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Master's thesis

Spatial Pair Partitions and Applications to Finite Quantum Spaces

submitted by

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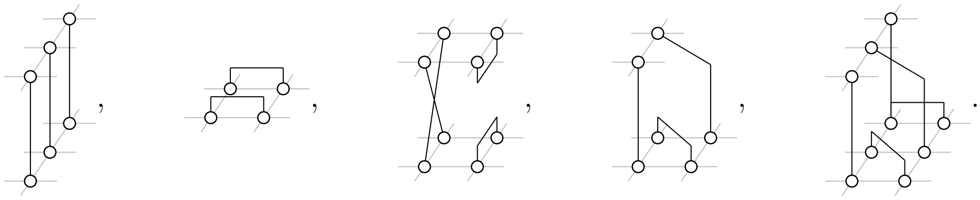
Abstract

In 2016, Cébron and Weber introduced spatial partition quantum groups as a generalization of easy quantum groups. These are compact matrix quantum groups whose intertwiners are indexed by categories of three-dimensional partitions instead of two-dimensional ones.
We study examples of categories of spatial pair partitions and their corresponding quantum groups. In particular, we show that the quantum group associated to the category of all spatial pair partitions is isomorphic to the projective orthogonal group. Further, we generalize combinatorial methods for partitions to the setting of spatial partitions. This allows us to find an explicit description of a category of spatial partitions linked to quantum symmetries of finite quantum spaces.

Contents

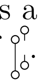
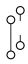
1 Introduction

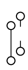
In this thesis we study spatial partitions and their associated compact matrix quantum groups. These were first introduced by Cébron-Weber in [CW22] under the name spatial partition quantum groups and they generalize easy quantum groups from [BS09] by replacing partitions with three-dimensional spatial partitions.
A spatial partition $p \in P^{(m)}(k,l)$ is a partition of the set $S = \{1,\dots,k+l\} \times \{1,\dots,m\}$ into disjoint subsets. These can be visualized by drawing points for elements in S and connecting them with lines if they are in the same subset, e.g.



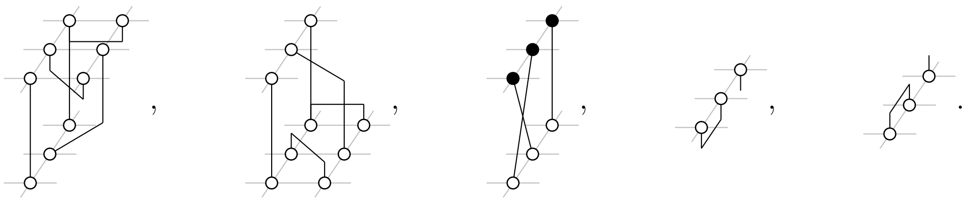
Further, one can define a tensor product, an involution and a composition for spatial partitions. A category of spatial partitions $C \subseteq P^{(m)}$ is then a set of spatial partitions which is closed under these operations and which contains so called base partitions. To each category C of spatial partitions Cébron and Weber associated a corresponding compact matrix quantum group. These are called spatial partition quantum groups and are defined via Tannaka-Krein duality by associating linear maps T_p to every spatial partition $p \in C$. Alternatively, Cébron and Weber gave a more concrete description of spatial partition quantum groups in terms of generators and relations associated to spatial partitions.
An example of a category of spatial partitions is the set $P_2^{(2)}$ of spatial pair partitions on two levels. In [CW22], Cébron-Weber computed multiple subcategories of $P_2^{(2)}$ and gave a list of generators of this category. However, it remained open which quantum group is associated to the category $P_2^{(2)}$. In this thesis we continue this work and answer the question regarding the quantum group associated to $P_2^{(2)}$.

Theorem 1 (Thm. 4.29)**.** *The quantum group associated to the category of spatial partitions $P_2^{(2)}$ is isomorphic to the projective orthogonal group PO_n.*

In addition, we consider a construction from Cébron and Weber which associates a quantum group $\mathring{G} \subseteq O_n^+$ to a quantum group $G \subseteq O_{n^2}^+$ satisfying the relations of .
By adding the partition  to $P_2^{(2)}$, we obtain the following result.

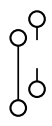
Theorem 2 (Thm. 5.16)**.** *Let $G \subseteq O_{n^2}^+$ be the quantum group associated to the category of spatial partitions generated by $P_2^{(2)}$ and . Then $C(G) \cong C(B_n)$ and $C(\mathring{G}) \cong C(B_n)$, where $B_n \subseteq O_n$ is the bistochastic group.*

In the second half of this thesis we consider applications of spatial partition quantum groups to finite quantum spaces. Cébron and Weber gave an example of a finite quantum space (B,ψ), where its quantum automorphism group $G^+(B,\psi)$ is a spatial partition quantum group corresponding to a category C_B. This category is a category of colored spatial partitions and is generated by



Our next main result answers another question from Cébron and Weber, which asks for a complete description of all partitions $p \in C_B$. In particular, we give an explicit construction of the following isomorphism between categories of colored spatial partitions.

Theorem 3 (Thm. 7.19)**.** *The category C_B is isomorphic to the category of colored non-crossing partitions $NC_{\circ\bullet}$ as categories of colored spatial partitions.*

In order to prove this theorem, we generalize techniques from classical partitions to the case of spatial partitions. This includes eliminating colors of partitions and rotating partitions such that these have only lower points.
In the following we give a short overview on how this thesis is structured. We begin in Section 2 with some preliminaries regarding compact matrix quantum groups, partitions and easy quantum groups. Then we introduce spatial partitions and spatial partition quantum groups in Section 3. In Section 4 we come to our first own results. After summarizing some facts about spatial pair partitions from [CW22], we present two new subcategories $C_\oplus, C_\ominus \subseteq P_2^{(2)}$ which are obtained by labeling points with $\oplus$ and $\ominus$ in an alternating way. Further, we prove our first main result by computing the quantum group associated to $P_2^{(2)}$. Then we use the results from Section 4 in Section 5 to compute the quantum group associated to the category generated by $P_2^{(2)}$ and  and apply the construction from Cébron and Weber to this quantum group. In Section 6 we generalize techniques from classical partitions to spatial partitions, which allow us to reduce categories of colored spatial partitions to the case without colors and with only lower points. Finally, we consider finite quantum spaces and the category C_B in Section 7. The results from Section 6 are then used to characterize the partitions $p \in C_B$ and to construct an isomorphism between the category C_B and the category $NC_{\circ\bullet}$ of colored non-crossing partitions.

2 Preliminaries

In the following we start with some preliminaries regarding compact matrix quantum groups, partitions and easy quantum groups. This section does not contain any new results and most of the definitions and statements can be found in [Tim08], [Web17a] or [Web06]. Throughout the rest of this thesis we will use the notation $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1 Compact Matrix Quantum Groups

We begin with the definition of compact matrix quantum groups. These were first defined by Woronowicz in [Wor87] under the name compact matrix pseudogroups and generalize compact subgroups of $\mathrm{GL}(n, \mathbb{C})$.

Definition 2.1 (Compact matrix quantum group)**.** Let A be a unital C^*-algebra and $u \in M_n(A)$. The pair $G = (A, u)$ is called *compact matrix quantum group* of size n if

1. A is generated by the entries u_{ij} $(1 \leq i, j \leq n)$,
2. u and u^T are invertible,
3. there exists a $*$-homomorphism $\Delta\colon A \to A \otimes_{\min} A$ with
$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

In this case, the C^*-algebra A is denoted by $C(G)$.

Further, one can also generalize the notion of subgroup to the setting of compact matrix quantum groups.

Definition 2.2 (Quantum subgroup)**.** Let $G = (A, u)$ and $H = (B, v)$ be two compact matrix quantum groups of size n. Then H is a *quantum subgroup* of G and we write $H \subseteq G$ if there exists a $*$-homomorphism $\varphi\colon A \to B$ with $\varphi(u_{ij}) = v_{ij}$ for $1 \leq i, j \leq n$.

Next we give some examples of compact matrix quantum groups.

Example 2.3.

1. Let $G \subseteq \mathrm{GL}(n, \mathbb{C})$ be a compact group. Then the algebra of continuous functions $C(G)$ is a commutative C^*-algebra, which gives rise to a compact matrix quantum group. In this setting, the u_{ij} correspond to the coefficient functions of the underlying matrices. Examples of such compact matrix groups are the group of orthogonal matrices O_n and the group of unitary matrices U_n.
2. In [Wan95], Wang introduced the free orthogonal quantum group O_n^+ and the free unitary quantum group U_n^+. These can be defined via the universal unital C^*-algebras
$$C(O_n^+) := C^*(u_{ij}, 1 \leq i, j \leq n \mid u \text{ is orthogonal}),$$
$$C(U_n^+) := C^*(u_{ij}, 1 \leq i, j \leq n \mid u \text{ and } u^T \text{ are unitary}).$$
Here, a matrix $u \in M_n(A)$ is unitary if $uu^* = u^*u = 1$ in $M_n(A)$. Similarly, u is orthogonal if it is unitary and each u_{ij} is self-adjoint. Note that $C(O_n^+)$ and $C(U_n^+)$ are not commutative for $n \geq 2$ and do not come from a classical group in these cases.
3. Another compact matrix quantum group is the quantum permutation group S_n^+, which was defined by Wang in [Wan98] as the quantum automorphism group of n points. It is given by the universal unital C^*-algebra
$$C(S_n^+) := C^*(u_{ij}, 1 \leq i, j \leq n \mid u \text{ is a magic unitary}),$$
where a matrix $u \in M_n(A)$ is a magic unitary if its entries are projections and sum up to 1 in each row and in each column. In the case of $A = \mathbb{C}$, magic unitaries are exactly permutation matrices.

In the first example, we described how classical groups give rise to compact matrix quantum groups where $C(G)$ is a commutative C^*-algebra. The following proposition shows that every compact matrix quantum group where $C(G)$ is commutative comes from a classical group.

Proposition 2.4. *Let $G = (A, u)$ be a compact matrix quantum group such that A is commutative. Then* $\operatorname{Spec} A$ *is a compact matrix group with $A \cong C(\operatorname{Spec} A)$. The group operation is defined by*

$$g * h := (g \otimes h) \circ \Delta$$

for $g, h \in \operatorname{Spec} A$.

One can directly check that $\operatorname{Spec} A$ is a compact space with $A \cong C(\operatorname{Spec} A)$ using the Gelfand-Naimark theorem and that $\operatorname{Spec} A \hookrightarrow \mathrm{GL}(n, \mathbb{C})$, $\varphi \mapsto \varphi(u)$ respects the group operation. However, the complete proof requires more theory about compact quantum groups which, can for example be found in [Tim08] and [Web17a]. The next proposition justifies Definition 2.2 and shows that in the commutative setting the definition of quantum subgroup agrees with the classical definition.

Proposition 2.5. *Let $G = (A, u)$ and $H = (B, v)$ be compact matrix quantum groups such that H is a quantum subgroup of G via the $*$-homomorphism $\varphi\colon A \to B$. Further, assume A and B are commutative such that* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *are compact groups. Then $i\colon \operatorname{Spec} B \to \operatorname{Spec} A$, $g \mapsto g \circ \varphi$ is an injective continuous group homomorphism and we can identify* $\operatorname{Spec} B$ *with a compact subgroup of* $\operatorname{Spec} A$.

The proof of the previous proposition is straightforward and the statement can be checked using Proposition 2.4.

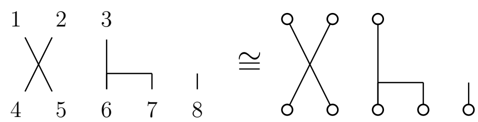
2.2 Partitions

In the following we introduce partitions and categories of partitions as purely combinatorial objects. These will then be used in Section 2.3 to define a special class of compact matrix quantum groups. The following definitions regarding partitions can also be found in [BS09], [Web17a] and [TW18].

Definition 2.6 (Partition)**.** Let $k, l \in \mathbb{N}_0$. A *partition* on k upper and l lower points is a partition of the set $\{1, ..., k, k+1, ...k+l\}$ into disjoint subsets called blocks. Denote with $P(k, l)$ the set of all partitions on k upper and l lower points and with $P := \bigcup_{k,l} P(k, l)$ the set of all partitions.

Given a partition $p \in P(k, l)$, we can visualize p by drawing a row of k upper points and a row of l lower points. Then one can number these points from 1 to $k+l$ and connect them according to the blocks of p. However, we will not number the points explicitly and only draw dots. The following is an example of a partition which is contained in $P(3, 5)$:

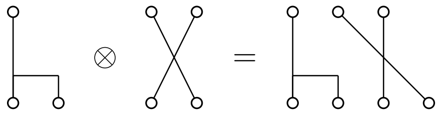
$$\{1, 5\}, \{2, 4\}, \{3, 6, 7\}, \{8\} \cong$$



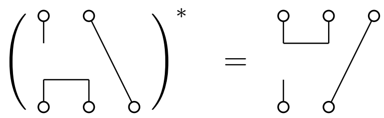
Further, one can define the following operations on partitions.

Definition 2.7 (Partition operations)**.**

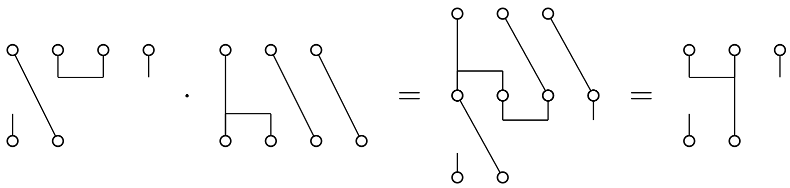
1. Let $p \in P(k_1, l_1)$ and $q \in P(k_2, l_2)$. Then the *tensor product* $p \otimes q \in P(k_1 + k_2, l_1 + l_2)$ is obtained by writing the partitions p and q next to each other. For example:



2. Let $p \in P(k, l)$. Then the *involution* $p^* \in P(l, k)$ is obtained by swapping the upper and lower points. For example:



3. Let $p \in P(k, l)$ and $q \in P(m, k)$. Then the *composition* $pq \in P(m, l)$ is obtained by first writing q above p and joining the lower points of q with the upper points of p. Then any intermediate points and loops are removed such that only the upper points of q and the lower points of p are left. For example:



These operations lead to the definition of a category of partitions.

Definition 2.8 (Category of partitions)**.** A *category of partitions* is a subset $C \subseteq P$ which contains the base partitions $\in P(1, 1)$,  $\in P(0, 2)$ and which is closed under tensor products, involutions and compositions. In this case, we define $C(k, l) := C \cap P(k, l)$.

Next we give some examples of categories of partitions.

Example 2.9. The following sets are categories of partitions.

1. The set P of all partitions.

2. The set P_2 of pair partitions, where

$$P_2 := \{p \in P \mid \text{every block of } p \text{ has size two}\}.$$

3. The set NC of non-crossing partitions, where

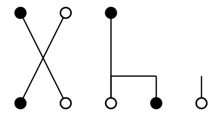
$$NC := \{p \in P \mid p \text{ can be drawn without crossing blocks}\}.$$

Further, let C_1 and C_2 be categories of partitions. Then $C_1 \cap C_2$ is again a category of partitions. In particular, the set $NC_2 := NC \cap P_2$ of non-crossing pair partitions is a category of partitions.

By intersecting categories, we can also define the smallest category containing some partitions $p_1, \ldots, p_n$.

Definition 2.10. Let $p_1, \dots, p_n \in P$. Then we define $\langle p_1, \dots, p_n \rangle$ as the intersection of all categories containing $p_1, \dots, p_n$. It is the category generated by $p_1, \dots, p_n$.

Remark 2.11. One can generalize partitions by coloring the points either black or white, for example:



In this case, the category operations stay the same except that the upper points of p and the lower points of q are required to have the same colors when composing two partitions p and q. Further, in a category of colored partitions the base partitions $|$ and $\sqcap$ are replaced by $\circ|\circ$, $\bullet|\bullet$, $\circ\sqcap\bullet$ and $\bullet\sqcap\circ$.

2.3 Easy Quantum Groups

Next we will define easy quantum groups. These were first introduced by Banica-Speicher in [BS09] and are compact matrix quantum groups defined by categories of partitions. This has the advantage that many properties of easy quantum groups can be reformulated as combinatorial properties of partitions and that the classification of easy quantum groups can be reduced to the classification of categories of partitions. An introduction to easy quantum groups can for example be found in [Web17a] and [Web06]. Before we come to the definition of easy quantum groups, we first have to introduce intertwiners.

