UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 Mathematik

Von Neumann algebras and zero sets of Bergman spaces

Bachelor thesis

submitted by

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on 24.11.2021

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1 Introduction

In this thesis, we want to get some condition on if some subset of $\mathbb{D}$ is a Bergman zero set (i.e., there is a Bergman space function vanishing on this set) We first sketch the ingredients.

1.1 Bergman Spaces

The Bergman spaces A^2_α are spaces of holomorphic functions in the unit disk square integrable with respect to the measure $(1-|z|^2)^\alpha dxdy$ with $z = x+iy$. Meaning $f \in A^2_\alpha$, if

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha dxdy < \infty.$$

The Bergman spaces where first defined by Stefan Bergman in the book [6]. These spaces are known to be notoriously difficult, despite there relativly easy definition. This definition is at the first glance quite similar to that of the Hardy space H^2, which is the space of holomorphic functions in the unit disk that satisfy

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^2 d\theta \right)^{\frac{1}{2}} < \infty.$$

But in many aspects, these spaces behave very differently.

1.2 Zero sets

A set S is said to be a zero set for a space of functions if for S and $\alpha > -1$ there is a nonzero function $f \in A^2_\alpha$ with $f(S) \equiv 0$. The set S has to be countable for the Bergman and Hardy space because otherwise the identity theorem for holomorphic functions implies that the only holomorphic function vanishing on S is the zero function. Thus, we can write $S = \{z_n | n \in \mathbb{N}\}$. For the Hardy space, the answer is well known. The Blaschke condition

$$\sum_{n\in\mathbb{N}} (1-|z_n|) < \infty$$

is equivalent to the existence of a nonzero function $f \in H^2$ vanishing on S. In fact such a function can be constructed explicitly with the help of Blaschke products. These are functions of the form

$$\mathbb{D} \to \mathbb{D}, z \mapsto z^m \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}$$

(see [9] Chapter 2 and 10).

1.3 Main result

Finding such a condition for the Bergman spaces is difficult and seems out of reach at the moment (see [13] Chapter 4). We will restrict ourselves to orbits of Fuchsian groups (these are discrete subgroups of the 2×2 matrices that act on the upper half plane $\mathbb{H}$ and the unit disk $\mathbb{D}$ via the automorphism group). One thing to note regarding Fuchsian groups is the so-called covolume. This is the area with respect to the hyperbolic area of a fundamental domain. A fundamental domain is the smallest connected domain that tesselates the upper half plane with the action of the Fuchsian group. The main result of this thesis is the following theorem; it will be proved in Chapter 6.

Theorem 6.14. *Let Γ be a Fuchsian group and let $O_1, \ldots, O_n$ be disjoint orbits in $\mathbb{D}$ of Γ. The condition*

$$s > 1 + \frac{4\pi}{covolume(\Gamma)} \sum_{i=1}^{n} \frac{k_i}{|stab_i|},$$

with $stab_i$ being the stabilizers of O_i, is a necessary and sufficient condition for the existence of a function $f \in A^2_{s-2} \setminus \{0\}$ with zero of order at least k_i on O_i.

We do not use traditional Bergman space theory to prove this result, but rather use von Neumann algebras. These are certain $*$-subalgebras of the operator algebra $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Von Neumann algebras have a variety of applications, for example in quantum field theory, knot theory and free probability. This is quite astonishing because these two topics do not have much in common. Jones [19] combined these two theories and obtain this exciting theorem above. Another point to note here is that at a first glance, the von Neumann theory is hidden entirely in the proof. Jones had proven before some theorems in other fields of mathematics with the help of von Neumann algebras which do not have much in common with them (see [18]). To combine these two concepts, we use the so-called group von Neumann algebra $vN_\omega(\Gamma)$. This is the von Neumann algebra generated by the action of the Fuchsian group on the Bergman space. We are also going to compare this to the result of Korenblum [20] and Seip [28], [29] who got results regarding zero sets, but using so-called asymptotic κ-densities, which is the more traditional approach.
One drawback of the approach using von Neumann algebras is that it does not produce an explicit function vanishing on the given subset. This can be done using cusp forms and for orbits of $PSL(2, \mathbb{Z})$. It is a known procedure and discribed by Jones [19]. The Fuchsian group $PSL(2, \mathbb{Z})$ is the so called modular group and cusp forms are in a sense invariant functions with respect to the action of the modular group. This only works for $PSL(2, \mathbb{Z})$ and not for general Fuchsian groups. By multiplying with a cusp form vanishing on the orbit, one gets a Bergman space function vanishing on the orbit. This technique has nothing in common with the argument using von Neumann algebras before. This is a powerful tool we use to generate these vanishing Bergman space functions and also we answer the question if all Bergman space functions can be expressed as a product of a cusp form and another function from a Bergman space with another weight.
At the end, we look at some more von Neumann theory that is specific to this case. We will look at the commutant of $vN_\omega(\Gamma)$ (these are all operators that commute with all elements of $vN_\omega(\Gamma)$).

1.4 Overview of this work

This work is addressed to students that know the basics in functional analysis and complex analysis. So, we will introduce all von Neumann algebra theory and Fuchsian groups that is necessary, and we will give further references for these topics.
Before we prove the main theorem, we discuss the basics that we need to use to prove the main theorem in Chapter 6. We will begin with the basics of Bergman spaces, such as all weighted Bergman spaces are reproducing kernel Hilbert spaces. This will be an important element in the proof of the main theorem. We do not need any sophisticated Bergman space theory for the proof in Chapter 6.
In Chapter 3, we will discuss basic results concerning Fuchsian groups that will help us to prove our main theorem in Chapter 6 and the results in Chapter 8. We will focus on the action of a Fuchsian group on the upper half plane and the Bergman spaces.
The chapter after that will discuss the basics on von Neumann algebras, especially on group von Neumann algebras that are generated by the action of a Fuchsian group on the upper half plane.
Continuing with von Neumann algebras, we will construct in Chapter 5 the von Neumann dimension, which will help us to measure the size of a Hilbert space on which a von Neumann algebra acts (more precisely a type II_1 factor). The von Neumann dimension will be central in the following chapters. Therefore, we will prove some essential properties of the von Neumann dimension. Unfortunately, there are two definitions of von Neumann dimension. The first definition is due to Murray and von Neumann. We will mostly use the definition of Jones [19], which measures in a sense the size of the image of the Hilbert space in the direct sum of the

GNS construction of the von Neumann algebra. Both of these definitions will be important later on. To prove some properties it is advantages to use one or the other definition. One difficulty of the definition of Murray and von Neumann is that most sources use the definition of Jones [19], in recent years almost all of them. Thus, it is quite difficult to find some good sources on some of the properties.
At that point, we can use the results of the previous chapters and prove our main theorem. This will be rather easy because most of the work has been done in the previous chapters, especially in Chapter 5. We also will look at some other approaches to this problem and look at some more classical results.
Chapter 7 will give us another view on our main question, but we will restrict our attention to $PSL(2,\mathbb{Z})$ and not arbitrary Fuchsian groups. There we will construct a Bergman space function that vanishes on an orbit of a $PSL(2,\mathbb{Z})$. For that, we will not use von Neumann theory, but cusp forms, which are special modular forms.
In the last chapter, we turn our attention to the commutant of the group von Neumann algebra $vN_\omega(\Gamma)$ in the case that the commutant also is a type II_1 factor. We will get a condition for the existence of a trace and separating vector, this vector will link the group von Neumann algebra with its commutant. An element $\mu \in \mathcal{H}$ in a Hilbert space $\mathcal{H}$ on which a von Neumann algebra M acts is called trace vector if it satisfies

$$\langle x\mu, \mu\rangle = \alpha tr(x) \text{for all } x \in M.$$

An element $\mu \in \mathcal{H}$ is called separating, if for $x \in M$ we have $x\xi = 0 \implies x = 0$. Thus, trace vector describes the trace on a von Neumann algebra with the inner product on the Hilbert space. And a separating vector embed M into $\mathcal{H}$ via the injective map $x \mapsto x\xi$. A vector This theorem is an existence result. An explicit construction of such a vector remains an open problem. We do not get any properties of this vector from this proof. One probably has to consider other ways of proving this theorem to get any information on this trace and separating vector.

2 Bergman Spaces

In this chapter, we will go over the basics for Bergman spaces; we essentially only need that they are Hilbert spaces and have a reproducing kernel on them. In the first section, we will look at the general theory of reproducing kernels. These will be important at the end of this chapter and for the proof of our main theorem. After this, we define the weighted Bergman spaces on the unit disk and prove that these are reproducing kernel Hilbert spaces. At the end of this chapter, we will port everything from the unit disk to the upper half plane with the Cayley transform.
For the basic Bergman space theory, the books from Duren and Schuster [10] gives a great introduction and Hedenmalm, Korenblum, and Zhu [13] go way deeper into the topic, it is therefore a great further reading, if one already knows the basics. Paulsen and Raghupathi [24] shows a nice introduction in the theory of reproducing kernel Hilbert spaces and give a good overview to the topic.

2.1 Basics on reproducing kernels

Before we define Bergman spaces and look at some basic results, we have to study the more general case of reproducing kernel Hilbert spaces. We will see later that Bergman spaces are of this form and inherit some nice properties. Before we go into Bergman spaces, we define and look at some properties of the more general reproducing kernel Hilbert spaces. The vector space $\mathcal{F}(X, Y)$ is the set of all maps from X to Y. (for more information, see [24])

Definition 2.1. *Let X be a set, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A vector subspace $\mathcal{H}$ of $\mathcal{F}(X, \mathbb{F})$ is called a reproducing kernel Hilbert space (RKHS), if it admits the following conditions*

1. *$\mathcal{H}$ has an inner product $\langle \cdot, \cdot \rangle$ which makes into a Hilbert space*
2. *the linear evaluation functional $ev_x : \mathcal{H} \to \mathbb{F}, f \mapsto f(x)$ is continuous for every $x \in X$.*

Reproducing kernel Hilbert spaces have a wide variety of applications, for example in machine learning and other spaces of holomorphic functions, e.g., Hardy and Bergman spaces (see [24]). With the Riesz representation theorem, we get a unique vector ε_x for every $x \in X$ with

$$f(x) = \langle f, \varepsilon_x \rangle \qquad \forall f \in \mathcal{H}.$$

This brings us to the definition of the kernel function.

Definition 2.2. *The kernel function on a RKHS is given by*

$$K(z, w) = \varepsilon_w(z)$$

with $\varepsilon_w \in \mathcal{H}$ as in the consideration above.

One important thing to note is that this kernel function is unique. The existence of such a function is important in the following and in general (not only in Bergman space theory).
With the following theorem, we can calculate this kernel function explicitly. A version of this theorem can be found in [24].

Theorem 2.3. *For every reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ the kernel function is given by*

$$K(z, w) = \sum_{n=1}^{\infty} e_n(z)\overline{e_n(w)}$$

where $(e_n)_{n\in\mathbb{N}}$ is any orthonormal basis of $\mathcal{H}$.

Proof. Let $K(z,w) = \varepsilon_w(z)$ be the reproducing kernel. Since $\varepsilon_w \in \mathcal{H}$ and

$$\varepsilon_w = \sum_{n=0}^{\infty} \langle \varepsilon_w, e_n \rangle e_n = \sum_{n=0}^{\infty} \overline{e_n(w)} e_n$$

the series $\sum_{n=0}^{\infty} \overline{e_n(w)} e_n$ converges. Further, $(e_n(w))_n \in \ell^2$. Plugging in z we have

$$\varepsilon_w(z) = \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z).$$

□

2.2 Definition of Bergman space

We will first define the (weighted) Bergman space on the unit disk $\mathbb{D}$. This is the standard way of defining the Bergman and Hardy spaces. With that at hand we can define the Bergman space over the upper half plane $\mathbb{H}$ via the Cayley transform, but more on that later.

Definition 2.4. *The (weighted) Bergman space $A^2_\alpha(\mathbb{D})$ with $\alpha > -1$ is the space of holomorphic functions on the disk circle that are square integrable with respect to the measure $(1-|z|^2)^\alpha dxdy$. I.e. f is holomorphic on $\mathbb{D}$ and satisfies*

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha dxdy < \infty$$

Note that we write x and y for the real part and imaginary part of z respectively. It can be helpful to write A^2_α as A^2_{s-2}, why this is advantageous will become clear, when we look at Bergman spaces on the upper half plane.
If one puts $d\nu_s = (1-|z|^2)^{s-2}dxdy$, then $A^2_\alpha(\mathbb{D})$ is a subspace of $L^2(\mathbb{D}, \nu_s)$. This will be beneficial, when proving that the Bergman spaces are reproducing kernel Hilbert spaces. For $\alpha = s-2$, $A_\alpha(\mathbb{D})$ inherits some properties from $L^2(\mathbb{D}, \nu_\alpha)$, but we will not use them very much. In the next section, we will prove that the Bergman spaces are all Hilbert spaces with inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z)\, \overline{g(z)}\, (1-|z|^2)^\alpha dxy.$$

This will be an important byproduct when we show that the Bergman spaces are all RKHS.

2.3 Kernel Function

To see that the Bergman spaces are RKHS, we will look at first at point evaluations. With the continuity of the point evaluation functions, we get the completeness of A^2_α as a closed subspace of $L^2(\mathbb{D}, \nu_\alpha)$. After that we will construct the kernel function on A^2_α with the help of Theorem 2.3. A version for the unweighted Bergman space can be found in [10], but we use here a different proof.

Lemma 2.5. *Let $K \subset \mathbb{D}$ be a compact set. Set $0 < \varepsilon < \inf_{x \in K} d(x, \partial\mathbb{D})$. Then there is a $M > 0$ such that for every $z \in K$*

$$|f(z)| \leq \frac{1}{\varepsilon\sqrt{\pi M}} \|f\|$$

for all $f \in A_\alpha(\mathbb{D})$. Hence, point evaluation is a bounded linear functional for all $z \in \mathbb{D}$ and $\alpha > -1$.

Proof. Let K, ε be as in the lemma. Because $\varepsilon < \inf_{x\in K} d(x, \partial\mathbb{D})$ we have $K_\varepsilon = \{z \in \mathbb{D} | d(z,K) \leq \varepsilon\} \subset \mathbb{D}$. Since K_ε is compact (as a closed bounded set) there is a $M > 0$ with $M \leq \inf_{x\in K_\varepsilon}(1-|x|^2)^\alpha$. Let $z \in K$. We have

$$f(a) = \frac{1}{\pi r^2}\int_{D_r(a)} f(z)dxdy.$$

for some $r > 0$. This formula follows from the Cauchy integral formula and integration in polar coordinates

$$\begin{aligned}\frac{1}{\pi r^2}\int_{D_r(a)} f(z)dxdy &= \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi} f(a+se^{i\theta})sd\theta ds\\ &= \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi}\frac{f(a+se^{i\theta})}{se^{i\theta}}sie^{i\theta}d\theta\frac{s}{i}ds\\ &= \frac{1}{\pi r^2}\int_0^r\int_{\partial D_s(a)}\frac{f(\zeta)}{\zeta-a}d\zeta\frac{s}{i}ds\\ &= \frac{1}{\pi r^2}\int_0^r 2\pi f(a)sds\\ &= f(a).\end{aligned}$$

Let $a \in K$, we have with the above

$$\begin{aligned}|f(a)|^2 &= \left|f^2(a)\right|\\ &= \left|\frac{M}{M\pi\varepsilon^2}\int_{D_\varepsilon(a)} f^2(z)dxdy\right|\\ &\leq \frac{1}{M\pi\varepsilon^2}\int_{D_\varepsilon(a)} |f(z)|^2 Mdxdy\\ &\leq \frac{1}{M\pi\varepsilon^2}\int_{D_\varepsilon(a)} |f(z)|^2(1-|z|^2)^\alpha dxdy\\ &\leq \frac{1}{M\pi\varepsilon^2}\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha dxdy = \|f\|^2.\end{aligned}$$

□

Duren and Schuster [10] approach the following corollary in the same manner, but for the unweighted Bergman space. This shows that the weighted Bergman spaces are Hilbert spaces. We only need to show that $A^2_\alpha \subset L^2(\mathbb{D},\nu_s)$ is closed, since $L^2(\mathbb{D},\nu_s)$ is a Hilbert space.

Corollary 2.6. *Norm convergence implies uniform convergence on each compact subset of* $\mathbb{D}$*. Furthermore, the Bergman spaces are complete.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in A^2_α, i.e., $\|f_n - f\| \xrightarrow{(n\to\infty)} 0$. Let $K \subset \mathbb{D}$ be compact, then there is some $0 < \varepsilon < 1$ with $K \subset B_\varepsilon(0)$. Thus, by Theorem 2.5 there is some $M > 0$ with

$$|f_n(z) - f_m(z)| \leq (M\pi)^{-\frac{1}{2}}\varepsilon^{-1}\|f_n - f_m\| \qquad \forall z \in K$$

So f_n is a uniform Cauchy sequence on K.
For completeness, we first note that $L^2(\mathbb{D},\nu_s)$ is complete. Thus, we only need to show that every convergent sequence in A^2_α converges to a holomorphic function. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in A^2_α norm convergent to a function $f \in L^2(\mathbb{D},\nu_s)$. There is a subsequence $(f_{n_k})_k$ of $(f_n)_n$ converging to f almost everywhere. Since $(f_n)_n$ is also a Cauchy sequence, that converges uniformly on compact sets. Thus, $(f_n)_n$ converges locally uniform to some analytic function g. We have $f(z) = g(z)$ almost everywhere. Therefore, A^2_α is closed and hence complete. □

Combining Corollary 2.6 and Theorem 2.5, we get that the Bergman spaces are reproducing kernel Hilbert spaces.
To obtain this kernel function, we also have to look at an orthonormal basis for A^2_α. We will use the gamma function $\Gamma(n+1) = \int_0^\infty t^n e^{-t} dt$. This is well known as a generalization for the factorial, since $\Gamma(n+1) = n!$ for all natural numbers. Another function used in the following proof is the Beta function $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$. It is linked to the gamma function $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ (For more information see [4]).
Following Jones [19], we find an orthonormal basis for A^2_α.

Lemma 2.7. *An orthonormal basis for A^2_α is given by*

$$e_n(z) = \sqrt{\frac{\Gamma(n+s)}{4\pi\Gamma(n+1)\Gamma(s-1)}} z^n$$

(with $\alpha = s-2$).

Proof. For the orthogonality, we calculate for $n \neq m$

$$\begin{aligned}
\langle z_n, z_m \rangle &= \int_0^1 \int_0^{2\pi} e_n \overline{e_m} (1-r^2)^\alpha r d\theta dr \\
&= \int_0^1 \int_0^{2\pi} r^n e^{in\theta} r^m e^{-im\theta} (1-r^2)^\alpha r d\theta dr \\
&= \int_0^1 r^{n+m+1} (1-r^2)^\alpha \int_0^{2\pi} e^{i(n-m)\theta} d\theta dr \\
&= 0.
\end{aligned}$$

Thus, we only have to calculate the norm of z^n to get that $\|e_n\| = 1$.

$$\|z^n\|^2 = \int_{\mathbb{D}} |z|^{2n} (1-|z|^2)^{s-2} dudv = \int_{\mathbb{D}} r^{2n} (1-r^2)^{s-2} dudv = \int_0^{2\pi} \int_0^1 r^{2n+1} (1-r^2)^{s-2} dr d\theta$$

With the substitution $t = r^2$, follows

$$4\pi \int_0^1 t^n (1-t)^{s-2} dt = 4\pi\beta(n+1, s-1) = 4\pi \frac{\Gamma(n+1)\Gamma(s-1)}{\Gamma(n+s)}.$$

Therefore, $\langle e_n, e_m \rangle = \delta_{n,m}$.
We now need to show, that the $(e_n)_n$ span the whole space. This is equivalent to Parseval's equation

$$\sum_{n=0}^\infty |\langle f, e_n \rangle|^2 = \|f\|_2^2$$

Let $s_n(z) = \sum_{k=0}^n a_k z^k$ be the partial Taylor series of f on $D_\varepsilon(0)$ for $0 < \varepsilon < 1$. We then have

$$\int_{D_\varepsilon(0)} |s_n(z)|^2 (1-|z|^2)^{s-2} dxdy = \sum_{k=0}^n |a_k|^2 \langle z^k, z^k \rangle_\varepsilon = \sum_{k=0}^n 4\pi \frac{\Gamma(k+1)\Gamma(s-1)}{\Gamma(k+s)} |a_k|^2 \varepsilon^{2(k+1)}.$$

with $\langle f, g \rangle_\varepsilon = \int_{\mathbb{D}_\varepsilon(0)} f(z)\overline{g(z)}(1-|z|^2)^\alpha dxdy$. Since $s_n(z)$ converges uniformly on $D_\varepsilon(0)$, we have with monotone convergence

$$\int_{D_\varepsilon(0)} |f(z)|^2 (1-|z|^2)^\alpha dxdy = \sum_{k=0}^\infty 4\pi \frac{\Gamma(k+1)\Gamma(s-1)}{\Gamma(k+s)} |a_k|^2 \varepsilon^{2(k+1)}.$$

Further we let $\varepsilon \to 1$ to get the desired result. Hence $(e_n)_n$ form a basis of A^2_α. □

The proof above also shows, that the measure $(1-r^2)^\alpha r d\theta dr$ is finite.
Notice that we can also write $e_n(z)$ as $\sqrt{\frac{s-1}{4\pi}\frac{s(s+1)\dots(s+n-1)}{n!}}w^n$, since n is always a natural number. We can now use Theorem 2.3 to obtain the kernel function.

