priori, the image of b^* only needs to lie in $C^{n+1}(A)$, but it turns out that it is actually contained in C^{n+1}_λ, so the way we wrote it makes sense and we again get a well-defined cochain complex. By the same standard procedure as for Hochschild (co)homology we obtain the **cyclic cohomology**

$$HC^n(A) := \frac{\ker(b^* : C^n_\lambda(A) \to C^{n+1}_\lambda(A))}{\operatorname{im}(b^* : C^{n-1}_\lambda(A) \to C^n_\lambda(A))}.$$

Example 5.3. *We have*

$$HC^0(A) = HH^0(A) = \{f : A \to \mathbb{C} \mid f(ab) = f(ba) \quad \forall a, b \in A\},$$

the space of traces on A.

As we already remarked above, $C^\bullet_\lambda(A)$ is a subcomplex of $C^\bullet(A)$, so we get a short exact sequence of complexes:

$$0 \to C^\bullet_\lambda(A) \to C^\bullet(A) \to C^\bullet_\lambda(A)/C^\bullet(A) \to 0.$$

By standard homological algebra this short exact sequence induces a long exact sequence of cohomology groups:

$$\cdots \to HC^n(A) \to HH^n(A) \to H^n((C/C_\lambda)^\bullet) \to HC^{n+1}(A) \to \dots \tag{5.2.1}$$

As it turns out $HC^{n-1}(A)$ and $H^n((C/C_\lambda)^\bullet)$ are isomorphic ([11, Chapter 3.7]). Thus 5.2.1 can be written as follows:

$$\cdots \to HC^n(A) \to HH^n(A) \to HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \to \dots$$

The degree two map S is called **periodicity operator**. By applying S successively we get the following two diagrams:

$$HC^0(A) \xrightarrow{S} HC^2(A) \xrightarrow{S} \dots \xrightarrow{S} HC^{2n}(A) \xrightarrow{S} \dots$$
$$HC^1(A) \xrightarrow{S} HC^3(A) \xrightarrow{S} \dots \xrightarrow{S} HC^{2n+1}(A) \xrightarrow{S} \dots$$

Definition 5.4. *We define the* ***periodic cohomology*** *of an unital *-algebra as the direct limits of the above diagrams:*

$$HP^0(A) = \varinjlim HC^{2n}(A) \ , \quad HP^1(A) = \varinjlim HC^{2n+1}(A)$$

5.3 The Index Pairing

In this section we define a pairing between K-theory and cyclic cohomology

$$\langle \cdot, \cdot \rangle : \ K_0(A) \times \bigoplus_{n \geq 0} HC^{2n}(A) \to \mathbb{C}$$

as follows. Let f be a cyclic $2n$-cocycle representing a class in $HC^{2n}(A)$ and $p \in M_k(A)$ for some $k \in \mathbb{N}$ a projection representing a class in $K_0(A)$. Then we put

$$\langle [p], [f] \rangle \coloneqq \frac{1}{n!} \hat{f}(p \otimes \cdots \otimes p).$$

By $\hat{f}$ we mean the extension of f to $M_k(A) \cong M_k(\mathbb{C}) \otimes A$ in the following way:

$$\hat{f}(m_0 \otimes a_0 \otimes \cdots \otimes m_k \otimes a_k) = \mathrm{Tr}(m_0 m_1 \dots m_k) f(a_0 \otimes \cdots \otimes a_k)$$

The pairing also satisfies $\langle [p], [f] \rangle = \langle [p], S[f] \rangle$, with the periodicity operator S. Therefore it induces a well-defined pairing between K-theory and periodic cyclic cohomology.

$$\langle \cdot, \cdot \rangle : \; K_0(A) \times HP^0(A) \to \mathbb{C}$$

We also have a pairing between $K_1(A)$ and $HP^1(A)$ that is similarly defined but we will not make use of it.
In this chapter we have consistently used tensor powers of an algebra as chain groups, but in the coming chapters we will most often switch to using cartesian powers and multilinear functionals for ease of notation. Note that linear functionals on $A^{\otimes n}$ are equivalent to multilinear functionals on A^n by the universal property of the tensor product.

6. Fredholm Modules and the Chern Character

In this chapter we will define Fredholm modules and show a way to construct a cohomology class from a Fredholm module. This will allow us to refine the index pairing, that was introduced in the last chapter. We follow [7, Chapter 4.1].

6.1 Fredholm Modules

Definition 6.1. *Let A be a *-algebra (over $\mathbb{C}$). An even* ***Fredholm module*** *over A consists of:*

- *A *-representation on a Hilbert space $\pi : A \to B(H)$*
- *A selfadjoint operator $F \in B(H)$*
  such that $F^2 = \mathrm{id}_H$ and $[F, \pi(a)] \in \mathcal{K}(H)$ for all $a \in A$
- *A selfadjoint operator $\gamma \in B(H)$ (grading operator) such that*

$$\gamma^2 = \mathrm{id}_H, \qquad [\pi(a), \gamma] = 0 \quad \forall a \in A \text{ and } \quad F\gamma = -\gamma F$$

We denote the ± 1-eigenspaces of γ by $H^{\pm}$. It holds that $H = H^+ \oplus H^-$. The restriction of π to $H^{\pm}$ are $\pi^{\pm} : A \to H^{\pm}$.
We call a Fredholm module (A, π, H, F) ***p-summable*** *if we have*

$$[F, \pi(a)] \in \mathcal{L}^p(H) \quad \text{for all } a \text{ in } A.$$

And respectively ***p+-summable****, if $[F, \pi(a)] \in \mathcal{L}^{p+}(H)$ for all a in A.*

In the presence of a grading we need to redefine commutator and trace. An operator T on H is graded if it commutes ($\deg(T) = 0$) or anticommutes ($\deg(T) = 1$) with the grading operator. By this definition we have $\deg(\pi(a)) = 0$ for all $a \in A$. The graded commutator is defined as

$$[T, U]_S = TU - (-1)^{\deg(T)\deg(U)} UT$$

for graded operators. The definition can be linearly extended to all of $B(H)$ since every operator can be written as a sum of a degree 0 and a degree 1 operator. We will make use of the shorthand notation $dT \coloneqq [F, T]_S$.
The graded trace is given by

$$\mathrm{Tr}_S(T) = \frac{1}{2} \mathrm{Tr}(\gamma F dT),$$

whenever the right-hand side is defined.
We will provide an example of a Fredholm module in the next chapter, after we have constructed the algebra of our system.

6.2 The Chern Character

One particular use of Fredholm modules is that they produce elements of cyclic and periodic cohomology.
Let (A, π, H, F) be an even Fredholm module that is n-summable for $n \in \mathbb{N}$ an odd integer.

Proposition 6.2. *The map*

$$\tau_n : C_n(A) \to \mathbb{C}$$
$$\tau_n(a_0, \dots a_n) = \operatorname{Tr}(\gamma a_0 da_1 \dots da_n)$$

gives an element in $HC^n(A)$. The application of the trace on the right-hand side is justified by the n-summability and Hölders inequality.

By Lemma 3.3 (i) we know that if a Fredholm module is n-summable, it is also $(n+2m)$-summable for all $m \in \mathbb{N}$. So by the above proposition we get a sequence of elements $(\tau_{n+2m})_{m\in\mathbb{N}}$ in even cyclic cohomology. We might hope that this sequence is compatible with the periodicity operator S such that we get an element of $HP^0(A)$. This is not quite the case, but the following lemma tells us that it only fails up to a scalar:

Lemma 6.3. *Let n be an odd integer and (A, π, H, F) an even Fredholm module, that is n-summable. Then we have:*

$$-\frac{2}{n+2} S\tau_n = \tau_{n+2}$$

Hence we see that the following sequence is compatible with the periodicity operator:

$$(\operatorname{Ch}^m(A, \pi, H, f))_{m\in\mathbb{N}} := \left((-1)^{n/2}\left(\frac{n}{2}\right)! \tau_{n+2m}\right)_{m\in\mathbb{N}}$$

This enables us to finally define the Chern character of a Fredholm module.