Definition 2.12 (Intertwiners)**.** Let $G = (A, u)$ be a compact matrix quantum group of size n and let $k, l \in \mathbb{N}_0$. Then

$$\mathrm{Hom}_G(k, l) := \{T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l} \text{ linear} \mid T u^{\otimes k} = u^{\otimes l} T\}$$

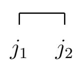
is the set of *intertwiners* between $u^{\otimes k}$ and $u^{\otimes l}$. Here, $u^{\otimes m} \in M_{n^m}(A)$ denotes the m-fold Kronecker product of u and we identify each T with an A-valued matrix by $\mathbb{C} \cong \mathbb{C}1 \subseteq A$.

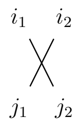
The idea of easy quantum groups is to assign linear maps T_p to partitions p and consider quantum groups with intertwiners given by these linear maps. For the following definitions, we fix some $n \in \mathbb{N}$ and denote with $e_1, \dots, e_n$ the standard basis of $\mathbb{C}^n$. Further, a multi-index will denote a tuple $(i_1, \dots, i_m)$ with $1 \leq i_k \leq n$ for $1 \leq k \leq m$.

Definition 2.13. Let $p \in P(k, l)$ be a partition and let $i = (i_1, \dots, i_k)$, $j = (j_1, \dots, j_l)$ be multi-indices. Further, assign the indices $i_1, \dots, i_k$ to the upper points and $j_1, \dots, j_l$ to the lower points of p. Then

$$\delta_p(i, j) := \begin{cases} 1 & \text{the indices agree inside all blocks,} \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.14. Consider the partitions $|$, $\sqcap$ and $\times$. Then the previous definition yields

 : $\delta_{|}(i_1, j_2) = \delta_{i_1 j_1}$,  : $\delta_{\sqcap}(j_1, j_2) = \delta_{j_1 j_2}$,



$$: \quad \delta_{\times}(i_1, i_2, j_1, j_2) = \delta_{i_1 j_2}\delta_{i_2 j_1}.$$

Using the definition of δ_p, we can now associate linear maps T_p to partitions.

Definition 2.15. Let $p \in P(k,l)$ be a partition. Then define the linear map $T_p \colon (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ by

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_j \delta_p(i,j) e_{j_1} \otimes \cdots \otimes e_{j_l},$$

where $i = (i_1, \ldots, i_k)$ and $j = (i_1, \ldots, j_l)$ are multi-indices.

Note that the mapping T_p can be identified with a $n^l \times n^k$-matrix with entries $(T_p)_{ji} = \delta_p(i,j)$. After introducing the linear maps T_p, we can finally define easy quantum groups.

Definition 2.16 (Easy quantum group)**.** A compact matrix quantum group G is called *easy quantum group* if $S_n \subseteq G \subseteq O_n^+$ and there exists a category of partitions $C \subseteq P$ such that

$$\mathrm{Hom}_G(k,l) = \mathrm{span}\{T_p \mid p \in C(k,l)\}.$$

Example 2.17. The following table contains some easy quantum groups and their corresponding categories of partitions. More examples of easy quantum groups can be found in [Web13].

Easy quantum group	Category of partitions
S_n	all partitions P
S_n^+	non-crossing partitions NC
O_n	pair partitions P_2
O_n^+	non-crossing pair partitions NC_2

Remark 2.18. Note that one can generalize the definition of easy quantum groups to the case of categories of colored partitions. Then one obtains compact matrix quantum groups $G \subseteq U_n^+$ called unitary easy quantum groups, which were first defined by Tarrago-Weber in [TW16].

Next we come to Tannaka-Krein duality and how to construct easy quantum groups from categories of partitions. In this context, consider some compact matrix quantum group G. Then its intertwiners are linked to finite-dimensional representations and can be organized into a concrete monoidal W^*-category in the sense of Woronowicz. Conversely, Woronowicz proved a Tannaka-Krein duality theorem in [Wor88], which states that compact matrix quantum groups are in 1-1 correspondence with concrete monoidal W^*-categories. Therefore, compact matrix quantum groups can be uniquely reconstructed from such categories. We refer to [Mal18] for more details on the reconstruction without the use of much category theory language.
In the setting of partitions, this result gives a 1-1 correspondence between categories of partitions and easy quantum groups via the linear maps T_p. These linear maps generate a concrete monoidal W^*-category for which there exists a unqiue easy quantum group with intertwiners T_p. Further, this easy quantum group can be

described in terms of generators and relations obtained from the mappings T_p. In the following we define these relations and state how easy quantum groups can be defined via universal C^*-algebras. More details on this construction and a proof of Proposition 2.20 can be found in [Web17b] and [Web17c].

Definition 2.19. Let A be a C^*-algebra with matrix $u \in M_n(A)$ and let $p \in P(k,l)$ be a partition. Then define the relations $R(p)$ as

$$\forall i, j\colon \sum_g \delta_p(g,j) u_{g_1 i_1} ... u_{g_k i_k} = \sum_h \delta_p(i,h) u_{j_1 h_1} ... u_{j_l h_l},$$

where i, j, g, h are multi-indices.

Note that these are exactly the defining equations such that T_p intertwines $u^{\otimes k}$ and $u^{\otimes l}$. Using the previous relations, one can define an easy quantum group with intertwiners given by T_p.

Proposition 2.20. *Let $C = \langle p_1, \dots, p_m \rangle \subseteq P$ be a category of partitions. Further, define A as the universal unital C^*-algebra generated by the entries of a $n \times n$-matrix u such that*

1. $u_{ij} = u_{ij}^*$ *and* $\sum_{k=1}^n u_{ik} u_{jk} = \sum_{k=1}^n u_{ki} u_{kj} = \delta_{ij}$ *for all* $1 \leq i, j \leq n$,
2. $R(p_i)$ *holds for all* $1 \leq i \leq m$.

Then $G = (A, u)$ is the unique compact matrix quantum group with

$$\mathrm{Hom}_G(k,l) = \mathrm{span}\{T_p \mid p \in C(k,l)\}.$$

3 Spatial Partition Quantum Groups

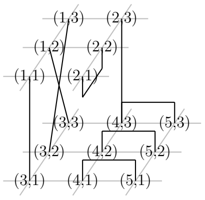
In this section we introduce spatial partitions and spatial partition quantum groups. These were first defined by Cébron-Weber in [CW22] and generalize easy quantum groups by replacing classical partitions with spatial partitions. First we consider spatial partitions before we come to spatial partition quantum groups and how these are constructed from categories of spatial partitions. Note that this section contains no new results and only summarizes parts of [CW22].

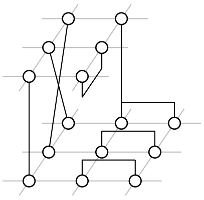
3.1 Spatial Partitions

We begin with the definition of spatial partitions.

Definition 3.1 (Spatial partition)**.** Let $k, l \in \mathbb{N}_0$ and $m \in \mathbb{N}$. A *spatial partition* is a partition of the set $\{1, \dots, k, k+1, \dots, k+l\} \times \{1, \dots, m\}$ into disjoint subsets called blocks. The first component is divided into k upper and l lower points whereas the second component consists of m levels. Denote with $P^{(m)}(k,l)$ the set of all spatial partitions with k upper points, l lower points and m levels. Further, define $P^{(m)} := \bigcup_{k,l} P^{(m)}(k,l)$ as the set of all spatial partitions on m levels.

Similar to classical partitions, we can visualize spatial partitions by drawing upper and lower points and then connecting points which are in the same block by lines. In the case of multiple levels, we place them from the front to the back such that we obtain a three-dimensional picture.

{(1,1), (3,1)}, {(1,2), (3,3)},
{(3,2), (1,3)}, {(2,1), (2,2)},
{(4,1), (5,1)}, {(4,2), (5,2)},
{(2,3), (4,3), (5,3)}
$\cong$


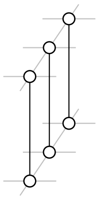
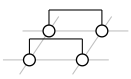
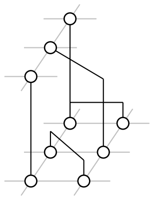
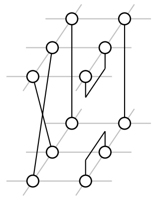
$\cong$


The previous spatial partition has $m = 3$ levels and $k = 2$ upper and $l = 3$ lower points per level. Before we come to further examples of spatial partitions, we first define amplifications and π-graded partitions.

Definition 3.2 (Amplification and graded partitions)**.**

1. Let $p \in P$ be a partition. Then denote with $p^{(m)} \in P^{(m)}$ the *amplification* of p, which is obtained by repeating p on each of the m levels.
2. Let π be a partition of $\{1, \ldots, m\}$ and $p \in P^{(m)}$ be a spatial partition. We say p is π-*graded* if for all points (x_1, y_1) and (x_2, y_2) which are in the same block in p holds that their levels y_1 and y_2 are in the same block in π.

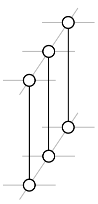
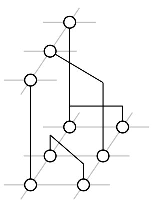
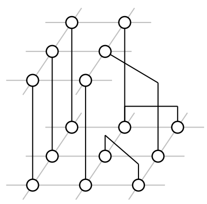
The following are some more examples of spatial partitions:

$p_1 =$  $p_2 =$  $p_3 =$  $p_4 =$ 

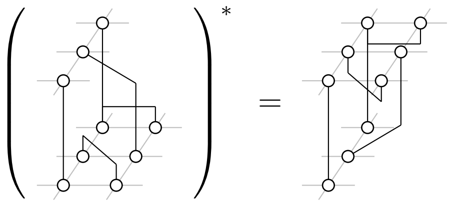
Here $p_1 =$ $^{(3)}$ and $p_2 =$ $^{(2)}$ are amplifications of the partitions and . Further, p_3 is a $\{1, 2\}\{3\}$-graded partition since there are blocks connecting level 1 with level 2 but there are no blocks connecting level 3 with level 1 or level 2.
As for classical partitions, we can define a tensor product, an involution and a composition for spatial partitions, which leads to the definition a category of spatial partitions.

Definition 3.3 (Spatial partition operations)**.**

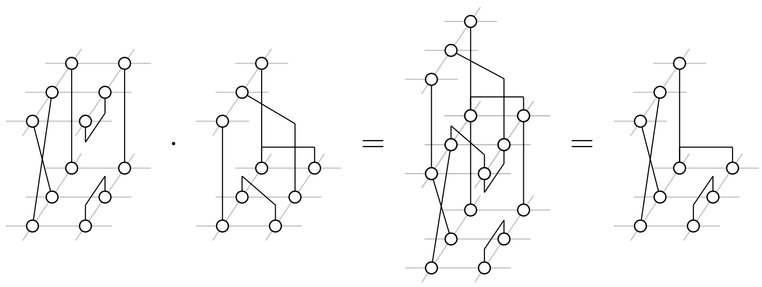
1. Let $p \in P^{(m)}(k_1, l_1)$ and $q \in P^{(m)}(k_2, l_2)$. Then the *tensor product* $p \otimes q \in P^{(m)}(k_1 + k_2, l_1 + l_2)$ is obtained by placing both spatial partitions next to each other. For example:

 $\otimes$  $=$ 

2. Let $p \in P^{(m)}(k,l)$. Then the *involution* $p^* \in P^{(m)}(l,k)$ is obtained by swapping the upper and lower points of p. For example:



3. Let $p \in P^{(m)}(k_1,l_1)$ and $q \in P^{(m)}(k_2,l_2)$ with $k_1 = l_2$. Then the *composition* $pq \in P^{(m)}(k_2,l_1)$ is obtained by first writing q above p and merging the lower points of q with the upper points of p. Then all intermediate points and possible loops are removed and the remaining blocks are simplified. For example:



Definition 3.4 (Category of spatial partitions)**.** A *category of spatial partitions* is a subset $C \subseteq P^{(m)}$ which is closed under tensor products, involutions and compositions and which contains the base partitions $\mid^{(m)} \in P^{(m)}(1,1)$ and $\sqcap^{(m)} \in P^{(m)}(0,2)$.

One can directly check that the following examples are categories of spatial partitions. We refer again to [CW22] for more details.

1. Let $C \subseteq P$ be a category of partitions. Then C is also a category of spatial partitions on one level ($m = 1$). Hence, spatial partitions generalize classical partitions.

2. The set
$$P_2^{(m)} := \{p \in P^{(m)} \mid \text{each block of } p \text{ has size } 2\}$$
is a category of spatial partitions, which generalizes pair partitions.

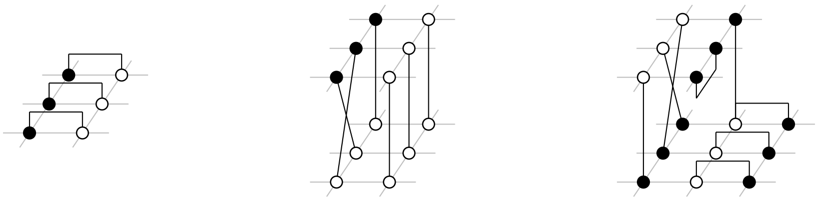
3. Let $C \subseteq P$ be a category of partitions and define
$$[C]^{(m)} := \{p^{(m)} \mid p \in C\} \subseteq P^{(m)}.$$
Then $[C]^{(m)}$ is a category of spatial partitions called the amplification of C. Note that $[P]^{(m)} \neq P^{(m)}$ since latter contains all spatial partitions on m levels.

4. Let π be a partition of $\{1,\ldots,m\}$. Then the set of all π-graded partitions on m levels
$$P_\pi^{(m)} := \{p \in P^m \mid p \text{ is } \pi\text{-graded}\}$$
is a category of spatial partitions.

Note that the intersection of categories of spatial partitions is again a category of spatial partitions. This allows us to define the smallest category containing some spatial partitions as in the case of classical partitions.

Definition 3.5. Let $p_1, \ldots, p_n \in P^{(m)}$ be spatial partitions and define $\langle p_1, \ldots, p_n \rangle$ as the intersection of all categories containing $p_1, \ldots, p_n$. It is the category *generated* by $p_1, \ldots, p_n$.

Remark 3.6. Similar to classical partitions, one can generalize spatial partitions by coloring the points either black or white. In this case, we require again that colors are compatible when composing partitions and that categories contain $\overset{\circ}{\underset{\circ}{|}}{}^{(m)}$, $\overset{\bullet}{\underset{\bullet}{|}}{}^{(m)}$, $\sqcap_{\circ\bullet}^{(m)}$ and $\sqcap_{\bullet\circ}^{(m)}$ as base partitions instead of $\overset{\circ}{\underset{\circ}{|}}{}^{(m)}$ and $\sqcap_{\circ\circ}^{(m)}$. Further, in the case of multiple levels, the points $(i,1), \ldots, (i,m)$ have to share the same color for each $1 \leq i \leq k+l$. The following partitions are examples of colored spatial partitions:

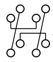


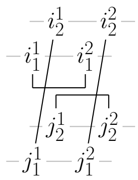
3.2 Spatial Partition Quantum Groups

Next we want to define spatial partition quantum groups and describe how these can be constructed in terms of generators and relations. As in the case of classical partitions and easy quantum groups in Section 2.2 and Section 2.3, spatial partition quantum groups will be compact matrix quantum groups with intertwiners indexed by spatial partitions. To define such quantum groups, one first needs to assign linear maps S_p to spatial partitions. This is done similar to the construction of the mappings T_p for classical partitions. But before we can define these mappings S_p, we first need to introduce some notation.
For the following definitions fix some $n, m \in \mathbb{N}$ where n has the form $n = n_1 \cdots n_m$. Denote with $\ker(n_1, \ldots, n_m)$ the partition of $\{1, \ldots, m\}$ where i and j are in the same block if $n_i = n_j$. A multi-index will be a tuple $(i_1, \ldots, i_m)$ where $1 \leq i_k \leq n_k$ for all $1 \leq k \leq m$. Further, a spatial multi-index of length l will denote a tuple $(i^1, \ldots, i^l)$ where each $i^k = (i^k_1, \ldots, i^k_m)$ is a multi-index for all $1 \leq k \leq l$.
Next, we need to choose a basis $\{e_i\}$ of $\mathbb{C}^n = \mathbb{C}^{n_1 \cdots n_m}$ indexed by all multi-indices $i = (i_1, \ldots, i_m)$. Note that on a technical level the following definitions depend on this basis. However, different choices will result in isomorphic quantum groups. Using these notations, we can now define δ_p, associate linear maps S_p to spatial partitions and define the class of spatial partitions quantum groups.