Theorem 2.8. *The kernel function in A^2_α is given by*

$$K(z,w) = \frac{s-1}{4\pi}(1-z\overline{w})^{-s}.$$

Proof. The Taylor series of $(1-x)^{-s}$ in $x=0$ is

$$\sum_{n=0}^{\infty} \frac{s(s+1)\dots(s+n-1)}{n!}x^n.$$

This converges uniformly on compact subsets on $\mathbb{D}$. We therefore have $(1-x)^{-s} = \sum_{n=0}^{\infty} \frac{s(s+1)\dots(s+n-1)}{n!}$. Thus, we can conclude

$$\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)} = \frac{s-1}{4\pi}\sum_{n=0}^{\infty}\frac{s(s+1)\dots(s+n-1)}{n!}(z\overline{w})^n = \frac{s-1}{4\pi}(1-z\overline{w})^{-s}.$$

□

2.4 Transforming to the upper half plane

One can define the (weighted) Bergman spaces A^2_α similarly on the upper half plane $\mathbb{H} = \{x+iy | y>0\}$.

Definition 2.9. *The Bergman space on the upper half plane $A^2_\alpha(\mathbb{H})$ is the Hilbert space of holomorphic functions that are square integrable with respect to the measure $y^{s-2}dxdy$. This means f satisfies*

$$\int_{\mathbb{H}} |f(z)|^2 y^{s-2} dxdy < \infty.$$

Here we can see, why the notation A^2_{s-2} is quite useful.
We will also define the measure $\mu_s = y^{s-2}dxdy$ on $\mathbb{H}$ and ν_s on $\mathbb{D}$, like in Section 2.2.
We transform everything from the unit disk to the upper half plane with the Cayley transform. This biholomorphic function is well known and often used for this type of tasks. Why we transform everything to the upper half plane will be clear when talking about Fuchsian groups, which act on the upper half plane. With these two representations of essentially the same Bergman space, we can choose a domain that fits our needs best and makes computations much easier. For example, finding an orthonormal basis in the space $A_\alpha(\mathbb{D})$ is easier than in $A_\alpha(\mathbb{H})$. But everything concerning Fuchsian groups and the hyperbolic geometry is nicer on the upper half plane.

Theorem 2.10. *The Cayley transform*

$$C(z) = \frac{z-i}{z+i}$$

maps the upper half plane to the unit disk. Its inverse is

$$C^{-1}(w) = \frac{w+1}{i(w-1)}.$$

Proof. Obviously, C and C^{-1} are holomorphic on $\mathbb{H}$ and $\mathbb{D}$, respectively. One can verify easily that C^{-1} is the inverse of C. It also holds that for every $z \in \mathbb{H}$ and $w \in \mathbb{D}$:

$$\operatorname{Im} \frac{w+1}{i(w-1)} = \operatorname{Re} \frac{w+1}{1-w} = \operatorname{Re} \frac{(1+w)(1-\overline{w})}{|1-w|^2} = \frac{1-|w|^2}{|1-w|^2} > 0$$

$$\left| \frac{z-i}{z+i} \right|^2 = \frac{x^2+y^2-2y+1}{x^2+y^2+2y+1} < 1$$

□

In the following, we will use the branch of logarithm

$$\log(z) = \int_\kappa \frac{1}{w} dw + \frac{i\pi}{2},$$

with κ being the line from i to z. To see, that this actually defines a branch of logarithm we describe the line from i to z by the map

$$[0,1] \to \mathbb{C}, t \mapsto r(t)e^{i\Theta(t)}$$

with $r(t)$ and Θ real valued. Then we have

$$\begin{aligned}
\exp\left(\int_0^1 \frac{(r'(t) + ir(t)\Theta'(t))e^{i\Theta(t)}}{r(t)e^{i\Theta(t)}} dt + \frac{i\pi}{2} \right) &= \exp\left(\int_0^1 \frac{r'(t)}{r(t)} + i\Theta'(t)dt + \frac{i\pi}{2} \right) \\
&= \exp\left(\log r(1) - \log r(0) + i\Theta(1) - i\Theta(0) + \frac{i\pi}{2} \right) \\
&= \exp\left(\log|z| + i\Theta(1)\right) \\
&= |z| \exp(i\Theta(1)) = z.
\end{aligned}$$

Jones [19] describes a link between the Bergman spaces on the upper half plane and the unit disk. This theorem transforms functions from the unit disk to the upper half plane.

Theorem 2.11. *The mapping* $f \mapsto \left(\frac{2}{z+i}\right)^s f\left(\frac{z-i}{z+i}\right)$ *defines a unitary between* $A^2_{s-2}(\mathbb{D})$ *to* $A^2_{s-2}(\mathbb{H})$.

Proof. Linearity is clear. Note that the Jacobian determinant of the Cayley transform is given by

$$\left|C'(z)\right|^2 = \left| \frac{2}{(z+i)^2} \right|^2 = \frac{1}{4} \left| \frac{2}{z+i} \right|^4.$$

With the transformation theorem follows

$$\begin{aligned}
\|f\|^2_{\mathbb{D}} &= \int_{\mathbb{D}} |f(z)|^2 \, 4(1-|z|^2)^{s-2} dxdy \\
&= \int_{\mathbb{H}} \left| f\left(\frac{z-i}{z+i}\right)\right|^2 \left(1 - \left|\frac{z-i}{z+i}\right|^2\right)^{s-2} \left|\frac{2}{z+i}\right|^4 dxdy \\
&= \int_{\mathbb{H}} \left| f\left(\frac{z-i}{z+i}\right)\right|^2 \left(\frac{x^2+y^2+2y+1-x^2-y^2+2y-1}{|z+i|^2}\right)^{s-2} \left|\frac{2}{z+i}\right|^4 dxdy \\
&= \int_{\mathbb{H}} \left| f\left(\frac{z-i}{z+i}\right)\right|^2 \left(\frac{4y}{|z+i|^2}\right)^{s-2} \left|\frac{2}{z+i}\right|^4 dxdy \\
&= \int_{\mathbb{H}} \left| f\left(\frac{z-i}{z+i}\right)\right|^2 y^{s-2} \left|\frac{2}{z+i}\right|^{2s-4} \left|\frac{2}{z+i}\right|^4 dxdy \\
&= \int_{\mathbb{H}} \left| f\left(\frac{z-i}{z+i}\right)\right|^2 y^{s-2} \left|\frac{2}{z+i}\right|^{2s} dxdy \\
&= \int_{\mathbb{H}} \left| f\left(\frac{z-i}{z+i}\right) \left(\frac{2}{z+i}\right)^s \right|^2 y^{s-2} dxdy = \left\| \left(\frac{2}{z+i}\right)^s f\left(\frac{z-i}{z+i}\right) \right\|^2_{\mathbb{H}}.
\end{aligned}$$

Thus, the map is an isometry. Surjectivity follows by using the inverse of the Cayley transform. □

With the theorem above, we can now get the kernel function on the upper half plane.

Theorem 2.12. *The map*

$$K_{\mathbb{H}}(z,w) = \frac{2^s}{(i(\overline{w}-z))^s}$$

is the reproducing kernel of $A^2_{s-2}(\mathbb{H})$.

Proof. We have for every $f \in A^2_{s-2}(\mathbb{D})$ with the reproducing kernel property of $K_{\mathbb{D}}$ and the transformation theorem

$$f(C(w)) = \int_{\mathbb{H}} \overline{K_{\mathbb{D}}(C(z), C(w))} f(C(z)) \left| \frac{2}{z+i} \right|^{2s} y^{-s} dxdy.$$

With that follows

$$\begin{aligned} f(C(w)) \left(\frac{2}{w+i} \right)^s &= \left(\frac{2}{w+i} \right)^s \int_{\mathbb{H}} \overline{K_{\mathbb{D}}(C(z), C(w))} f(C(z)) \left| \frac{2}{z+i} \right|^{2s} y^{-s} dxdy \\ &= \int_{\mathbb{H}} \overline{K_{\mathbb{D}}(C(z), C(w))} \left(\frac{2}{w+i} \right)^s f(C(z)) \left| \frac{2}{z+i} \right|^{2s} y^{-s} dxdy \\ &= \left\langle K_{\mathbb{D}}(C(z), C(w)) \left(\frac{2}{z+i} \right)^s \overline{\left(\frac{2}{w+i} \right)}^s, f(C(z)) \left(\frac{2}{z+i} \right)^s \right\rangle. \end{aligned}$$

Thus, by Theorem 2.11, $K_{\mathbb{D}}(C(z), C(w)) \left(\frac{2}{z+i} \right)^s \overline{\left(\frac{2}{w+i} \right)}^s$ is a reproducing kernel in $A^2_{s-2}(\mathbb{H})$.It also follows

$$\begin{aligned} K_{\mathbb{D}}(C(z), C(w)) \left(\frac{2}{z+i} \right)^s \overline{\left(\frac{2}{w+i} \right)}^s &= \left(1 - \frac{z-i}{z+i} \overline{\left(\frac{w-i}{w+i} \right)} \right)^{-s} \left(\frac{2}{z+i} \right)^s \overline{\left(\frac{2}{w+i} \right)}^s \\ &= \left(1 - \frac{z-i}{z+i} \frac{\overline{w}+i}{\overline{w}-i} \right)^{-s} \left(\frac{2}{z+i} \right)^s \left(\frac{2}{\overline{w}-i} \right)^s \\ &= \left(\frac{2i(\overline{w}-z)}{(z+i)(\overline{w}-i)} \right)^{-s} \left(\frac{2}{z+i} \right)^s \left(\frac{2}{\overline{w}-i} \right)^s \\ &= 2^s \left(i(\overline{w}-z) \right)^{-s}. \end{aligned}$$

Thus $K_{\mathbb{H}}(z,w)$ fulfills the reproducing kernel condition, thus $K_{\mathbb{H}}$ is the reproducing kernel by the uniqueness of the reproducing kernel. □

Everything done in this section can be made in reverse, i.e., from the upper half plane to the unit disk, in essentially the same manner as done above. Moreover, this porting can be done with every domain Ω conformal to the unit disk, i.e., every simply connected domain except $\mathbb{C}$ itself. For that we have to use some conformal map φ that maps $\mathbb{D}$ to Ω instead of the Cayley transform. From Liouville's theorem, it follows directly that $A_\alpha(\mathbb{C})$ only contains the constant zero function. For more details, see [10].

3 Fuchsian Groups

Fuchsian groups are in a sense discrete subgroups of the automorphism group of the upper half plane. The action of the Fuchsian group on the upper half plane and the Bergman spaces will help us obtain our main result, since the zero sets we are looking at are orbits of Fuchsian groups. In this chapter, we discuss some properties of Fuchsian groups which we need in the following and also look at the actions on the Bergman space as well as on the upper half plane. This is a well-understood topic, but we do not need much of it, so this is just a brief introduction to this topic.
We used basic literature of Dal'Bo [8], which is more concentrated on hyperbolic geometry and Tsuji [31] which describes in the last chapter Fuchsian groups acting on the unit disk, but this is the same since the unit disk and upper half plane are conformally equivalent.

3.1 Basics

We will first look at the Möbius transformations on the upper half plane $\mathbb{H}$. They are given by

$$z \mapsto \frac{az+b}{cz+d} \qquad \qquad \forall a,b,c,d \in \mathbb{R} \text{ and } ad-bc>0$$

and these are all automorphisms of the upper half plane into itself. The map from $GL(\mathbb{R},2)$ (i.e., the invertible real 2×2 matrices) to the automorphisms on $\mathbb{H}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$$

is a surjective group homomorphism. This map is clearly not injective because for every $\lambda \in \mathbb{R}_+$ is

$$\frac{\lambda az+\lambda b}{\lambda cz+\lambda d} = \frac{az+b}{cz+d}$$
$$\text{and}$$
$$\frac{-az-b}{-cz-d} = \frac{az+b}{cz+d}$$

To solve this first "problem", we need to look at a subgroup of $GL(2,\mathbb{R})$ namely the special linear group.

Definition 3.1. *The subgroup*

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; ad-bc=1 \right\}$$

of $GL(2,\mathbb{R})$ is called the special linear group (short $SL(2,\mathbb{R})$).

The second "problem" implies that $A \in SL(2,\mathbb{R})$ and $-A \in SL(2,\mathbb{R})$ correspond to the same Möbius transformation. To solve this, we introduce the projective special linear group.

Definition 3.2. *The projective special linear group or short $PSL(2,\mathbb{R})$ is the quotient $SL(2,\mathbb{R})/\{\pm I\}$, with I being the identity matrix.*

In the following $g \in PSL(2,\mathbb{R})$ with representative

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In the following we identify $PSL(2,\mathbb{R})$ with the Möbius transforms on the upper half plane.

Lemma 3.3. *The map $PSL(2,\mathbb{R}) \to Aut(\mathbb{H}), g \mapsto \frac{az+b}{cz+d}$ is an isomorphsim where $Aut(\mathbb{H})$ are the conformal maps from the upper half plane onto itself.*

Proof. Surjectivity is clear. For injectivity, let $g \in PSL(2,\mathbb{R})$ with $\frac{az+b}{cz+d} = z$. Then, $az + b = cz^2 + dz$ and therefore $c = 0$, $b = 0$ and $d = a$. Since, $\det(g) = 1$ we have $d = a = 1$ or $d = a = -1$. Thus, $g = \pm I$. Since, $I = -I$ in $PSL(2,\mathbb{R})$ we have that injectivity. $\square$

Now we can define what we mean by Fuchsian group.

Definition 3.4. *A Fuchsian group is a discrete subgroup of $PSL(2,\mathbb{R})$.*

From now on, if not stated otherwise Γ is a Fuchsian group. Important examples for Fuchsian groups are $PSL(2,\mathbb{Z})$. Another important concept in this regard is the fundamental domain. A fundamental domain is in a sense the smallest domain in $\mathbb{H}$ that spans everything with the help of the action of G (see [8]).

Definition 3.5. *A closed connected subset F of $\mathbb{H}$ is called fundamental domain, if it satisfies the following conditions:*

(i) $int(F) \neq \emptyset$

(ii) $\bigcup_{\gamma\in\Gamma} \gamma F = \mathbb{H}$

(iii) $int(F) \cap int(\gamma F) = \emptyset$ *for all* $\gamma \in \Gamma \setminus \{\gamma\}$

Dal'Bo [8, Ch.2.3] constructed a fundamental domain for every Fuchsian group. The fundamental domain tessellates the whole plane. This can be helpful, if we look at some Γ invariant function. In this case, we only need to look at some fundamental domain and not the whole plane. Furthermore, a fundamental domain gives a sense of how big the underlying Fuchsian group is. This can be seen, when we define the covolume of a Fuchsian group. But to make the area invariant under the action one has to use the hyperbolic area instead of the Lebesgue area. The hyperbolic area is the measure $\mu := \frac{dxdy}{y^2}$ on $\mathbb{H}$. For this we have to first look at the imaginary part of after the translation on the upper half plane (see [19]).

Proposition 3.6. *For $g \in PSL(2,\mathbb{R})$ and $z \in \mathbb{H}$ we have* $\operatorname{Im}(g(z)) = \frac{\operatorname{Im}(z)}{|cz+d|^2}$.

Proof. Let be g and z be fixed as in the assertion. Then

$$\begin{aligned} g(z) &= \frac{az+b}{cz+d} = \frac{ax+b+iay}{cx+d+icy} = \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{ac\,|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2} \\ &= \frac{ac\,|z|^2 + adx + bcx + bd}{|cz+d|^2} + i\frac{ady - bcy}{|cz+d|^2}. \end{aligned}$$

Here, $ady - bcy = y$, as the determinant of g is equal to 1. This ends the proof. $\square$

The next proposition shows a glimpse on why this is.

Proposition 3.7. *The measure μ is invariant under the action of a Fuchsian group Γ. This means*

$$\int_{gB} \frac{dxdy}{y^2} = \int_B \frac{dxdy}{\operatorname{Im}(g(z))^2}$$

for every $g \in \Gamma$, $f \in L^1(\mathbb{H},\mu)$ and $B \subset \mathbb{H}$ Borel set.

Proof. The Jacobian determinate is (with the determinat of g being 1)

$$|g'(z)|^2 = \left|\frac{ad-bc}{(cz+d)^2}\right|^2 = \frac{1}{|cz+d|^4}.$$

With the transformation theorem follows

$$\begin{aligned}\int_{gB}\frac{dxdy}{y^2} &= \int \mathbb{1}_{gB}(z)\frac{dxdy}{\operatorname{Im}(z)^2}\\ &= \int \mathbb{1}_{B}(g(z))\frac{1}{|cz+d|^4}\frac{dxdy}{\operatorname{Im}(g(z))^2}\\ &= \int \mathbb{1}_{B}(g(z))\frac{dxdy}{\operatorname{Im}(z)^2}\\ &= \int_B \frac{dxdy}{\operatorname{Im}(z)^2}\end{aligned}$$

□

The following lemma by Beardon [5] lets us define an important measure of Fuchsian groups and their action on the upper half plane.

Lemma 3.8. *For two fundamental domains F, F' of a Fuchsian group follows*

$$\int_F \frac{dxdy}{y^2} = \int_{F'}\frac{dxdy}{y^2}.$$

Proof. We write $\mu[F] := \int_F \frac{dxdy}{y^2}$. With the invariance of the measure of $\frac{dxdy}{y^2}$ with respect to the action of the Fuchsian group follows (Proposition 3.7):

$$\mu[F] = \mu\left[F\cap\left(\bigcup_{\gamma\in\Gamma}\gamma F'\right)\right] = \mu\left[\bigcup_{\gamma\in\Gamma}(F\cap\gamma F')\right] = \mu\left[\bigcup_{\gamma\in\Gamma}(\gamma^{-1}F\cap F')\right] = \mu[F']$$

□

The covolume describes in a sense the size of the Fuchsian group as well. But if the Fuchsian group is bigger, then its covolume is smaller, since the "tiles" that tessellates the upper half plane are smaller.

Definition 3.9. *The covolume(Γ) for a Fuchsian group Γ is the hyperbolic area of a fundamental domain F, i.e.*

$$\int_F \frac{dxdy}{y^2}.$$

Note that the definition of *covolume* does not depend on the choice of fundamental domain, only on the Fuchsian group Γ.