Definition 6.4. *Let (A, π, H, F) be an even Fredholm module that is n-summable. The **Chern character** of (A, π, H, F) is the element of periodic cyclic cohomology given by the equivalence class of $\operatorname{Ch}^m(A, \pi, H, f)$ for an arbitrary $m \in \mathbb{N}$ with $m \geq n$.*

So by Section 5.3 we have a pairing between Fredholm modules and K-theory. There is another construction which pairs Fredholm modules and K-theory. To define this pairing, we need to introduce the concept of a Fredholm operator
(compare [10, Chapter 25]).

Definition 6.5. *An operator $F \in B(H)$ is called a **Fredholm operator** if one of the following equivalent conditions holds:*

- $\dim(\ker(T)) < \infty$ *and* $\dim(H/\operatorname{im}(T)) < \infty$
- *There exist $S \in B(H)$ and $K_1, K_2 \in K(H)$ such that $ST = \operatorname{id} + K_1$ and $TS = \operatorname{id} + K_2$ (T is invertible up to a compact operator.)*

*The first characterization allows us to define the **Index** of a Fredholm operator*

$$\operatorname{Ind}(T) = \dim(\ker(T)) - \dim(H/\operatorname{im}(T)).$$

The second characterization gives us the following lemma, which we will later need in our final proof of the integrality of σ_H:

Lemma 6.6. *Let T be a Fredholm operator and K a compact operator. Then $T+K$ is also a Fredholm operator with the same index as T. ("The Fredholm index is invariant under compact perturbation")*

Now we come to the connection between Fredholm modules and Fredholm operators. The operator F itself is not Fredholm, but it "turns" other operators into Fredholm operators.

Proposition 6.7. *Let (A, π, H, F) be a Fredholm module and $q \in \mathbb{N}$. We define the following Fredholm module over $M_q(A)$ by*

$$H_q = H \otimes \mathbb{C}^q, \quad F_q = F \otimes I_q, \quad \pi_q = \pi \otimes \mathrm{id}_{\mathbb{C}^q}.$$

Let p be a projection in $M_q(A)$. Then the operator

$$\pi_q^-(p) F_q \pi_q^+ : \pi_q(p)^+ H \to \pi_q(p)^- H$$

is a Fredholm operator. Its index does not depend on the representative of the class of p in $K_0(A)$. Thus this induces a map

$$\phi : K_0(A) \to \mathbb{Z}, \quad \phi([p]) = \mathrm{Ind}(\pi_q^-(p) F_q \pi_q^+).$$

The map ϕ a special case of the Kasparov product of KK-theory:

$$K_0(A) \times K^*(A) = KK(\mathbb{C}, A) \times KK(A, \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$$

Where $K^*(A)$ is a particular abelian group formed by equivalence classes of Fredholm modules.
The relation of the two pairings is given by the following theorem.

Theorem 6.8. *In the setting as before, we have for $[p] \in K_0(A)$*

$$\langle [p], \mathrm{Ch}(A, \pi, H, F) \rangle = \phi([p]).$$

The main thing to take away from this chapter is that Fredholm modules give periodic cyclic cohomology classes and the pairing between $HP^0(A)$ and $K_0(A)$, which can a priori yield any real number, will always be an integer in the case that the cohomology class comes from a Fredholm module.

7. Application to Physics: The Quantum Hall Effect

This chapter is the heart of this thesis. We will introduce the quantum Hall effect and give a proof of the quantization of the Hall conductance according to the paper of Bellisard, van Elst and Schulz-Baldes [2].

7.1 The Classical Quantum Hall Effect

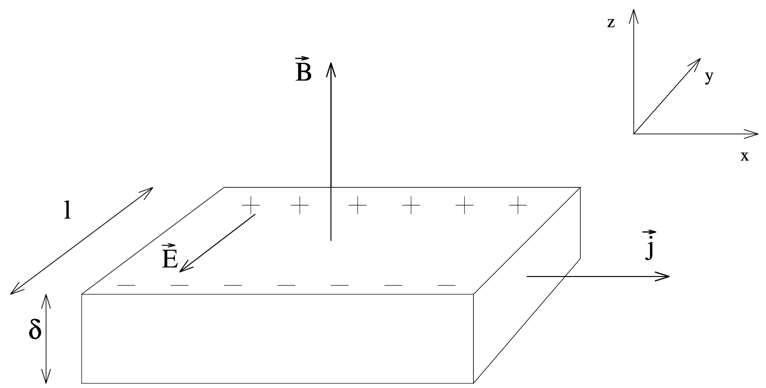

Figure 1: Schematic experimental set-up of the Hall effect.
This Figure is taken from [2, Chapter 2.1].

The setting of the Hall effect is a thin rectangular plate, assumed to lie in the xy-plane, on which electrons are confined to be. By applying an external, constant, homogeneous electric field E in the y-direction, the electrons form a current moving along this direction. In other words consider a standard conductor. We now additionally place a constant, homogenous, magnetic field B perpendicular to the plane. Thus the Lorentz force pushes the electrons into the x-direction. The current density into the x-direction j turns out to be

$$j = \frac{en\delta}{B} E =: \sigma_H E.$$

Here n denotes the volume density of electrons, e the elementary charge, δ the plate thickness and σ_H is called **Hall conductance**. If we further define the **filling factor** ν and the **von Klitzing constant** R_H (h is the Planck constant)

$$\nu = \frac{hn\delta}{Be}, \qquad R_H = \frac{h}{e^2}$$

we easily see the relation

$$\sigma_H = \frac{\nu}{R_H}.$$

Thus the Hall conductance is simply a rescaled version of the filling factor. So, why do we define all these quantities if they are essentially the same? Because in the regime of low temperatures, low plate thickness and high magnetic field we observe that the Hall conductance depends on the filling factor in a rather peculiar way.

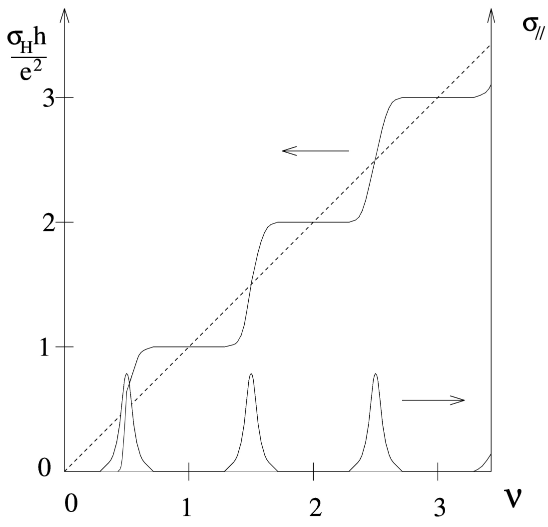

Figure 2: Dependence of the Hall conductance on the filling factor.
The dashed line shows the classical behaviour. $\sigma_{//}$ is shown in arbitrary units.
This Figure is taken from [2, Chapter 2.1].

Instead of a simple line, we see that σ_H repeatedly jumps from one integer[1] multiple of $\frac{1}{R_H}$ to the next and then stays constant until the next jump occurs, as can be seen in Figure 2. The Hall conductance stays constant whenever the **direct conductance** $\sigma_{//}$, the conductance in the direction of the electron flow, vanishes. The relationship between the two conductances is interesting, but will not be covered here.
The reason for the integrality of the Hall conductance, meaning that it always seems to be an integer multiple of the von Klitzing constant, is a deep theoretical problem. We are going to present an argument via the theory of noncommutative geometry. In the following sections we show how to associate a C*-algebra to this setup. The Hall conductance is simply the Chern character of a certain Fredholm module over this algebra applied to an eigenprojection multiplied by R_H, so we will show the integrality of σ_H by extending the result of Theorem 6.8.