Definition 3.7. Let $p \in P^{(m)}(k,l)$ be a spatial partition and let $i = (i^1, \ldots, i^k)$, $j = (j^1, \ldots, j^l)$ be spatial multi-indices. Assign the indices i^h_g to the upper points (h,g) and the indices j^h_g to the lower points $(k+h, g)$. Then

$$\delta_p(i,j) := \begin{cases} 1 & \text{the indices agree inside all blocks,} \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.8. Consider the spatial partition $p =$ . Then δ_p from Definition 3.7 is given by



$$: \qquad \delta_p(i,j) = \delta_{i_1^1 i_1^2} \cdot \delta_{j_2^1 j_2^2} \cdot \delta_{i_2^1 j_1^1} \cdot \delta_{i_2^2 j_1^2},$$

where $i = ((i_1^1, i_2^1), (i_1^2, i_2^2))$ and $j = ((j_1^1, j_2^1), (j_1^2, j_2^2))$ are spatial multi-indices. This is similar to the case of classical partitions in Definition 2.13 and Example 2.14.

Next we associate linear maps S_p, which are defined with the help of δ_p from Definition 3.7, to spatial partitions.

Definition 3.9. Let $p \in P^{(m)}(k,l)$ be a spatial partition. Then define the linear map $S_p \colon (\mathbb{C}^{n_1 \cdots n_m})^{\otimes k} \to (\mathbb{C}^{n_1 \cdots n_m})^{\otimes l}$ by

$$S_p(e_{i^1} \otimes \cdots \otimes e_{i^k}) = \sum_j \delta_p(i,j) e_{j^1} \otimes \cdots \otimes e_{j^l},$$

where $i = (i^1, \ldots, i^k)$ and $j = (i^1, \ldots, j^l)$ are spatial multi-indices. The mapping S_p can be identified with the $n^l \times n^k$-matrix with entries $(S_p)_{ji} = \delta_p(i,j)$.

Using these mappings S_p, we can now introduce spatial partition quantum groups similar to easy quantum groups in Definition 2.16.

Definition 3.10 (Spatial partition quantum group)**.** A compact matrix quantum group $G = (A, u)$ is called *spatial partition quantum group* if $G \subseteq O^+_{n_1 \cdots n_m}$ and there exists a $\ker(n_1, \ldots, n_m)$-graded category of spatial partitions $C \subseteq P^{(m)}$ such that

$$\mathrm{Hom}_G(k,l) = \mathrm{span}\{S_p \mid p \in C(k,l)\}.$$

Note that $u \in M_{n_1 \cdots n_m}(A)$ and we can index the entries of u by multi-indicies $i = (i_1, \ldots, i_m)$ and $j = (j_1, \ldots, j_m)$ when using the basis $\{e_i\}$ of $\mathbb{C}^{n_1 \cdots n_m}$.

Remark 3.11. The main difference in the construction of the mappings S_p compared to the mappings T_p from Section 2.3 lies in the use of spatial multi-indices. This results in more possibilities to define the mappings S_p by making use of the factorization $n = n_1 \cdots n_m$. Further, in the case of amplifications of classical partitions $p \in P$ the mappings $S_{p^{(m)}}$ and T_p agree. Hence, spatial partition quantum groups generalize easy quantum groups.

Remark 3.12. As in the case of easy quantum groups, one can generalize Definition 3.10 to quantum groups $G \subseteq U^+_{n_1 \cdots n_m}$ by using colored spatial partitions as in Remark 3.6. More details regarding this case can be found in [CW22].

Recall from Section 2.3 that easy quantum groups can be constructed from given categories of partitions by using a Tannaka-Krein duality theorem. A similar theorem allows the construction of spatial partition quantum groups from categories of spatial partitions. In the following we show how these spatial partition quantum groups can be described in terms of generators and relations, similar to Proposition 2.20 in Section 2.3. More details and a proof of the corresponding Tannaka-Krein theorem and the following construction can be found in [CW22]. We begin by associating relations to spatial partitions.

Definition 3.13. Let A be a C^*-algebra with matrix $u \in M_{n_1 \cdots n_m}(A)$ and let $p \in P^{(m)}(k,l)$ be a spatial partition. Then define the relations $R(p)$ as

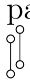
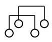
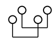
$$\forall i,j\colon \sum_g \delta_p(g,j) u_{g^1 i^1} \dots u_{g^k i^k} = \sum_h \delta_p(i,h) u_{j^1 h^1} \dots u_{j^l h^l},$$

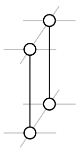
where i,j,g,h are spatial multi-indices.

As for classical partitions, one can check that these are exactly the defining equations such that S_p intertwines $u^{\otimes k}$ and $u^{\otimes l}$, i.e.

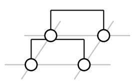
$$S_p u^{\otimes k} = u^{\otimes l} S_p$$

when identifying S_p with an A-valued matrix by $\mathbb{C} \cong \mathbb{C}1 \subseteq A$.

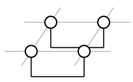
Example 3.14. The following list contains some spatial partitions and corresponding relations from [CW22]. Note that the relations for  are always satisfied and that the partitions  and  correspond exactly to the orthogonality relations for u.



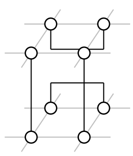
$$u_{(i_1,i_2)(j_1,j_2)} = u_{(i_1,i_2)(j_1,j_2)}$$



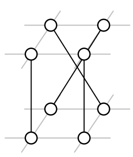
$$\sum_{k_1,k_2} u_{(i_1,i_2)(k_1,k_2)} u_{(j_1,j_2)(k_1,k_2)} = \delta_{i_1 j_1} \delta_{i_2 j_2}$$



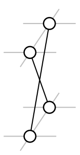
$$\sum_{k_1,k_2} u_{(k_1,k_2)(i_1,i_2)} u_{(k_1,k_2)(j_1,j_2)} = \delta_{i_1 j_1} \delta_{i_2 j_2}$$



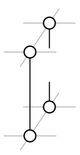
$$\delta_{i_2 i_4} \sum_k u_{(i_1,k)(j_1,j_2)} u_{(i_3,k)(j_3,j_4)} = \delta_{j_2 j_4} \sum_k u_{(i_1,i_2)(j_1,k)} u_{(i_3,i_4)(j_3,k)}$$



$$u_{(i_1,i_2)(j_1,j_2)} u_{(i_3,i_4)(j_3,j_4)} = u_{(i_1,i_4)(j_1,j_4)} u_{(i_3,i_2)(j_3,j_2)}$$



$$u_{(i_1,i_2)(j_1,j_2)} = u_{(i_2,i_1)(j_2,j_1)}$$



$$\sum_k u_{(i_1,k)(j_1,j_2)} = \sum_k u_{(i_1,i_2)(j_1,k)}$$

Using the relations associated to spatial partitions, we can now define a spatial partition quantum group for a given category of spatial partitions.

Proposition 3.15. *Let $n = n_1 \cdots n_m$ and let $C = \langle p_1, \ldots, p_m \rangle \subseteq P^{(m)}$ be a $\ker(n_1, \ldots, n_m)$-graded category of spatial partitions. Further, define A as the universal unital C^*-algebra generated by the entries u_{ij} of an $n \times n$-matrix u such that*

1. *$u_{ij} = u_{ij}^*$ and $\sum_k u_{ik}u_{jk} = \sum_k u_{ki}u_{kj} = \delta_{ij}$ for all multi-indices i, j,*
2. *$R(p_i)$ holds for all $1 \leq i \leq m$.*

Then $G = (A, u)$ is the unique compact matrix quantum group with intertwiners

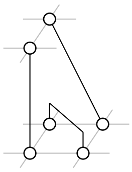
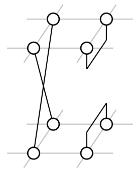
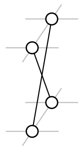
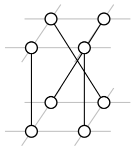
$$\mathrm{Hom}_G(k, l) = \mathrm{span}\{S_p \mid p \in C(k, l)\}.$$

4 Spatial Pair Partitions on Two Levels

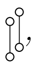
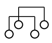
In this section we consider the category $P_2^{(2)}$ of spatial pair partitions on two levels. This is one of the simplest categories of spatial partitions and was already studied by Cébron-Weber in [CW22]. Here we continue this work and present two new subcategories $C_\oplus, C_\ominus \subseteq P_2^{(2)}$. Further, we provide the first proof that the quantum group associated to $P_2^{(2)}$ is a classical compact group which is isomorphic to the projective orthogonal group PO_n. In particular, this answers an open question from [CW22], which asks for a concrete description of the quantum group associated to $P_2^{(2)}$.

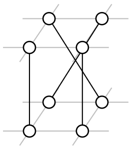
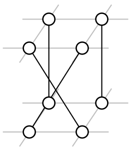
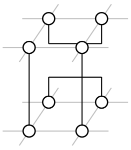
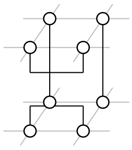
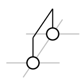
4.1 The Category $P_2^{(2)}$

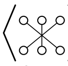
We begin with the combinatorics of the category of spatial pair partitions $P_2^{(2)}$. Recall that $P_2^{(2)}$ consists of all spatial pair partitions on two levels, e.g.

 ,  ,  ,  , ...

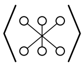
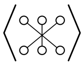
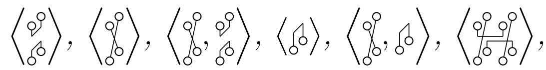
In [CW22], Cébron-Weber gave another description of this category in terms of generators.

Proposition 4.1. *The category $P_2^{(2)}$ of spatial pair partitions is generated by the base partitions* *,*  *and*

 ,  ,  ,  ,  .

Further, Cébron and Weber considered subcategories of $P_2^{(2)}$. In the case of classical partitions, the only subcategories of P_2 are NC_2, P_2 and $\langle$  $\rangle$. However, in the case of spatial pair partitions Cébron and Weber presented a larger (incomplete) list of subcategories.

Proposition 4.2. *The following are distinct subcategories of* $P_2^{(2)}$*:*

- *the amplifications* $C^{(2)}$ *for* $C \in \{NC_2, P_2, \langle$  $\rangle\}$,
- *the products* $C_1 \times C_2$ *for* $C_1, C_2 \in \{NC, P_2, \langle$  $\rangle\}$,
- *the categories* ,
- *the category* $P_2^{(2)}$.

The proof of the previous proposition can be found in [CW22] and uses the following properties of spatial pair partitions.

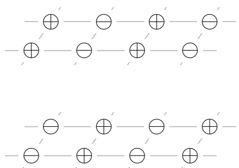
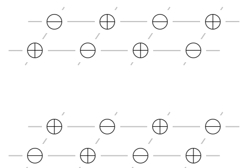
- *Respecting levels*: No block in the partition connects both levels.
- *Level symmetric*: The partition is symmetric with respect to swapping both levels.
- *Non-diagonal*: There are no diagonal blocks between the two levels of the partition, i.e. if $(a, 1)$ and $(b, 2)$ are in a block, then $a = b$.
- *Non-vice-versa*: No opposite points of the partition form a block, i.e. if $(a, 1)$ and $(b, 2)$ are in a block, then $a \neq b$.
- *Even*: The number of upper and lower points $k + l$ is even.

The idea of the proof of Proposition 4.2 is to show that combinations of the previous properties define subcategories of $P_2^{(2)}$ and can be used to distinguish the categories from Proposition 4.2.
In the following we present two new subcategories of $P_2^{(2)}$ which are distinct from the ones in Proposition 4.2. In [Web13], Weber described a category of classical pair partitions $C \subseteq P_2$ which is obtaind by first labeling the points with $\oplus$ and $\ominus$ in an alternating way:

$$\begin{array}{cccc} \oplus & \ominus & \oplus & \ominus \\ \\ \ominus & \oplus & \ominus & \oplus \end{array}$$

Then a partition is contained in the category C if every block connects exactly a $\oplus$ with a $\ominus$. This construction can be generalized to the case of spatial pair partitions. However, we now obtain two categories $C_\oplus$ and $C_\ominus$ depending on the alignment of the labelings of the two levels.

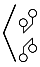
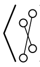
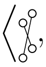
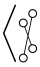
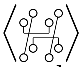
Definition 4.3. The sets $C_\oplus$ and $C_\ominus$ are defined by all partitions $p \in P^{(2)}(k, l)$ where $k + l$ is even and every block in p connects a $\oplus$ with a $\ominus$. Here, the labelings of the points are given by

$C_\oplus$:  , $C_\ominus$:  .

Remark 4.4. The proof that $C_\oplus$ and $C_\ominus$ are categories of spatial partitions is similar to the case of the category C in [Web13]. In particular, the tensor product requires an even number of points and the composition is checked by chasing blocks and keeping track of the partity.

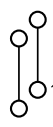
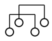
Further, we obtain the following result.

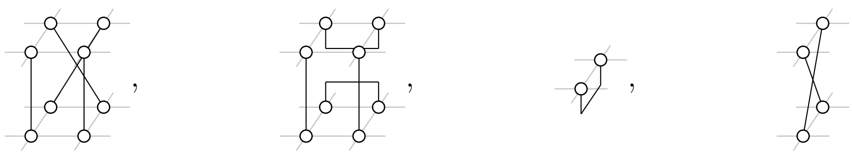
Proposition 4.5. *The categories $C_\oplus$, $C_\ominus$ and the categories form Proposition 4.2 are distinct.*

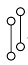
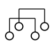
Proof. Because of the different labelings, one can easily find partitions which distinguish $C_\oplus$ and $C_\ominus$. Therefore, we have to show that the categories $C_\oplus$ and $C_\ominus$ are not contained in the list of Proposition 4.2. Note that $C_\oplus$ and $C_\ominus$ allow blocks between the two levels such that these are not an amplification or a product of classical partitions. Further, one checks that $C_\oplus$ and $C_\ominus$ are not level symmetric, which distinguishes them from $\langle$  $\rangle$, $\langle$  $\rangle$, $\langle$ ,  $\rangle$, $\langle$  $\rangle$ and $\langle$ ,  $\rangle$. Finally, the generator of $\langle$  $\rangle$ is not compatible with the labelings on the first level and the categories $C_\oplus$ and $C_\ominus$ do not contain every pair partition on two levels. □

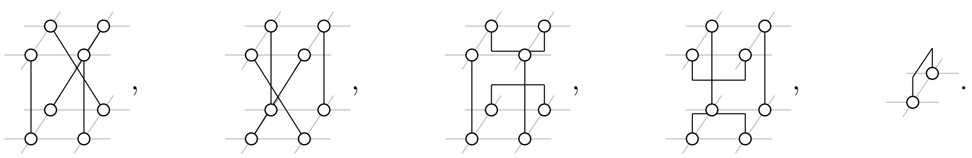
4.2 The Quantum Group Associated to $P_2^{(2)}$

Next we want to define the quantum group associated to $P_2^{(2)}$ in terms of generators and relations as in Proposition 3.15. However, we first need to reformulate the generators from Proposition 4.1 in order to obtain simpler relations.

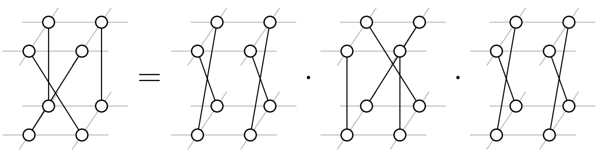
Lemma 4.6. *The category $P_2^{(2)}$ of spatial pair partitions is generated by the base partitions* ,  *and*

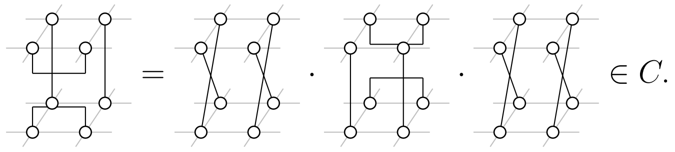


Proof. Denote with C the category generated by the partitions from the statement and recall from Proposition 4.1 that the category $P_2^{(2)}$ is generated by the base partitions ,  and



Using the partition , we obtain

 $\in C,$



Hence, $P_2^{(2)} \subseteq C$. Conversely, we have $C \subseteq P_2^{(2)}$ because each generator of C is a pair partition on two levels. $\square$

In order to define the quantum group associated to $P_2^{(2)}$, we need the relations corresponding to the generators from the previous lemma. Except for , these can all be found in Example 3.14 and in [CW22]. Therefore, we need to compute the relations corresponding to as defined in Definition 3.13.