Example 3.10. *For $\Gamma = PSL(2,\mathbb{Z})$ a fundamental domain is given by*

$$F := \left\{z\in\mathbb{H} \mid |z|>1 \text{ and } |x|\le\frac{1}{2}\right\}.$$

Figure 1 shows the tesselation of the fundamental domain F. It was created using the fact, that for an element $x\in\partial F$ we have $g(x)\in\partial gF$ for some $g\in PSL(2,\mathbb{Z})$, since g is a homeomorphism and that the maps $z\mapsto z+1$ and $z\mapsto -\frac{1}{z}$ generate $PSL(2,\mathbb{Z})$ (see [2]). The figure is slightly inaccurate, since one cannot use all points on the boundary and cannot form all combinations of the two maps, therefore it is not complete but gives a good impression on how these fundamental domains can look like.

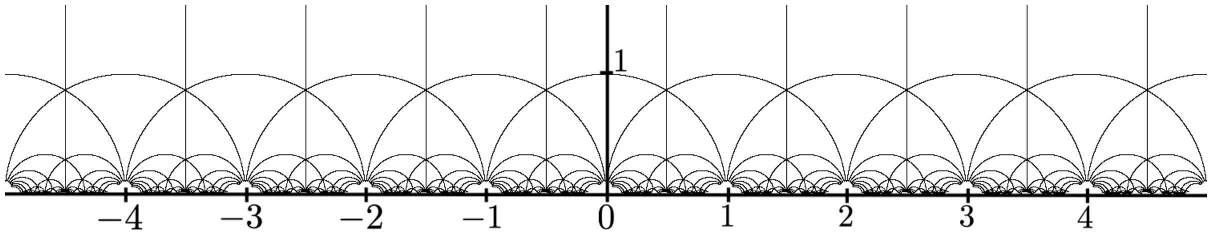

Figure 1: fundamental domain of $PSL(2,\mathbb{Z})$

This follows by using the translations

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which generate $PSL(2,\mathbb{Z})$ and translating every point of $\mathbb{H}$ to F (for more details see[11]). To calculate the covolume of Γ we just integrate over the fundamental domain.

$$\int_F \frac{dxdy}{y^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} \frac{dxdy}{y^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} -\frac{1}{y}\Big|_{\sqrt{1-x^2}}^{\infty} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x)|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi}{3}$$

In the following, we demand also that *covolume*(Γ) is finite for every Fuchsian group. This need not be the case (see for example [31]).
Akemann [1] showed all Fuchsian groups have a nice property that will later be essential when it comes to von Neumann algebras. This is a quite algebraic and technical proof, we will miss out.

Theorem 3.11. *All Fuchsian goups are icc. (infinite conjugacy classes, i.e., for all $g \in \Gamma \setminus \{e\}$ the set $\{hgh^{-1} | h \in \Gamma\}$ is infinite).*

3.2 $PSL(2,\mathbb{R})$ on Bergman space

$PSL(2,\mathbb{R})$ not only acts on the upper half plane, but also on the Bergman spaces A^2_{s-2}. In the following, we will use the branch of logarithm

$$\log(cz+d) = \int_\kappa \frac{c}{cw+d} dw + \log(ci+d)$$

with κ being the line from i to z and $\log(ci+d)$ some branch of logarithm, which cannot be the same for every $c, d \in \mathbb{R}$. This is a branch of logarithm, since $\log(cw+d)$ is an antiderivative of $\frac{c}{cw+d}$. We showed, that this is a branch of logarithm in the last chapter.
Following Jones [19], we define the following unitary operator. This operator will be refined in the next two chapter and will then be our main tool.

Theorem 3.12. *The map*

$$\hat{\pi}_s(g^{-1})(f)(z) = \frac{1}{(cz+d)^s} f(g(z))$$

defines a unitary operator on $L^2(\mathbb{H}, \mu_s)$. Furthermore $\hat{\pi}_s(g^{-1})(f)$ is holomorphic if f is holomorphic.

Proof. Linearity is clear. Let $f, h \in L^2(\mathbb{H}, \mu_s)$. With the transformation theorem it follows:

$$\begin{aligned} \langle f, h \rangle &= \int_{\mathbb{H}} f(z)h(z)y^s \frac{dxdy}{y^2} \\ &= \int_{\mathbb{H}} f(g(z))h(g(z)) \operatorname{Im}(g(z))^{s-2} \frac{1}{|cz+d|^4} dxdy \\ &= \int_{\mathbb{H}} f(g(z))h(g(z)) \frac{y^s}{|cz+d|^{2s}} \frac{dxdy}{y^2} = \langle \hat{\pi}_s(g^{-1}f), \hat{\pi}_s(g^{-1}h) \rangle \end{aligned}$$

Thus, $f \mapsto \hat{\pi}_s(g^{-1}f)$ is an isometry on $L^2(\mathbb{H}, \mu_s)$.
We now need to prove, that $\hat{\pi}_s$ is surjective, but since $\hat{\pi}_s(g^{-1})(\hat{\pi}_s(g)(f)) = f$ this is clear. Hence $\hat{\pi}_s(g^{-1})$ defines a unitary on $L^2(\mathbb{H}, \mu_s)$.
The holomophy of $\hat{\pi}_s(g^{-1})(f)$ is clear, if f is homomorphic, since multiplication and composition of holomorphic function is holomorphic. $\square$

4 Von Neumann algebras

We will first look at some basic properties of general von Neumann algebras. With this knowledge, we dive deeper into group von Neumann algebras and which group properties imply certain properties of the group von Neumann algebra. We then want to categorize factors further. Therefore, we look at projections and what they tell us about the von Neumann algebra, they belong to. We are then ready to define the types I,II and III, the type II factor being the one of interest for us.
This chapter is mostly based on the works of Jones [15], Peterson [25] and Anantharaman and Popa [3].

4.1 The Basics

Before we can define von Neumann algebras, we have to look at some topologies on $\mathcal{B}(\mathcal{H})$.

Definition 4.1. *The weak operator topology is induced by the seminorms $T \mapsto |\langle T\xi, \nu\rangle|$ for all $\xi, \nu \in \mathcal{H}$.*
The strong operator topology is induced by the seminorms $T \mapsto \|T\xi\|$ for all $\xi \in \mathcal{H}$.

For more details, see Peterson [25]. We now can define von Neumann algebras.

Definition 4.2. *A von Neumann algebra is a $*$-subalgebra of the bounded operators $\mathcal{B}(\mathcal{H})$ over some Hilbert space $\mathcal{H}$ that is closed in the weak operator topology and contains the identity $1_{\mathcal{B}(\mathbb{H})}$.*

If we look at the definition of von Neumann algebra, it is quite similar to that of a C^*-algebra, except that a C^*-algebra is closed under the norm topology. Notice also that every C^*-algebra can be identified with a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ via the second Gelfand-Naimark theorem. This difference seems like a small detail at the first glance, but changes quite a lot. For example, in von Neumann theory projections play a vital role in describing the nature of a von Neumann algebra, as we will see later. In C^*-algebras it is not the case. On the other hand it is easy to find some isomorphisms between C^*-algebras, but this is way more difficult for von Neumann algebras.
In the following, let M be a von Neumann algebra. There are many types of von Neumann algebras. The most important for us are factors. Following Peterson [25], we define the commutant and center of von Neumann algebras.

Definition 4.3.
(i) Let $\mathcal{H}$ be some Hilbert space and $N \subset \mathcal{B}(\mathcal{H})$. Then $N' = \{y \in \mathcal{B}(\mathcal{H}) | xy = yx \text{ for all } x \in N\}$ is the commutant of N.

(ii) The center of a set N is $N \cap N'$.

(iii) A von Neumann algebra M is called a factor if the center of M is trivial (i.e., the center is equal to $\mathbb{C}1$).

There are also subtypes of factors, but we will discuss them later in Section 4.4.
We will now define some important notions that will be useful when we look at group von Neumann algebras. It also will be essential, when we look at the Gelfand-Naimark-Segal construction in 5.1. Furthermore, these definitions are important in other operator algebras such as C^* algebras.

Definition 4.4. *Let M be a von Neumann algebra and φ a functional on M.*

(i) φ is called positive if $\varphi(x^*x) \geq 0$ for all $x \in M$.

(ii) If φ is positive and $\varphi(1) = 1$, then φ is a state.

(iii) *If φ is positive and satisfies*

$$\varphi(x^*x) = 0 \implies x = 0,$$

then φ is faithful.

(iv) *We call φ a trace, if it is positive and for all $x, y \in M$ we have $\varphi(xy) = \varphi(yx)$. If in addition φ is a state, φ is called a tracial state.*

For more details, see [15], [25] and [3]. The most known example of a faithful tracial state is the normalized trace on the matrix algebra A_N of $N \times N$ matrices. (The normalized trace here means $\frac{1}{N}tr_N(\cdot)$)

4.2 Group von Neumann algebra

The group von Neumann algebra is one of the central tools to prove our main theorem and is also in other circumstances quite useful. The goal is essentially to describe the action of a Fuchsian group on the Bergman space with the help of a von Neumann algebra on it. This von Neumann algebra will be the group von Neumann algebra of this Fuchsian group.
We will in the following only consider discrete groups, if not stated otherwise. For a group Γ we will construct a von Neumann algebra on $\ell^2(\Gamma)$, with $\ell^2(\Gamma)$ being the square summable functions on Γ. We call $(\delta_g)_{g \in G}$ the natural orthonormal basis for $\ell^2(\Gamma)$ with

$$\delta_g(h) = \begin{cases} 1 & \text{if } h = g \\ 0 & \text{otherwise} \end{cases}.$$

Obviously, this is an orthonormal basis.
We have to consider some sort of twisting of the action. This means we multiply by a factor of modulus one in a specific manner. It is described by so called 2-cocycles (see [23]).

Definition 4.5. *A 2-cocycle on Γ is a map $\omega : \Gamma \times \Gamma \to \mathbb{T}$ ($\mathbb{T} = \{x \in \mathbb{C} | \, |x| = 1\}$) that satisfies*

(i) *$\omega(g, h)\omega(gh, k) = \omega(h, k)\omega(g, hk)$ for all $g, h, k \in \Gamma$,*

(ii) *$\omega(g, e) = \omega(e, g) = 1$ for all $g \in \Gamma$.*

With a 2-cocycle we can define the action of the group Γ on $\ell^2(\Gamma)$. This action will later generate the group von Neumann algebra.

Definition 4.6. *Let ω be a 2-cocycle on Γ and λ_ω and ρ_ω the linear operators on $\ell^2(\Gamma)$ with*

$$\begin{aligned} \lambda_\omega(h)\delta_g &= \omega(h, g)\delta_{hg} \\ \rho_\omega(h)\delta_g &= \omega(gh^{-1}, h)\delta_{gh^{-1}}. \end{aligned}$$

λ_ω and ρ_ω are called the left regular representation and right regular representation of Γ on $\ell^2(\Gamma)$ respectively.

λ_ω and ρ_ω form a so called projective representation of the group Γ. This means that the map $g \mapsto \lambda_\omega(g)$ forms a group homomorphism modulo multiplication by a complex number of modulus one. For $g, h, k \in \Gamma$ we get

$$\begin{aligned} \lambda_\omega(g)\lambda_\omega(h)\delta_k &= \lambda_\omega(g)\omega(h, k)\delta_{hk} \\ &= \omega(g, hk)\omega(h, k)\delta_{gh,k} \\ &= \omega(g, h)\omega(gh, k)\delta_{(gh)k} \\ &= \omega(g, h)\lambda_\omega(gh)\delta_k \end{aligned}$$

Showing $\lambda_\omega(g)\lambda_\omega(h) = \omega(g, h)\lambda(gh)$. In a similar manner follows $\rho_\omega(g)\rho_\omega(h) = \omega(g, h)\rho_\omega(gh)$.
Following Anantharaman and Popa [3], we now define the group von Neumann algebra as the weak closure of the span of actions of the group. It will be essential in the following chapters.

Definition 4.7. *The von Neumann algebra $vN_\omega(\Gamma)$ generated by $\lambda_\omega(\Gamma)$, i.e., the weak operator closure of the span of the $\lambda_\omega(\Gamma)$, is called the (left) group von Neumann algebra of Γ.*

Similarly one can define the right group von Neumann algebra $R_\omega(\Gamma)$ with ρ instead of λ.

Lemma 4.8. *Let $h, g \in \Gamma$. We have for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$*

$$\lambda_\omega(h)\rho_\omega(g) = \alpha\rho_\omega(g)\lambda_\omega(h).$$

Proof. Let $h, k, g \in \Gamma$.

$$\begin{aligned}\lambda_\omega(h)\rho_\omega(g)\delta_k &= \omega(kg^{-1}, g)\lambda_\omega(h)\delta_{kg^{-1}}\\ &= \omega(kg^{-1}, g)\omega(h, kg^{-1})\delta_{hkg^{-1}}\\ &= \omega(kg^{-1}, g)\omega(h, kg^{-1})\omega(hkg^{-1}, g)^{-1}\rho_\omega(g)\delta_{hk}\\ &= \omega(kg^{-1}, g)\omega(h, kg^{-1})\omega(hkg^{-1}, g)^{-1}\omega(h, k)^{-1}\rho_\omega(g)\lambda_\omega(h)\delta_k\end{aligned}$$

□

But we will not go more into detail.
Furthermore there is also a tracial state on $vN_\omega(\Gamma)$, but more on that in the next example.

Example 4.9. *Let e be the neutral element of Γ. Note that $vN_\omega(\Gamma)$ acts naturally on the Hilbert space $\ell^2(\Gamma)$. Then the map*

$$\tau(x) = \langle x\delta_e, \delta_e\rangle$$

is a faithful tracial state. By $\langle\cdot,\cdot\rangle$ we mean the inner product on $\ell^2(\Gamma)$. Obviously τ is positive.
Faithful:
Let $x \in vN_\omega(\Gamma)$ with $\tau(x^*x) = 0$ then $0 = \langle x^*x\delta_e, \delta_e\rangle = \langle x\delta_e, x\delta_e\rangle$ hence $x\delta_e = 0$. Now, we want to show that $x = 0$. We first look at $\lambda_\omega(h)$:

$$\lambda_\omega(h)\delta_g = \omega(g, g^{-1})^{-1}\lambda_\omega(h)\rho_\omega(g^{-1})\delta_e = \omega(g, g^{-1})^{-1}\alpha\rho_\omega(g^{-1})\lambda_\omega(h)\delta_e = 0$$

with α like in Lemma 4.8. Since, the left regular representations generate $vN_\omega(\Gamma)$ $x\delta_g = 0$ for all $x \in vN_\omega(\Gamma)$ Thus, $x = 0$. State:

$$\tau(1) = \langle 1\delta_e, \delta_e\rangle = \langle\delta_e, \delta_e\rangle = 1$$

For $g \in \Gamma$ follows

$$\tau(\lambda_\omega(g)) = \langle\lambda_\omega(g)\delta_e, \delta_e\rangle = \langle\delta_g, \delta_e\rangle = \begin{cases}1 & \text{for } g = e\\ 0 & \text{otherwise.}\end{cases}$$

Further we have

$$\tau(\lambda_\omega(g)\lambda_\omega(h)) = \tau(\omega(g, h)\lambda_\omega(gh)) = \begin{cases}1 & \text{for } gh = 1\\ 0 & \text{otherwise}\end{cases} = \tau(\omega(h, g)\lambda_\omega(hg)) = \tau(\lambda_\omega(h)\lambda_\omega(g)).$$

Thus, τ is a trace.

We will see this nice appearance of the tracial state above again in the so called Gelfand-Naimark-Segal construction. With this construction one can show that every tracial state is of this form, but this is not of interest for us. It is essentially the same for von Neumann algebras as for C^* algebras, but more on that later.
There is also another way to define the left regular representation, which can be found in the literature (see [15]). The following lemma shows that these two definitions are the same.

Lemma 4.10. *The action of $\lambda_\omega(g)$ can also be expressed via*

$$\lambda_\omega(h)f(k) = \omega(h, h^{-1}k)f(h^{-1}k)$$

for all $f \in \ell^2(\Gamma)$.

Proof. For every $g, h, k \in \Gamma$ we have

$$\lambda_\omega(h)\delta_{h^{-1}g}(k) = \omega(h, h^{-1}g)\delta_g(k) = \omega(h, h^{-1}g)\delta_{h^{-1}g}(h^{-1}k).$$

The above equation remains valid for linear combinations and limits of $\{\delta_a | a \in \Gamma\}$. Therefore, the claim follows. □

Following Omland [23], we define what it means to be ω-regular (ω is a 2-cocycle). When using 2-cocycles we get that if two elements $g, h \in \Gamma$ commute, then $\lambda_\omega(g)$ and $\lambda_\omega(h)$ need not to commute. This will be helpful, when proving that if Γ is icc, then $vN_\omega(\Gamma)$ is a factor. Proving this assertion, will be our goal in the rest of this section.

Definition 4.11. *(i) We say that an element g is ω-regular, if $\omega(g, h) = \omega(h, g)$ for all h that commute with g. Moreover, if an element g commutes with h all elements of the conjugacy classes commute. In this sense, it is beneficial to talk about ω-regular conjugacy classes rather then ω-regular elements.*

(ii) (Γ, ω) satisfies Kleppner's condition, if every ω-regular conjugacy class is infinite.

Remark 4.12. *(i) Note that if a group Γ is icc, then for all 2-cocycles ω (Γ, ω) satisfies the Kleppner's condition.*

(ii) We will later also prove, that if (Γ, ω) satisfies the Kleppner's condition, then $vN_\omega(\Gamma)$ is a factor. Hence, if Γ is icc, then $vN_\omega(\Gamma)$ is a factor for all 2-cocycles ω.

Omland [23] builds a function on a regular conjugacy class that will be very useful when deciding if a group von Neumann algebra is a factor or not.