[1]As we remarked in the introduction, we will only describe the integer quantum Hall effect that neglects any interaction between electrons. For temperatures in the millikelvin regime, this approximation is no longer justified and a more difficult behaviour occurs: the fractional quantum Hall effect. The jump size is no longer restricted to be an integer, but can also take some rational values. The precise nature of the numbers that can occur as jump sizes is a topic of current research.

7.2 Magnetism in Quantum Mechanics

We will first describe the n-dimensional setting and later specialize to two dimensions. There are two, in general different, objects, that are called momentum. The well known kinetical momentum, which simply is the product of mass and velocity $p_{k,j} = m\dot{x_j}$ and the canonical momentum, that arises in the Lagrangian description of classical mechanics $p_{c,j} = \frac{\partial \mathcal{L}}{\partial \dot{x_j}}$. Here $\mathcal{L} = T - V$ denotes the Lagrangian of the system, the difference between kinetic and potential energy. If the potential energy does not depend on $\dot{x_j}$ the two notions coincide because we have

$$p_{c,j} = \frac{\partial \mathcal{L}}{\partial \dot{x_j}} = \frac{\partial T}{\partial \dot{x_j}} = \frac{\partial (1/2m\dot{x_j}^2)}{\partial \dot{x_j}} = m\dot{x_j} = p_{k,j}.$$

But this is not the case in the situation of the Hall effect. The Lagrangian for a particle of mass m and charge q in the presence of a magnetic field B is given by

$$\mathcal{L} = m\frac{\|x\|^2}{2} + q\langle \dot{x}, A\rangle.$$

The magnetic vector potential A is (non uniquely) defined by the property $\nabla \times A = B$. Hence we have

$$p_{c,j} = m\dot{x_j} + qA_j = p_{k,j} + qA_j.$$

Now we have to switch from the classical to the quantum description. We will not worry about the details of “quantizing a classical system" and simply declare that we have to replace $p_{k,j}$ by the unbounded operator $-i\frac{\partial}{\partial x_j}$ acting on $L^2(\mathbb{R}^n)$ with a suitable domain[2]. Thus we have to replace $p_{c,j}$ by $-i\frac{\partial}{\partial x_j} + qA_j$.
To lighten notation and specialize to the two-dimensional case with $q = -e$ and $\vec{\mathbf{B}} = Be_z$, we denote the operators associated to the canonical momentum to x and y-direction by p_x and p_y. Written out:

$$p_x = -i\frac{\partial}{\partial x} - eA_x, \quad p_y = -i\frac{\partial}{\partial y} - eA_y$$

Now we come to the crucial observation

$$[p_x, p_y] = p_x p_y - p_y p_x = ie\frac{\partial A_y}{\partial x} - ie\frac{\partial A_x}{\partial y} = +ieB.$$

So, in the presence of a non-vanishing magnetic field, the momentum operators no longer commute and the size of the magnetic field measures the amount of noncommutativity. Thus – if we denote by U, V the unitary translation operators into x and y-direction – we have (by a Baker Campbell Hausdorff type formula) $UV = qVU$, with $q = \mathrm{e}^{ieB}$. They exactly fulfil the relations of the noncommutative torus, a very interesting object, which is the topic of our next section.
To conclude this section: The two-dimensional Hamiltonian in the presence of a magnetic field with strength B, associated potential A and no external potential takes the form

$$H_B = \frac{p_x^2 + p_y^2}{2m} = \frac{(p_{k,1} + qA_x)^2 + (p_{k,2} + qA_y)^2}{2m}. \tag{7.2.1}$$

[2] In this section, we set $\hbar = 1$

7.3 The Noncommutative Torus

In this section we will give a definition of the noncommutative torus and survey some known results, including its K-theory and cohomology.

Definition 7.1. *Fix $\theta \in \mathbb{R}$. The* ***noncommutative torus*** *is the universal C*-algebra generated by two unitaries u, v and the relation $uv = e^{2\pi i\theta} vu$.*

$$\mathcal{A}_\theta = C^*(u, v \text{ unitaries} \mid uv = e^{2\pi i\theta} vu)$$

The following map from $A_{\theta=0}$ to $C(\mathbb{T})$ with $\mathbb{T} = S^1 \times S^1$, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

$$\Phi : \mathcal{A}_{\theta=0} \to C(\mathbb{T})$$
$$u \mapsto ((z_1, z_2) \mapsto z_1), \quad v \mapsto ((z_1, z_2) \mapsto z_2)$$

is an isomorphism. This is the reason, why we call $\mathcal{A}_{\theta\neq 0}$ noncommutative tori.
In the case $\theta \notin \mathbb{Z}$ we can give an explicit representation of $\mathcal{A}_\theta$ on $L^2(\mathbb{R}/\mathbb{Z})$. We work with $\mathbb{R}/\mathbb{Z}$ instead of S^1, because it will be more convenient later.

Lemma 7.2. *Let U, V be the following operators*

$$U : L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z}), \quad (Uf)(t) = e^{2\pi i\theta} f(t)$$
$$V : L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z}), \quad (Vf)(t) = f(t - \theta)$$

Then the map

$$\Phi : \mathcal{A}_\theta \to C^*(U, V) \subseteq B(L^2(S^1)) \quad u \mapsto U,\ v \mapsto V$$

*is a *-representation.*

Proof. We have $(U(Vf))(t) = e^{2\pi i(t-\theta)} f(t-\theta)$ and $(V(Uf))(t) = e^{2\pi it} f(t-\theta)$. Since U and V are furthermore unitary, they satisfy the relations of the noncommutative torus, hence we get a (surjective) map Φ from $\mathcal{A}_\theta$ to $C^*(U, V) \subseteq B(L^2(S^1))$. □

It will later turn out that this map is actually an isomorphism.

Definition 7.3. *Let A be an unital C*-algebra, G a discrete*[3] *group and $\alpha : G \to \mathrm{Aut}(A)$ a group homomorphism. Then the* ***crossed product*** *of A and G with respect to α is*

$$A \rtimes_\alpha G = C^*(a \in A,\ \textit{relations of } A, u_g \textit{ unitaries } \forall g \in G \mid 1_A = u_e,$$
$$u_g u_h = u_{gh}, u_g^* = u_{g^{-1}}, (\alpha(g))(a) = u_g a u_g^* \ \forall g, h \in G, a \in A)$$

We see that in the case $G = \mathbb{Z}$ such an α is completely determined by the image of $1_\mathbb{Z}$ under α. The important thing for us is that $\mathcal{A}_\theta$ can be written as the crossed product $\mathcal{A}_\theta = C(S^1) \rtimes_\alpha \mathbb{Z}$, with $\alpha : \mathbb{Z} \to C(S^1)$, $u \mapsto e^{2\pi i\theta} u$, where u is an unitary generating $C(S^1)$. We want to calculate the K-theory of $\mathcal{A}_\theta$. This can be done with the help of the following theorem by Pimsner and Voiculescu that gives an exact sequence for the K-theory of a crossed product with $\mathbb{Z}$.

[3]The definition can be generalized to locally compact groups.

Theorem 7.4. *([16]) Let A be a C^*-algebra and $\alpha : \mathbb{Z} \to \mathrm{Aut}(A)$ a group homomorphism. Denote $i : A \to A \rtimes_\alpha \mathbb{Z}$, $i(a) = au_0$. Then the following sequence is exact:*

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{K_0(id_A - \alpha)} & K_0(A) & \xrightarrow{K_0(i)} & K_0(A \rtimes_\alpha \mathbb{Z}) \\
\psi \uparrow & & & & \downarrow \phi \\
K_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow{K_1(i)} & K_1(A) & \xleftarrow{K_1(id_A - \alpha)} & K_1(A)
\end{array}$$

The maps ψ and ϕ can be constructed by a "Toeplitz extension", but we will not need an explicit description of them.

Now we want to apply this result to our specific situation and calculate the K-theory of $\mathcal{A}_\theta = C(S^1) \rtimes_\alpha \mathbb{Z}$.