Lemma 4.7. *Let A be a C^*-algebra and $u \in M_{n^2}(A)$. Then the partition corresponds to the relations*

$$\forall j_1, j_2 \colon \sum_k u_{(k,k)(j_1,j_2)} = \delta_{j_1 j_2}.$$

Proof. By Definition 3.13 and by using the notation $i^1 = (i_1, i_2)$ and $g^1 = (g_1, g_2)$, the partition corresponds to the relations

$$\forall i_1, i_2 \colon \sum_{g_1, g_2} \delta(g_1, g_2) u_{(g_1,g_2)(i_1,i_2)} = \delta(i_1, i_2).$$

Labeling the upper points of with i_1, i_2, we obtain $\delta(i_1, i_2) = \delta_{i_1 i_2}$. Therefore, the relations have the form

$$\forall i_1, i_2 \colon \sum_{g_1, g_2} \delta_{g_1 g_2} u_{(g_1,g_2)(i_1,i_2)} = \delta_{i_1 i_2},$$

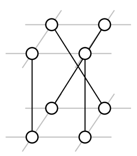
which can be relabeled and simplified to

$$\forall j_1, j_2 \colon \sum_k u_{(k,k)(j_1,j_2)} = \delta_{j_1 j_2}.$$

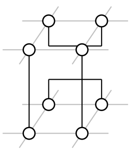
$\square$

Given the relations corresponding to the generators of $P_2^{(2)}$, we can now use Proposition 3.15 to define the spatial partition quantum group associated to $P_2^{(2)}$ via a universal C^*-algebra.

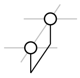
Definition 4.8. Let $n \in \mathbb{N}$ and let $G = (A, u)$ be the spatial partition quantum group of size n^2 associated to the category of spatial partitions $P_2^{(2)}$. Then A is given by the universal unital C^*-algebra generated by self-adjoint elements $u_{(i_1,i_2)(j_1,j_2)}$ $(1 \leq i_1, i_2, j_1, j_2 \leq n)$ such that the matrix $u = (u_{(i_1,i_2)(j_1,j_2)}) \in M_{n^2}(A)$ is orthogonal and the following relations hold:



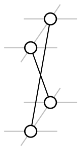
$$R_1 : \; u_{(i_1,i_2)(j_1,j_2)} u_{(i_3,i_4)(j_3,j_4)} = u_{(i_1,i_4)(j_1,j_4)} u_{(i_3,i_2)(j_3,j_2)},$$



$$R_2: \ \delta_{i_2 i_4} \sum_k u_{(i_1,k)(j_1,j_2)} u_{(i_3,k)(j_3,j_4)} = \delta_{j_2 j_4} \sum_k u_{(i_1,i_2)(j_1,k)} u_{(i_3,i_4)(j_3,k)},$$



$$R_3: \ \sum_k u_{(k,k)(j_1,j_2)} = \delta_{j_1 j_2},$$



$$R_4: \ u_{(i_1,i_2)(j_1,j_2)} = u_{(i_2,i_1)(j_2,j_1)}.$$

Since the description of A from the previous definition is rather abstract, Cébron-Weber asked in [CW22] if the quantum group associated to $P_2^{(2)}$ is related to other known quantum groups. Throughout the rest of Section 4, we will answer this question and prove in Theorem 4.29 that the quantum group associated to $P_2^{(2)}$ is a classical group which is isomorphic to the projective orthogonal group PO_n. We begin with the observation that the C^*-algebra A from Definition 4.8 is commutative such that $\mathrm{Spec}\, A$ is a classical group. Then we present the isomorphism $\mathrm{Aut}(M_n(\mathbb{C})) \cong PU_n$ in Section 4.3 and some new arguments in Section 4.4, which allow us to prove Theorem 4.29.

Proposition 4.9. *The C^*-algebra A from Definition 4.8 is commutative.*

Proof. It is sufficient to show commutativity on the generators. With relations R_1 and R_4 from Definition 4.8 follows

$$\begin{aligned}
u_{(i_1,i_2)(j_1,j_2)} u_{(i_3,i_4)(j_3,j_4)} &= u_{(i_1,i_4)(j_1,j_4)} u_{(i_3,i_2)(j_3,j_2)} && (R_1) \\
&= u_{(i_4,i_1)(j_4,j_1)} u_{(i_2,i_3)(j_2,j_3)} && (R_4) \\
&= u_{(i_4,i_3)(j_4,j_3)} u_{(i_2,i_1)(j_2,j_1)} && (R_1) \\
&= u_{(i_3,i_4)(j_3,j_4)} u_{(i_1,i_2)(j_1,j_2)} && (R_4).
\end{aligned}$$

□

Remark 4.10. Proposition 4.9 and Proposition 3.15 imply that $\mathrm{Spec}\, A$ is a compact matrix group with $A \cong C(\mathrm{Spec}\, A)$. Further, recall that the group operation $*$ is given by

$$\varphi_1 * \varphi_2 = (\varphi_1 \otimes \varphi_2) \circ \Delta.$$

In particular, one checks that $\mathrm{id} \in \mathrm{Spec}\, A$ is given by

$$\mathrm{id}(u_{(i_1,i_2)(j_1,j_2)}) = \delta_{i_2 j_2}.$$

4.3 The Isomorphism $\mathrm{Aut}(M_n(\mathbb{C})) \cong PU_n$

We continue with the definition of the groups PO_n and PU_n before we come to the isomorphism $\mathrm{Aut}(M_n(\mathbb{C})) \cong PU_n$. This isomorphism is already known and will be used in Section 4.4 to prove Theorem 4.29.

Definition 4.11 (Center)**.** Let G be a group. The *center* of G is given by

$$Z(G) := \{a \in G : ab = ba \ \forall b \in G\}.$$

Definition 4.12. Define the

- projective unitary group $PU_n := U_n/Z(U_n)$,
- projective orthogonal group $PO_n := O_n/Z(O_n)$.

Note that $Z(U_n)$ and $Z(O_n)$ are closed normal subgroups such that PU_n and PO_n are again compact Hausdorff groups. Next we compute the centers $Z(U_n)$ and $Z(O_n)$. This can be done with the help of Schur's lemma, which can for example be found in [Hal03].

Lemma 4.13 (Schur's lemma)**.** *Let $\varphi\colon G \to \mathrm{GL}(\mathbb{C}, n)$ be an irreducible representation of a group G and let $T \in M_n(\mathbb{C})$ such that $T\varphi(g) = \varphi(g)T$ for all $g \in G$. Then $T = \alpha I$ for some $\alpha \in \mathbb{C}$.*

Lemma 4.14. *$Z(U_n) = \{\alpha I : \alpha \in \mathbb{C}, |\alpha| = 1\}$ and $Z(O_n) = \{\pm I\}$.*

Proof. Let $Q \in Z(U_n)$. Since the identity representation of U_n is irreducible, Schur's lemma implies $Q = \alpha I$ for some $\alpha \in \mathbb{C}$. Because Q is unitary, we obtain $|\alpha| = 1$. The other inclusion follows directly since $(\alpha I)Q = \alpha Q = Q(\alpha I)$ for all $Q \in U_n$. The statement for $Z(O_n)$ can be shown similarly. □

Now we come to the isomorphism $\mathrm{Aut}(M_n(\mathbb{C})) \cong PU_n$, where $\mathrm{Aut}(M_n(\mathbb{C}))$ denotes the set of $*$-automorphisms of $M_n(\mathbb{C})$. First we show that each $*$-automorphism of $M_n(\mathbb{C})$ is inner. Note that this result holds more generally for $\mathrm{Aut}(B(H))$ (see [Bla05, Example II.5.5.14]).

Lemma 4.15. *Every $*$-automorphism $T \in \mathrm{Aut}(M_n(\mathbb{C}))$ has the form $T(A) = QAQ^*$ for some unitary $Q \in U_n$.*

Proof. The statement is a direct consequence of [Mur90, Example 5.1.1]. In the following we sketch this argument in our setting. Consider a $T \in \mathrm{Aut}(M_n(\mathbb{C}))$. Then T is surjective and defines an irreducible representation of the C^*-algebra $M_n(\mathbb{C})$ on $\mathbb{C}^n$. Next choose some unit vector $x \in \mathbb{C}^n$. Then x is cyclic for this representation and

$$\varphi\colon M_n(\mathbb{C}) \to \mathbb{C}, \quad A \mapsto \langle T(A)x, x\rangle$$

defines a pure state. Since all pure states of $M_n(\mathbb{C})$ are given by vectors, there exists a unit vector $y \in \mathbb{C}^n$ such that $\varphi(A) = \langle Ay, y\rangle$. Now consider the identity representation of $M_n(\mathbb{C})$ on $\mathbb{C}^n$, which is irreducible. Then y is a cyclic vector for this representation with

$$\langle T(A)x, x\rangle = \langle Ay, y\rangle \quad \forall A \in M_n(\mathbb{C}).$$

Therefore, the representation T is unitarily equivalent to the identity representation and there exists a unitary $Q \in U_n$ such that $T(A) = QAQ^*$ for all $A \in M_n(\mathbb{C})$. □

Using the previous lemmas, we can now prove the isomorphism $\mathrm{Aut}(M_n(\mathbb{C})) \cong PU_n$.

Lemma 4.16. *$\mathrm{Aut}(M_n(\mathbb{C})) \cong PU_n$ as groups.*

Proof. Consider the map $\Phi\colon U_n \to \mathrm{Aut}(M_n(\mathbb{C}))$, $Q \mapsto (A \mapsto QAQ^*)$. It is a group homomorphism since

$$\Phi(Q_1Q_2)(A) = (Q_1Q_2)A(Q_1Q_2)^* = Q_1Q_2AQ_2^*Q_1^* = (\Phi(Q_1) \circ \Phi(Q_2))(A)$$

for all $A \in M_n(\mathbb{C})$. Further, it is surjective by Lemma 4.15. It remains to show that $\ker\Phi = Z(U_n)$ such that

$$\mathrm{Aut}(M_n(\mathbb{C})) \cong U_n/\ker\Phi = PU_n.$$

Let $Q \in \ker\Phi$. Then $\Phi(Q) = \mathrm{id}$ such that

$$A = \Phi(Q)(A) = QAQ^* \quad (A \in U_n).$$

Hence, $QA = AQ$ for all $A \in U_n$ and thus $Q \in Z(U_n)$. Now, let $Q \in Z(U_n)$. Then $Q = \alpha I$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ by Lemma 4.14. This implies

$$\Phi(Q)(A) = (\alpha I)A(\alpha I)^* = \alpha\overline{\alpha}A = A$$

such that $\Phi(Q) = \mathrm{id}$ and $Q \in \ker\Phi$. □

Remark 4.17. The isomorphism from Lemma 4.16 is given by

$$(A \mapsto QAQ^*) \in \mathrm{Aut}(M_n(\mathbb{C})) \quad \longleftrightarrow \quad [Q] = \{\alpha Q : |\alpha| = 1\} \in PU_n.$$

4.4 Computing the Group Associated to $P_2^{(2)}$

In the following we show that $\mathrm{Spec}\,A$ from Definition 4.8 is isomorphic to the projective orthogonal group PO_n and we prove Theorem 4.29. This is done by constructing an injective group homomorphism $\mathrm{Spec}\,A \hookrightarrow \mathrm{Aut}(M_n(\mathbb{C})) \cong PU_n$. Then its image is computed, which is isomorphic to PO_n. We begin with three technical lemmas, which will be used throughout this section.

Lemma 4.18. *Let $1 \le a_1, a_2, a_3, a_4 \le n$ and $1 \le i, j \le n$. Further, let A be a C^*-algebra and $u \in M_{n^2}(A)$ which satisfies the relations R_2 from Definition 4.8. Then*

$$\sum_k u_{(a_1,k)(a_2,i)}u_{(a_3,k)(a_4,j)} = \delta_{ij}\sum_k u_{(a_1,k)(a_2,1)}u_{(a_3,k)(a_4,1)}.$$

Proof. A direct computation using the relations R_2 shows

$$\begin{aligned}
\sum_k u_{(a_1,k)(a_2,i)}u_{(a_3,k)(a_4,j)} &= \delta_{11}\sum_k u_{(a_1,k)(a_2,i)}u_{(a_3,k)(a_4,j)} \\
&= \delta_{ij}\sum_k u_{(a_1,1)(a_2,k)}u_{(a_3,1)(a_4,k)} && (R_2) \\
&= \delta_{ij}\left(\delta_{11}\sum_k u_{(a_1,1)(a_2,k)}u_{(a_3,1)(a_4,k)}\right) \\
&= \delta_{ij}\left(\delta_{11}\sum_k u_{(a_1,k)(a_2,1)}u_{(a_3,k)(a_4,1)}\right) && (R_2) \\
&= \delta_{ij}\sum_k u_{(a_1,k)(a_2,1)}u_{(a_3,k)(a_4,1)}.
\end{aligned}$$

□

Lemma 4.19. *Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $T(A) = QAQ^*$ for some $Q \in M_n(\mathbb{C})$. Then $T(e_{ij})_{kl} = Q_{ki}\overline{Q_{lj}}$, where e_{ij} denotes a matrix unit.*

Proof. A direct computation shows

$$T(e_{ij})_{kl} = (Qe_{ij}Q^*)_{kl} = \sum_{r,s} Q_{kr}(e_{ij})_{rs}Q^*_{sl} = \sum_{r,s} Q_{kr}\delta_{ir}\delta_{js}Q^*_{sl} = Q_{ki}Q^*_{jl} = Q_{ki}\overline{Q_{lj}}.$$

□

Lemma 4.20. *Let $Q \in M_n(\mathbb{C})$ be an orthogonal matrix. Then the elements $q_{(i,j)(k,l)} := Q_{ik}Q_{jl}$ satisfy the relations R_1, R_2, R_3 and R_4 from Definition 4.8.*

Proof. This statement was already observed by Cébron and Weber and can be found in a different form in [CW22]. For the convinience of the reader, we present the complete proof in the following.

(R_1) Since the entries of Q commute, we obtain

$$\begin{aligned} q_{(i_1,i_2)(j_1,j_2)}q_{(i_3,i_4)(j_3,j_4)} &= Q_{i_1j_1}Q_{i_2j_2}Q_{i_3j_3}Q_{i_4j_4} \\ &= Q_{i_1j_1}Q_{i_4j_4}Q_{i_3j_3}Q_{i_2j_2} \\ &= q_{(i_1,i_4)(j_1,j_4)}q_{(i_3,i_2)(j_3,j_2)}. \end{aligned}$$

(R_2) Since Q is orthogonal and its entries commute, we obtain

$$\begin{aligned} \delta_{i_2i_4}\sum_k q_{(i_1,k)(j_1,j_2)}q_{(i_3,k)(j_3,j_4)} &= \delta_{i_2i_4}\sum_k Q_{i_1j_1}Q_{kj_2}Q_{i_3j_3}Q_{kj_4} \\ &= \delta_{i_2i_4}\left(\sum_k Q_{kj_2}Q_{kj_4}\right)Q_{i_1j_1}Q_{i_3j_3} \\ &= \delta_{i_2i_4}\delta_{j_2j_4}Q_{i_1j_1}Q_{i_3j_3} \\ &= \left(\sum_k Q_{i_2k}Q_{i_4k}\right)\delta_{j_2j_4}Q_{i_1j_1}Q_{i_3j_3} \\ &= \delta_{j_2j_4}\sum_k Q_{i_1j_1}Q_{i_2k}Q_{i_3j_3}Q_{i_4k} \\ &= \delta_{j_2j_4}\sum_k q_{(i_1,i_2)(j_1,k)}q_{(i_3,i_4)(j_3,k)}. \end{aligned}$$

(R_3) Since Q is orthogonal, we obtain

$$\sum_k q_{(k,k)(j_1,j_2)} = \sum_k Q_{kj_1}Q_{kj_2} = \delta_{j_1j_2}.$$

(R_4) Since the entries of Q commute, we obtain

$$q_{(i_1,i_2)(j_1,j_2)} = Q_{i_1j_1}Q_{i_2j_2} = Q_{i_2j_2}Q_{i_1j_1} = q_{(i_2,i_1)(j_2,j_1)}.$$

□

Now we come to the main part of this section. In the following denote with A the C^*-algebra from Definition 4.8. First we show that its generators $u_{(i,j)(k,l)}$ satisfy the matrix unit relations.

Lemma 4.21. *Define $E_{ij} \in M_n(A)$ by $(E_{ij})_{kl} = u_{(k,l)(i,j)}$. Then E_{ij} satisfies the matrix unit relations*

1. $E_{ij}^* = E_{ji}$,
2. $E_{i_1j_1}E_{i_2j_2} = \delta_{j_1i_2}E_{i_1j_2}$.