Proposition 4.13. *Let C be a conjugacy class. If C is ω-regular, then there is a function $f : \Gamma \to \mathbb{C}$ with*

(i) $0 \notin f(C)$

(ii) $f(hgh^{-1}) = \omega(h, g)\omega(hgh^{-1}, h)^{-1}f(g)$

Proof. Define for a fixed $k \in C$

$$f(g) = \begin{cases} \omega(h, k)\omega(hkh^{-1}, h)^{-1} & \text{if } g \in C \text{ and } g = hkh^{-1} \\ 0 & \text{otherwise} \end{cases}$$

We have to first prove the well definedness of f. For that let $hkh^{-1} = h'kh'^{-1}$. For this we only need to show

$$\omega(h, k)\omega(hkh^{-1}, h)^{-1} = \omega(h', k)\omega(h'kh'^{-1}, h')^{-1}.$$

First observe with Definition 4.5 follows

$$\omega(h^{-1}, hkh^{-1})\omega(kh^{-1}, h') = \omega(h^{-1}, hkh^{-1}h')\omega(hkh^{-1}, h') = \omega(h^{-1}, h'k)\omega(h'kh'^{-1}, h'). \quad (1)$$

We have

$$kh^{-1}h' = h^{-1}(hkh^{-1})h' = h^{-1}(h'kh'^{-1})h' = h^{-1}h'k.$$

Thus, k commutes with $h^{-1}h'$ and since C is ω-regular it follows $\omega(h^{-1}h', k) = \omega(k, h^{-1}h')$. Using this and Definition 4.5 we have

$$\omega(k, h^{-1})\omega(kh^{-1}, h') \overset{4.5}{=} \omega(k, h^{-1}h')\omega(h^{-1}, h') = \omega(h^{-1}h', k)\omega(h^{-1}, h') \overset{4.5}{=} \omega(h^{-1}, h'k)\omega(h', k). \tag{2}$$

Combining (1) and (2) yields

$$\omega(h^{-1}, hkh^{-1})\omega(h', k) = \omega(k, h^{-1})\omega(h'kh'^{-1}, h')$$

Further we have again by using Definition 4.5

$$\begin{aligned}
\omega(h^{-1}, hkh^{-1}) &\overset{4.5}{=} \omega(h^{-1}, hkh^{-1})\omega(kh^{-1}, h)\omega(kh^{-1}, h)^{-1} \\
&= \omega(h^{-1}, hk)\omega(hkh^{-1}, h)\omega(kh^{-1}, h)^{-1} \\
&\overset{4.5}{=} \omega(h^{-1}, hk)\omega(h, k)\omega(h, k)^{-1}\omega(hkh^{-1}, h)\omega(kh^{-1}, h)^{-1} \\
&= \omega(k, h^{-1})\omega(kh^{-1}, h)\omega(h, k)^{-1}\omega(hkh^{-1}, h)\omega(kh^{-1}, h)^{-1} \\
&= \omega(k, h^{-1})\omega(h, k)^{-1}\omega(hkh^{-1}, h)
\end{aligned}$$

From that follows by combining the last two results

$$\omega(hkh^{-1}, h)\omega(h', k) = \omega(h, k)\omega(h'kh'^{-1}, h'),$$

which implies

$$\omega(h, k)\omega(hkh^{-1}, h)^{-1} = \omega(h', k)\omega(h'kh'^{-1}, h')^{-1}.$$

It remains to show that (ii) holds for all $g \in C$ and not only for the specific k choosen in the definition of f. Let $g \in C$, then there is a $h \in \Gamma$ with $g = hkh^{-1}$.

$$f(g) = f(hkh^{-1}) = \omega(h, k)\omega(hkh^{-1}, h)^{-1} = \omega(h, k)\omega(g, h)^{-1}$$

By using Definition 4.5 we get

$$\begin{aligned}
&\omega(lgl^{-1}, l)\omega(lh, k)\omega(lgl^{-1}, lh)^{-1}\omega(h, k)^{-1} \\
&= \omega(lgl^{-1}, l)\omega(lh, k)\omega(lgl^{-1}, lh)^{-1}\omega(h, k)^{-1}\omega(l, hk)^{-1}\omega(l, hk) \\
&\overset{4.5}{=} \omega(lgl^{-1}, l)\omega(lh, k)\omega(lgl^{-1}, lh)^{-1}\omega(l, h)^{-1}\omega(lh, k)^{-1}\omega(l, hk) \\
&\overset{4.5}{=} \omega(lgl^{-1}, l)\omega(lh, k)\omega(lgl^{-1}, l)^{-1}\omega(lg, h)^{-1}\omega(lh, k)^{-1}\omega(l, hk) \\
&= \omega(lg, h)^{-1}\omega(l, hk) \\
&\overset{\text{def. of } g}{=} \omega(lg, h)^{-1}\omega(l, gh) \\
&= \omega(lg, h)^{-1}\omega(l, g)^{-1}\omega(l, g)\omega(l, gh) \\
&\overset{4.5}{=} \omega(g, h)^{-1}\omega(l, gh)^{-1}\omega(l, g)\omega(l, gh) \\
&= \omega(g, h)^{-1}\omega(l, g)
\end{aligned}$$

Finally we get

$$\begin{aligned}
f(lgl^{-1}) &\overset{\text{def. of } g}{=} f(lhkh^{-1}l^{-1}) \\
&\overset{\text{def. of } f}{=} \omega(lh, k)\omega(lhkh^{-1}l^{-1}, lh)^{-1} \\
&\overset{\text{def. of } g}{=} \omega(lh, k)\omega(lgl^{-1}, lh)^{-1} \\
&= \omega(lgl^{-1}, l)\omega(lh, k)\omega(lgl^{-1}, lh)^{-1}\omega(h, k)^{-1}\omega(lgl^{-1}, l)^{-1}\omega(h, k) \\
&\overset{(*)}{=} \omega(g, h)^{-1}\omega(l, g)\omega(lgl^{-1}, l)^{-1}\omega(h, k) \\
&\overset{\text{def. of } f}{=} \omega(l, g)\omega(lgl^{-1}, l)^{-1}f(g)
\end{aligned}$$

where we used in $(*)$ the equation above. □

Lemma 4.14. *For $h \in \Gamma$ we have*

$$\lambda_\omega(h)^* = \omega(h^{-1}, h)^{-1}\lambda_\omega(h^{-1})$$

Proof. Let $g, h, k \in \Gamma$.

$$\begin{aligned}
\langle \delta_k, \omega(h^{-1}, h)^{-1})\lambda_\omega(h^{-1})\delta_g\rangle &= \omega(h^{-1}, h)\omega(h^{-1}, g)^{-1}\langle \delta_k, \delta_{h^{-1}g}\rangle \\
&= \begin{cases} \omega(h^{-1}, h)\omega(h^{-1}, hk)^{-1} & \text{if } hk = g \\ 0 & \text{otherwise} \end{cases} \\
&= \omega(h^{-1}, h)\omega(h^{-1}, hk)^{-1}\langle \delta_{hk}, \delta_g\rangle \\
&\overset{\text{def } \lambda_\omega(h)}{=} \omega(h^{-1}, h)\omega(h^{-1}, hk)^{-1}\omega(h, k)^{-1}\langle \lambda_\omega(h)\delta_k, \delta_g\rangle \\
&\overset{4.5}{=} \omega(h^{-1}, h)\omega(h^{-1}, h)^{-1}\omega(e, k)^{-1}\langle \lambda_\omega(h)\delta_k, \delta_g\rangle \\
&= \langle \lambda_\omega(h)\delta_k, \delta_g\rangle
\end{aligned}$$

Thus, $\lambda_\omega(h)^* = \omega(h^{-1}, h)^{-1}\lambda_\omega(h^{-1})$. □

Combining the results from above, we now can prove that the group von Neumann algebra is a factor if and only if (Γ, ω) satisfies Kleppner's condition. Note that all Fuchsian groups are icc by Theorem 3.11. Thus, this implies Kleppner's condition and therefore all group von Neumann algebras that are generated by Fuchsian groups are factors. Why this is important will be clear, when we discuss the von Neumann dimension in Chapter 5. In [23] one can find this theorem, but it is proven slightly differently.

Theorem 4.15. *The group von Neumann algebra $vN_\omega(\Gamma)$ is a factor, if and only if (Γ, ω) satisfies Kleppner's condition.*

Proof. $\Longleftarrow$: Let x be in the center of $vN_\omega(\Gamma)$. Notice first that $\rho_\omega(h)\delta_e = \lambda_\omega(h)^*\delta_e$ on ω-regular conjugacy classes. Therefore,

$$\begin{aligned}
x\delta_e(hgh^{-1}) &\overset{\text{def. } \rho_\omega}{=} \omega(hgh^{-1}, h)^{-1}x\rho_\omega(h)\delta_e(hg)) \\
&\overset{\text{def. } \lambda_\omega}{=} \omega(hgh^{-1}, h)^{-1}\omega(h, g)^{-1}x\rho_\omega(h)\lambda_\omega(h)\delta_e(g) \\
&\overset{x \text{ in center}}{=} \omega(hgh^{-1}, h)^{-1}\omega(h, g)^{-1}\lambda_\omega(h)x\rho_\omega(h)\delta_e(g) \\
&= \omega(hgh^{-1}, h)^{-1}\omega(h, g)^{-1}\lambda_\omega(h)x\lambda_\omega(h)^*\delta_e(g) \\
&\overset{x \text{ in center}}{=} \omega(hgh^{-1}, h)^{-1}\omega(h, g)^{-1}\lambda_\omega(h)\lambda_\omega(h)^*x\delta_e(g) \\
&\overset{4.14}{=} \omega(hgh^{-1}, h)^{-1}\omega(h, g)^{-1}\omega(h^{-1}, h)^{-1}\lambda_\omega(h)\lambda_\omega(h^{-1})x\delta_e(g) \\
&= \omega(hgh^{-1}, h)^{-1}\omega(h, g)^{-1}x\delta_e(g)
\end{aligned}$$

Thus $x\delta_e$ is a function like in Proposition 4.13. As $|x\delta_e|$ is constant on ω regular conjugacy classes, it can only be non-zero on finite conjugacy classes, since $x\delta_e \in \ell^2(G)$. But Γ has no finite ω-regular conjugacy classes except for the trivial, thus $x\delta_e = \alpha\delta_e$ for some $\alpha \in \mathbb{C}$. With that $vN_\omega(\Gamma)$ is a factor.
$\Longrightarrow$: Assume that Γ does not satisfy Kleppner's condition. Then there is a finite non trivial ω-regular conjugacy class $C \subset \Gamma$. The map

$$g : \ell^2(\Gamma) \to \ell^2(\Gamma), g = \sum_{c \in C} f(c)\lambda_\omega(c)$$

with f being a function like in Proposition 4.13, is in the center of $vN_\omega(\Gamma)$ and $g \notin \mathbb{C}1$. To see this first notice that for $h \in C$

$$\begin{aligned}
g\lambda_\omega(h) &= \sum_{c\in C} f(c)\lambda_\omega(c)\lambda_\omega(h) \\
&\overset{c\in C}{=} \sum_{c'\in C} f(hc'h^{-1})\lambda_\omega(hc'h^{-1})\lambda_\omega(h) \\
&\overset{4.13(ii)}{=} \sum_{c'\in C} \omega(h,c')\omega(hc'h^{-1},h)^{-1} f(c')\lambda_\omega(hch^{-1})\lambda_\omega(h) \\
&\overset{\text{def. of } \lambda_\omega}{=} \sum_{c'\in C} \omega(h,c')f(c')\lambda_\omega(hc') \\
&\overset{4.10+4.5}{=} \sum_{c'\in C} \omega(h^{-1},h)\lambda_\omega(h)f(c')\lambda_\omega(c') \\
&\overset{\text{def. of } g}{=} \omega(h,h^{-1})\lambda_\omega(h)g
\end{aligned}$$

Thus the center of $vN_\omega(\Gamma)$ is not trivial and therefor it is not a factor. □

Recall, that that a group Γ is icc (infinite conjugacy class), if for all $g \neq e$ the conjugacy class $\{h^{-1}gh | h \in \Gamma\}$ is infinite. With Theorem 4.15 follows that if Γ is icc, $vN_\omega(\Gamma)$ is a factor for every 2-cocycle ω. We are going to use this implication if we look at the action of Fuchsian goups on Bergman spaces in more detail.

4.3 Projections

Projections play a major role in von Neumann algebras, unlike in C^* algebras. The following remark describes why that is the case. In essence, we know the entire von Neumann algebra, if we know their projections. We will look at some properties of these projections and will later on characterize factors with their projections.
One can associate von Neumann algebras to measurable functions in some measure space, as C^* algebras to continuous functions. This is the so called Borel functional calculus. I.e., commutative von Neumann algebras are essentially measurable functions and for every normal element in a von Neumann algebra there is a measurable functional calculus. With this association, the projections in the von Neumann algebra get mapped to characteristic functions, since the characteristic functions describe all measurable functions (they are the pointwise closure of the span). Thus, we can see that the von Neumann algebra can be studied entirely by looking at the projections. For more details on this, see [3]. We define an order on the projections on $\mathcal{B}(\mathcal{H})$ by

$$p \leq q \Leftrightarrow p\mathcal{H} \subset q\mathcal{H}$$

(see [15]). We refer to the orthogonal projection onto $p\mathcal{H} \cap q\mathcal{H}$ as $p \wedge q$ and write $p^\perp$ for $1-p$. Continuing in this manner we write $p \vee q$ for the orthogonal projection onto $\overline{p\mathcal{H} + q\mathcal{H}}$ or $(p^\perp \wedge q^\perp)^\perp$. Following Peterson [25], we look at some properties of projections.

Definition 4.16. *Let M be a von Neumann algebra and p, q be projections in M.*

(i) A bounded operator u is a partial isometry, if $u = uu^*u$.

(ii) p, q are called equivalent and write $p \sim q$, if there is some partial isometry $u \in M$ with $p = uu^*$ and $q = u^*u$.

(iii) For a von Neumann algebra M, a non zero projection $p \in M$ is called minimal, if $(q \leq p) \implies (q = 0$ or $q = p)$.

Note also that it is important that p, q, u are in M, thus for the relation between p and q it is important in which algebra they lie. The next example demonstrates this. Thus, the existence of a minimal projection in M is just dependent of the structure of M.

Example 4.17. *If we look at the von Neumann algebra M in the 4×4 matrices $M_4(\mathbb{C})$ generated by*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then A is clearly minimal in M, but A is also clearly not minimal in $M_4(\mathbb{C})$.

Following Peterson [25], we define, what it means for a projection to be finite or infinite. This is also dependent on the von Neumann algebra in which the projection lie for the same reasons.

Definition 4.18. *A projection p is called*

- *finite, if there is no projection $q < p$ with $p \sim q$*

- *infinite, if p is not finite*

Note here that if the dimension of $p\mathcal{H}$ is finite, then p is finite. The converse of this statement needs not to be true. A similar observation can be made for infinite projections.

Proposition 4.19. *If a projection $p\mathcal{B}(\mathcal{H})$ is infinite, then there are projections $q, r \in \mathcal{B}(\mathcal{H})$ with $q \sim r \sim p$ with $p = q + r$.*

Proof. Obviously $p\mathcal{H}$ is infinite dimensional, so there is a subspace K of $p\mathcal{H}$ isomorphic to $p\mathcal{H}$ and $K^{\perp}$ isomorphic to K. The orthogonal projection q on K is equivalent to p and $1 - q$ is equivalent to p. □

4.4 Types of Factors

Factors can be categorized into three different types named type I, II and III. These differ in the structure of their projections and therefore in their whole "appearance". One can define these types just on properties of their projections, but this would not be helpful in getting some intuition on these types and their behavior (for a definition based just on projections, see [25]). There are also subtypes of these types, which we will discuss briefly. The next example shows that there are indeed different types of von Neumann algebras. Remember that there is a trace in a group von Neumann algebra. The following example can be found in [3].

Example 4.20. *We want to show that there is no tracial state on $\mathcal{B}(\mathcal{H})$ for every infinite dimensional Hilbert space $\mathcal{H}$. Suppose there is a trace tr on $\mathcal{B}(\mathcal{H})$. The identity id is infinite in $\mathcal{B}(\mathcal{H})$, then there is p, q with $p \sim id \sim q$ and $id = p + q$. Then there are partial isometries u_1 and u_2 with $u_1u_1^* = p$, $u_1^*u_1 = id$, $u_2u_2^* = q$ and $u_2^*u_2 = id$. We have then*

$$1 = tr(id) = tr(p) + tr(q) = tr(u_1u_1^*) + tr(u_2u_2^*) = tr(u_1^*u_1) + tr(u_2^*u_2) = 2$$

This is a contradiction, thus there is no trace on $\mathcal{B}(\mathcal{H})$.

We begin by the simplest of all of them: the type I factor. We will not go deeper into type I factors, but include them for completeness. Following Anantharaman and Popa [3] we define type I factors.

Definition 4.21. *A factor is called a type I factor, if it is isomorphic to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

This is also equivalent to the existence of a minimal projection (see [25]).
If the Hilbert space $\mathcal{H}$ from the definition above has dimension $n := \dim \mathcal{H} \in [0, \infty]$, one says M is a type I_n factor.
We now come to the most important type, the type II factor. This and especially the II_1 type is the one that we are focusing on. It will help us prove the main theorem, and we will define the von Neumann dimension for these types of factors. The type I and II_1 factors are also known as finite types. These are special because they have a tracial state defined on them, that is essential for the construction in the next chapter (see [15]). Following Jones [15], we define type II factors.

Definition 4.22. *Let M be a factor. If M has finite projections, but no minimal projection, it is a type II factor. We distinguish two subcategories*

(i) M is a II_1, if there is a tracial state on M.

(ii) A type II_∞ factor, if the identity is infinite.

Notice that there cannot be a tracial state if the identity is infinite (see Example 4.20). With this definition, it follows immediately that $vN_\omega(\Gamma)$ is a type II_1 factor (see Example 4.9). We will focus on this type of factor. Moreover, the normalized trace on a type II_1 factor is unique. (see [15]). We will later consider von Neumann algebras M that act on some Hilbert space $\mathcal{H}$. Therefore we have to define what this actually means. In [15] we can find a definition of M-module.

Definition 4.23. *Let M be a type II_1 factor. We call a Hilbert space together with a unital $*$-homomorhpism from M to a type II_1 factor on $\mathcal{H}$ an M-module.*

We write $x\xi$ for the action of $x \in M$ on $\xi \in \mathcal{H}$. Similarly, we mean by $M\xi = \{x\xi | x \in M\}$. To compare the actions of two von Neumann algebras on two Hilbert spaces, one uses the notion of unitarily equivalence. Following Jones [15], we define what it means for two actions to be equivalent.

Definition 4.24. *Let M, N be von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. We call the action of M on $\mathcal{H}$ and N on $\mathcal{K}$ unitarily equivalent, if there is a unitary $u : \mathcal{H} \to \mathcal{K}$ with $M \to N, x \mapsto uxu^*$ being an isomorphism between von Neumann algebras.*

The next definition will become important in the next chapter. It will help us show that the two definitions of von Neumann dimension are the same. A definition can be found in [19] and [15].

Definition 4.25. *Let M be a von Neumann algebra that acts on a Hilbert space $\mathcal{H}$, then we call a vector $\xi \in \mathcal{H}$ cyclic if $M\xi$ is dense in $\mathcal{H}$.*
We call a vector $\xi \in \mathcal{H}$ separating, if for $x \in M$ we have $x\xi = 0 \implies x = 0$.

The following lemma can be found in [30] and will help us, when we look at some properties of the von Neumann dimension. It is also a vital part of the well definedness of the definition of von Neumann dimension from Murray and von Neumann. But this will be the easy part to show the well definedness, but more on that in the next chapter.

Lemma 4.26. *For a von Neumann algebra M on $\mathcal{H}$ the projection p onto $M\xi$ for some $\xi \in M$ belongs to M'.*

Proof. Let $x \in M$. We have to show $xp = px$. With the orthogonal decomposition $\overline{M\xi} \oplus (M\xi)^\perp$ it suffices to show the assertion for elements of the form $y\xi + \nu$ with $y \in M$ and $\nu \in (M\xi)^\perp$. Notice also, that since M is a $*$-algebra, $x\nu \in (M\xi)^\perp$. We have

$$px(y\xi + \nu) = pxy\xi + px\nu = xy\xi = xpy\xi = xpy\xi + xp\nu = xp(y\xi + \nu).$$

□

The connection between the lemma above and below will be clear in the next chapter. It says that a cyclic vector is also a separating vector for the commutant, thus if we have a cyclic and separating vector for M, then M and M' are almost the same, because a separating and cyclic vector describes the von Neumann algebra entirely, this will come apparent in the next chapter.

Lemma 4.27. *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A vector $\xi \in \mathcal{H}$ is cyclic for M if and only if ξ is separating for M'.*

Proof. $\Longrightarrow$: Let ξ be cyclic, then $\overline{M\xi} = \mathcal{H}$.
Suppose: There is a $x \in M'$ with $x \neq 0$ and $x\xi = 0$, i.e. ξ is not separating.
Since $x \neq 0$ and $M\xi$ is dense in $\mathcal{H}$, there is a $y \in M$ with $xy\xi \neq 0$. It follows $0 = yx\xi = xy\xi \neq 0$ is a contradiction.
$\Longleftarrow$: Let ξ be separating for M'.
Suppose: ξ is not cyclic for M.
With Lemma 4.26 the projection p onto $(M\xi)^{\perp}$ is in M'. Since ξ is not cyclic, p is non zero. But $p\xi = 0$ and $p \neq 0$. This is a contradiction to ξ being separating for M'. $\square$

We will now note that there is a third type of factor, but this is not important for our considerations, so we will not go deeper into it. It is only mentioned for completeness.

Definition 4.28. *A factor M is said to be a type III factor, if M is not a type I or type II factor.*

5 Von Neumann Dimension and Gelfand-Naimark-Segal construction

In this chapter, we will do the Gelfand-Naimark-Segal construction. Using the GNS construction, we define the von Neumann dimension, which measures the size of M-modules with respect to the GNS construction. At the end, we prove some properties of the von Neumann dimension and calculate the von Neumann dimension of the weighted Bergman spaces with the group von Neumann algebra of a Fuchsian group.
This chapter is mostly based on [3] for the GNS construction and [15] for the von Neumann dimension.