Proposition 7.5. *We have $K_0(\mathcal{A}_\theta) = K_1(\mathcal{A}_\theta) = \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. First we need to calculate the induced maps $K_0(\alpha)$ and $K_1(\alpha)$.
Let $[p] \in H(A)$ with $(p_{i,j})_{i,j=1\dots n} \in M_n(A)$. Then we have

$$H(\alpha)([p]) = [(\alpha(p_{i,j})_{i,j}] = [(e^{2\pi i\theta} p_{i,j})_{i,j}] = [e^{2\pi i\theta} p] = [p].$$

The last equality holds, since $\gamma : [0,1] \to M_n(A), t \mapsto e^{2\pi i\theta t} p$ is a homotopy between p and $e^{2\pi i\theta} p$. Therefore we have $H(\alpha) = \mathrm{id}_{H(A)}$ and by functoriality of the Grothendieck construction also $K_0(\alpha) = \mathrm{id}_{K_0(A)}$.
Now $K_1(\alpha)$: Let $[u] \in K_1(A)$ with $(u_{i,j})_{i,j=1,\dots n} \in U_n(A)$. Then we have

$$K_1(\alpha)([u]) = [(\alpha(u_{i,j}))_{i,j}] = [(e^{2\pi i\theta} I_n)u] = [u].$$

The last equality holds, since $e^{2\pi i\theta} I_n \in U_0^\infty(A)$. A path connecting the identity I_n with $e^{2\pi i\theta} I_n$ is for example $\gamma : [0,1] \to M_n(A), t \mapsto e^{2\pi i\theta t} I_n$. Hence we get $K_1(\alpha) = \mathrm{id}_{K_1(A)}$.
Therefore the Pimsner-Voiculescu sequence takes the form:

$$\begin{array}{ccccc}
K_0(C(S^1)) & \xrightarrow{\quad 0 \quad} & K_0(C(S^1)) & \xrightarrow{K_0(i)} & K_0(C(S^1) \rtimes_\alpha \mathbb{Z}) \\
\psi \uparrow & & & & \downarrow \phi \\
K_1(C(S^1) \rtimes_\alpha \mathbb{Z}) & \xleftarrow{K_1(i)} & K_1(C(S^1)) & \xleftarrow{\quad 0 \quad} & K_1(C(S^1))
\end{array}$$

The zero maps allow us to extract two short exact sequences. Applying the result of Example 4.5(ii) : $K_0(C(S^1)) = \mathbb{Z} = K_1(C(S^1))$, they take the following form:

$$0 \to \mathbb{Z} \to K_0(\mathcal{A}_\theta) \to \mathbb{Z} \to 0$$
$$0 \to \mathbb{Z} \to K_1(\mathcal{A}_\theta) \to \mathbb{Z} \to 0$$

Since exact sequences of free abelian groups always split, we see that:

$$K_0(\mathcal{A}_\theta) = K_1(\mathcal{A}_\theta) = \mathbb{Z} \oplus \mathbb{Z}$$

□

We still need to find suitable representatives of the generators of $K_0(A)$. To do this we need to make some preparatory definitions. From now on we will also assume that $\theta \notin \mathbb{Q}$. Let us first define the following *-subalgebra of $\mathcal{A}_\theta$:

$$B_\theta \coloneqq \Big\{ \sum_{n,m \in \mathbb{Z}} a_{n,m} u^n v^m : (|n|^k + |m|^k)|a_{n,m}| \text{ is bounded for all } k > 0 \Big\} \subseteq \mathcal{A}_\theta$$

Since in particular B_θ contains finite linear combinations of u, v, u^*, v^*, we see that it is dense in $\mathcal{A}_\theta$. The map

$$\tau\colon B_\theta \to \mathbb{C},$$
$$\sum_{n,m\in\mathbb{Z}} a_{n,m}u^n v^m \mapsto a_{0,0}$$

can be shown to be a positive linear functional of norm 1. We also have

$$\tau((u^m v^n)^*(u^m v^n)) = \tau((v^{-n}u^{-m})(u^m c^n)) = \tau(1) = \tau((u^m v^n)(u^m v^n)^*).$$

So by linearity the identity $\tau(x^*x) = \tau(xx^*)$ holds for a dense *-subalgebra and hence τ is a tracial state. This trace can be used to show that $\mathcal{A}_\theta$ is simple and one can furthermore show that τ is the only trace on $\mathcal{A}_\theta$ (see [8]). Note that both of these properties are very different in the commutative case[4]! The simplicity of $\mathcal{A}_\theta$ implies that the map from $\mathcal{A}_\theta$ to $C^*(U,V)$ in Lemma 7.2 is injective and hence bijective. This will be used in the following result by Rieffel, that helps us to find explicit representatives of $K_0(\mathcal{A}_\theta)$.

Proposition 7.6. *([18]) Let θ be in $\mathbb{R}\setminus\mathbb{Q}$.*

For all $\alpha \in (\mathbb{Z}+\theta\mathbb{Z})\cap[0,1]$ there exists a projection $P \in \mathcal{A}_\theta$ such that $\tau(P) = \alpha$.

The projection satisfying the above for $\alpha = \theta$ is called the Rieffel projection.

Proof. First note that $e^{2\pi i\theta} = e^{2\pi i\phi}$ for $\theta \equiv \phi \bmod 1$, hence we can restrict θ to be in $(0,1)$. Furthermore we have $\mathcal{A}_\theta \cong \mathcal{A}_{1-\theta}$, implemented by the map $U \mapsto \tilde{U}, V \mapsto \tilde{V}$ for generators U, V of $\mathcal{A}_\theta$ and $\tilde{U}, \tilde{V}$ for $\mathcal{A}_{1-\theta}$. Hence we will fix $\theta \in (0, \frac{1}{2})$.
We prove this result for the explicit representation of $\mathcal{A}_\theta$ on $L^2(\mathbb{R}/\mathbb{Z})$ from Lemma 7.2. Denote by M_f the multiplication operator for $f \in C(\mathbb{R}/\mathbb{Z})$. Let us first construct the Rieffel projection P by making the following ansatz:

$$P = M_g V + M_f + M_h V^*$$

The three properties $P^* = P$, $P^2 = P$ and $\tau(P) = \theta$ will put conditions on f, g and h. To determine these conditions, we will make use of the following commutation relations:

$$(V(M_f h))(t) = (V(fh))(t) = f(t-\theta)h(t-\theta) = (M_{Vf}h)(t-\theta) = (M_{Vf}Vh)(t)$$
$$\Rightarrow VM_f = M_{Vf}V$$

In the same way, we have $V^*M_f = M_{V^*f}V^*$.
Checking the relation $P = P^*$

$$M_gV + M_f + M_hV^* = V^*M_{\bar{g}} + M_{\bar{h}} + VM_{\bar{h}} = M_{V^*\bar{g}}V^* + M_{\bar{f}} + M_{V\bar{h}}V$$

Comparing the coefficients gives us $f = \bar{f}$ and $g = V\bar{h} \iff h = V^*\bar{g}$.
Checking the relation $P = P^2$
Using the commutation relation and $h = V^*\bar{g}$ we obtain:

$$P^2 = M_{gVg}V^2 + M_{V^*\bar{g}(V^*)^2\bar{g}}(V^*)^2 + M_{|g|^2+f^2+|V^*g|^2} + M_{(Vf+f)g}V + M_{(f+V^*f)V^*\bar{g}}V^*$$

[4]Every open subset of $\mathbb{T}$ yields an ideal of $C(\mathbb{T})$ by considering the functions, that vanish on this subset, so $\mathcal{A}_{\theta=0}$ is extremely far from being simple. We also have an abundance of traces, since every element of $C(\mathbb{T})$ gives a trace by integrating against it.