Proof. The statement follows from direct computations using R_1, R_2, R_3 and R_4:

1. $$(E_{ij}^*)_{kl} = ((E_{ij})_{lk})^* = u^*_{(l,k)(i,j)} = u_{(l,k)(i,j)} \overset{(R_4)}{=} u_{(k,l)(j,i)} = (E_{ji})_{kl}$$

2. $$\begin{aligned}(E_{i_1j_1}E_{i_2j_2})_{kl} &= \sum_r (E_{i_1j_1})_{kr}(E_{i_2j_2})_{rl} \\ &= \sum_r u_{(k,r)(i_1,j_1)}u_{(r,l)(i_2,j_2)} \\ &= \sum_r u_{(k,r)(i_1,j_1)}u_{(l,r)(j_2,i_2)} && (R_4) \\ &= \delta_{j_1i_2}\sum_r u_{(k,r)(i_1,1)}u_{(l,r)(j_2,1)} && (4.18) \\ &= \delta_{j_1i_2}\sum_r u_{(k,r)(i_1,1)}u_{(r,l)(1,j_2)} && (R_4) \\ &= \delta_{j_1i_2}\sum_r u_{(k,l)(i_1,j_2)}u_{(r,r)(1,1)} && (R_1) \\ &= \delta_{j_1i_2}u_{(k,l)(i_1,j_2)}\sum_r u_{(r,r)(1,1)} \\ &= \delta_{j_1i_2}(E_{i_1j_2})_{kl} && (R_3)\end{aligned}$$

□

Next we explore some properties of characters $\varphi \in \operatorname{Spec} A$ using Lemma 4.21 and the relations of A. These properties will then be used to construct an injective group homomorphism $\operatorname{Spec} A \to \operatorname{Aut}(M_n(\mathbb{C}))$.

Lemma 4.22. *Let $\varphi \in \operatorname{Spec} A$. Then φ induces a $*$-homomorphism $\varphi\colon M_n(A) \to M_n(\mathbb{C})$, $(a_{ij}) \mapsto (\varphi(a_ij))$. Thus, in the notation of Lemma 4.21 we have $\varphi(E_{ij}) \in M_n(\mathbb{C})$ with $\varphi(E_{ij})_{kl} = \varphi(u_{(k,l)(i,j)})$.*

1. *Denote with e_{ij} the standard matrix units in $M_n(\mathbb{C})$. Then there exists a $*$-isomorphism $T_\varphi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with $T_\varphi(e_{ij}) = \varphi(E_{ij})$.*
2. *There exists an orthogonal $Q_\varphi \in M_n(\mathbb{C})$ such that $T_\varphi(A) = Q_\varphi A Q_\varphi^*$. In particular, $\varphi(u_{(i,j)(k,l)}) = T_\varphi(e_{kl})_{ij} = (Q_\varphi)_{ik}(Q_\varphi)_{jl}$.*

Proof.

1. By Lemma 4.21, E_{ij} satisfies the matrix unit relations such that $\varphi(E_{ij})$ also satisfies these relations. Since $M_n(\mathbb{C})$ is simple and finite dimensional with basis e_{ij}, there exists a $*$-isomorphism $T_\varphi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with $T_\varphi(e_{ij}) = \varphi(E_{ij})$.

2. By Lemma 4.15 there exists an unitary $\widetilde{Q}$ such that $T_\varphi(A) = \widetilde{Q}A\widetilde{Q}^*$. Since $\widetilde{Q}$ is unitary, we find i_0, j_0 with $\widetilde{Q}_{i_0j_0} \neq 0$. Write $\widetilde{Q}_{i_0j_0}$ in polar coordinates as $\widetilde{Q}_{i_0j_0} = re^{i\theta}$ and define $Q := e^{-i\theta}\widetilde{Q}$ such that $Q_{i_0j_0} = r \in \mathbb{R}^+$.

   Now, let $1 \leq i, j, k, l \leq n$. By definition

$$\varphi(u_{(i,j)(k,l)}) = \varphi(E_{kl})_{ij} = T_\varphi(e_{kl})_{ij} \overset{4.19}{=} \widetilde{Q}_{ik}\overline{\widetilde{Q}_{jl}} = Q_{ik}\overline{Q_{jl}}.$$

   This implies

$$\varphi(u_{(k,i_0)(l,j_0)}) = Q_{kl}\overline{Q_{i_0j_0}} = Q_{kl} \cdot r.$$

   Since $u_{(k,i_0)(l,j_0)}$ is self-adjoint, $\varphi(u_{(k,i_0)(l,j_0)})$ is real such that $Q_{kl} \in \mathbb{R}$. Hence, Q has only real entries and is orthogonal. In particular, the previous equation simplifies to

$$\varphi(u_{(i,j)(k,l)}) = T_\varphi(e_{kl})_{ij} = Q_{ik}Q_{jl}.$$

□

Lemma 4.23. *The mapping* Spec $A \to \mathrm{Aut}(M_n(\mathbb{C}))$, $\varphi \mapsto T_\varphi$ *is an injective group homomorphism.*

Proof. Let $\varphi_1, \varphi_2 \in \operatorname{Spec} A$ and $1 \leq i, j, k, l \leq n$. Then

$$(T_{\varphi_1} \circ T_{\varphi_2})(a) = Q_{\varphi_1}Q_{\varphi_2}a(Q_{\varphi_2})^*(Q_{\varphi_1})^* = (Q_{\varphi_1}Q_{\varphi_2})a(Q_{\varphi_1}Q_{\varphi_2})^*$$

such that

$$\begin{aligned}
T_{\varphi_1*\varphi_2}(e_{ij})_{kl} &= (\varphi_1 * \varphi_2)(u_{(k,l)(i,j)}) \\
&= \sum_{r,s} \varphi_1(u_{(k,l)(r,s)})\varphi_2(u_{(r,s)(i,j)}) \\
&= \sum_{r,s} (Q_{\varphi_1})_{kr}(Q_{\varphi_1})_{ls} \cdot (Q_{\varphi_2})_{ri}(Q_{\varphi_2})_{sj} \\
&= \left(\sum_r (Q_{\varphi_1})_{kr}(Q_{\varphi_2})_{ri}\right) \cdot \left(\sum_s (Q_{\varphi_1})_{ls}(Q_{\varphi_2})_{sj}\right) \\
&= (Q_{\varphi_1}Q_{\varphi_2})_{ki}(Q_{\varphi_1}Q_{\varphi_2})_{lj} \\
&\overset{4.19}{=} (T_{\varphi_1} \circ T_{\varphi_2})(e_{ij})_{kl}.
\end{aligned}$$

Hence, $T_{\varphi_1*\varphi_2}(e_{ij}) = (T_{\varphi_1} \circ T_{\varphi_2})(e_{ij})$ for all matrix units e_{ij}. Since the e_{ij} form a basis, we obtain $T_{\varphi_1*\varphi_2} = T_{\varphi_1} \circ T_{\varphi_2}$ such that the mapping is a group homomorphism. Now, let $T_{\varphi_1} = T_{\varphi_2}$. Then

$$\varphi_1(u_{(ij)(kl)}) = T_{\varphi_1}(e_{kl})_{ij} = T_{\varphi_2}(e_{kl})_{ij} = \varphi_2(u_{(ij)(kl)}).$$

Since the $u_{(i,j)(k,l)}$ generate A, it follows that $\varphi_1 = \varphi_2$. Thus, the mapping is injective.

□

Remark 4.24. By composing the mapping Spec $A \to \mathrm{Aut}(M_n(\mathbb{C}))$ from Lemma 4.23 with the isomorphism $\mathrm{Aut}(M_n(\mathbb{C})) \cong PU_n$ from Lemma 4.16, we obtain an injective group homomorphism $\Phi\colon \operatorname{Spec} A \to PU_n$, $\varphi \mapsto [Q_\varphi]$.

Next we compute the image of the mapping $\Phi\colon \operatorname{Spec} A \to PU_n$, which is isomorphic to $\operatorname{Spec} A$.

Lemma 4.25. *Let* $\Phi\colon \operatorname{Spec} A \to PU_n$ *be the homomorphism from Remark 4.24. Then* $\operatorname{Im} \Phi = \{[Q] : Q \text{ is orthogonal}\}$.

Proof. The inclusion "$\subseteq$" holds since Q_φ is orthogonal by Lemma 4.22. Now consider the other inclusion. Let Q be orthogonal and define $q_{(i,j)(k,l)} = Q_{ik}Q_{jl}$. Then $q_{(i,j)(k,l)}$ satisfies the relations R_1, R_2, R_3 and R_4 by Lemma 4.20. By the universal property of A, there exists a $*$-homomorphism $\varphi\colon A \to \mathbb{C}$ with $\varphi(u_{(i,j)(k,l)}) = Q_{ik}Q_{jl}$. Further, define $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $T(A) = QAQ^*$. We show that $T = T_\varphi$ such that $[Q] = [Q_\varphi] \in \operatorname{Im} \Phi$. It holds

$$T(e_{ij})_{kl} \overset{4.19}{=} Q_{ki}Q_{lj} = \varphi(u_{(k,l)(i,j)}) = T_\varphi(e_{ij})_{kl}.$$

Hence, $T(e_{ij}) = T_\varphi(e_{ij})$ for all matrix units such that $T = T_\varphi$. □

Lemma 4.26. $\operatorname{Spec} A \cong PO_n$ *as groups.*

Proof. Consider $O_n \subseteq U_n$ and the canonical projection $\pi\colon U_n \to PU_n$ with $\ker \pi = Z(U_n)$. Then $\operatorname{Im} \pi|_{O_n} = \{[Q] : Q \text{ is orthogonal}\}$ such that

$$\operatorname{Spec} A \cong \operatorname{Im} \pi|_{O_n} \cong O_n/(\ker \pi|_{O_n}) = O_n/(Z(U_n) \cap O_n) = O_n/Z(O_n) = PO_n$$

by Lemma 4.25. □

Remark 4.27. Denote with $\Psi\colon \operatorname{Spec} A \to PO_n$ the isomorphism from Lemma 4.26. Then

$$\Psi(\varphi) = [Q_\varphi] \qquad \forall \varphi \in \operatorname{Spec} A,$$

where Q_φ is defined in Lemma 4.22. Conversely, the inverse $\Psi^{-1}\colon PO_n \to \operatorname{Spec} A$ is again a group homomorphism and Lemma 4.22 implies

$$\Psi^{-1}([Q])(u_{(i,j)(k,l)}) = Q_{ik}Q_{jl}.$$

Hence, the isomorphism $\operatorname{Spec} A \cong PO_n$ can be written as

$$[Q] \in PO_n \quad \longleftrightarrow \quad (u_{(i,j)(k,l)} \mapsto Q_{ik}Q_{jl}) \in \operatorname{Spec} A.$$

It remains to show that the previous isomorphism is an isomorphism of compact groups.

Lemma 4.28. $\operatorname{Spec} A \cong PO_n$ *as compact groups.*

Proof. We have to show that the isomorphism $\Psi\colon \operatorname{Spec} A \cong PO_n$ from Remark 4.27 is a homeomorphism. Since PO_n is compact and $\operatorname{Spec} A$ is Hausdorff, it is sufficient to show that $\Psi^{-1}\colon PO_n \to \operatorname{Spec} A$ is continuous. Further, PO_n is equipped with the quotient topology such that we can consider the continuity of $\widetilde{\Psi} := \Psi^{-1} \circ \pi$ instead. Here, $\pi\colon O_n \to PO_n$ denotes the canonical projection.
Because O_n and $\operatorname{Spec} A$ are topological groups and the topology on O_n is induced by a metric, it is enough to show sequential continuity at the identity. Let (Q_m) be a sequence which converges to $\mathrm{id} \in O_n$. Then we have to show that $\widetilde{\Psi}(Q_m)$

converges pointwise to the character $\widetilde{\Psi}(\mathrm{id})$, which is again the identity on $\operatorname{Spec} A$. By Remark 4.10, this identity is given by

$$\widetilde{\Psi}(\mathrm{id})(u_{(i,j)(k,l)}) = \delta_{ik}\delta_{jl},$$

which maps the generators $u_{(i,j)(k,l)}$ to the coefficients of the identity matrix. Futher, we have

$$\widetilde{\Psi}(Q_m)(u_{(i,j)(k,l)}) = \Psi([Q_m])(u_{(i,j)(k,l)}) = (Q_m)_{ik}(Q_m)_{jl}$$

by Remark 4.27. Since Q_m converges to $\mathrm{id} \in O_n$, we obtain $(Q_m)_{ij} \to \mathrm{id}_{ij} = \delta_{ij}$ as $m \to \infty$. Therefore,

$$\widetilde{\Psi}(Q_m)(u_{(i,j)(k,l)}) \to \delta_{ik}\delta_{jl} = \widetilde{\Psi}(\mathrm{id})(u_{(i,j)(k,l)}) \qquad (m \to \infty).$$

□

Using the previous lemma, we can finally prove the following theorem.

Theorem 4.29. *Let G be the quantum group associated to the category of spatial partitions $P_2^{(2)}$. Then G corresponds to a compact matrix group which is isomorphic to the projective orthogonal group PO_n.*

Proof. Let $G = (A, u)$ be the spatial partition quantum group of size n^2 associated to the category of spatial partitions $P_2^{(2)}$. Then A is given by the universal C^*-algebra from Definition 4.8. By Proposition 4.9 and Remark 4.10, G corresponds to the classical compact matrix group $\operatorname{Spec} A$. Lemma 4.28 then shows that $\operatorname{Spec} A$ is isomorphic to the projective orthogonal group PO_n as compact groups. □

Remark 4.30. Let us comment on possible generalizations of Theorem 4.29. One possibility is to consider the unitary quantum group $G \subseteq U_{n^2}^+$ associated to the category $P_{\circ\bullet,2}^{(2)}$ of colored spatial pair partitions on two levels. It is obtained by adding the partition to the generators from Lemma 4.6. However, the partition makes the generators $u_{(i,j)(k,l)}$ self-adjoint such that we are again in the orthogonal case and obtain PO_n.
Another possibility is to consider $P_2^{(m)}$ for $m > 2$. A first step would be to generalize Lemma 4.6 and find generators of $P_2^{(m)}$. However, we have to leave this question regarding the generators of $P_2^{(m)}$ and the corresponding quantum group open.

5 Bistochastic Subgroups

In this section we use the new results from Section 4 to show that the bistochastic group B_n corresponds to the category of spatial partitions generated by $P_2^{(2)}$ and . This result is again new and allows us to apply a construction from [CW22] which associates a quantum groups $\mathring{G} \subseteq O_n^+$ to special quantum groups $G \subseteq O_{n^2}^+$.

5.1 Bistochastic Groups

We begin with the definition of bistochastic matrices and their corresponding classical groups. These definitions and more information on bistochastic quantum groups can for example be found in [BS09] and [Web13].

Definition 5.1. Let $Q \in O_n$ be an orthogonal matrix.

1. The matrix Q is called bistochastic' if $\sum_k Q_{kj} = \sum_k Q_{ik}$ for all $1 \leq i, j \leq n$. In this case, denote the sum with r_Q.
2. The matrix Q is called bistochastic if Q is bistochastic' and $r_Q = 1$.

Definition 5.2. Define the

- bistochastic' group $B'_n := \{Q \in O_n \mid Q \text{ is bistochastic'}\}$,
- bistochastic group $B_n := \{Q \in O_n \mid Q \text{ is bistochastic}\}$,
- projective bistochastic' group $PB'_n := B'_n/\{\pm I\}$.

In the following we show that these sets are indeed compact groups.

Lemma 5.3. *The sets B'_n and B_n are closed subgroups of O_n. In particular, $r_{Q_1Q_2} = r_{Q_1}r_{Q_2}$ for all $Q_1, Q_2 \in B'_n$.*

Proof. The sets B'_n and B_n are closed since the defining equations are continuous and preserved under limits. Further, r_{Q^T} is again independent of the row/column and given by $r_{Q^T} = r_Q$. Thus, B'_n and B_n contain inverses. It remains to show that B'_n and B_n are closed under multiplication. Consider some $1 \leq j \leq n$. Then a direct computation shows that

$$\sum_k (Q_1Q_2)_{kj} = \sum_{k,l} (Q_1)_{kl}(Q_2)_{lj} = \sum_l (Q_2)_{lj} \sum_k (Q_1)_{kl} = r_{Q_1} \sum_l (Q_2)_{lj} = r_{Q_1}r_{Q_2}.$$

In particular, the sum is independent of the column j. A similar calculation shows that

$$\sum_k (Q_1Q_2)_{ik} = r_{Q_1}r_{Q_2}.$$

This implies that $r_{Q_1Q_2}$ is well-defined and given by $r_{Q_1}r_{Q_2}$. Therefore, B'_n and B_n are closed under multiplication. □

Remark 5.4. Since O_n is a compact group, the previous lemma implies that B'_n and B_n are again compact groups. Further, $\{\pm I\}$ is a closed normal subgroup of B'_n such that PB'_n is a compact Hausdorff group.