5.1 Gelfand-Naimark-Segal construction

Following Anantharaman and Popa [3] we construct for a von Neumann algebra and tracial state $\varphi : M \to \mathbb{C}$ a Hilbert space M on which M acts. We will use the well known Gelfand-Naimark-Segal construction or short GNS construction. This construction comes from the theory of C^*-algebras, and we will use some C^*-theory since von Neumann algebras are $*$-subalgebras of the C^*-algebra $\mathcal{B}(\mathcal{H})$. The map

$$\langle x, y\rangle = \varphi(y^* x)$$

defines a semi-definite sesquilinear form on $M \times M$. To turn $\langle \cdot, \cdot\rangle$ into a scalar product we have to consider the set $I := \{x \in M | \langle x, x\rangle = 0\}$. We can use the Cauchy-Schwarz inequality to get for every $x \in I$ and $y \in M$

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle = 0.$$

It follows immediately for $x, z \in I$

$$\langle \lambda x + z, \lambda x + z\rangle = |\lambda|^2 \langle x, x\rangle + \lambda\langle x, z\rangle + \overline{\lambda}\langle z, x\rangle + \langle z, z\rangle = 0.$$

Thus, I is a closed linear subspace in M. With basic algebra, we get that M/I is also an algebra. If we factor out I we get a scalar product on M/I with

$$\langle x + I, y + I\rangle = \varphi(y^* x)$$

For well definedness consider for $x, \tilde{x} \in I$ and $y, \tilde{y} \in M$

$$\langle y + x, \tilde{y} + \tilde{x}\rangle = \langle y, \tilde{y}\rangle + \langle x, \tilde{y}\rangle + \langle y, \tilde{x}\rangle + \langle x, \tilde{x}\rangle = \langle y, \tilde{y}\rangle.$$

We will denote the norm induced by φ as $\|\cdot\|_\varphi$.
The operator $L_x : M/I \to M/I, y + I \mapsto xy + I$ is a bounded linear operator.

$$\|L_x(y + I)\|_\varphi^2 = \|xy + I\|_\varphi = \varphi(y^* x^* xy) \leq \|x^* x\| \varphi(y^* y) = \|x\|^2 \|y + I\|_\varphi^2$$

The inequality above shows also that L_x is well-defined. (We have $\|x^* x\| 1 - x^* x$ is positive and therefore $\|x^* x\| y^* y - y^* x^* xy$ is positive. This follows from C^* theory. (see [3])). Moreover, if we set $\xi_\varphi = 1 + I$, we have

$$\langle L_x \xi_\varphi, \xi_\varphi\rangle_\varphi = \langle x + I, \xi_\varphi\rangle_\varphi = \varphi(1^* x) = \varphi(x).$$

We see that ξ_φ is cyclic for L_M.

Definition 5.1. *The triplet $(L., \overline{M/I}, \xi_\varphi)$ is called the Gelfand-Naimark-Segal construction or short the GNS construction and is denoted $L^2(M)$. Here, $\overline{M/I}$ is the completion of M/I.*

The operator L_x extends to a bounded operator on $L^2(M)$. If φ is faithful, then the map $M \to \mathcal{B}(L^2(M)), x \mapsto L_x$ is injective and $L^2(M)$ is isomorphic to the completion of M with respect to the norm $\|\cdot\|_\varphi$. The Hilbert space $L^2(M)$ is going to be our baseline to measure Hilbert spaces on which M acts.
If we look at group von Neumann algebras, we see that they are already in the "GNS-construction form", but for general von Neumann algebras this need not to be the case.

5.2 Definition and basic properties

We now focus on type II_1 factors that act on separable Hilbert spaces. The von Neumann dimension discussed in this section describes the Hilbert space with respect to the action of the type II_1 factor acting on it and not the underlying field. The isometry in the next theorem will help us compare the M-module $\mathcal{H}$ and $L^2(M)$.
We construct this isometry as follows. The following theorem can be found in [19] and [30].

Theorem 5.2. *For a separable M-module $\mathcal{H}$, there exists an isometry*

$$u : \mathcal{H} \to \bigoplus_{n=1}^{\infty} L^2(M)$$

with u being M-linear (i.e. $ux = (x \otimes 1)u$).

Proof. M acts on $\mathcal{H} \oplus (\oplus_{n=1}^{\infty} L^2(M))$ via

$$x(\xi, \oplus_{n=1}^{\infty} m_n) = (x\xi, \otimes_{n=1}^{\infty} x m_n)$$

The projections $p = 1 \oplus 0$ and $q = 0 \oplus 1$ commute with the action of M, thus $p, q \in M'$. There is a partial isometry $v \in M'$ with $v^* v = p$ and $vv^* \leq q$ since q is infinite in M'.

$$v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{B}(\bigoplus_{n=1}^{\infty} L^2(M), \mathcal{H}), c \in \mathcal{B}(\mathcal{H}, \bigoplus_{n=1}^{\infty} L^2(M)), d \in \mathcal{B}(\bigoplus_{n=1}^{\infty} L^2(M))$. It follows

$$v^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p = v^* v = \begin{pmatrix} a^* a + c^* c & a^* b + c^* b \\ b^* a + d^* c & b^* b + d^* d \end{pmatrix}$$

Thus, $b^* b + d^* d = 0$ hence $b = d = 0$. In the same manner follows $a = 0$, when we use q. We have further

$$v^* v = \begin{pmatrix} c^* c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p$$

Thus, $c^* c = 1$ (and similarly follows $cc^* \leq 1$) hence c is an isometry. Now, set $u = c$. We also have, since $v \in M'$

$$\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x \otimes 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \otimes 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}$$

Hence $ux = (x \otimes 1)u$. □

This isometry is not unique. It is the central element in defining the von Neumann dimension. Following Jones [19], we now define with it the von Neumann dimension.

Definition 5.3. *The von Neumann dimension of a M-module $\mathcal{H}$ is defined as*

$$\dim_M(\mathcal{H}) = tr_{L^2}(uu^*)$$

with u being the isometry from Theorem 5.2.

We mean by tr_{L^2} the trace of the commutant M' on $\bigoplus_{n=1}^{\infty} L^2(M)$, but normalized in such a way that the projections on one of the $L^2(M)$ is one. The von Neumann dimension can attain every non-negative real value and ∞. The definition of von Neumann dimension is independent of u. Suppose there is another unitary v satisfying the conditions in Theorem 5.2. Then u and v are unitarily equivalent with respect to the commutant M' on $\bigoplus_{n=1}^{\infty} L^2(M)$. Thus, the traces are the same.

Remark 5.4. *The above definition differs from the original Murray and von Neumann definition. In chapter X of [22] Murry and von Neumann took any nonzero element ξ in an M-module $\mathcal{H}$ and get the orthogonal projections p and q onto $\overline{M\xi}$ and $\overline{M'\xi}$ respectively. Then the ratio $\frac{tr_M(p)}{tr_{M'}(q)}$ is independent of ξ. To show this fact is rather difficult and elaborate, we will therefore skip this proof and assume it is independent. These two definitions of von Neumann dimension are the same, but to prove that we have to first look at some basic properties of the von Neumann dimension.*

The idea behind the von Neumann dimension is the same as the classical dimension concept. Here, we do not want to count how many copies of the underlying field are contained in the Hilbert space that we are investigating, instead we use the GNS construction as our base measure. Therefore, when using $\mathbb{C}$ as our von Neumann algebra we get the classical dimension. The difference here is that the von Neumann dimension can be any non-negative real number and infinity, instead of only the natural numbers and infinity. Before we conduct basic properties of the von Neumann dimension, we have to get a good description of the commutant M' acting on $L^2(M)$. We will look at a more general case, instead of only looking at the GNS construction, we look at all M-modules $\mathcal{H}$ that have a cyclic and separating vector. In the following, $\Omega \in \mathcal{H}$ is this cyclic and separating vector. Then the trace on M is given by $tr(x) = \langle x\Omega, \Omega\rangle$ (see [15]). We define the antilinear unitary involution J. Following Jones [16], we define the map $J : \mathcal{H} \to \mathcal{H}$, that we need for proving some properties of the von Neumann dimension.

Definition 5.5. *We define the map $J : \mathcal{H} \to \mathcal{H}$ as the extension of the antilinear isometry*

$$J : \mathcal{H} \to \mathcal{H}, J(x\Omega) = x^*\Omega$$

for all $x \in M$.

The map J is an isometry, since it holds for $x, y \in M$

$$\langle J(x\Omega), J(y\Omega)\rangle = tr(yx^*) = tr(x^*y) = \langle y, x\rangle$$

and $\Omega \in \mathcal{H}$ is cyclic and separating. We now can describe the commutant of M with the help of the map J (see [16]).

Proposition 5.6. *Let M be a von Neumann algebra acting on $\mathcal{H}$. We have*

$$JMJ = M'$$

Proof. We first show $JMJ \subset M'$. First observe for every $x \in M$ and $a \in M'$ we have

$$JxJ(a\Omega) = J(xa^*\Omega) = ax^*\Omega = aJ(x\Omega) = aJxJ\Omega.$$

Since the map $a \mapsto a\Omega$ is injective as Ω is separating we get JxJ commutes with a.
Secondly, we are going to prove $Jx\Omega = x^*\Omega$ for all $x \in M'$. Let $y \in M$.

$$\langle Jx\Omega, y\Omega\rangle = \langle Jy\Omega, JJx\Omega\rangle = \langle Jy\Omega, x\Omega\rangle = \langle y^*\Omega, x\Omega\rangle = \langle \Omega, yx\Omega\rangle = \langle \Omega, xy\Omega\rangle = \langle x^*\Omega, y\Omega\rangle.$$

Thus, since $M\Omega$ is dense it follows $Jx\Omega = x^*\Omega$.
Thirdly, we prove that $x \mapsto \langle x\Omega, \Omega\rangle$ is a trace on M'. Let $x, y \in M'$.

$$\langle xy\Omega, \Omega\rangle = \langle y\Omega, x^*\Omega\rangle = \langle y\Omega, Jx\Omega\rangle = \langle x\Omega, Jy\Omega\rangle = \langle x\Omega, y^*\Omega\rangle = \langle yx\Omega, \Omega\rangle$$

Thus, this map defines a trace with separating vector Ω. We get with our first assertion $JM'J \subset M$. All in all, we have

$$M' = JJM'JJ \subset JMJ \subset M'$$

Thus, $JMJ = M$. □

We can now show a bunch of properties of the von Neumann dimension, which we will use in the next chapters. At first, we discuss the ones that we can prove with our definition of the von Neumann dimension. With these properties, we see that the two definitions of von Neumann dimension coincide and prove some properties with the Murray and von Neumann definition. We will need all the properties in the next or last chapter in one form or the other. Some elementary properties of the von Neumann dimension that Jones [15] describes are the following.

Proposition 5.7. *Let M be a II_1 factor and $\mathcal{H}$ be a M-module. Then*

(i) $\dim_M(\mathcal{H}) < \infty$ *if and only if M' is a II_1 factor.*

(ii) $\dim_M(\mathcal{H}) = \dim_M(\mathcal{K})$, *if and only if the actions of M on $\mathcal{H}$ and M on $\mathcal{K}$ are unitarily equivalent.*

(iii) $\dim_M(\mathcal{H}) = 0 \Longleftrightarrow \mathcal{H} = \{0\}$

(iv) *For a projection $p \in M$ we have* $\dim_M(L^2(M)p) = tr_M(p)$.

(v) *For a projection $p \in M$ we have* $\dim_{pMp}(p\mathcal{H}) = tr_M(p)^{-1} \dim_M(\mathcal{H})$ *if* $tr_M(p) \neq 0$.

(vi) $\dim_M\left(\bigoplus_{i\in\mathbb{N}} \mathcal{H}_i\right) = \sum_{i\in\mathbb{N}} \dim_M(\mathcal{H}_i)$

Let in the following be M' a II_2 factor.

(vii) $(\dim_M \mathcal{H})(\dim_{M'} \mathcal{H}) = 1$.

(viii) *There exists a cyclic $\xi \in \mathcal{H}$ if* $\dim_M \mathcal{H} \leq 1$.

(ix) *There exists a separating vector if* $\dim_M(\mathcal{H}) \geq 1$.

Proof.

(i) $\Longrightarrow$: Let $\dim_M(\mathcal{H}) < \infty$. Let u be as in Theorem 5.2, then $uu^* \in M'$ on $\bigoplus_{n=1}^{\infty} L^2(M)$. M on $\mathcal{H}$ is unitarily equivalent to the M on $uu^* \bigoplus_{n=1}^{\infty} L^2(M)$. The commutant M' on $uu^* \bigoplus_{n=1}^{\infty} L^2(M)$ is given by the commutant $uu^*\hat{M}'$ with $\hat{M}'$ being the commutant on $\bigoplus_{n=1}^{\infty} L^2(M)$.
$\Longleftarrow$: Let M' be a II_1 factor. Since uu^* is an element of M', then $tr_{L^2}(uu^*) < \infty$, since M' is a finite subalgebra of the commutant of $\bigoplus_{n=1}^{\infty}$.

(ii) $\Longrightarrow$: Let $\dim_M(\mathcal{H}) = \dim_M(\mathcal{K})$. Then $tr_{L^2}(uu^*) = tr_{L^2}(vv^*)$ with u and v being the unitaries as in Theorem 5.2 for $\mathcal{H}$ and $\mathcal{K}$ respectively. Thus, $uu^* \bigoplus_{n=1}^{\infty} L^2(M)$ and $vv^* \bigoplus_{n=1}^{\infty} L^2(M)$ are unitarily equivalent, with this follows the assertion.
$\Longleftarrow$: Let be the action of M on $\mathcal{H}$ and $\mathcal{K}$ unitarily equivalent. Then there is a unitary u with $u : \mathcal{H} \to \mathcal{K}$ as in Definition 4.24. Let v be as in Theorem 5.2 for M and $\mathcal{K}$. Then uv is a unitary as in Theorem 5.2 for M and $\mathcal{H}$. We have finally

$$\dim_M(\mathcal{H}) = tr_{L^2}(uv(uv)^*) = tr_{L^2}((uv)^*uv) = tr_{L^2}(v^*u^*uv) = tr_{L^2}(v^*v) = \dim_M(\mathcal{K})$$

(iii) This follows immediately from (ii).

(iv) Define $u(v) = v \oplus \bigoplus_{n\in\mathbb{N}} 0$. Then u is a unitary and $uu^* = JqJ \oplus \bigoplus_{n\in\mathbb{N}} 0$. Thus $tr_L^2(uu^*) = tr(q)$.

(v) We will first prove the assertion for $\mathcal{H} = L^2(M)q$ with $q \leq p$. The map $L^2(pMp) \to pL^2(M)p, px p\Omega \mapsto p(x\Omega)p$ defines a unitary. Thus the action of pMp on $pL^2(M)pq = pL^2(M)q$ is unitarily equivalent to the action of pMp on $L^2(pMp)q$. With (ii) and (iv) we have

$$\begin{aligned}\dim_{pMp}(pL^2(M)q) &= \dim_{pMp}(L^2(pMp)q) = tr_{pMp}(q)\\ &= tr_M(p)^{-1}tr_M(q) = tr_M(p)^{-1}\dim_M(L^2(M)q)\end{aligned}$$

For arbitrary $\mathcal{H}$, M on $\mathcal{H}$ is unitarily equivalent to M on $q\left(\bigoplus n \in \mathbb{N} L^2(M)\right)$ with q being some projection (q is i.e. uu^* with the unitary u from Theorem 5.2). One can write q as an orthogonal sum of projections q_i with $q_i \leq p$. Then with the case $\mathcal{H} = L^2(M)q$ follows the assertion.

(vi) Choose a M-linear isometry u_i for all $\mathcal{H}_i$ like in Theorem 5.2 in such a way that $u_i\mathcal{H}_i \perp u_j\mathcal{H}_j$ for all $i \neq j$. Define the M-linear isometry

$$u = \sum_{i\in\mathbb{N}} u_i.$$

Now u is a M-linear isometry from $\bigoplus_{i\in\mathbb{N}} \mathcal{H}_i \to \bigoplus_{n=1}^{\infty} L^2(M)$. With the definition of von Neumann dimension follows

$$\dim_M\left(\bigoplus_{i\in\mathbb{N}} \mathcal{H}_i\right) = tr_{L^2}(uu^*) = tr_{L^2}\left(\sum_{i\in\mathbb{N}} u_iu_i^*\right) = \sum_{i\in\mathbb{N}} tr_{L^2}(u_iu_i^*) = \sum_{i\in\mathbb{N}} \dim_M(\mathcal{H}_i)$$

(vii) On $L^2(M)$ is $\dim_M(\mathcal{H})\dim_{M'}(\mathcal{H}) = 1$. If $\mathcal{H}$ is of the form $L^2(M)p$, then by (iv) and (v) we get

$$\dim_{pMp}(L^2(M)p)\dim_{M'p}(L^2(M)p) = tr_M(p)\dim_M(L^2(M))tr_M(p)^{-1}\dim_M(L^2(M)) = 1$$

For $\mathcal{K} = \bigoplus_{n=1}^{k} \mathcal{H}$ follows with (iv)

$$\dim_{M\otimes 1}(\mathcal{K}) = k\dim_M(\mathcal{H})$$

and with (v)

$$\dim_{(M\otimes 1)'}(\mathcal{K}) = k^{-1}\dim_{M'}(\mathcal{H})$$

With $\mathcal{H}$ of the form $L^2(M)p$ follows the assertion. We can express every $\mathcal{K}$ in this form since uu^* with u from the Theorem 5.2 is a projection in the type II_1 M'. Further, by picking a projection q with the same element p on the diagonal with $tr_{M'}(uu^*) = tr_{M'}(q)$ we get, that uu^* and q are unitarily equivalent, hence $\mathcal{K}$ is of the desired form.

(viii) Let $\dim_M(\mathcal{H}) \leq 1$, then the action of M on $\mathcal{H}$ is equivalent to the action of M on $L^2(M)p$ for some projection p. Since $L^2(M)$ is the GNS construction of M there is a cyclic vector.

(ix) Let $\dim_M(\mathcal{H}) \geq 1$, then $\dim_{M'}(\mathcal{H}) \leq 1$ by part (vii), thus there is a cyclic vector ξ for M' by (viii). But ξ is then a separating vector for M by Lemma 4.27.

□

We will now prove that our definition of von Neumann dimension and that of Murray and von Neumann coincide. This is not a well documented fact because the definition of Murray and von Neumann is rarely used anymore. The literature where this definition is mentioned (see [15],[19]) lack this proof. After completing this task, we can prove that (viii) and (ix) in the proposition above are equivalences.

Lemma 5.8. *Let M be a type II_1 factor, $\mathcal{H}$ a M-module and $\xi \in \mathcal{H}$. If the commutant M' is a type II_1 factor, then we have*

$$\dim_M(\mathcal{H}) = \frac{tr_M(q)}{tr_{M'}(p)}$$

with p and q being the projections onto $\overline{M\xi}$ and $\overline{M'\xi}$ respectively.