Equating this with $P = M_g V + M_f + M_h V^*$ and again comparing coefficients, yields the following conditions:

$$gVg = 0, \quad (Vf + f)g = g,$$
$$|g| + |V^* g|^2 + f^2 = f$$

We can find functions f and g that satisfy these constraints. Let $\varepsilon > 0$ be such that $\theta + \varepsilon < \frac{1}{2}$ and define:

$$f(t) = \begin{cases} \frac{t}{\epsilon} & \text{for } 0 \le t \le \epsilon \\ 1 & \text{for } \varepsilon \le t \le \theta \\ \frac{\theta+\varepsilon-t}{\varepsilon} & \text{for } \theta \le t \le \theta + \varepsilon \\ 0 & \text{for } \theta + \varepsilon \le t \le 1 \end{cases}, \quad g(t) = \begin{cases} \sqrt{f(t) - f^2(t)} & \text{for } \theta \le t \le \theta + \varepsilon \\ 0 & \text{else} \end{cases}$$

One can check that all three conditions are satisfied, hence $P = M_g V + M_f + M_{V^* g} V^*$ is a projection. Now we have to confirm that $\tau(P) = \theta$. By a density argument, one can show that $\tau(\sum_{n\in\mathbb{Z}} M_{f_n} V^n) = \int_0^1 f_0(t)dt$ and hence

$$\tau(P) = \tau(M_{V^*\bar{g}} V^{-1} + M_f V^0 + M_g V^1) = \int_0^1 f(t)dt = \frac{\varepsilon}{2} + \theta - \varepsilon + \frac{\varepsilon}{2} = \theta.$$

Now for a general $\xi = n + m\theta \in (\mathbb{Z} + \theta\mathbb{Z}) \cap [0, 1]$, we note that $C^*(U, V^m) \subseteq \mathcal{A}_\theta$ gives us a subalgebra, that is isomorphic to $\mathcal{A}_{m\theta}$. Hence, we can apply the above construction in $\mathcal{A}_{m\theta}$ and get a projection, that has as trace the fractional part of $m\theta$. Since this is exactly ξ, we establish that the image of τ is all of $(\mathbb{Z} + \theta\mathbb{Z}) \cap [0, 1]$. □

With some more work one can show that [1], [p] are generators for $K_0(\mathcal{A}_\theta)$, but we will not do this here[5].
Proposition 7.6 also constitutes one part of the proof that two irrational rotation algebras $\mathcal{A}_\theta$ and $\mathcal{A}_\varphi$ are isomorphic if and only if $\theta \equiv \pm\varphi \mod 1$. The second part consists of embedding $\mathcal{A}_\theta$ into a so-called AF algebra. These **a**pproximately **f**inite C*-algebras are those C*-algebras, that can be constructed as a direct limit of finite-dimensional C*-algebras. In our case, the choice of finite-dimensional C*-algebras depends on the continued fraction expansion of θ. For the proof we refer to [8, Chapter 6.3,6.4,6.5].
We now turn to cohomology, where we face the following problem: Although K-theory and cyclic cohomology are defined in the same way for *-algebras without any further structure and for C*-algebras, K-theory works best for C*-algebras and (this version of) cohomology for plain *-algebras[6]. One way to deal with this is by identifying suitable *-algebras of a C*-algebra such that the inclusion induces an isomorphism on K-theory and then work with this *-subalgebra for the cohomological aspects. An important class of such *-algebras is given in the following definition.

[5] This would require a more precise study of the short exact sequence $0 \to \mathbb{Z} \xrightarrow{K_0(i)} K_0(\mathcal{A}_\theta) \xrightarrow{\phi} \mathbb{Z} \to 0$. Since $K_0(i)(1_\mathbb{Z}) = 1_{\mathcal{A}_\theta}$ we would need to show that $\Phi([p]) = 1_\mathbb{Z}$, to establish our claim.

[6] The reason that cyclic cohomology does not work that well for C*-algebras is mainly that our tensor product does not incorporate the topological structure, see Chapter 8 for more on this.

Definition 7.7. *An unital subalgebra $B \subseteq A$ of an unital C*-algebra is said to be* ***stable under holomorphic functional calculus****, if for all $b \in B$ and holomorphic functions f on a neighbourhood of $\mathrm{sp}(a)$, we have $f(a) \in B$.*

Lemma 7.8. *Let $B \subseteq A$ be an unital *-subalgebra of an unital C*-algebra, with inclusion map $i : B \to A$. If B is stable under holomorphic functional calculus, then $K_0(i)$ is a group isomorphism.*

The main point for us is that $B_\theta \subset \mathcal{A}_\theta$ is dense and stable under holomorphic functional calculus[7], hence $K_0(B_\theta) = \mathbb{Z} \oplus \mathbb{Z}$. Such *-subalgebras, that are stable under holomorphic functional calculus are the noncommutative analogue of a smooth manifold structure on the C*-algebra. In the literature on noncommutative differential geometry the focus lies on the smooth structure and therefore one often calls B_θ the noncommutative torus instead of $\mathcal{A}_\theta$ (for example in [7], [22] and [11]).
Let us now talk about the cohomology of B_θ for θ irrational. While we have $HH^0(B_\theta) = HC^0(B_\theta) = \mathbb{C}[\tau]$ (since τ is the unique trace on $\mathcal{A}_\theta$ and therefore also on B_θ), the higher Hochschild cohomology groups are quite intricate. The behaviour depends on a number theoretic property of θ, called the Diophantine condition. Details can be found in [7, Chapter 3.2]. Surprisingly the periodic case is much easier since the cyclic cohomology stabilizes extremely early: The vector spaces are finite-dimensional and we can even find a nice basis in both cases. We define the following maps:

$$\delta_1, \delta_2 : B_\theta \to B_\theta$$
$$\delta_1\Big(\sum_{k,l\in\mathbb{Z}} a_{k,l}U^kV^l\Big) = \sum_{k,l\in\mathbb{Z}} k a_{k,l}U^kV^l, \quad \delta_2\Big(\sum_{k,l\in\mathbb{Z}} a_{k,l}U^kV^l\Big) = \sum_{k,l\in\mathbb{Z}} l a_{k,l}U^kV^l$$

These allow us to construct the 1-cocycles φ_1, φ_2:

$$\varphi_1, \varphi_2 : B_\theta^2 \to \mathbb{C}$$
$$\varphi_1(a_0, a_1) = \tau(a_0\delta_1(a_1)), \quad \varphi_2(a_0, a_1) = \tau(a_0\delta_2(a_1))$$

and also the 2-cocycle $\tau_2 : B_\theta^3 \to \mathbb{C},\ \tau_2(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2)))$.
Now we are able to state Connes' result on the periodic cohomology of B_θ; recall that S is the periodicity operator from Section 5.2.

Theorem 7.9. *([6]) $HP^0(B_\theta) = HC^2(B_\theta) \cong \mathbb{C}^2$, with basis $[S\tau], [\tau_2]$.*
$HP^1(B_\theta) = HC^2(B_\theta) \cong \mathbb{C}^2$ with basis $[\varphi_1], [\varphi_2]$.

Using these generators for cohomology and denoting $[P]$ for the projection in B_θ, that corresponds to the Rieffel projection under the isomorphism $K_0(B_\theta) \cong K_0(A_\theta)$, the index pairing takes the following form:

$$\langle [1], [S\tau] \rangle = 1,\ \langle [1], [\tau_2] \rangle = 0,\ \langle [P], [S\tau] \rangle = \theta,\ \langle [P], [\tau_2] \rangle = 1$$

In particular we have $\langle K_0(B_\theta), [\tau_2] \rangle \subseteq \mathbb{Z}$.

[7]This is the reason, why we defined B_θ the way we did and not just as finite linear combinations, which would have been sufficient for everything thus far.