In the rest of this section, we present a short own proof of the isomorphism $PB'_n \cong B_n$ as groups. However, this result is not new and is for example stated as $B'_n \cong B_n \times \mathbb{Z}_2$ in [BS09].

Lemma 5.5. *Let $Q \in B'_n$. Then $r_Q = 1$ or $r_Q = -1$.*

Proof. Let $Q \in B'_n$. Since Q is orthogonal, we have $Q^{-1} = Q^T$ and $r_Q = r_{Q^{-1}}$. This implies

$$(r_Q)^2 = r_Q \cdot r_{Q^{-1}} = r_{QQ^{-1}} = r_I = 1.$$

Hence, $r_Q = 1$ or $r_Q = -1$. □

Proposition 5.6. *$PB'_n \cong B_n$ as groups.*

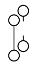
Proof. Consider the mapping $\Phi\colon B'_n \to B_n$, $Q \mapsto r_Q Q$, which is well-defined because $r_{r_Q Q} = r_Q \cdot r_Q = 1$. Then Φ is a group homomorphism since

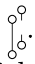
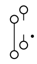
$$(r_{Q_1}Q_1)(r_{Q_2}Q_2) = r_{Q_1}r_{Q_2}Q_1Q_2 = r_{Q_1Q_2}(Q_1Q_2).$$

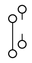
Further, $r_Q = 1$ for $Q \in B_n$ such that $\Phi(Q) = Q$ for all $Q \in B_n$. Therefore, Φ is surjective. Finally, one checks that $\ker \Phi = \{\pm I\}$ since $r_Q \in \{\pm 1\}$. Hence,

$$PB'_n = B'_n / \ker \Phi \cong \operatorname{Im} \Phi = B_n.$$

□

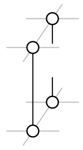
5.2 The Quantum Group Associated to $P_2^{(2)}$ and 

Next we use the results from Section 4 to compute the spatial partition quantum group associated to the category of spatial partitions generated by $P_2^{(2)}$ and . According to Proposition 3.15, it can be defined via the universal C^*-algebra with the relations from Definition 4.8 and the relations associated to . Latter relations can be found in Example 3.14 and [CW22].

Definition 5.7. The quantum group $G \subseteq O^+_{n^2}$ associated to the category of spatial partitions generated by $P_2^{(2)}$ and  is given by

$$C(G) := C^*(u_{(i_1,i_2)(j_1,j_2)} \mid u \text{ orthogonal}, R_1, R_2, R_3, R_4, R_5).$$

Here, the relations R_1, R_2, R_3 and R_4 are from Definition 4.8 and R_5 is given by



$$: \qquad \sum_k u_{(i_1,k)(j_1,j_2)} = \sum_k u_{(i_1,i_2)(j_1,k)}.$$

Remark 5.8. Consider the quantum group G from the previous definition and denote with H the quantum group from Definition 4.8 which is associated to $P_2^{(2)}$. Since the generators of $C(G)$ also satisfy the relations R_1, R_2, R_3 and R_4, there exists a surjective $*$-homomorphism $C(H) \to C(G)$ mapping generators to generators. Thus, $G \subseteq H$ is a quantum subgroup in the sense of Definition 2.2. This implies that G also corresponds to a classical compact matrix group and we can identify $\operatorname{Spec} C(G)$ with a subgroup of $\operatorname{Spec} C(H)$ by Proposition 2.5.

Remark 5.9. In the notation of the previous remark, we can use the inclusion $\operatorname{Spec} C(G) \hookrightarrow \operatorname{Spec} C(H)$ to define the matrix Q_φ from Lemma 4.22 for $\varphi \in \operatorname{Spec} C(G)$ such that

$$\varphi(u_{(i,j)(k,l)}) = (Q_\varphi)_{ik}(Q_\varphi)_{jl}.$$

Similarly, we can restrict the isomorphism $\Psi\colon \operatorname{Spec} C(H) \to PO_n$, $\varphi \mapsto [Q_\varphi]$ from Remark 4.27 to an injective goup homomorphism $\Psi\colon \operatorname{Spec} C(G) \to PO_n$.

Throughout the rest of this section, denote with $G = (A, u)$ the quantum group from Definition 5.7. In the following we compute the image of the homomorphism Ψ from Remark 5.9 in order to show $\operatorname{Spec} A \cong B_n$.

Lemma 5.10. *Let $\varphi \in \operatorname{Spec} A$. Then Q_φ is bistochastic'*

Proof. Let $\varphi \in \operatorname{Spec} A$. By applying φ on both sides of R_5, we obtain

$$\sum_k (Q_\varphi)_{i_1 j_1} (Q_\varphi)_{k j_2} = \sum_k \varphi(u_{(i_1,k)(j_1,j_2)}) = \sum_k \varphi(u_{(i_1,i_2)(j_1,k)}) = \sum_k (Q_\varphi)_{i_1 j_1} (Q_\varphi)_{i_2 k}$$

for all $1 \leq i_1, i_2, j_1, j_2 \leq n$. Since Q_φ is orthogonal, there exist some i_1, j_1 such that $Q_{i_1 j_1} \neq 0$. Dividing by $Q_{i_1 j_1}$ yields

$$\sum_k (Q_\varphi)_{k j_2} = \sum_h (Q_\varphi)_{i_2 k}$$

for all $1 \leq i_2, j_2 \leq n$. Hence, Q_φ is bistochastic'. □

Lemma 5.11. *Let Q be bistochastic'. Then there exists a $\varphi \in \operatorname{Spec} A$ with $[Q_\varphi] = [Q]$ in PO_n.*

Proof. The proof is similar to the proof of Lemma 4.25. Let Q be bistochastic' and define $q_{(i,j)(k,l)} := Q_{ik} Q_{jl}$. Then $q_{(i,j)(k,l)}$ is orthogonal and satisfies R_1, R_2, R_3 and R_4 by Lemma 4.20. Further,

$$\sum_k \underbrace{q_{(i_1,k)(j_1,j_2)}}_{=Q_{i_1 j_1} Q_{k j_2}} = Q_{i_1 j_1} \sum_k Q_{k j_2} = Q_{i_1 j_1} \sum_k Q_{i_2 k} = \sum_k \underbrace{q_{(i_1,i_2)(j_1,k)}}_{=Q_{i_1 j_1} Q_{i_2 k}}$$

because Q is bistochastic'. Hence, $q_{(i,j)(k,l)}$ also satisfies R_5. By the universal property of A, there exists a character $\varphi \in \operatorname{Spec} A$ with $\varphi(u_{(i,j)(k,l)}) = q_{(i,j)(k,l)}$.
As in the proof of Lemma 4.25, we define $T \in \operatorname{Aut}(M_n(\mathbb{C}))$ by $T(A) = QAQ^*$ and compute that $T = T_\varphi$ such that $[Q] = [Q_\varphi]$ in PO_n. □

Lemma 5.12. $\operatorname{Spec} A \cong B_n$ *as compact groups.*

Proof. By Remark 5.9, the mapping $\Psi\colon \operatorname{Spec} A \to PO_n$, $\varphi \mapsto [Q_\varphi]$ is an injective group homomorphism. Further, Lemma 5.10 and Lemma 5.11 imply that

$$\operatorname{Im} \Psi = \{[Q] \in PO_n \mid Q \text{ is bistochastic'}\} \subseteq PO_n.$$

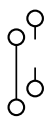
Similar to Lemma 4.26, one shows that $\operatorname{Im} \Psi \cong PB'_n$. Further, $PB'_n \cong B_n$ by Proposition 5.6 such that

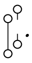
$$\operatorname{Spec} A \cong \operatorname{Im} \Psi \cong PB'_n \cong B_n$$

as groups. Denote this isomorphism with $\phi\colon B_n \to \operatorname{Spec} A$. Then it satisfies $\phi(Q)(u_{(i,j)(k,l)}) = Q_{ik} Q_{jl}$ by Remark 5.9.
It remains to show that ϕ is a homeomorphism. Since both $\operatorname{Spec} A$ and B_n are compact Hausdorff spaces, it is enough to show that ϕ is continuous. In addition, $\operatorname{Spec} A$ and B_n are metrizable topological groups such that it is sufficient to show sequential continuity at the identity. Therefore, let (Q_m) be a sequence in B_n converging to $\mathrm{id} \in B_n$. Then

$$\phi(Q_m)(u_{(i,j)(k,l)}) = Q_{ik} Q_{jl} \to \delta_{ik} \delta_{jl} \quad (m \to \infty)$$

for all generators $u_{(i,j)(k,l)}$. Hence, $\phi(Q_m) \to \mathrm{id} \in \operatorname{Spec} A$ as $m \to \infty$. Thus, ϕ is continuous. □

Remark 5.8 and the previous lemma show that the quantum group G associated to $P_2^{(2)}$ and  is a classical group which is isomorphic to the bistochastic group B_n. In the following we lift this isomorphism to an isomorphism $C(G) \cong C(B_n)$, which will be applied to $C(\mathring{G})$ in Section 5.3.

Lemma 5.13. *Let G be the quantum group associated to the category of spatial partitions generated by $P_2^{(2)}$ and* *. Further, denote with $q_{ij} \in C(B_n)$ the matrix coefficient functions. Then $C(G) \cong C(B_n)$ via $u_{(i,j)(k,l)} \mapsto q_{ik}q_{jl}$.*

Proof. As in the previous lemmas, we denote the C^*-algebra $C(G)$ with A. Then the isomorphism $\phi\colon B_n \to \operatorname{Spec} A$ from Lemma 5.12 induces an isomorphism

$$C(\operatorname{Spec} A) \to C(B_n), \quad f \mapsto f \circ \phi.$$

Using the Gelfand-Naimark theorem, we obtain the isomorphism

$$\widehat{\phi}\colon A \to C(B_n), \quad a \mapsto (Q \mapsto \phi(Q)(a)).$$

In particular, we have

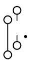
$$\widehat{\phi}(u_{(i,j)(k,l)})(Q) = \phi(Q)(u_{(i,j)(k,l)}) = Q_{ik}Q_{jl}$$

for all $Q \in B_n$. Using the matrix coefficient functions $q_{ij} \in C(B_n)$, we can write the previous equation as

$$\widehat{\phi}(u_{(i,j)(k,l)}) = q_{ik}q_{jl}.$$

□

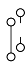
5.3 Constructing Quantum Subgroups of O_n^+

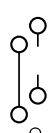
Next we present a construction from [CW22], which allows us to construct quantum groups $\mathring{G} \subseteq O_n^+$ from quantum groups $G \subseteq O_{n^2}^+$ satisfying the relations corresponding to .

Definition 5.14. Let $G \subseteq O_{n^2}^+$ be a compact matrix quantum group generated by elements $u_{(i,j)(k,l)}$. Then define

$$\mathring{u}_{ij} := \sum_k u_{(i,k)(j,1)}$$

and let $C(\mathring{G}) \subseteq C(G)$ be the C^*-algebra generated by the elements $\mathring{u}_{ij}$.

Proposition 5.15. *Let $G = (A, u)$ be a compact matrix quantum group such that $G \subseteq O_{n^2}^+$ and u satisfies the relations corresponding to* *. Then $C(\mathring{G})$ gives rise to a compact matrix quantum group $\mathring{G} \subseteq O_n^+$.*

In order to prove this proposition, one uses the relations of  to check that the comultiplication $\Delta\colon C(G) \to C(G) \otimes C(G)$ restricts to $\Delta\colon C(\mathring{G}) \to C(\mathring{G}) \otimes C(\mathring{G})$ with

$$\Delta(\mathring{u}_{ij}) = \sum_k \mathring{u}_{ik} \otimes \mathring{u}_{kj}.$$

Further, the matrix $\mathring{u} = (\mathring{u}_{ij})$ is orthogonal such that $\mathring{G} := (C(\mathring{G}), \mathring{u})$ is a compact matrix quantum group. Additionally, Cébron and Weber showed that if $S_n \subseteq G$ and G satisfies specific relations, then one obtains $S_n \subseteq \mathring{G} \subseteq S_n^+$.
Now consider the quantum group G associated to $P_2^{(2)}$ and the partition $\overset{\circ\circ}{\underset{\circ\circ}{|}}$. Then we can apply Proposition 5.15 to G to obtain a quantum group $\mathring{G}$. Using the results from Section 5.2, we can now prove the following theorem.

Theorem 5.16. *Let $G \subseteq O_{n^2}^+$ be the quantum group associated to the category of spatial partitions generated by $P_2^{(2)}$ and $\overset{\circ\circ}{\underset{\circ\circ}{|}}$. Then $C(G) \cong C(B_n)$ and $C(\mathring{G}) \cong C(B_n)$, where $B_n \subseteq O_n$ is the bistochastic group.*

Proof. Denote with $q_{ij} \in C(B_n)$ the matrix coefficient functions and with $u_{(i,j)(k,l)} \in C(G)$ the generators of $C(G)$. Then we obtain an isomorphism $\Phi\colon C(G) \to C(B_n)$, $u_{(i,j)(k,l)} \mapsto q_{ik}q_{jl}$ by Lemma 5.13. It remains to show that $C(\mathring{G}) \cong C(B_n)$. Since $C(\mathring{G}) \subseteq C(G)$, we can restrict the isomorphism Φ to an injective $*$-homomorphism $\mathring{\Phi}\colon C(\mathring{G}) \to C(B_n)$. Then one computes

$$\mathring{\Phi}\,(\mathring{u}_{ij}) = \mathring{\Phi}\left(\sum_k u_{(i,k)(j,1)}\right) = \sum_k \Phi(u_{(i,k)(j,1)}) = \sum_k q_{ij}q_{k1} = q_{ij}\sum_k q_{k1} = q_{ij}$$

such that $\mathring{\Phi}$ is surjective. Thus, $\mathring{\Phi}\colon C(\mathring{G}) \to C(B_n)$ is an isomorphism. □

Remark 5.17. The previous theorem shows that the construction from Cébron and Weber yields again $C(\mathring{G}) \cong C(B_n)$ when applied to the quantum group G with $C(G) \cong C(B_n)$. Note that one could try to generalize this theorem to the case of the quantum bistochastic group B_n^+ as defined in [Web13]. Then one might find a spatial partition quantum group $G \subseteq O_{n^2}^+$ with $C(G) \cong C(B_n^+)$ and $C(\mathring{G}) \cong C(B_n^+)$.

6 Transforming Spatial Partitions

In this section we consider techniques for transforming categories of spatial partitions. However, it turns out that categories of spatial partitions are too restrictive and we need to drop the base partitions. Therefore, we first introduce pre-categories and corresponding homomorphisms in Section 6.1. Using the language of pre-categories, we then consider the case of removing colors of spatial partitions in Section 6.2. In particular, we show that some pre-categories of colored spatial partitions can be characterized by its version without colors. Finally, we consider rotations of spatial partitions in Section 6.3 and generalize results from Gromada to the setting of spatial partitions. This allows us to characterize pre-categories of spatial partitions by its partitions with only lower points. See [Gro18] for more information on the classical case. In the following we denote with $P_{\circ\bullet}^{(m)}$ the category of all colored spatial partitions on m levels.

6.1 Pre-Categories and Homomorphisms

We begin with the definition of pre-categories of spatial partitions and some related constructions.

Definition 6.1 (Pre-category)**.** Let $C \subseteq P_{\circ\bullet}^{(m)}$ be a set of spatial partitions. Then C is a *pre-category* of spatial partitions if it is closed under involutions, tensor products and compositions.

Definition 6.2 (Closure and product of pre-categories)**.**

1. Let $p_1, \ldots, p_n \in P_{\circ\bullet}^{(m)}$ be spatial partitions. Then denote with $\operatorname{cl}\{p_1, \ldots, p_n\}$ the *closure* of $\{p_1, \ldots, p_n\}$, which is the smallest pre-category containing p_1, $\ldots$, p_n.

2. Let $C_1 \subseteq P_{\circ\bullet}^{(m_1)}$, $C_2 \subseteq P_{\circ\bullet}^{(m_2)}$ be pre-categories of spatial partitions. Then define their *product*

$$C_1 \times C_2 := \{(p, q) \mid p \in C_1, q \in C_2\} \subseteq P_{\circ\bullet}^{(m_1+m_2)}.$$

Here (p, q) denotes the spatial partition containing p at the first m_1 and q at the last m_2 levels. Note that $C_1 \times C_2$ is again a pre-category of spatial partitions. Compare this to [CW22], where this construction is described in more detail for categories of partitions.