Proof. First observe that by Theorem 4.26 the projections p and q are elements of M and M' respectively. We prove the assertion first for the case $\mathcal{H} = L^2(M)q$ for a projection $q \in M$. Then $\dim_M(\mathcal{H}) \leq 1$, and there is a cyclic vector $\xi \in \mathcal{H}$ by Proposition 5.7. Since ξ is cyclic, the projection p onto $\overline{M\xi} = \mathcal{H}$ is id. With that follows

$$\frac{tr_M(q)}{tr_{M'}(p)} = tr_M(q) = \dim_M(L^2(M)q) = \dim_M(\mathcal{H}).$$

Before we proceed, we have to prove that the Murray and von Neumann definition is additive for direct sums. Let $\mathcal{H}$ be an arbitrary M-module and s be a projection in M. Define $\xi = (0, \mu) \in \mathcal{H} \oplus L^2(M)q$ and μ be cyclic vector for $L^2(M)s$. Then p is the projection onto $\overline{M\xi} = \{0\} \times \overline{M\mu} = \{0\} \times L^2(M)q$ and q is the projection onto $\overline{M'\xi} = \{0\} \times \overline{M'\mu}$. Let now be $\mathcal{H}$ be an arbitrary M-module and q a projection in M. Then M on $\mathcal{H}$ is equivalent to M on $\bigoplus_{j=1}^k L^2(M)q$ for some projection $q \in M$ and $k \in \mathbb{N}$. This fact follows in the same way as in the proof of Proposition 5.7 (vii) and $k < \infty$ since M' is a type II_1 factor and with Proposition 5.7 (ii). Then M on $\bigoplus_{j=1}^k L^2(M)q$ acts diagonally, this means M has the from

$$\begin{pmatrix} u & 0 & 0 & \dots & 0 \\ 0 & u & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u & 0 \\ 0 & 0 & \dots & 0 & u \end{pmatrix}$$

for $u \in M$. Thus the commutant M' for M on $\bigoplus_{j=1}^k L^2(M)p$ is

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k,1} & x_{k,1} & \dots & x_{k,k} \end{pmatrix}$$

for $x_{i,j}$ in M' on $L^2(M)q$. Now define $\xi = (\xi_1, 0, \dots, 0) \in \bigoplus_{j=1}^k L^2(M)q$ with $\xi_1 \in L^2(M)q$ cyclic. If we look at $M'\xi$ we get

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k,1} & x_{k,1} & \dots & x_{k,k} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} x_{1,1}\xi_1 \\ \vdots \\ x_{k,1}\xi_1 \end{pmatrix}$$

Thus $\overline{M'\xi} = \bigoplus_{j=1}^k \overline{M'\xi_1}$. Let s be the projection onto $\overline{M'\xi}$, then s is in M and has trace $tr_M(s) = tr_M(q)$. If we now look at $M\xi$ we get

$$\begin{pmatrix} u & 0 & 0 & \dots & 0 \\ 0 & u & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u & 0 \\ 0 & 0 & \dots & 0 & u \end{pmatrix} \begin{pmatrix} \xi_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u\xi_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Thus $\overline{M\xi} = \overline{M\xi_1} \oplus \bigoplus_{j=1}^{k-1}\{0\}$. So we get the projection onto $\overline{M\xi}$

$$p = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

For the trace of p it follows $tr_M(p) = \frac{1}{k}$. All in all we get

$$\frac{tr_M(s)}{tr_M(p)} = \frac{tr_M(q)}{\frac{1}{k}} = k\, tr_M(q)$$

The assertion follows now from Proposition 5.7. □

We can now show the equivalence of (viii) and (iv) of Proposition 5.7 with the help of the von Neumann dimension definition of Murray and von Neumann. This will be especially important in the last chapter, when we show the existence of a wandering and tracelike vector.

Lemma 5.9. *For a M-module $\mathcal{H}$ with M' being a type II_1 factor, then*

(i) there is a cyclic vector for M if and only if $\dim_M \mathcal{H} \leq 1$,

(ii) there is a separating vector if and only if $\dim_M \mathcal{H} \geq 1$.

Proof.

(i) Let ξ be a cyclic vector, then $\overline{M\xi} = \mathcal{H}$. If we use the Murray and von Neumann definition of von Neumann dimension, then the projection p must be equal to id and $tr_{M'}(p) = 1$. Since $tr_M(q) \leq 1$, it follows

$$\dim_M(\mathcal{H}) = \frac{tr_M(q)}{tr_{M'}(p)} \leq 1$$

The other implication was proven in Proposition 5.7 (viii).

(ii) Let ξ be a separating vector, then ξ is a cyclic vector for M'. Thus $\dim_{M'}(\mathcal{H}) \leq 1$, by (i). With (vii) of Proposition 5.7 we then have $\dim_M(\mathcal{H}) \geq 1$.

□

For the next proposition, we need the definition of Murray and von Neumann again. It will help us construct a wandering subspace in the next chapter. We need this wandering subspace to show that there is a function vanishing if and only if the von Neumann dimension of the Bergman space is greater than one. Following Jones [19], we show the following proposition.

Proposition 5.10. *Let M be a type II_1 factor with M' be a type II_1 factor. If p is a projection in M with $p\xi = \xi$, then $\dim_M(\overline{M\xi}) \leq tr_M(p)$.*

Proof. We are going to use the Murray and von Neumann definition of von Neumann dimension. W.l.o.g: $\overline{M\xi} = \mathcal{H}$ (we can restrict the action of M on $\mathcal{H}$ to of $\overline{M\xi}$).
Now let $\mathcal{H} = \overline{M\xi}$. This yields

$$M'\xi = M'p\xi = pM'\xi \subset p\mathcal{H}.$$

Thus, $tr_M(q) \leq tr_M(p)$. And therefore

$$\dim_M(\overline{M\xi}) = \frac{tr_M(q)}{tr_{M'}(1)} = tr_M(q) \leq tr_M(p)$$

□

The next remark does not fit into our main goal. Nevertheless, it is an important question concerning subfactors, Fuchsian groups and von Neumann dimension. It fits nicely to the topic of the last three chapters. Therefore, we look at a short overview of this question that include the so called Hecke group. These Hecke groups are special Fuchsian groups.

Remark 5.11. *Another question arising from the theory of von Neumann dimension is the dimension of subfactors. If we take a subfactor N of a type II_1 factor M, then $L^2(M)$ is an N-module by left multiplication. One defines the index $[M : N] = \dim_N(L^2(M))$. Jones [17] showed that the index can only attain the values $\left\{4\cos^2\left(\frac{\pi}{4}\right); n = 3, 4, \dots\right\} \cup [4, \infty]$. These values are the squares of the Hecke group generators (see [12]). The Hecke groups are Fuchsian groups generated by*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

for $\lambda \in \mathbb{R}$. Only for $\lambda \in \left\{2\cos\left(\frac{\pi}{4}\right); n = 3, 4, \dots\right\} \cup [2, \infty)$ the generated group is discrete, thus a Fuchsian group. This is remarkable, but the connection between the two is unknown to this day.

Jones [19] describes another way to calculate the von Neumann dimension. This will be helpful, when calculating the von Neumann dimension of the weighted Bergman spaces. It will close the gap between this and the previous chapters. We introduce some new concepts and some necessary results.

Lemma 5.12. *Let Γ be a discrete icc group and $\gamma \mapsto v_\gamma$ a projective unitary group representation on $\mathcal{H}$. If there is a projection q on $\mathcal{H}$ such that*

$$v_\gamma q v_\gamma^{-1} \perp q \forall \gamma \in \Gamma, \gamma \neq 1 \text{ and } \sum_{\gamma \in \Gamma} v_\gamma q v_\gamma^{-1} = id$$

then there is a unitary $u : \mathcal{H} \to \ell^2(\Gamma) \otimes q\mathcal{H}$ with $u v_\gamma u^{-1} = \lambda_\gamma^\omega \otimes id$ for all $\gamma \in \Gamma$.

Proof. Let $\{e_n | n \in \mathbb{N}\}$ be an orthonormal basis of $q\mathcal{H}$. Then $\{v_\gamma e_n | \gamma \in \Gamma, n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$. Obviously the $v_\gamma e_n$ has unit norm. For all $\gamma, \mu \in \Gamma$ with $\gamma \neq \mu$ and all $n, m \in \mathbb{N}$ follows

$$\langle v_\gamma e_n, v_\mu e_m \rangle = \langle v_{\mu^{-1}\gamma} e_n, e_m \rangle = \langle v_{\mu^{-1}\gamma} q v_{\mu^{-1}\gamma}^{-1} (v_{\mu^{-1}\gamma} e_n), e_m \rangle = 0.$$

For $m \neq n$ $v_\gamma e_n$ and $v_\gamma e_m$ are orthogonal since v_γ is an unitary. And $v_\gamma e_n$ form a basis, because for all $x \in \mathcal{H}$ is

$$x = \sum_{\gamma \in \Gamma} v_\gamma q v_\gamma^{-1} x.$$

As e_n is a orthonormal basis for $q\mathcal{H}$ there are $\alpha_{\gamma,n}$ with $q v_\gamma^{-1} x = \sum_{n \in \mathbb{N}} \alpha_{\gamma,n} e_n$. Combining yields

$$x = \sum_{\gamma \in \Gamma} v_\gamma \left(\sum_{n \in \mathbb{N}} \alpha_{\gamma,n} e_n \right) = \sum_{\gamma \in \Gamma} \sum_{n \in \mathbb{N}} \alpha_{\gamma,n} v_\gamma e_n.$$

Now set $u(v_\gamma e_n) = \delta_\gamma \otimes e_n$ with δ_γ being the characteristic function of $\{\gamma\}$. □

There is also another definition of trace, in the concept of positive operators on some Hilbert space and not as trace in a von Neumann algebra. We will use a different notation for those two concepts. In the next theorems, we use it to obtain a way to calculate the von Neumann dimension. Calculating the von Neumann dimension with our definition or with the definition of Murray and von Neumann is difficult, because finding an isometry u like in Theorem 5.2 or determining the trace of the projections is difficult.

Definition 5.13. *For a Hilbert space $\mathcal{H}$ with orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and a positive operator $x \in \mathcal{B}(\mathcal{H})$, the trace of x is defined as*

$$Tr_{\mathcal{B}(\mathcal{H})}(x) = \sum_{n\in\mathbb{N}} \langle xe_n, e_n\rangle.$$

The definition above is independent from the choice of the orthonormal basis. The trace vector will play a major role in the next chapter and especially in the last chapter. We need it also, to calculate the von Neumann dimension of the weighted Bergman spaces. More information on trace vectors can be found in [19].

Definition 5.14. *Let M be a von Neumann algebra on $\mathcal{H}$. A vector $\mu \in \mathcal{H}$ is called a trace vector for M, if there is some $\alpha \in \mathbb{C}$ with*

$$\langle x\mu, \mu\rangle = \alpha tr(x) \text{for all } x \in M.$$

The concept of trace vectors will come up at different points later on and can describe the von Neumann algebra completely. It will also give us a link between the von Neumann algebra and its commutant in certain circumstances. Another common way of defining a trace vector is the following: If $\langle ab\mu, \mu\rangle = \langle ba\mu, \mu\rangle$ for all $a, b \in M$, then μ is a trace vector (see for example [19], but there are both definitions stated). These two definitions are the same. Before we can move on we have to define the tensor product of two type II_1 factors (for more details see [3]).

Definition 5.15. *Let M and N be type II_1 factors and $\mathcal{H}$ and $\mathcal{K}$ be M-modules and N-modules respectivly. The tensor product $M \otimes N$ acts on the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ as follows*

$$(x \otimes y)(\xi \otimes \mu) = (x\xi) \otimes (y\mu)$$

Jones [19] combined Lemma 5.12 and the notion of trace vectors to calculate the von Neumann dimension.

Theorem 5.16. *Let γ, v, q and u as in Lemma 5.12. If p is a projection on $\mathcal{H}$ commuting with v_γ for all γ, then the action of Γ on $\mathcal{H}$ makes it into a $vN_\omega(\Gamma)$-module. Then one has*

$$\dim_{vN_\omega(\Gamma)} \mathcal{H} = Tr_{\mathcal{B}(\mathcal{H})}(pqp) = Tr_{\mathcal{B}(\mathcal{H})}(qpq)$$

Proof. The commutant of $vN_\omega(\Gamma)$ on $\ell^2(\Gamma) \otimes q\mathcal{H}$ is $M' := vN_\omega(\Gamma)' \otimes \mathcal{B}(q\mathcal{H})$. Note that ε_{id} is a trace vector for $vN_\omega(\Gamma)'$. For some positive $x \in M'$ follows

$$Tr_{M'}(x) = \sum_{n\in\mathbb{N}} \langle x(\delta_{id} \otimes e_n), \delta_{id} \otimes e_n\rangle = Tr_{\mathcal{B}(\ell^2(\Gamma)\otimes q\mathcal{H})}(exe)$$

with e being the orthogonal projection onto $\delta_{id} \otimes q\mathcal{H}$. Up maps $\mathcal{H}$ to $\ell^2(\Gamma) \otimes q\mathcal{H}$ isometrically and is $vN_\omega(\Gamma)$-linear Thus,

$$\dim_{vN_\omega(\Gamma)} p\mathcal{H} = Tr_{B(\ell^2(\Gamma)\otimes p\mathcal{H})}(eUpU^*e) = Tr_{B(\ell^2(\Gamma)\otimes p\mathcal{H})}(U^*eUpU^*eU) = Tr_{B(\ell^2(\Gamma)\otimes p\mathcal{H})}(qpq).$$

□

With the theorem above, we now can finally determine the von Neumann dimension of the Bergman spaces. With respect to a Fuchsian group Jones [19] uses the projection onto a fundamental domain and the identity. By the use of the unitary (projective) representation $\hat{\pi}_s$ we can identify the weighted Bergman spaces as $vN_\omega(\Gamma)$ modules for a Fuchsian group Γ. The 2-cocycle is defined by the branch of logarithm used in Theorem 3.12.

Theorem 5.17. *The von Neumann dimension of A^2_{s-2} is*

$$dim_{vN_\omega(\Gamma)}(A^2_{s-2}) = \frac{s-1}{4\pi} covolume(\Gamma).$$

Proof. We want to use Theorem 5.16. Let F be the fundamental domain of Γ in the unit disk. Set q as the restriction of functions to the fundamental domain and p as the identity, then we have

$$\pi(\gamma)q\pi(\gamma)^{-1}\perp q \;\forall\gamma\in\Gamma\setminus\{e\} \qquad\qquad \sum_{\gamma\in\Gamma}\pi(\gamma)q\pi(\gamma)^{-1}=id.$$

Obviously, p commutes with every $\pi(\gamma)$.
Thus, the assumptions of Theorem 5.16 are fulfilled, we can continue with

$$Tr_{\mathcal{B}(\mathcal{H})}(pqp)=\sum_{n\in\mathbb{N}}\langle pqpe_n,e_n\rangle=\sum_{n\in\mathbb{N}}\langle qe_n,e_n\rangle=\sum_{n\in\mathbb{N}}\int_F|e_n(z)|^2\,4(1-|z|^2)^{s-2}dxdy$$

with $(e_n)_n$ being the orthonormal basis of A^2_{s-2} from Lemma 2.7. We can interchange summation and integration, since everything is positive.

$$\begin{aligned}Tr_{\mathcal{B}(\mathcal{H})}(pqp)&=\int_F\sum_{n\in\mathbb{N}}|e_n(z)|^2\,4(1-|z|^2)^{s-2}dxdy\\&=\int_F\sum_{n\in\mathbb{N}}\frac{s-1}{4\pi}\frac{s(s+1)\dots(s+n-1)}{n!}\,|z|^{2n}\,4(1-|z|^2)^{s-2}dxdy\\&=\frac{s-1}{4\pi}\int_F\sum_{n\in\mathbb{N}}\frac{s(s+1)\dots(s+n-1)}{n!}\,|z|^{2n}\,4(1-|z|^2)^{s-2}dxdy\end{aligned}$$

We have $\sum_{n\in\mathbb{N}}\frac{s(s+1)\dots(s+n-1)}{n!}\,|z|^{2n}=(1-|z|^2)^{-s}$. So

$$\begin{aligned}Tr_{\mathcal{B}(\mathcal{H})}(pqp)&=\frac{s-1}{4\pi}\int_F 4(1-|z|^2)^{-2}dxdy\\&=\frac{s-1}{4\pi}covolume(\Gamma)\end{aligned}$$

□

This result is essential in the next chapter, and gives us a criterion for the existence of a function vanishing on the orbit of a Fuchsian group.
In the next example we explicitly calculate the von Neumann dimension of the weighted Bergman spaces with the Fuchsian group $PSL(2,\mathbb{Z})$. We need this result in Chapter 7, when we construct a Bergman space function vanishing on the orbit of $PSL(2,\mathbb{Z})$. The calculation is straight forward, and we just calculate the covolume of $PSL(2,\mathbb{Z})$ by integrating over the fundamental domain.

Example 5.18. *For the Fuchsian group $PSL(2,\mathbb{Z})$ the von Neumann dimension of A^2_{s-2} is given by*

$$dim_{vN_\omega(PSL(2,\mathbb{Z}))}A^2_{s-2}=\frac{s-1}{4\pi}\frac{\pi}{3}=\frac{s-1}{12}$$

(see Example 3.10).

At the end of this chapter, we discuss some remarks concerning the action of a Fuchsian group on the Bergman space. It is fairly important, for the following chapters, but one can follow the next chapters if one only knows that the action is a projective representation of $PSL(2,\mathbb{R})$.

Remark 5.19. *For $s\in\mathbb{R}_+$ the map $\hat{\pi}_s$ is a projective unitary representation of $SL(2,\mathbb{R})$. If s is not an integer we might have to use a 2-cocycle. For positive even integers s the map $\hat{\pi}_s$ defines a unitary representation of $PSL(2,\mathbb{R})$ because $\hat{\pi}_s(-id)=\hat{\pi}_s(id)$. (Recall $\hat{\pi}_s(g)(f)(z)=\frac{1}{(cz+d)^s}f(g(z))$ with $g\in\Gamma, f\in A^2_{s-2}(\mathbb{H})$ see Theorem 3.12) Otherwise, we only get a projective representation, for this we have to use 2-cocycles. Notice that this 2-cocycles depends on the chosen branch of logarithm. For more details, see [19].*

6 Bergman space function vanishing on orbit

In this chapter, we prove the main theorem. Firstly, we prove it for fix point free Fuchsian groups. We will use most of the results from the previous sections. Furthermore, we look at some more traditional approach to this question. We prove our many result in the most general case.
This chapter is manly based on [19] for the proof of our main theorem. The more traditional approach can be found in [13].

6.1 Fixpoint free setting

Before we can study conditions that imply the existence of a function $f \in A^2_{s-2}$ that vanishes on an orbit of a Fuchsian group, we have to first introduce some notions and some other results, such as the concept of wandering vector which will pop up later on many times.

Definition 6.1. *Let π be a representation of a group Γ on a Hilbert space $\mathcal{H}$.*

(i) A wandering vector $\xi \in \mathcal{H} \setminus \{0\}$ for π satisfies $\langle \pi(\gamma)\xi, \xi \rangle = 0$ for all $\gamma \in \Gamma \setminus \{e\}$.

(ii) Similarly a wandering subspace V is orthogonal to its translates or more precisely

$$\pi(\gamma)V \perp V \textit{for all } \gamma \in \Gamma \setminus \{e\}$$

Jones [15] links the concept of wandering vector and trace vector. This is a subtle detail in the proof at the end of this section. It will also come into play a major role in the last chapter.

Theorem 6.2. *Let π be a projective representation of a group Γ on a Hilbert space $\mathcal{H}$. If ξ is a wandering vector for π then ξ is a trace vector for the von Neumann algebra M generated by $\pi(\Gamma)$. In this case, M on $\overline{M\xi}$ is isomorphic to the group von Neumann algebra $vN_\omega(\Gamma)$ acting on $L^2(vN_\omega(\Gamma))$ with ω being the 2-cocycle of the projective representation.*

Proof. Let ξ be a wandering vector for π. Then the map $M \to \mathbb{C}, x \mapsto \frac{1}{\|\xi\|^2}\langle x\xi, x \rangle$ defines a tracial state on M. This can be proven in the same way as in Example 4.9. Thus, by the uniqueness of the trace ξ is a trace vector for M (see [3]).
With Proposition 5.7 follows M on $\overline{M\xi}$ and M on $L^2(M)$ are unitarily equivalent. Furthermore, the actions of $\pi(g)$ on $L^2(M)$ and $\lambda_\omega(g)$ on $L^2(vN_\omega(\Gamma))$ are the same for all $g \in \Gamma$. □

For the end of this section, we will essentially only work with orderable Fuchsian groups. These will allow us to find a wandering subspace of sufficient size that we need to show that there is no vanishing function if the Bergman space is not big enough.

Definition 6.3. *We call a Fuchsian group Γ orderable, if there is a total order $<$ with*

$$g < h \iff kg < kh \qquad \forall k \in \Gamma$$

Example 6.4. *Let $\pi : PSL(2,\mathbb{Z}) \to PSL(2,\mathbb{Z}/2\mathbb{Z})$ be the canonical projection (we map everything componentwise). Then the group $\Gamma(2) = \ker \pi$ is a Fuchsian group (as a subgroup of a Fuchsian group). It has index 6 in $PSL(2\mathbb{Z})$ and has finite covolume (for more information see [8, Ch. 2.3.2])*

We are going to restrict us to orderable subgroups of Fuchsian group. A necessary result of Hoare, Karrass, and Solitar [14] is the following proposition, which allows us to restrict ourselves to orderable subgroups by stating the existence of them. We will not go into detail, but rather build on the results of [14] because it is a rather algebraic consideration, which does not help to understand the topic in more detail. It is also important that this orderable subgroup has finite index, but more on that later in the proof of the main theorem.