Proof. Explicitly $S\tau$ acts as $(S\tau)(a_0, a_1, a_2) = \tau(a_0 a_1 a_2)$. Thus we have:

$\langle [1], [S\tau] \rangle = (S\tau)(1) = \tau(1 \cdot 1 \cdot 1) = 1,$ since τ is unital.
$\langle [P], [S\tau] \rangle = (S\tau)(P) = \tau(P^3) = \tau(P) = \theta,$ by construction of the Rieffel projection P.
$\langle [1], [\tau_2] \rangle = \tau_2(1) = \tau(\delta_1(1)\delta_2(1) - \delta_2(1)\delta_1(1)) = \tau(0) = 0.$

The calculation for $\langle [P], [\tau_2] \rangle = 1$ is more difficult and we again refer to [6]. $\square$

To put the results from this section into the context of this thesis, we need to point out that $\mathcal{A}_\theta$ is not the C*-algebra, that we will associate to our system in the next section. Even though this is not the case, there seems to be a connection between the two algebras. Both of them are constructed as a crossed product, they both are tracial and there will also be an analogue of the cocycle τ_2, that will be extremely important. In the case that our system has a lattice symmetry, the two algebras are Morita equivalent, according to [7, Chapter 4.6]. However, for a completely general disordered system, I have not been able to find a statement on the precise relationship in the literature. Nevertheless, I hope that this section will turn out to be helpful in the understanding of the coming material and will remedy the lack of proofs in this thesis so far.

7.4 The Observable Algebra

In this section we will sketch how to construct a C*-algebra from our system, according to [2, Chapter 3.5,3.6]. For more information on the general framework we refer to [1]. Firstly one constructs a compact space Ω, that is given as a strong closure of a certain set of operators. This space models the translation symmetry of our problem. We have a group action of $\mathbb{R}^2$ on Ω and use it to twist the multiplication and involution on $\mathcal{A}_0 = C_c(\Omega \times \mathbb{R}^2)$. The strength of the magnetic field influences how much the multiplication twist occurs, thus our resulting algebra will depend on B. For each $\omega \in \Omega$ we get a representation of $\mathcal{A}_0$ on $L^2(\mathbb{R}^2)$, denoted by π_ω. These representations allow us to define a C*-norm on $\mathcal{A}_0$ in the following way:

$$\|A\| = \sup_{\omega \in \Omega} \|\pi_\omega(A)\|_2$$

Finally we complete $C_c(\Omega \times \mathbb{R}^2)$ with respect to this norm. The resulting C*-algebra will be denoted by $\mathcal{A}$ and this is the C*-algebra, which we associate to our system. The representations π_ω can be extended to $\mathcal{A}$. We need to define several further structures on $\mathcal{A}$. Firstly a trace: Let P be a measure on Ω, that is invariant under the action of $\mathbb{R}^2$ and ergodic with respect to this action. We define for $A \in \mathcal{A}_0$

$$T(A) = \int_\Omega A(\omega, 0) dP(\omega).$$

This definition can be extended to $\mathcal{A}$. The map T is tracial and independent of the choice of P. It allows us to apply the GNS-construction (see [4, Chapter II.6.4]). The weak closure of the image of $\mathcal{A}$ under the GNS-representation is by definition a von Neumann algebra, denoted by $\mathcal{W}$ (the enveloping von Neumann algebra of $\mathcal{A}$). It is important to extend our C*-algebra to $\mathcal{W}$ since we will later need eigenprojections of an element of $\mathcal{A}$, but they are only guaranteed to live in $\mathcal{W}$ since we apply measurable functional calculus.

Actually we need these eigenprojections to belong to a noncommutative Sobolev space $\mathcal{S}$. Analogously to the classical Sobolev space[8], we define an inner product on $\mathcal{A}_0$ which also accounts for derivatives:

$$\langle A, B\rangle := T(A^*B) + T(\partial_1 A^* \partial_1 B + \partial_2 A^* \partial_2 B)$$

With $\partial_j A(w,x) = ix_j A(w,x)$ for $A \in \mathcal{A}_0$. The completion of $\mathcal{A}_0$ under this inner product is the noncommutative Sobolev space $\mathcal{S}$.
To conclude this section, the objects, we have defined, satisfy the following chain of inclusions:

$$\mathcal{A}_0 \subseteq \mathcal{A} \subseteq \mathcal{W} \subseteq B(L^2(\mathbb{R}^2))$$
$$\mathcal{A}_0 \subseteq \mathcal{S} \subseteq \mathcal{W} \subseteq B(L^2(\mathbb{R}^2))$$

7.5 A Fredholm Module over $\mathcal{A}_0$

Now that we have introduced our algebra, we are able to define the following collection of Fredholm modules, parameterized by Ω:

Example 7.10. *Choose A to be $\mathcal{A}_0$, $\hat{\mathcal{H}} = \mathcal{H}^+ \oplus \mathcal{H}^- = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$. For $\omega \in \Omega$, take as the representation of $\mathcal{A}_0$ on $\hat{\mathcal{H}}$ the direct sum of π_ω with itself:*

$$\hat{\pi}_\omega(A) = \begin{pmatrix} \pi_\omega(A) & 0 \\ 0 & \pi_\omega(A) \end{pmatrix}$$

We furthermore choose $F = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix}$, with $u = \frac{X_1 + iX_2}{|X_1 + iX_2|}$ and X_1, X_2 the two components of the position operator on $L^2(\mathbb{R}^2)$.
The grading operator is simply given by $\begin{pmatrix} \mathrm{id}_{L^2(\mathbb{R}^2)} & 0 \\ 0 & -\mathrm{id}_{L^2(\mathbb{R}^2)} \end{pmatrix}$.

In the next section we will relate the Chern character of these Fredholm modules to the Hall conductance.

7.6 The Integrality of σ_H

In this section we show how the tools we have developed so far can be used to show the integrality of the Hall conductance. The main formula, that we need to connect physics with math is the Kubo formula. For a derivation, we refer to [2, Chapter 4.2] and simply state the result here. Let H be a Hamiltonian of the form

$$H = H_B + V \tag{7.6.1}$$

with V a bounded potential, i.e. $V \in L^\infty(\mathbb{R}^2)$ and H_B from 7.2.1. The first summand is simply a free Hamiltonian in the presence of a magnetic field and V models all effects that we have not considered, for example thermal fluctuations, external perturbations

[8] by this we mean $H^1 = W^{1,2}$

and impurities. The fact that all that follows can be proven without putting any restrictions on V, apart from the boundedness, shows the enormous robustness of the quantum Hall effect. Besides its robustness, the quantum Hall effect can also be measured extremely precisely. This actually comes from a non-vanishing potential; as long as the disorder and randomness are not too big they "localize" the otherwise spread out quantum states. These two aspects explain the importance of the quantum Hall effect in physics. For a physical treatment of the precision and robustness of the quantum Hall effect see for example [21, Chapter 2.2].
But now back to the Kubo formula and mathematics: We denote by μ the Fermi level, a physical quantity, that parametrizes the energy distribution in a material and $P_\mu := \chi_{(-\infty,\mu]}(H)$ the spectral projection onto energies less than the Fermi level.

Proposition 7.11. *(Kubo formula) If $\mu \notin \operatorname{spec}(H)$ and $P_\mu \in \mathcal{S}$, we have in the zero temperature limit:*

$$\sigma_H = \frac{2\pi i}{R_H} T(P_\mu[\partial_1 P_\mu, \partial_2 P_\mu]) \tag{7.6.2}$$