Remark 6.3. Note that a set $C \subseteq P^{(m)}$ or $C \subseteq P_{\circ\bullet}^{(m)}$ is a category of (colored) spatial partitions if it is a pre-category of spatial partitions and contains the corresponding base partitions. These base partitions are different if we see C as category with colors or without colors. By using the language of pre-categories, we unify this setting and can regard $P^{(m)}$ as a sub-pre-category of $P_{\circ\bullet}^{(m)}$.

Next we define homomorphisms between pre-categories of spatial partitions.

Definition 6.4 (Homomorphism of pre-categories)**.** Let $C_1 \subseteq P_{\circ\bullet}^{(m_1)}$ and $C_2 \subseteq P_{\circ\bullet}^{(m_2)}$ be pre-categories of spatial partitions. A function $f\colon C_1 \to C_2$ is a *homomorphism of pre-categories* if

1. $f(p) \in C_2(k, l)$ for all $k, l \in \mathbb{N}_0$ and $p \in C_1(k, l)$,

2. $f(p^*) = f(p)^*$ for all $p \in C_1$,

3. $f(p \otimes q) = f(p) \otimes f(q)$ for all $p, q \in C_1$,

4. $f(pq) = f(p)f(q)$ for all $p, q \in C_1$ which are composable. Here we also require that $f(p)$ and (q) are composable.

Definition 6.5 (Isomorphic pre-categories)**.**

1. Let C_1 and C_2 be two pre-categories of spatial partitions. Then C_1 and C_2 are *isomorphic* and we write $C_1 \cong C_2$ if there exists a bijective homomorphism $f\colon C_1 \to C_2$. In this case, the inverse $f^{-1}\colon C_2 \to C_1$ is again a homomorphism.

2. Let $C_1 \subseteq P_{\circ\bullet}^{(m_1)}$ and $C_2 \subseteq P_{\circ\bullet}^{(m_2)}$ be two categories of colored spatial partitions. Then C_1 and C_2 are *isomorphic as categories of colored partitions* if there exists an isomorphism f of pre-categories which respects the colored base partitions, i.e.

$$f\left(\substack{\circ\\|\\\circ}^{(m_1)}\right) = \substack{\circ\\|\\\circ}^{(m_2)}, \quad f\left(\substack{\bullet\\|\\\bullet}^{(m_1)}\right) = \substack{\bullet\\|\\\bullet}^{(m_2)}, \quad f\left(\sqcap_{\bullet\circ}^{(m_1)}\right) = \sqcap_{\bullet\circ}^{(m_2)}, \quad f\left(\sqcap_{\circ\bullet}^{(m_1)}\right) = \sqcap_{\circ\bullet}^{(m_2)}.$$

Next we give some examples of homomorphisms of pre-categories.

Example 6.6. One can directly check that the following mappings are homomorphisms of pre-categories.

1. The identity $\mathrm{id}\colon C \to C$ and the composition of homomorphisms are homomorphisms.
2. Let $f\colon C_1 \to C_2$ be a homomorphism and let $D \subseteq C_1$ be a pre-category. Then the restriction $f|_D\colon D \to C_2$ is a homomorphism.
3. Any mapping $f\colon C \to C$ which permutes the levels is a homomorphism. In particular, these mappings are isomorphisms.
4. The projections $\pi_1\colon C_1 \times C_2 \to C_1$, $(p,q) \mapsto p$ and $\pi_2\colon C_1 \times C_2 \to C_2$, $(p,q) \mapsto q$ are homomorphisms.
5. Let $f\colon C \to D_1$ and $g\colon C \to D_2$ be homomorphisms. Then $(f,g)\colon C \to D_1 \times D_2$, $p \mapsto (f(p), g(p))$ is a homomorphism.
6. Let $f\colon C_1 \to C_2$ and $g\colon D_1 \to D_2$ be homomorphisms. Then $(f \times g)\colon C_1 \times D_1 \to C_2 \times D_2$, $(p,q) \mapsto (f(p), g(q))$ is a homomorphism.

Next we show that images and pre-images of pre-categories under homomorphisms are again pre-categories and that the closure operation is compatible with homomorphisms.

Proposition 6.7. *Let $f\colon C_1 \to C_2$ be a homomorphism of pre-categories.*

1. *If $D \subseteq C_1$ is a pre-category, then the image $f(D) \subseteq C_2$ is a pre-category.*
2. *If $D \subseteq C_2$ is a pre-category, then the pre-image $f^{-1}(D) \subseteq C_1$ is a pre-category.*
3. *If $D \subseteq C_1$ with $D = \mathrm{cl}\,\{p_1, \dots, p_n\}$, then $f(D) = \mathrm{cl}\,\{f(p_1), \dots, f(p_n)\}$.*

Proof. The statements 1. and 2. follow directly from the definitions since D is closed under all operations and f respects these operations. For 3. we have

$$\mathrm{cl}\,\{f(p_1), \dots, f(p_n)\} \subseteq f(D)$$

since $f(D)$ is a pre-category by 1. and contains $f(p_1), \dots, f(p_n)$. Conversely, we have

$$D = \mathrm{cl}\,\{p_1, \dots, p_n\} \subseteq f^{-1}(\mathrm{cl}\,\{f(p_1), \dots, f(p_n)\})$$

since $f^{-1}(\mathrm{cl}\,\{f(p_1), \dots, f(p_n)\})$ is a pre-category by 2. and contains $p_1, \dots, p_n$. This implies $f(D) \subseteq \mathrm{cl}\,\{f(p_1), \dots, f(p_n)\}$. □

Next we introduce the graph of a homomorphism, which will be used in Section 7.

Definition 6.8 (Graph)**.** Let $f\colon C_1 \to C_2$ be a homomorphism of pre-categories. Then the *graph* of f is defined by

$$\Gamma_f := \{(p, f(p)) \mid p \in C_1\} \subseteq C_1 \times C_2.$$

Proposition 6.9. *Let $f\colon C_1 \to C_2$ be a homomorphism of pre-categories. Then Γ_f is a pre-category and $\Gamma_f \cong C_1$.*

Proof. Γ_f is the image of $(\mathrm{id}, f)\colon C_1 \to C_1 \times C_2$, which is a homomorphism by Example 6.6 and which is a pre-category by Proposition 6.7. Further, the inverse of (id, f) is given by the projection π_1 onto the first component. Hence, $\Gamma_f \cong C_1$. □

6.2 Removing Colors

When working with classical partitions, one can compose partitions with in order to color points arbitrarily (see [TW18]). We now generalize this technique to spatial partitions and allow the case where the levels get twisted when changing the color of points. Further, we show that pre-categories which allow changing colors can be characterized by its colorless version.

Definition 6.10. Let $q \in P_{\circ\bullet}^{(m)}(1,1)$. We say q can *swap colors* if $q \cdot q^* = {}^{(m)}$ and $q^* \cdot q = {}^{(m)}$. In particular, the upper points of q have to be colored black and the lower points have to be colored white.

For example, the following partitions can swap colors:

Next we define a mapping S_q which uses a partition q to remove the colors of spatial partitions.

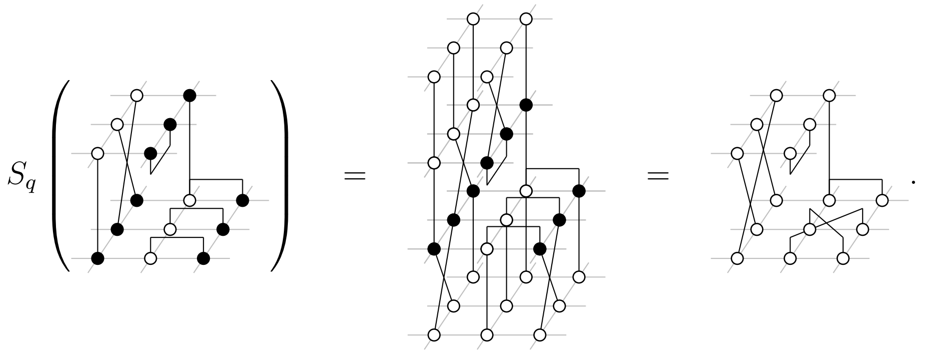
Definition 6.11. Let $q \in P_{\circ\bullet}^{(m)}(1,1)$ be a spatial partition which can swap colors.

1. Let $p \in P_{\circ\bullet}^{(m)}(k,l)$ and denote with $x_i \in \{\circ, \bullet\}$ for $1 \le i \le k$ the coloring of the k upper points and with $y_i \in \{\circ, \bullet\}$ for $1 \le i \le l$ the coloring of the l lower points. Then define $s_\circ := {}^{(m)}$, $s_\bullet := q$ and

$$s_\uparrow(p) := s_{x_1} \otimes \cdots \otimes s_{x_k} \in P^{(m)}(k,k),$$
$$s_\downarrow(p) := s_{y_1} \otimes \cdots \otimes s_{y_l} \in P^{(m)}(l,l).$$

2. Define $S_q \colon P_{\circ\bullet}^{(m)} \to P_{\circ\bullet}^{(m)}$, $p \mapsto s_\downarrow(p) \cdot p \cdot s_\uparrow(p)^*$. By definition, the number of points and coloring of $s_\downarrow(p)$ and $s_\uparrow(p)^*$ match the number of points and coloring of p such that the compositions are well-defined.

Remark 6.12. The mapping S_q uses the partition q to change the color of every point to white. Hence, it removes the colors of partitions. However, it might permute the levels depending on q. In particular, S_q will not necessarily preserve the base partitions. Consider for example $q =$. Then



In the following we want to prove that the mapping S_q from Definition 6.11 is a homomorphism. In particular, this shows that the colorless version $S_q(C)$ of a pre-category C is again a pre-category. But first, we need some identities involving the partitions $s_\uparrow(p)$ and $s_\downarrow(p)$.

Proposition 6.13. *Let $q \in P_{\circ\bullet}^{(m)}$ be a spatial partition which can swap colors. Then the following statements hold for $s_\uparrow$ and $s_\downarrow$:*

1. *Let $p_1, p_2 \in P_{\circ\bullet}^{(m)}$. If p_1 and p_2 have the same upper colors, then $s_\uparrow(p_1) = s_\uparrow(p_2)$. Similarly, if p_1 and p_2 have the same lower colors, then $s_\downarrow(p_1) = s_\downarrow(p_2)$.*

2. *Let $p \in P_{\circ\bullet}^{(m)}$. Then $s_\uparrow(p) = s_\downarrow(p^*)$ and $s_\downarrow(p) = s_\uparrow(p^*)$.*

3. *Let $p_1, p_2 \in P_{\circ\bullet}^{(m)}$. Then $s_\uparrow$ and $s_\downarrow$ respect the tensor product:*

$$s_\uparrow(p_1 \otimes p_2) = s_\uparrow(p_1) \otimes s_\uparrow(p_2), \qquad s_\downarrow(p_1 \otimes p_2) = s_\downarrow(p_1) \otimes s_\downarrow(p_2).$$

4. *Let $p_1, p_2 \in P_{\circ\bullet}^{(m)}$. If the following compositions with p_2 are well-defined, then*

$$\begin{aligned} s_\uparrow(p_1) \cdot s_\uparrow(p_1)^* \cdot p_2 &= p_2, & p_2 \cdot s_\uparrow(p_1) \cdot s_\uparrow(p_1)^* &= p_2, \\ s_\uparrow(p_1)^* \cdot s_\uparrow(p_1) \cdot p_2 &= p_2, & p_2 \cdot s_\uparrow(p_1)^* \cdot s_\uparrow(p_1) &= p_2. \end{aligned}$$

   The same equations also hold for $s_\downarrow$.

Proof. Statement 1 follows directly from the definitions of $s_\uparrow$ and $s_\downarrow$ since these only depend on the color of the corresponding points. Further, statement 1 directly implies statement 2 when applied to p^*.
For statement 3 consider $p_1 \in P_{\circ\bullet}^{(m)}(k_1, l_1)$ and $p_2 \in P_{\circ\bullet}^{(m)}(k_2, l_2)$ with upper colors $x_1, \dots, x_{k_1}$ and $y_1, \dots, y_{k_2}$. Then $p_1 \otimes p_2$ has upper colors $x_1, \dots, x_{k_1}, y_1, \dots, y_{k_2}$ and

$$s_\uparrow(p_1 \otimes p_2) = s_{x_1} \otimes \cdots \otimes s_{x_{k_1}} \otimes s_{y_1} \otimes \cdots \otimes s_{y_{k_2}} = s_\uparrow(p_1) \otimes s_\uparrow(p_2).$$

The same argument also applies to $s_\downarrow$ when considering the colors of the lower points.
It remains to prove statement 4. Let $p_1 \in P_{\circ\bullet}^{(m)}(k, l)$ with upper colors $x_1, \dots, x_k$. Then

$$\begin{aligned} s_\uparrow(p) \cdot s_\uparrow(p)^* &= (s_{x_1} \otimes \cdots \otimes s_{x_k}) \cdot (s_{x_1} \otimes \cdots \otimes s_{x_k})^* \\ &= (s_{x_1} \otimes \cdots \otimes s_{x_k}) \cdot (s_{x_1}^* \otimes \cdots \otimes s_{x_k}^*) \\ &= (s_{x_1} s_{x_1}^*) \otimes \cdots \otimes (s_{x_k} s_{x_k}^*) \end{aligned}$$

and similarly

$$s_\uparrow(p)^* \cdot s_\uparrow(p) = (s_{x_1}^* s_{x_1}) \otimes \cdots \otimes (s_{x_k}^* s_{x_k}).$$

Now consider some x_j with $1 \le j \le k$. If x_j is white, then $s_{x_j} = \overset{\circ}{\underset{\circ}{|}}^{(m)}$ and $x_j x_j^* = x_j^* x_j = \overset{\circ}{\underset{\circ}{|}}^{(m)}$. Otherwise $s_{x_j} = q$, which implies $s_{x_j} s_{x_j}^* = \overset{\circ}{\underset{\circ}{|}}^{(m)}$ and $s_{x_j}^* s_{x_j} = \overset{\bullet}{\underset{\bullet}{|}}^{(m)}$ by the definition of q. In both cases $s_\uparrow(p) \cdot s_\uparrow(p)^*$ and $s_\uparrow(p)^* \cdot s_\uparrow(p)$ are a tensor product of colored identity partitions such that

$$\begin{aligned} s_\uparrow(p_1) \cdot s_\uparrow(p_1)^* \cdot p_2 &= p_2, & p_2 \cdot s_\uparrow(p_1) \cdot s_\uparrow(p_1)^* &= p_2, \\ s_\uparrow(p_1)^* \cdot s_\uparrow(p_1) \cdot p_2 &= p_2, & p_2 \cdot s_\uparrow(p_1)^* \cdot s_\uparrow(p_1) &= p_2, \end{aligned}$$

whenever the corresponding composition with $p_2 \in P_{\circ\bullet}^{(m)}$ is well-defined. The same argument also applies to $s_\downarrow$ when using the lower colors. □

Proposition 6.14. *The mapping* $S_q\colon P_{\circ\bullet}^{(m)} \to P_{\circ\bullet}^{(m)}$, $p \mapsto s_\downarrow(p)\cdot p \cdot s_\uparrow(p)^*$ *is a homomorphism.*

Proof. Let $p \in P_{\circ\bullet}^{(m)}(k,l)$. Then $S_q(p) \in P_{\circ\bullet}^{(m)}(k,l)$ since $s_\uparrow(p)^* \in P_{\circ\bullet}^{(m)}(k,k)$ and $s_\downarrow(p) \in P_{\circ\bullet}^{(m)}(l,l)$. It remains to check the category operations:

1. Let $p \in P_{\circ\bullet}^{(m)}$. Then

$$S_q(p)^* = [s_\downarrow(p)\cdot p\cdot s_\uparrow(p)^*]^* = s_\uparrow(p)\cdot p^*\cdot s_\downarrow(p)^* \overset{6.13.2}{=} s_\downarrow(p^*)\cdot p^*\cdot s_\uparrow(p^*)^* = S_q(p^*).$$

2. Let $p_1, p_2 \in P_{\circ\bullet}^{(m)}$. Then

$$\begin{aligned} S_q(p_1)\otimes S_q(p_2) &= [s_\downarrow(p_1)\cdot p_1\cdot s_\uparrow(p_1)^*]\otimes[s_\downarrow(p_2)\cdot p_2\cdot s_\uparrow(p_2)^*] \\ &= [s_\downarrow(p_1)\otimes s_\downarrow(p_2)]\cdot[p_1\otimes p_2]\cdot[s_\uparrow(p_1)\otimes s_\uparrow(p_2)]^* \\ &= s_\downarrow(p_1\otimes p_2)\cdot[p_1\otimes p_2]\cdot s_\uparrow(p_1\otimes p_2)^* && (6.13.3) \\ &= S_q(p_1\otimes p_2). \end{aligned}$$

3. Let $p_1, p_2 \in P_{\circ\bullet}^{(m)}$ be composable. Then $S_q(p_1)$ and $S_q(p_2)$ are composable since all points are colored white and S_q preserves the number of points. Further,

$$\begin{aligned} S_q(p_1)\cdot S_q(p_2) &= [s_\downarrow(p_1)\cdot p_1\cdot s_\uparrow(p_1)^*]\cdot[s_\downarrow(p_2)\cdot p_2\cdot s_\uparrow(p_2)^*] \\ &= s_\downarrow(p_1p_2)\cdot p_1\cdot s_\uparrow(p_1)^*\cdot s_\uparrow(p_1)\cdot p_2\cdot s_\uparrow(p_1p_2)^* && (6.13.1) \\ &= s_\downarrow(p_1p_2)\cdot p_1\cdot p_2\cdot s_\uparrow(p_1p_2)^* && (6.13.4) \\ &= S_q(p_1\cdot p_2). \end{aligned}$$

□

Since the mapping S_q is a homomorphism, the image $S_q(C)$ is again a pre-category by Proposition 6.7. In the following we show that a pre-category C containing the partition q is completely characterized by the pre-category $S_q(C)$.