Proposition 6.5. *Let* Γ *be a Fuchsian Group. Then there is a orderable subgroup* Ψ *with index* $[\Gamma : \Psi] < \infty$.

Proof. By Hoare, Karrass, and Solitar [14] Γ has a surface group of finite index or Γ is a free product of cyclic groups. Surface groups are bi-orderable, except for the Klein bottle group and the projective plane group (see [7]). The later two being only left orderable, thus in this case Γ has an orderable subgroup of finite index.
If Γ is a free product of cyclic groups, then Γ is free and by [7] it is left orderable. □

The next ingredient for our result is the following proposition, which lets us remove a zero at a specific point and does not touch the zeros at other points. This will be important in the next lemma, which allows us to create a wandering subspace of sufficient size. This lemma can be found in [19] but with a different proof.

Proposition 6.6. *If* f *is a nonzero function in* A^2_{s-2} *with a zero of order* k *at* w, *then* $(z-w)^{-j}f(z) \in A^2_{s-2}$ *for all* $1 \leq j \leq k$

Proof. Let $\varepsilon > 0$ with $\overline{D_\varepsilon(w)} \subset \mathbb{H}$ and $1 \leq j \leq k$. Since f is holomorphic, the map $(z-w)^{-j}f(z)$ is bounded on $D_\varepsilon(w)$ by some $m > 0$. In $\mathbb{H} \setminus D_\varepsilon(w)$ $(z-w)^{-j}$ is bounded by ε^{-j}. Combining everything, we have

$$\begin{aligned}\int_{\mathbb{H}} \left|(z-w)^j f(z)\right|^2 \frac{dxdy}{y^{2-s}} &= \int_{D_\varepsilon(w)} \left|(z-w)^j f(z)\right|^2 \frac{dxdy}{y^{2-s}} + \int_{\mathbb{H}\setminus D_\varepsilon(w)} \left|(z-w)^j f(z)\right|^2 \frac{dxdy}{y^{2-s}} \\ &\leq m \int_{D_\varepsilon(w)} \frac{dxdy}{y^{2-s}} + \varepsilon^{-j} \int_{\mathbb{H}} |f(z)|^2 \frac{dxdy}{y^{2-s}} < \infty\end{aligned}$$

Thus, $(z-w)^{-j}f(z) \in A^2_{s-2}$. □

Jones [19] constructed a wandering subspace with the help of a function that vanishes on orbits of an orderable Fuchsian group. This is the central lemma for the sufficient condition for our main theorem. The proof of the sufficient condition requires most of the work. The rest will be a combination of the results from previous chapters.

Lemma 6.7. *Let* Γ *be an orderable Fuchsian group and* $O_1, \dots, O_n$ *be disjoint orbits of* Γ. *If there is a function* $f \in A^2_{s-2} \setminus \{0\}$ *with a zero of order at least* k_i *on all points of* O_i *then there is a wandering subspace* W *of dimension* $\sum_{i=1}^n k_i$ *for the representation* π_s *and* $\pi_s(\gamma)(f) \in W^\perp$ *for all* $\gamma \in \Gamma$.

Proof. Fix $z_i \in O_i$ for every $1 \leq i \leq n$. Define the subspaces

$$\begin{aligned} U &= \{\xi | \xi^{(j)}(\gamma(z_i)) = 0 \text{ for } \gamma \leq e \text{ for all i and } 0 \leq j < k_i\} \\ V &= \{\xi | \xi^{(j)}(\gamma(z_i)) = 0 \text{ for } \gamma < e \text{ for all i and } 0 \leq j < k_i\}\end{aligned}$$

We obviously have $U \subset V$. To show $V \neq U$, let $\xi \in U$. We will use Proposition 6.6 to remove the zero at z_i. If ξ has a zero of order k in z_i, then $(z-z_i)^{-k}\xi(z)$ has no zero at z_i, but it is still in V. Thus, $(z-z_i)^{-k}\xi(z) \in V$ and $(z-z_i)^{-k}\xi(z) \notin U$.
We are going to show now that $W := U^\perp \cap V$ is the wandering subspace. Let $\xi \in W$, then for some $g < e$ and $h \leq e$ we have

$$gh \leq ge = g < e$$

With that follows

$$\pi_s(g^{-1})\xi(h(z_i)) = \frac{1}{(cz+d)^s}\xi(gh(z_i)) = 0$$

for all i. Thus $\pi_s(g^{-1})\xi \in U$ and

$$\langle \pi_s(g^{-1})\xi, \mu \rangle = 0$$

for all $\mu \in W$, which means W is wandering. $\pi_s(g)(f) \in W^\perp$ follows from $\pi_s(g)(f) \in U$.
We only need to show that W has dimension $\sum_{i=1}^n k_i$. W.lo.g.: the order of zeros at z_i of f is k_i (if this is not the case use Proposition 6.6 to lower the number of zeros)
The functions

$$\frac{f}{(z-z_i)^j}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$ are in V by Proposition 6.6. Let $a_{i,j} \in \mathbb{C}$ such that

$$\sum_{i,j} a_{i,j} \frac{f}{(z-z_i)^j} = g \in U$$

This means that the function $\frac{g}{f}$ is holomorphic in each z_i, thus all the $a_{i,j}$ are zero. Thus the $\frac{f}{(z-z_i)^j}$ are linearly independent in V/U. Since $V/U \simeq W$, W has at least dimension $\sum_{i=1}^n k_i$. □

With the lemma from above, Jones [19] states necessary and sufficient conditions for the existence of a Bergman space function vanishing on the orbit. This is a rather short proof for this result because most of the work has been outsourced to the previous chapters or the results from immediately above. We only need to combine all the results from earlier.

Theorem 6.8. *Let Γ be a fixed point free Fuchsian group. There is a function $f \in A^2_\alpha \setminus \{0\}$ with $f(\Gamma(z)) \equiv 0$, if and only if*

$$\alpha > \frac{4\pi}{covolume(\Gamma)} - 1.$$

Proof. $\Longrightarrow$: Let $\alpha > \frac{4\pi}{covolume(\Gamma)} - 1$, then by Proposition 5.7 $\dim_{vN_\omega(\Gamma)} \overline{vN_\omega(\Gamma)\varepsilon_z} \leq 1$ with ε_z being the reproducing kernel for z. Thus there is some non zero $f \in A^2_\alpha$ orthogonal to $\overline{vN_\omega(\Gamma)\varepsilon_z}$ (This follows from Proposition 5.17). This means f is orthogonal to $\pi_s(\gamma)\varepsilon_z$ for all γ.

$$f(\gamma(z)) = \langle f, \varepsilon_{\gamma(z)} \rangle = \langle f, \pi_s(\gamma)\varepsilon_z \rangle = 0$$

So f vanishes on $\Gamma(z)$.
$\Longleftarrow$: There is a orderable subgroup Ψ of Γ with index $n < \infty$ by Proposition 6.5.
Suppose there is a function $f \in A^2_{s-2}$ with $s \leq 1 + \frac{4\pi}{covolume(\Gamma)}$ vanishes on the Γ-orbit of z
$\Gamma(z)$ is made up of n disjoint Ψ-orbits since Γ is supposed to be fixed point free. With Lemma 6.7 we get a n-dimensional wandering subspace W. Let $(\xi_i)_{i=1}^n$ be a orthonormal basis of W. Then all the ξ_i are trace vectors for the II_1 factor $vN_\omega(\Psi)$. Moreover the $vN_\omega(\Psi)$-modules $\overline{vN_\omega(\Psi)\xi_i}$ are orthogonal to each other and have von Neumann dimension one. That these $\overline{vN_\omega(\Psi)\xi_i}$ have von Neumann dimension one is clear, since all the ξ_i are wandering. The orthogonality follows from

$$\langle x\xi_i, y\xi_j \rangle = \langle y^* x\xi_i, \xi_j \rangle = 0$$

for $x, y \in vN_\omega(\Psi)$. With Proposition 5.7 follows

$$n \leq \dim_{vN_\omega(\Psi)} A^2_{s-2} = n \dim_{vN_\omega(\Gamma)} A^2_{s-2}$$

Thus $\dim_{vN_\omega(\Gamma)} A^2_{s-2} \geq 1$. Also with Lemma 6.7 follows that f is orthogonal to $vN_\omega(W)$ and $\dim_{vN_\omega(\Gamma)} vN_\omega(W) = 1$. So there can not be such a function f. □

The proof for the general main theorem will be very similar, except we have to account for the stabilizer that is here empty. For the necessary condition, we only needed basic von Neumann dimension theory and the reproducing kernel on the Bergman space. The other implication was much more difficult to prove and needed all the results stated in this chapter so far. Note also that we needed that the orderable subgroup has finite index, otherwise we could not do the estimate that finished essentially the proof. We also needed Theorem 6.2 for the properties of $\overline{vN_\omega(\Psi)}$.
We get an immediate result concerning Fuchsian groups and the Blaschke condition (see [19]).

Corollary 6.9. *For every Fuchsian group* Γ *with finite covolume and* $z \in \mathbb{H}$

$$\sum_{g\in\Gamma}(1-|g(z)|) \text{ diverges.}$$

Proof. If the Blaschke condition is satisfied, there is a Hardy space function $f \in H^2$ vanishing on the orbit of z. Since the Hardy space H^2 is contained in every Bergman space we have $f \in A^2_\alpha$ for all $\alpha > -1$. This is not possible, since there has to be some $\alpha_0 > -1$ with $f \in B^2_\alpha$ for all $\alpha > \alpha_0$. □

Thus, there is no Hardy space function that vanishes on an orbit of a Fuchsian group. Tsuji [31] describes this topic more in depth and in a more traditional way.
This is an unusual way to get a necessary and sufficient condition for the existence of a vanishing function in A^2_{s-2}. Usually one looks at so-called densities $D^+(S)$ for some $S \subseteq \mathbb{H}$.

6.2 Asymptotic density

In this section, we look into a more classical approach. The idea behind these so-called densities is that the change the Blaschke condition in such a way that it is true for the weighted Bergman spaces. We have to use more sophisticated methods to get our desired condition.
Furthermore, we will give a result for these densities that have only been proven by the results in the previous section.
Before we can define the asymptotic density, we have to think about the unit circle. An arc is the path of a continuous function $\gamma : [0,1] \to \mathbb{C}$. Thus an arc in the unit circle is just a connected closed subset. For a countable subset F, there are arcs $(I_n)_{n\in\mathbb{N}}$ with $\mathbb{T} = F \cup \bigcup_{n\in\mathbb{N}} I_n$. Following Hedenmalm, Korenblum, and Zhu [13] we define the Beurling-Carleson characteristic for a countable subset F of $\mathbb{T}$.

Definition 6.10. *Let* $F \subsetneq \mathbb{T}$ *be a countable subset. Then there are arcs* $(I_n)_{n\in\mathbb{N}}$ *with* $\mathbb{T} \setminus F = \bigcup_{n\in\mathbb{N}} I_n$. *We define the Beurling-Carleson characteristic of* F *by*

$$\kappa(F) = \sum_n |I_n| \log \frac{e}{|I_n|}$$

Another important concept are Stolz angles, these are often used to study limits from the inside of $\mathbb{D}$ to a point on the boundary. These are also often used in the theory of Hardy and Dirichlet spaces, so it is natural to consider them for Bergman space theory.

Definition 6.11. *(i) The Stolz angle at* $x \in \mathbb{T}$ *with aperture* $\alpha > 0$ *is defined by*

$$s_x = \left\{ z \in \mathbb{D}; \frac{|z-x|}{1-|z|} < 1+\alpha \right\}$$

(ii) For a finite $F \subset \mathbb{T}$ *the Stolz star domain is*

$$s_F = \bigcup_{x\in F} s_x$$

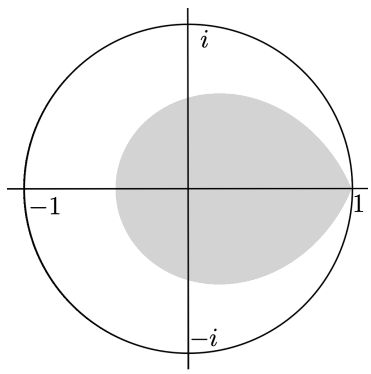

Figure 2: Stolz angle at 1

Figure 2 shows the Stolz angle at 1 with aperture $\frac{\pi}{2}$. In the following, we are only going to use Stolz angles with aperture $\frac{\pi}{2}$. The partial Blaschke sum of a sequence $A = (a_n)_{n\in\mathbb{N}}$ in $\mathbb{D}$ and some subset E of $\mathbb{D}$ is

$$\Sigma(A,E) = \frac{1}{2} \sum_{n\in\mathbb{N}; a_n \in E} 1 - |a_n|^2$$

The square in $\Sigma(A, E)$ changes nothing with respect to the Blaschke sum from above, since $|a_n| < 1$ and we have

$$(1 - |a_n|) \leq (1 - |a_n|^2) = (1 - |a_n|)(1 + |a_n|) \leq 2(1 - |a_n|).$$

Korenblum [20] and Seip [28],[29] stated a similar result using densities. But before we can state the result, we have to generalize and adjust the Blaschke sum.

Definition 6.12. *Let $A = (a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$.*

(i) The κ-density of A with respect to F is

$$D(A, s_F) = \frac{\Sigma(A, s_F)}{\kappa(F)}$$

(ii) The upper asymptotic κ-density $D^+(A)$ is defined by

$$D^+(A) = \limsup_{\kappa(F)\to\infty} D(A, s_F)$$

In [13] one can find the following theorem with a proof. It is in a sense similar to our main result, but is not restricted to Fuchsian groups. But it is much more difficult to calculate the upper asymptotic κ-density for general subsets of $\mathbb{D}$. We will see in a moment that we can calculate the asymptotic κ-density for the orbit of a Fuchsian group with the help of the next theorem.

Theorem 6.13. *Let $S \subset \mathbb{D}$. The condition*

$$D^+(S) \leq \frac{1+\alpha}{p}$$

is necessary and

$$D^+(S) < \frac{1+\alpha}{p}$$

is sufficient for S to be an A^p_α zero.

We will skip the proof of the theorem above because it is quite elaborate and uses more classical approaches. Also notice that we are only working with $p = 2$. We can now get the κ-density for the orbit of a fix point free Fuchsian group. We just use the theorem above the other way round as one would normally do.

Corollary 6.14. *If Γ and z are as in Theorem 6.8 then*

$$D^+(\Gamma(z)) = \frac{2\pi}{covolume(\Gamma)}$$

Proof. With Theorem 6.13 and Theorem 6.8 we get

$$D^+(\Gamma(z)) = \frac{1+\alpha}{2} = \frac{s-1}{2} = \frac{2\pi}{covolume(\Gamma)}$$

□

6.3 Addressing Fixpoints

In the first part of this chapter, we assumed that Γ has no fixed points in the orbit. We will now generalize the ideas from above to Fuchsian groups that have fixed points. The idea behind is, we just factor out the stabilizer group. When we do that, we have a fix point free group and can continue like we did before. At the end, we have to factor in the stabilizer.
Jones [19] proved the following proposition, which is a generalization of Theorem 6.8.

Theorem 6.15. *Let Γ be a Fuchsian group and let $O_1, \ldots, O_n$ be disjoint orbits in $\mathbb{D}$ of Γ. The condition*

$$s > 1 + \frac{4\pi}{covolume(\Gamma)} \sum_{i=1}^{n} \frac{k_i}{|stab_i|},$$

with $stab_i$ being the stabilizers of O_i, is a necessary and sufficient condition for the existence of a function $f \in A^2_{s-2} \setminus \{0\}$ with zero of order at least k_i on O_i.

Proof.
$\Longleftarrow$: Let Ψ be an orderable subgroup of Γ as in the proof of Theorem 6.8. First observe that the action of Γ and $\Gamma/stab_i$ create the same orbits. There are $\frac{[\Gamma:\Psi]}{|stab_i|}$ disjoint orbits of Ψ in O_i. There are $k_i \frac{[\Gamma:\Psi]}{|stab_i|}$ Ψ-orthogonal trace vectors by Lemma 6.7. Therefore

$$dim_{vN_\omega(\Psi)}(A^2_{s-2}) = \frac{s-2}{4\pi} covolume(\Gamma)[\Gamma : \Psi] \geq [\Gamma : \Psi] \sum_{i=1}^{n} \frac{v_i}{|stab_i|}$$

Like in the proof of Theorem 6.8 f itself is orthogonal to the $vN_\omega(\Psi)$-linear span. With that follows the assertion.
$\Longrightarrow$: Let $z_i \in O_i$ for $1 \leq i \leq n$ and $\varepsilon_i^j \in A^2_{s-2}$ with

$$\langle f, \varepsilon_i^j \rangle = f^{(j)}(z_i)$$

Let g_i be a generator for $stab_i$. Set the 2-cocycle ω in such a way that $\pi_s(g_i)^{|stab_i|} = 1$. We also have that $\pi_s(g_i)\varepsilon_i^j$ is multiple of ε_i^j. Thus ε_i^j is in the eigenspace of $\pi_s(g_i)$. Let $p \in vN_\omega(\Gamma)$ be the projection onto the image of $\pi_s(g_i)$, then p has trace $\frac{1}{|stab_i|}$. With Proposition 5.7 follows

$$dim_{vN_\omega(\Gamma)}(\overline{vN_\omega(\Gamma)\varepsilon_i^j}) \leq tr_{vN_\omega(\Gamma)}(p) = \frac{1}{|stab_i|}$$

Again by Proposition 5.7 follows

$$\sum_{i,j} dim_{vN_\omega(\Gamma)}(\overline{vN_\omega(\Gamma)\varepsilon_i^j}) \leq \sum_i \frac{k_i}{|stab_i|} \leq dim_{vN_\omega(\Gamma)}(A^2_{s-2})$$

Thus there is a $\xi \in A^2_{s-2}$ orthogonal to $vN_\omega(\Gamma)\varepsilon_i^j$ for all i, j. Therefore ξ is the desired function. ☐

We get a similar result to Theorem 6.8, this is only a more general proposition because, if Γ is fix point free, the stabilizer is only the neutral element. Thus, we get Theorem 6.8. Also, the corollaries drawn from Theorem 6.8 holds true in a similar manner.

Corollary 6.16. *If Γ and z are as in Theorem 6.15, then we have*

$$D^+(\Gamma(z)) = \frac{2\pi}{|stab_z|\, covolume(\Gamma)}$$

Proof. This follows in the same way as Corollary 6.14. ☐

7 Cusp forms

In this chapter, we briefly introduce modular forms and cusp forms. We go then straight to some examples of cusp forms, which we need for the considerations at the end of this chapter. In the second half, we consider the multiplication operator of a cusp form on the weighted Bergman spaces. These operators map a weighted Bergman space to another and is compatible with the action of the modular group. With these operators, we have finally a tool to explicitly construct a function vanishing on an orbit of $PSL(2, \mathbb{Z})$. Zagier [32] describes the general theory of modular forms and cusp forms. The second half of this chapter is mostly based on [19].

7.1 Basics on Cusp forms

Cusp forms are a special kind of modular form. Therefore, we first have to define what it means to be a modular form. Modular forms are in a sense invariant holomorphic functions with respect to the action of $PSL(2, \mathbb{Z})$. Notice also that $PSL(2, \mathbb{Z})$ is a Fuchsian group because it is a discrete subgroup with finite covolume. Following Jones [19], we define modular forms.

Definition 7.1. *A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying $f(g(z)) = (cz + d)^p f(z)$ for all $g \in PSL(2, \mathbb{Z})$ is called a modular from of weight p.*

We will use cusp forms to get a function vanishing on the orbit for certain values of s. But first we have to look at some properties of cusp forms. The next lemma is well-known and the key to the definition of cusp forms. It is i.e., the Fourier transformation. For more details see [21].