To connect this with the things, that we did in the last chapter, we notice that if we define

$$\begin{aligned} \tau_2 : \mathcal{A}_0^3 &\to \mathbb{C}, \\ (P,Q,R) &\mapsto 2\pi i T(P[\partial_1 Q, \partial_2 R]) \end{aligned}$$

the relation 7.6.2 can be expressed as $\sigma_H = \frac{1}{R_H}\tau_2(P_\mu, P_\mu, P_\mu)$.
Of course, we have simply rewritten the equation, but the advantage of it lies in the fact that τ_2 is actually a cyclic cocycle. Even more is true, τ_2 is in fact the Chern character of the Fredholm module from the last section. Its evaluation on a projection in $\mathcal{A}_0$ is therefore an integer and since P_μ is a projection it looks as if we had already achieved our goal. Unfortunately this is not the case because P_μ is not necessarily an element of $\mathcal{A}_0$.
Since H is an unbounded operator, it is in general not so easy to treat it in an operator algebraic framework because (concrete) C*-algebras (and thus also von Neumann algebras) contain by definition only bounded operators. The best situation is when the unbounded operator H is **affiliated** to the von Neumann algebra M, intuitively this means that we can approximate H arbitrarily well by elements of A. Mathematically this is the case when every unitary in the commutant of M commutes with H. If H is affiliated to M every spectral projection of H lies in M.[9] In [2, Chapter 3.6], the authors claim that the Hamiltonian from 7.6.1 is affiliated to our C*-algebra $\mathcal{A}$. According to their definition, this means that the resolvent of H lies in $\mathcal{A}$. I am not sure, but I suspect that this definition is equivalent to H being affiliated to the enveloping von Neumann algebra $\mathcal{A}'' = \mathcal{W}$ according to our definition. We would therefore have $P_\mu \in \mathcal{W}$.
Independent whether this is true or not, we have that under reasonable physical as-

[9] If A is affiliated to M, then $f(A)$ is also affiliated to M for all Borel measurable functions f [20, Chapter 9]. In the case that f is bounded, we have $f(A) \in B(H)$. But an affiliated operator, that lies in $B(H)$ is already an element M. Hence we have that all spectral projections $\chi_{(-\infty,a]}(A)$ lie in M, since characteristic functions are bounded.

sumptions[10] $P_\mu \in \mathcal{S}$. So in the following we will take it as a given.
In this section we will extend the results of the last chapter to projections in $\mathcal{S}$, which therefore proves the integrality of σ_H.
To this end, we have to quote three statements, that are necessary for the proof of our main theorem (see [2, Chapter 4.6,4.7]).
The first of them relates the Dixmier trace to the trace T, that we defined on $\mathcal{A}$. Recall that $dT = [F, T]_S$.

Theorem 7.12. *The Fredholm module $(\mathcal{A}_0, \hat{\mathcal{H}}, \hat{\pi}_\omega, F)$ defined in Example 7.10 is 2+-summable. For every $A \in \mathcal{A}_0$ the following formula holds:*

$$T(|\partial_1 A + \partial_2 A|^2) = \frac{2}{\pi} \operatorname{Tr}_{\text{Dix}}(|d\pi_\omega(A)|^2)$$

This formula can be continued to elements of the Sobolev space $\mathcal{S}$.
In particular we have that $d\pi_\omega(A) \in \mathcal{L}^{2+}\big(L^2(\mathbb{R}^2)\big)$ if $A \in \mathcal{S}$.

The second theorem allows us to calculate the cyclic cocycle τ_2 by integrating the graded trace over Ω.

Theorem 7.13. *For $A_0, A_1, A_2 \in \mathcal{A}_0$ we have the formula*

$$\int_\Omega dP(w) \operatorname{Tr}_S(\hat{\pi}_\omega(A_0)\hat{\pi}_\omega(A_1)\hat{\pi}_\omega(A_2)) = \tau_2(A_0, A_1, A_2) \tag{7.6.3}$$

The last fact, that we need is similar to Theorem 6.8. It relates the graded trace to the Fredholm index of an operator.

Proposition 7.14. *Let $P \in \mathcal{A}_0$ be a projection such that $d\hat{\pi}_\omega(A) \in \mathcal{L}^3(L^2(\mathbb{R}^2))$. Then $\pi_\omega(P)u|_{\pi_\omega(P)\mathcal{H}^-}$ is a Fredholm operator with index:*

$$\operatorname{Ind}\left(\pi_\omega(P)u|_{\pi_\omega(P)\mathcal{H}^-}\right) = \operatorname{Tr}_S\left(\hat{\pi}_\omega(P)\hat{\pi}_\omega(P)\hat{\pi}_\omega(P)\right)$$

Now we come to the main theorem of this thesis:

Theorem 7.15. *Let P be a projection in $\mathcal{S}$. Then we have*

$$\tau_2(P, P, P) = \operatorname{Ind}\left(\pi_\omega(P)u|_{\pi_\omega(P)\mathcal{H}^-}\right)$$

In particular $\tau_2(P, P, P)$ is an integer.

Proof. By the assumption $P \in \mathcal{S}$ and Theorem 7.12 we know that $d\pi_\omega(P) \in \mathcal{L}^{2+}\big(L^2(\mathbb{R}^2)\big)$, therefore we also get $d\hat{\pi}_\omega(P) \in \mathcal{L}^{2+}(\hat{\mathcal{H}})$. Since $\mathcal{L}^{2+} \subseteq \mathcal{L}^3$ holds, we have that $d\hat{\pi}_\omega(P) \in \mathcal{L}^3(\mathcal{H})$. So we can apply Proposition 7.14 and get $\operatorname{Tr}_S(\hat{\pi}_\omega(P)\hat{\pi}_\omega(P)\hat{\pi}_\omega(P)) = \operatorname{Ind}\left(\pi_\omega(P)u|_{\pi_\omega(P)\mathcal{H}^-}\right)$. Plugging this into 7.6.3 yields:

$$\tau_2(P, P, P) = \int_\Omega dP(w) \operatorname{Ind}\left(\pi_\omega(P)u|_{\pi_\omega(P)\mathcal{H}^-}\right)$$

Now we show that $\operatorname{Ind}\left(\pi_\omega(P)u|_{\pi_\omega(P)\mathcal{H}^-}\right)$ is P-almost surely independent of ω. Since orbits of the group action are invariant under this action, we have by ergodicity that the

[10] namely that the so-called "localization length" is finite. This is not necessary, but sufficient to show that $P_\mu \in \mathcal{S}$. For more on the details, see [2, Chapter 5.2].

probability of each orbit is either 0 or 1. Therefore we only need to show that the index is invariant under the action of $\mathbb{R}^2$. If we denote the translation by $a \in \mathbb{R}^2$ with T_a, we of course have that ω is translated to $T_a\omega$. The authors of [2] now claim that translating u by a changes it to $u + O\left(\frac{1}{|X|}\right)$, which supposedly implies that $\pi_\omega(P)u|_{\pi_\omega(P)\mathcal{H}^-}$ is changed to $\pi_{T_a\omega}(P)u|_{\pi_{T_a\omega}(P)\mathcal{H}^-} + K$, where K is a compact operator. Even though I spent a lot of time thinking about it, this step is unfortunately still not quite clear to me. However, if we accept the statement, the proof is immediate since by Lemma 6.6 the Fredholm index is invariant under compact perturbation. □

As a corollary, we get the integrality of the Hall conductance.

Corollary 7.16. *If the localization length is finite and $\mu \notin \operatorname{spec}(H)$, we have in the temperature zero limit:*

$$\sigma_H \in \frac{\mathbb{Z}}{R_H}$$

Proof. Apply Theorem 7.15 to the projection P_μ. By the Kubo formula, the result follows. □

8. Outlook

We have presented an introduction to noncommutative geometry and the interesting application towards the quantum Hall effect. For both topics, there are further directions to look at, that we have not covered here. In this last chapter we will give a non-comprehensive list of some of them.
As we have already noted, the fractional quantum Hall effect is still a topic of current research; an introduction to it is given in [2, Chapter 7] or [21, Chapter 3]. There are several explanation attempts for the fractional quantum Hall effect. This article [14] by Marcolli and Mathai gives some models based on noncommutative geometry.
Since the fractional quantum Hall effect cannot be explained without considering the interaction between the electrons, the mathematics that is used needs to be able to deal with quantum field theory and especially a possibly infinite number of particles. To do this, finitely summable Fredholm modules are insufficient; one needs a generalization of them, so-called θ-summable Fredholm modules. Related to this is an extension of cyclic cohomology, that is better suited for infinite-dimensional (topological) algebras, namely entire cyclic cohomology. A treatment of both of these concepts and the relation to quantum field theory can be found in [7, Chapter 4.7,4.8,4.9].
Besides the quantum Hall effect the methods and approach, that we have used here can also be applied in similar ways to many systems from solid state physics. Two sources, that discuss this general framework are [1] and [17]. The latter uses more heavy machinery from noncommutative geometry and also covers topological insulators, a fascinating and rather new topic.
From the purely mathematical side, we have only scratched the surface regarding noncommutative geometry. Besides the fact that a C*-algebra is a reasonable definition for a noncommutative space, there are also noncommutative versions of a (smooth) manifold, a Riemannian metric and a spin structure. All of this culminates in the concept of a spectral triple: This is essentially the same as a Fredholm module, but the operator F is no longer required to be bounded. To every compact Riemannian spin manifold (M, g), one can associate a spectral triple in the following way: Choose as the underlying *-algebra $A = C^\infty(M)$, as Hilbert space the space of L^2-spinors H and the Riemannian metric induces the canonical Levi-Civita connection, from which one can construct a so-called Dirac operator D, which is an unbounded operator on H. Then (A, H, D) is a spectral triple. The crucial point is that the converse also holds. If (A, H, D) is a commutative spectral triple, meaning that A is a commutative algebra and five further conditions are satisfied then the Connes reconstruction theorem guarantees the existence of an unique compact manifold together with a Riemannian and spin structure such that the above-described construction yields the spectral triple, we started with. For an introduction to spectral triples, we refer to [22] and [9, Chapter 9,10,11]. The latter also contains a proof of the reconstruction theorem.
Besides their rich mathematical structure, Connes also developed an approach to the Standard Model of physics based on a spectral triple, see [5].

Bibliography

[1] Jean Bellissard. "Gap Labelling Theorems for Schrödinger Operators". In: *From Number Theory to Physics* (1992). DOI: 10.1007/978-3-662-02838-4_12.

[2] Jean Bellissard, Andreas van Elst, and Hermann Schulz-Baldes. "The noncommutative geometry of the quantum Hall effect". In: *J. Math. Phys.* 35.10 (1994). Topology and physics, pp. 5373–5451. ISSN: 0022-2488. DOI: 10.1063/1.530758. URL: https://doi.org/10.1063/1.530758.

[3] Bruce Blackadar. *K-theory for operator algebras.* Second. Vol. 5. Mathematical Sciences Research Institute Publications. Cambridge University Press, Cambridge, 1998, pp. xx+300. ISBN: 0-521-63532-2.

[4] Bruce Blackadar. *Operator algebras.* Vol. 122. Encyclopaedia of Mathematical Sciences. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006, pp. xx+517. ISBN: 978-3-540-28486-4; 3-540-28486-9. DOI: 10.1007/3-540-28517-2. URL: https://doi.org/10.1007/3-540-28517-2.

[5] Ali H. Chamseddine and Alain Connes. "The spectral action principle". In: *Comm. Math. Phys.* 186.3 (1997), pp. 731–750. ISSN: 0010-3616. DOI: 10.1007/s002200050126. URL: https://doi.org/10.1007/s002200050126.

[6] Alain Connes. "Noncommutative differential geometry". In: *Inst. Hautes Études Sci. Publ. Math.* 62 (1985), pp. 257–360. ISSN: 0073-8301.

[7] Alain Connes. *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994, pp. xiv+661. ISBN: 0-12-185860-X.

[8] Kenneth R. Davidson. *C^*-algebras by example.* Vol. 6. Fields Institute Monographs. American Mathematical Society, Providence, RI, 1996, pp. xiv+309. ISBN: 0-8218-0599-1. DOI: 10.1090/fim/006. URL: https://doi.org/10.1090/fim/006.

[9] José M. Gracia-Bondía, Joseph C. Várilly, and Héctor Figueroa. *Elements of noncommutative geometry.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, 2001, pp. xviii+685. ISBN: 0-8176-4124-6. DOI: 10.1007/978-1-4612-0005-5. URL: https://doi.org/10.1007/978-1-4612-0005-5.

[10] Friedrich Hirzebruch and Winfried Scharlau. *Einführung in die Funktionalanalysis.* B. I.-Hochschultaschenbücher, No. 296*. Bibliographisches Institut, Mannheim-Vienna-Zurich, 1971, p. 178.

[11] Masoud Khalkhali. *Basic noncommutative geometry*. Second. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2013, pp. xviii+239. ISBN: 978-3-03719-128-6. DOI: 10.4171/128. URL: https://doi.org/10.4171/128.

[12] Giovanni Landi. *An introduction to noncommutative spaces and their geometries*. Vol. 51. Lecture Notes in Physics. New Series m: Monographs. Springer-Verlag, Berlin, 1997, pp. xii+200. ISBN: 3-540-63509-2.

[13] Tobias Mai and Moritz Weber. *Operator Algebras; lecture notes*. 2018. URL: https://www.math.uni-sb.de/ag/speicher/lehre/opalgsose18/FunktionalanalysisII.pdf.

[14] Matilde Marcolli and Varghese Mathai. "Towards the fractional quantum Hall effect: a noncommutative geometry perspective". In: *Noncommutative geometry and number theory*. Aspects Math., E37. Friedr. Vieweg, Wiesbaden, 2006, pp. 235–261. DOI: 10.1007/978-3-8348-0352-8_12. URL: https://doi.org/10.1007/978-3-8348-0352-8_12.

[15] Reinhold Meise and Dietmar Vogt. *Einführung in die Funktionalanalysis*. Vol. 62. Vieweg Studium: Aufbaukurs Mathematik [Vieweg Studies: Mathematics Course]. Friedr. Vieweg & Sohn, Braunschweig, 1992, pp. x+416. ISBN: 3-528-07262-8. DOI: 10.1007/978-3-322-80310-8. URL: https://doi.org/10.1007/978-3-322-80310-8.

[16] M. Pimsner and Dan Voiculescu. "Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras". In: *J. Operator Theory* 4.1 (1980), pp. 93–118. ISSN: 0379-4024.

[17] Emil Prodan and Hermann Schulz-Baldes. *Bulk and boundary invariants for complex topological insulators*. Mathematical Physics Studies. From K-theory to physics. Springer, [Cham], 2016, pp. xxii+204. ISBN: 978-3-319-29350-9; 978-3-319-29351-6. DOI: 10.1007/978-3-319-29351-6. URL: https://doi.org/10.1007/978-3-319-29351-6.

[18] Marc A. Rieffel. "C^*-algebras associated with irrational rotations". In: *Pacific J. Math.* 93.2 (1981), pp. 415–429. ISSN: 0030-8730. URL: http://projecteuclid.org/euclid.pjm/1102736269.

[19] Roland Speicher. *Noncommutative (algebraic) geometry; lecture notes*. 2017.

[20] Şerban Strătilă and László Zsidó. *Lectures on von Neumann algebras*. Revision of the 1975 original, Translated from the Romanian by Silviu Teleman. Editura Academiei, Bucharest; Abacus Press, Tunbridge Wells, 1979, p. 478. ISBN: 0-85626-109-2.

[21] David Tong. "Lectures on the Quantum Hall Effect". In: 2016. arXiv: 1606.06687 [hep-th].

[22] Joseph C. Várilly. *An introduction to noncommutative geometry*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2006, pp. viii+113. ISBN: 978-3-03719-024-1; 3-03719-024-8. DOI: 10.4171/024. URL: https://doi.org/10.4171/024.

[23] Moritz Weber. *C-*algebras and K-Theory; lecture notes.* 2012. URL: https://www.math.uni-sb.de/ag/speicher/weber_lehre_CsternAlgIIsose12.html.

[24] Moritz Weber. *Functional Analysis 1; lecture notes.* 2017. URL: https://www.math.uni-sb.de/ag/speicher/lehre/fawise1718/Funktionalanalysis.pdf.

[25] Niels E. Wegge-Olsen. *K-theory and C^*-algebras.* Oxford Science Publications. A friendly approach. The Clarendon Press, Oxford University Press, New York, 1993, pp. xii+370. ISBN: 0-19-859694-4.