Lemma 7.2. *Every modular form f can be expressed as*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n$$

with $q = e^{2\pi iz}$. The series above converges locally uniformly on $\mathbb{H}$.

Proof. Let f be a modular form, then we have

$$f(z + 1) = f(g(z)) = f(z)$$

with

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

There is a holomorphic function $h : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ with $f(z) = h(e^{2\pi iz})$. We let $h(z) = f\left(\frac{\log(z)}{2\pi i}\right)$ for some branch of logarithm depending on z. But h is independent from the chosen branch of logarithm, since let $\log'$ be another branch of logarithm, then there is a $k \in \mathbb{N}$ with $\log'(z) = \log(z) + 2\pi ik$. With the equation above we get inductively

$$f\left(\frac{\log'(z)}{2\pi i}\right) = f\left(\frac{\log(z) + 2\pi ik}{2\pi i}\right) = f\left(\frac{\log(z)}{2\pi i} + k\right) = f\left(\frac{\log(z)}{2\pi i}\right).$$

Thus, h is welldefined and holomorphic as a composition of holomorphic functions. Forming the Laurent series of g yields

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n$$

□

In the following let $q(z) = e^{2\pi i z}$. Following Jones [19], we define cusp forms. These differ from regular modular forms in the sense, that their q-extension from the theorem above starts by one instead of zero. We will use them to in a sense lift a Bergman space function to another weight and spread its zero on the orbit of $PSL(2, \mathbb{Z})$.

Definition 7.3. *A modular form of weight p f with representation*

$$f = \sum_{n=1}^{\infty} a_n q^n$$

and $|f(z)| \operatorname{Im}(z)^{\frac{p}{2}}$ bounded on $\mathbb{H}$ is called cusp from of weight p.

Cusp forms fulfil a certain boundness condition. It will be important for the boundness of the multiplication operator defined in the next section. For more details, see [19].

Lemma 7.4. *For a cusp form f of weight p there exists a constant $c \in \mathbb{R}$ with*

$$|f(z)| \leq c(\operatorname{Im}(z))^{-\frac{p}{2}}$$

Proof. We have for every $g \in PSL(2, \mathbb{Z})$

$$|f(g(z))| \operatorname{Im}(g(z))^{\frac{p}{2}} = |cz + d|^p |f(z)| \frac{\operatorname{Im}(z)^{\frac{p}{2}}}{|cz + d|^p} = |f(z)| \operatorname{Im}(z)^{\frac{p}{2}}.$$

It is enough to look at a fundamental domain and not the whole plane, since we can translate every point to the fundamental domain. Since $|f(z)| \operatorname{Im}(z)$ is bounded by definition, we have proven the assertion. □

We will have a look at some well-known examples of cusp forms. All the following examples will be used in the next section. The j-invariant is only a modular form of weight 0, but the product of a modular form and a cusp form is a cusp form. One can find that these are cusp forms or modular forms in [32].

Example 7.5.

(i) The Eisenstein series

$$G_k(w) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m + nw)^{2k}}$$

is a cusp from of weight $2k$

(ii) The modular discriminant

$$\Delta(w) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

is a cusp form of weight 12.

(iii) The j-invariant

$$j(w) = 1728 \frac{g_2(w)^3}{\Delta(w)}$$

is a modular form of weight 0.

7.2 Cusp forms and Bergman spaces

In this section, we define first the multiplication operator M_f for some cusp form f on $L^2(\mathcal{H}, \frac{dxdy}{y^{2-s}})$ and the compatibility with the action of $PSL(2,\mathbb{Z})$ and holomorphic functions. We then can restrict M_f to the Bergman space and choose f in such a way that we can construct a function vanishing on an orbit of $PSL(2,\mathbb{Z})$. In [19], one finds the following proposition.

Proposition 7.6. *Let f be a cusp form of weight p. The multiplication operator*

$$M_f : L^2(\mathbb{H}, y^{s-2}dxdy) \to L^2(\mathbb{H}, y^{s+p-2}dxdy), g \mapsto fg$$

is bounded and satisfies

$$\hat{\pi}_{s+p}(\gamma^{-1})(M_f\xi) = M_f(\hat{\pi}_s(\gamma^{-1})\xi)$$

Furthermore $M_f^*(\xi)(z) = \operatorname{Im}(z)^p\overline{f(z)}\xi(z)$.

Proof. For $g \in L^2(\mathbb{H}, y^{s-2}dxdy)$ follows with Lemma 7.4

$$\|M_f g\|^2 = \int_{\mathbb{H}} |f(z)|^2 \, |g(z)|^2 \, y^{s+p-2}dxdy \leq c \int_{\mathbb{H}} |g(z)|^2 \, y^{s-2}dxdy$$

The second statement follows with

$$\hat{\pi}_{s+p}(\gamma^{-1})(M_f g)(z) = \frac{1}{(cz+d)^{s+p}} f(\gamma(z))g(\gamma(z)) = \frac{(cz+d)^p}{(cz+d)^{s+p}} f(z)g(\gamma(z)) = M_f(\hat{\pi}_s(\gamma^{-1}g)(z)$$

Let $h \in L^2(\mathbb{H}, y^{s-2}dxdy)$

$$\langle M_f g, h\rangle = \int_{\mathbb{H}} f(z)g(z)\overline{h(z)}y^{2+p-2}dxdy = \int_{\mathbb{H}} g(z)\overline{\overline{f(z)}h(z)}\operatorname{Im}(z)^p y^{s-2}dxdy$$

□

With the proposition above, we can define the operator T_f which is in essence the restriction of M_f to the Bergman spaces. Following Jones [19], we define the operator T_f.

Definition 7.7. *The operator $T_f : A^2_\alpha \to A^2_{\alpha+p}$ is given by*

$$T_f = M_f$$

We construct in the following the cusp form in the multiplication operator that helps us construct the desired Bergman space function at the end of this chapter. Now the functions from Example 7.5 come into play. This function described in [19] helps us construct a function vanishing on an orbit of $PSL(2,\mathbb{Z})$.

Lemma 7.8. *Let $z_0 \in \mathbb{H}$. Then the function*

$$h_r(z) = (j(z) - j(z_0))\Delta\eta(z)^r$$

(with $\eta(z) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$) is a cusp form of weight $12 + \frac{r}{2}$ and vanishes on the orbit of $PSL(2,\mathbb{Z})$ of z_0.

Proof. The j-invariant is a modular form of weight zero. Thus, multiplying by Δ turns it into a cusp form of weight 12. As $\eta(z)^{24} = \Delta(z)$ it is clear that $\eta(z)^r$ is a cusp form of weight $\frac{r}{2}$. □

With the help of the function h_r we can give an explicit function in A^2_{s-2} vanishing on an orbit of $PSL(2,\mathbb{Z})$ for $s > 13$ (see [19]).

Theorem 7.9. *Let $s > 13$ and z_0, h_r as in Lemma 7.8. Choose $t > 1$ and $r > 0$ such that $s = 12 + t + \frac{r}{2}$. Then $T_{h_r}f \in A^2_{s-2}$ for $f \in A^2_{t-2}$ vanishes on the orbit of $PSL(2,\mathbb{Z})$ of z_0*

Proof. The function h_r vanishes on the orbit of z_0 because

$$h_r(g(z_0)) = (cz + d)^{12+\frac{r}{2}} h(z_0) = 0$$

Thus, the assertion follows from Lemma 7.8 and Proposition 7.6. □

We now look at another application of cusp forms and von Neumann algebras. The underlying question is "how big is the image of T_f?". Jones [19] answered this question with the following proposition.

Proposition 7.10. *Let f be a cusp form. Then $\overline{T_f A^2_{s-2}}$ is an M-module with*

$$\dim_M(\overline{T_f A^2_{s-2}}) = \dim_M(A^2_{s-2})$$

Proof. First, observe that multiplication by f is injective. For this, we see that f has only countable many zeros. With this result, we get that if $fg = fh$ for two functions g and h, then $g = h$ except on the zeros of f. But since g and h are holomorphic they have to be the same. Thus, T_f is injective.
With the polar decomposition, we get a partial isometry U and a non-negative P with $T_f = UP$ and with $\ker U = \ker P = \ker T_f = \{0\}$. Furthermore, U has range $\overline{T_f A^2_{s-2}}$. Thus U is a unitary from A^2_{s-2} to $T_f A^2_{s-2}$. Now, the assertion follows from Proposition 5.7. □

The following corollary is trivial, if f has a zero at z, then ε_z is clearly orthogonal to fA^2_{s-2}. But if f has no zeros, we do not have a constructive proof (see [19]).

Corollary 7.11. *Let $s > 1$ and f be a cusp form of weight p. Then there is a $\xi \in A^2_{s+p-2}$ with*

$$\xi \perp f A^2_{s-2}$$

Proof. This follows easily from Lemma 7.10 with

$$\dim_M(A_{s+p-2}) > \dim_M(\overline{T_f A^2_{s-2}}) = \dim_M(A^2_{s-2}).$$

□

8 Tracelike and trace vectors

We will first introduce the Poincaré series, which will later be central for the definition of a tracelike vector. We use Toeplitz operators to get a dense subset in the commutant of $vN_\omega(\Gamma)$ with respect to the norm $\sqrt{tr_M(x^*x)}$. With this knowledge, we can prove that it is equivalent to be a trace vector and tracelike for the commutant. At the end, we come to the result that there is a wandering and tracelike vector if and only if $\dim_M(A^2_{s-2}) = 1$. It is quite an astonishing result because this vector is a link between M' and M, leading to the conclusion that these are isomorphic. To this day, an explicit construction of a wandering and tracelike vectors remains an open problem. We will prove, if $\dim_M(A^2_{s-2}) = 1$, then there is a wandering and tracelike vector a second time, but with the help of reproducing kernels and not with the argument using von Neumann dimension. This chapter is manly based on [19] and [26].
Following Jones [19], we show the existence of the Poincaré series. It is a well-known result, and we sum over all the translations a Bergman space function.

Proposition 8.1. *For $f \in A^2_{s-2}$ the Poincaré series $\sum_{g\in\Gamma} \frac{f(g(z))^2}{(cz+d)^{2s}}$ converges locally uniformly in $\mathbb{H}$ to a holomorphic function. Furthermore $\sum_{g\in\Gamma} \frac{|f(g(z))|^2}{|(cz+d)|^{2s}}$ converges locally uniformly in $\mathbb{H}$ to a continuous function.*

Proof. Let $F \subset \mathbb{H}$ be a fundamental domain of Γ and $D_r(z) \subset F$ a ball in F. Set $f_g(z) = (\pi_s(g^{-1})f)(z)^2$. Then

$$\|f\|_2^2 = \int_{\mathbb{H}} |f(z)|^2 \frac{dxdy}{y^{s-2}} = \sum_{g\in\Gamma} \int_F \left|(\pi_s(g^{-1})f)(z)\right| \frac{dxdy}{y^{s-2}} = \sum_{g\in\Gamma} \int_F |f_g(z)| \frac{dxdy}{y^{s-2}}.$$

With the holomophy of f_g follows with the mean value property similar as in Lemma 2.5

$$|f_g(z)| \leq c \int_F |f_g(z)| \frac{dxdy}{y^{s-2}} \qquad \text{for all } g \in \Gamma, z \in K$$

for some $c > 0$. Combing these two results and applying the Weierstraß M-test yields the uniform convergence on K of the two stated functions. By varying the fundamental domain (run over gF for all $g \in \Gamma$) we get the locally uniform convergence for both functions. □

We can now define the notion of tracelike and will link it to trace vectors of the commutant of $vN_\omega(\Gamma)$. Note also that $\text{Im}(z)^{-s}dxdy$ is the measure for A^2_{s-2}. This will later come in handy and is not a coincidence.

Definition 8.2. *We call a function $f \in A^2_{s-2}$ tracelike, if there exists a $c > 0$ such that*

$$\sum_{g\in\Gamma} \frac{|f(g(z))|^2}{|cz+d|^{2s}} = c\,\text{Im}(z)^{-s}$$

for all $z \in \mathbb{H}$.

We are going to look at the commutant of $vN_\omega(\Gamma)$. Denote $M := vN_\omega(\Gamma)$. The next definition is similar to Definition 7.7, but now we do not restrict ourselves to cusp forms.

Definition 8.3. *Let f be a L^∞ function on $\mathbb{H}$ that is invariant with respect to the action of a Fuchsian group Γ and P be the projection from $L^2(\mathbb{H})$ onto A^2_{s-2}. We define the Toeplitz operator T_f by*

$$T_f = PM_f : A^2_{s-2} \to A^2_{s-2}$$

for a L^∞ function f.

Rădulescu [27] showed in Chapter 3 that we can calculate the trace of a Toeplitz operator with a very nice formula. We will not prove this theorem because it is quite elaborate and needs much more theory of Toeplitz operators and is not helpful for the comprehension of our goal in this chapter.

Theorem 8.4. *For a Fuchsian group Γ with fundamental domain F and $f \in L^\infty(\mathbb{H})$ there is a constant c with*

$$tr_M(T_f) = c \int_F f(z) \frac{dxdy}{y^2}.$$

Radulescu [26] proved the following theorem that is the second ingredient for our existence prove at the end.

Proposition 8.5. *The set of Toeplitz operators is dense in M with respect to the norm $\|x\| = \sqrt{tr_M(x^*x)}$*

Following Jones [19], we show that a tracelike vector is a trace vector and visa versa. It is the first and major step to prove the existence of a tracelike vector in the next theorem.

Proposition 8.6. *A function $\xi \in A^2_{s-2}$ is tracelike, if and only if it is a trace vector for the commutant M.*

Proof. $\Longrightarrow$: Let $f \in A^2_{s-2}$ be tracelike. We have by the definition of tracelike for some $\xi \in A^2_{s-2}$

$$\begin{aligned}\langle T_f\xi, \xi\rangle &= \int_{\mathbb{H}} f(z) \left|\xi(z)\right|^2 \frac{dxdy}{y^{2-s}} = \int_F f(z) \sum_{g\in\Gamma} \left|(\pi_s(g^{-1})\xi)(z)\right|^2 \frac{dxdy}{y^{2-s}} \\ &= \int_F f(z) c y^{-s} \frac{dxdy}{y^{2-s}} = c\int_F f(z)\frac{dxdy}{y^2}.\end{aligned}$$

With Theorem 8.4 follows that ξ is tracelike.
$\Longleftarrow$: Let $f \in A^2_{s-2}$ be a trace vector. By Theorem 8.4 we have up to a multiplicative constant

$$tr_M(T_f) = \int_F f(z)\frac{dxdy}{y^2}$$

Thus since ξ is a trace vector, it is

$$\int_F f(z)\frac{dxdy}{y^2} = \langle T_f\xi, \xi\rangle = \int_{\mathbb{H}} f(z) \left|\xi(z)\right|^2 \frac{dxdy}{y^{2-s}} = \int_F f(z) \sum_{g\in\Gamma} \left|(\pi_s(g^{-1})\xi)(z)\right|^2 \frac{dxdy}{y^{2-s}} \quad (3)$$

for all $f \in L^\infty(F)$. By Proposition 8.1 the series $\sum_{g\in\Gamma} \left|(\pi_s(g^{-1})\xi)(z)\right|^2$ converges to a continuous function, this implies

$$\sum_{g\in\Gamma} \left|(\pi_s(g^{-1})\xi)(z)\right|^2 = \operatorname{Im}(z)^{-s}$$

up to a multiplicative constant for all $z \in F$. Otherwise the equality 3 would be false since f was arbitrary. By running through gF for $g \in \Gamma$ (like in Proposition 8.1) follows ξ is tracelike. □

The next proposition is a minor (but important) step that we need to prove the next theorem. Jones [19] stated this statement incorporated in the proof of the next theorem.

Proposition 8.7. *Let M be a von Neumann algebra and $\mathcal{H}$ be a M-module with* $\dim_M(\mathcal{H}) = 1$. *Then $x \in \mathcal{H}$ is a trace vector for M, if and only if x is a trace vector for M'.*

Proof. Since $\dim_M(\mathcal{H}) = 1$, the action of M on $\mathcal{H}$ is unitarily equivalent to M on $L^2(M)$. With the same arguments M' on $\mathcal{H}$ is unitarily equivalent to M' on $L^2(M')$. □

Jones [19] combines the notion of tracelike, wandering vector and give necessary and sufficient conditions for the existence of a tracelike function. It is a major result of this work, and most of the effort has been done above. Unfortunately, the proof of this theorem gives us no clue on how to construct this wandering and tracelike vector.

Theorem 8.8. *The existence of a tracelike function* $f \in A^2_{s-2}$ *is equivalent to* $s \leq 1+\frac{4\pi}{covolume(\Gamma)}$. f *is addition a wandering vector for* Γ, *if and only if* $s = 1 + \frac{4\pi}{covolume(\Gamma)}$.

Proof. By Proposition 8.6 the existence of a tracelike function is equivalent to the existence of a trace vector for $vN_\omega(\Gamma)'$. With Proposition 5.7 this is equivalent to $\dim_{vN_\omega(\Gamma)'}(A^2_{s-2}) \leq 1$. Again with Proposition 5.7 this is equivalent to $s \leq 1 + \frac{4\pi}{covolume(\Gamma)}$.
A trace vector $f \in A^2_{s-2}$ for $vN_\omega(\Gamma)$ is wandering if and only if $\dim_{vN_\omega(\Gamma)}(A^2_{s-2}) = 1$. This follows from Theorem 6.2. Furthermore by Proposition 8.7 f is also a trace vector for M. The second assertion now follows from Proposition 8.6. □

The theorem above is very important, because it means that $vN_\omega(\Gamma)$ is the same as its commutant $vN_\omega(\Gamma)'$. If one finds this wandering and tracelike vector, we have the link between these two factors. But up to this day no one has found a way to construct such a vector.
The implication $f \in A^2_{s-2}$ is wandering for $s = 1+\frac{4\pi}{covolume(\Gamma)}$, then f is tracelike, can be proven with the help of kernels. After a conversation with Alexandru Aleman, the following proof came to my mind.

Proof. Let $f \in A^2_{s-2}$ be a wandering vector.
W.l.o.g: $\|f\| = 1$ (otherwise use $\frac{f}{\|f\|}$)
Now $\pi_s(g)f$ forms an orthonormal basis of A^2_{s-2}. Orthogonality follows immediately with the definition of wandering. Let $g, h \in \Gamma$

$$\langle \pi_s(g)f, \pi_s(h)f\rangle = \omega(h, h^{-1})\langle \pi_s(h^{-1})\pi_s(g)f, f\rangle = \omega(h, h^{-1})\omega(h^{-1}, g)\langle \pi_s(h^{-1}g)f, f\rangle = 0.$$

Suppose for sake of contradiction that $\pi_s(g)f$ is not a basis. Then we can write $A^2_{s-2} = \overline{vN_\omega(\Gamma)f} \oplus (vN_\omega(\Gamma)f)^\perp$. By Proposition 5.7 $\dim_{vN_\omega(\Gamma)}(\overline{vN_\omega(\Gamma)f}) \geq 1$.
And $1 = \dim_{vN_\omega(\Gamma)}(A^2_{s-2}) = \dim_{vN_\omega(\Gamma)}(\overline{vN_\omega(\Gamma)f}) + \dim_{vN_\omega(\Gamma)}((vN_\omega(\Gamma)f)^\perp) > 1$ which is a contradiction.
With Theorem 2.3 and Theorem 2.12 follows

$$\frac{2^s}{(i(\overline{w} - z))^s} = \sum_{g\in\Gamma}(\pi_s(g)f)(z)\overline{(\pi_s(g)f)(w)}$$

Finally looking at $K(z, z)$ we get

$$\mathrm{Im}(z)^{-s} = \frac{2^s}{(i(\overline{z} - z))^s} = \sum_{g\in\Gamma}\left(((\pi_s(g)f)(z))\left(\overline{(\pi_s(g)f)(z)}\right)\right) = \sum_{g\in\Gamma}|\pi_s(g)f)(z)|^2$$

Thus f is tracelike. □

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