Remark 6.15. Let $q_1, q_2 \in P_{\circ\bullet}^{(m)}$ be partitions which can swap colors. In the following we use the notation $s_\uparrow^{q_1}, s_\downarrow^{q_1}$ and $s_\uparrow^{q_2}, s_\downarrow^{q_2}$ to indicate whether $s_\uparrow, s_\downarrow$ depend on q_1 or q_2.

Proposition 6.16. *Let* $C_1 \subseteq P_{\circ\bullet}^{(m_1)}$, $C_2 \subseteq P_{\circ\bullet}^{(m_2)}$ *be pre-categories of spatial partitions and let* $q_1 \in P_{\circ\bullet}^{(m_1)}$, $q_2 \in C_2$ *be spatial partitions which can swap colors. Then any homomorphism* $f\colon S_{q_1}(C_1) \to S_{q_2}(C_2)$ *can be lifted to a homomorphism* $\widetilde{f}\colon C_1 \to C_2$.

Proof. Let $f\colon S_{q_1}(C_1) \to S_{q_2}(C_2)$ be a homomorphism. Then define $\widetilde{f}\colon C_1 \to C_2$ by

$$\widetilde{f}(p) = s_\downarrow^{q_2}(p)^*\cdot f(S_{q_1}(p))\cdot s_\uparrow^{q_2}(p).$$

Here, the mapping $\widetilde{f}$ first removes the colors using S_{q_1}, then applies f and finally restores the colors using $s_\downarrow^{q_2}(p)^*$ and $s_\uparrow^{q_2}(p)$. Note that $q_2 \in C_2$ implies $\widetilde{f}(p) \in C_2$. Further, partitions in $S_{q_2}(C_2)$ have only white points and S_{q_1} and f are homomorphisms. Therefore, the colors and the number of points in the compositions match such that $\widetilde{f}$ is indeed well-defined.

Next we show that $\widetilde{f}$ is a homomorphism. Let $p \in C_1(k,l)$. Then $f(p) \in C_2(k,l)$ because S_{q_1} and f are homomorphisms and $s_{\downarrow}^{q_2}(p)^* \in C_2(l,l)$, $s_{\uparrow}^{q_2}(p) \in C_2(k,k)$. Further, note that $\widetilde{f}$ preserves the color of the points since $s_{\downarrow}^{q_2}(p)^*$ and $s_{\uparrow}^{q_2}(p)$ have the same lower and upper colors as p. It remains to check the category operations. This can be done similarly to Proposition 6.14 by using the relations from Proposition 6.13 and the fact that S_{q_1} and f are homomorphisms.

1. Let $p \in C_1$. Then

$$\begin{aligned}
\widetilde{f}(p)^* &= \left[s_{\downarrow}^{q_2}(p)^* \cdot f(S_{q_1}(p)) \cdot s_{\uparrow}^{q_2}(p)\right]^* && (\text{def. } \widetilde{f})\\
&= s_{\uparrow}^{q_2}(p)^* \cdot f(S_{q_1}(p))^* \cdot s_{\downarrow}^{q_2}(p)\\
&= s_{\uparrow}^{q_2}(p)^* \cdot f(S_{q_1}(p^*)) \cdot s_{\downarrow}^{q_2}(p)\\
&= s_{\downarrow}^{q_2}(p^*)^* \cdot f(S_{q_1}(p^*)) \cdot s_{\uparrow}^{q_2}(p^*) && (6.13.2)\\
&= \widetilde{f}(p^*). && (\text{def. } \widetilde{f})
\end{aligned}$$

2. Let $p_1, p_2 \in C_1$. Then

$$\begin{aligned}
&\widetilde{f}(p_1) \otimes \widetilde{f}(p_2)\\
&= \left[s_{\downarrow}^{q_2}(p_1)^* \cdot f(S_{q_1}(p_1)) \cdot s_{\uparrow}^{q_2}(p_1)\right] \otimes \left[s_{\downarrow}^{q_2}(p_2)^* \cdot f(S_{q_1}(p_2)) \cdot s_{\uparrow}^{q_2}(p_2)\right] && (\text{def. } \widetilde{f})\\
&= \left[s_{\downarrow}^{q_2}(p_1) \otimes s_{\downarrow}^{q_2}(p_2)\right]^* \cdot \left[f(S_{q_1}(p_1)) \otimes f(S_{q_1}(p_2))\right] \cdot \left[\cdot s_{\uparrow}^{q_2}(p_1) \otimes s_{\uparrow}^{q_2}(p_2)\right]\\
&= s_{\downarrow}^{q_2}(p_1 \otimes p_2)^* \cdot \left[f(S_{q_1}(p_1)) \otimes f(S_{q_1}(p_2))\right] \cdot s_{\uparrow}^{q_2}(p_1 \otimes p_2) && (6.13.3)\\
&= s_{\downarrow}^{q_2}(p_1 \otimes p_2)^* \cdot f(S_{q_1}(p_1 \otimes p_2) \cdot s_{\uparrow}^{q_2}(p_1 \otimes p_2)\\
&= \widetilde{f}(p_1 \otimes p_2). && (\text{def. } \widetilde{f})
\end{aligned}$$

3. Let $p_1, p_2 \in C_1$ be composable. Then $\widetilde{f}(p_1)$ and $\widetilde{f}(p_2)$ are composable because $\widetilde{f}$ preserves colors and the number of points. Further

$$\begin{aligned}
&\widetilde{f}(p_1) \cdot \widetilde{f}(p_2)\\
&= \left[s_{\downarrow}^{q_2}(p_1)^* \cdot f(S_{q_1}(p_1)) \cdot s_{\uparrow}^{q_2}(p_1)\right] \cdot \left[s_{\downarrow}^{q_2}(p_2)^* \cdot f(S_{q_1}(p_2)) \cdot s_{\uparrow}^{q_2}(p_2)\right] && (\text{def. } \widetilde{f})\\
&= s_{\downarrow}^{q_2}(p_1p_2)^* \cdot f(S_{q_1}(p_1)) \cdot s_{\uparrow}^{q_2}(p_1) \cdot s_{\uparrow}^{q_2}(p_1)^* \cdot f(S_{q_1}(p_2)) \cdot s_{\uparrow}^{q_2}(p_1p_2) && (6.13.1)\\
&= s_{\downarrow}^{q_2}(p_1p_2)^* \cdot f(S_{q_1}(p_1)) \cdot f(S_{q_1}(p_2)) \cdot s_{\uparrow}^{q_2}(p_1p_2) && (6.13.4)\\
&= s_{\downarrow}^{q_2}(p_1p_2)^* \cdot f(S_{q_1}(p_1p_2)) \cdot s_{\uparrow}^{q_2}(p_1p_2)\\
&= \widetilde{f}(p_1 \cdot p_2). && (\text{def. } \widetilde{f})
\end{aligned}$$

□

Proposition 6.17. *Let $C_1 \subseteq P_{\circ\bullet}^{(m_1)}$, $C_2 \subseteq P_{\circ\bullet}^{(m_2)}$ be pre-categories of spatial partitions and let $q_1 \in C_1$, $q_2 \in C_2$ be spatial partitions which can swap colors. If $S_{q_1}(C_1) \cong S_{q_2}(C_2)$, then $C_1 \cong C_2$.*

Proof. Let $S_{q_1}(C_1)$ and $S_{q_2}(C_2)$ be isomorphic. Then there exist two inverse homomorphisms $f\colon S_{q_1}(C_1) \to S_{q_2}(C_2)$ and $g\colon S_{q_2}(C_2) \to S_{q_1}(C_1)$. By Proposition 6.16, we can lift these homomorphisms to $\widetilde{f}\colon C_1 \to C_2$ and $\widetilde{g}\colon C_2 \to C_1$ defined by

$$\widetilde{f}(p) = s_{\downarrow}^{q_2}(p)^* \cdot f(S_{q_1}(p)) \cdot s_{\uparrow}^{q_2}(p),$$

$$\widetilde{g}(p) = s_{\downarrow}^{q_1}(p)^* \cdot g(S_{q_2}(p)) \cdot s_{\uparrow}^{q_1}(p).$$

Next we show that $\widetilde{f} \circ \widetilde{g} = \mathrm{id}_{C_1}$. Let $p \in C_2$. Using the fact that $\widetilde{g}$ preserves colors, we obtain

$$\begin{aligned}
S_{q_1}(\widetilde{g}(p)) &= s_{\downarrow}^{q_1}(\widetilde{g}(p)) \cdot \widetilde{g}(p) \cdot s_{\uparrow}^{q_1}(\widetilde{g}(p))^* && (\text{def. } S_{q_1}) \\
&= s_{\downarrow}^{q_1}(p) \cdot \widetilde{g}(p) \cdot s_{\uparrow}^{q_1}(p)^* && (6.13.1) \\
&= s_{\downarrow}^{q_1}(p) \cdot s_{\downarrow}^{q_1}(p)^* \cdot g(S_{q_2}(p)) \cdot s_{\uparrow}^{q_1}(p) \cdot s_{\uparrow}^{q_1}(p)^* && (\text{def. } \widetilde{g}) \\
&= g(S_{q_2}(p)). && (6.13.4)
\end{aligned}$$

Denote the previous equation with $(*)$. Then

$$\begin{aligned}
\widetilde{f}(\widetilde{g}(p)) &= s_{\downarrow}^{q_2}(\widetilde{g}(p))^* \cdot f(S_{q_1}(\widetilde{g}(p))) \cdot s_{\uparrow}^{q_2}(\widetilde{g}(p)) && (\text{def. } \widetilde{f}) \\
&= s_{\downarrow}^{q_2}(p)^* \cdot f(S_{q_1}(\widetilde{g}(p))) \cdot s_{\uparrow}^{q_2}(p) && (6.13.1) \\
&= s_{\downarrow}^{q_2}(p)^* \cdot f(g(S_{q_2}(p))) \cdot s_{\uparrow}^{q_2}(p) && (*) \\
&= s_{\downarrow}^{q_2}(p)^* \cdot S_{q_2}(p) \cdot s_{\uparrow}^{q_2}(p) && (f^{-1} = g) \\
&= s_{\downarrow}^{q_2}(p)^* \cdot s_{\downarrow}^{q_2}(p) \cdot p \cdot s_{\uparrow}^{q_2}(p)^* \cdot s_{\uparrow}^{q_2}(p) && (\text{def. } S_{q_2}) \\
&= p. && (6.13.4)
\end{aligned}$$

Hence, $\widetilde{f} \circ \widetilde{g} = \mathrm{id}_{C_1}$. By swapping the roles of C_1 and C_2, one can show in the same way that $\widetilde{g} \circ \widetilde{f} = \mathrm{id}_{C_1}$. Therefore, both homomorphisms are inverse and $C_1 \cong C_2$. □

Remark 6.18. Proposition 6.17 can be useful when characterizing a pre-category C since it reduces the problem to the characterization of the image $S_q(C)$. This image has only white points and might be simpler to describe. In particular, one does not have to check colors when composing partitions in $S_q(C)$. Further, note that the conditions $q_1 \in C_1$ and $q_2 \in C_2$ are necessary and Proposition 6.17 does not hold in general without this requirement.

6.3 Rotating Partitions

In the following we consider only partitions without colors. In the case of classical partitions, it is possible to rotate points from the upper row to the lower row and vice versa using the partitions $\vert$ and $\sqcap$ (see [Web17a]). We now generalize these rotation operations to spatial partitions and to the case where levels get twisted when rotating points.

Definition 6.19 (Rotations)**.**

1. Let $r \in P^{(m)}(0, 2)$. Then r can *rotate points* if

$$\left(r^* \otimes \vert^{(m)}\right) \cdot \left(\vert^{(m)} \otimes r\right) = \vert^{(m)}.$$

2. Let $r \in P^{(m)}(0, 2)$ be a spatial partition which can rotate points. Then define the *rotation* $\mathrm{Rot}_{\downarrow} \colon P^{(m)}(k, l) \to P^{(m)}(k-1, l+1)$ for $k \geq 1$ by

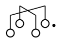
$$\mathrm{Rot}_{\downarrow}(p) = \left[\vert^{(m)} \otimes p\right] \cdot \left[r \otimes \left(\vert^{(m)}\right)^{\otimes(k-1)}\right]$$

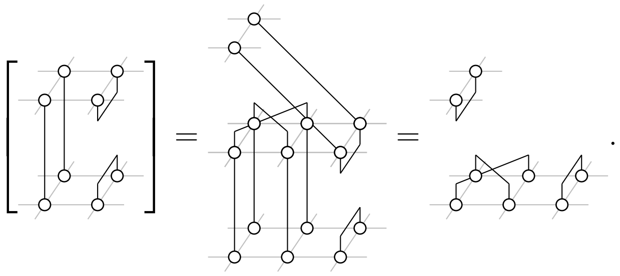
and $\mathrm{Rot}_{\uparrow} \colon P^{(m)}(k, l) \to P^{(m)}(k+1, l-1)$ for $l \geq 1$ by

$$\mathrm{Rot}_{\uparrow}(p) = \left[r^* \otimes \left(\vert^{(m)}\right)^{\otimes(l-1)}\right] \cdot \left[\vert^{(m)} \otimes p\right].$$

3. Let $C \subseteq P^{(m)}$ be a pre-category of spatial partitions. We say C can r-rotate points if it contains the identity $^{(m)}$ and a partition r which can rotate points.

Example 6.20. For example, the following partitions can rotate points:

Remark 6.21. The operation $\mathrm{Rot}_\downarrow$ moves the left-most points from the top row down to the bottom row. Similarly, $\mathrm{Rot}_\uparrow$ moves the left-most points from the bottom row up to the top row. However, $\mathrm{Rot}_\downarrow$ and $\mathrm{Rot}_\uparrow$ might permute the levels of the moved points depending on the partition r. Consider for example $r =$ . Then



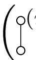
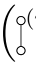
Remark 6.22. Let $r \in P^{(m)}$ be a partition which can rotate points. Since r permutes the levels when rotating points, we can associate a permutation $\sigma \in S_m$ to r. The defining property of r then implies that $\sigma^2 = \mathrm{id}$. This is exactly the case if every cycle in σ has length one or two. Therefore, every level either stays fixed or gets swapped with another level. In particular, the operations $\mathrm{Rot}_\uparrow$ and $\mathrm{Rot}_\downarrow$ are inverse.

Consider again classical partitions. Then it is known that a category is completely determined by its partitions with only lower points. More precisely, let C be a category of partitions and denote with C^{rot} the set of partitions with only lower points. Then one can equip C^{rot} with new operations such that there is a correspondence between categories C and their lower point versions C^{rot}. See [Gro18] for more information on this correspondence. In the following we will generalize this result to pre-categories of spatial partitions which can rotate points.

Definition 6.23 (Operations on lower points)**.** Let $r \in P^{(m)}$ be a spatial partition which can rotate points. Then we define the following operations on partitions with only lower points:

1. For $p \in P^{(m)}(0, n_1)$ and $q \in P^{(m)}(0, n_2)$ denote with $p \otimes q \in P^{(m)}(0, n_1 + n_2)$ the ususal *tensor product* of spatial partitions.

2. For $p \in P^{(m)}(0, n)$ with $n \geq 2$ define the *contraction* $\mathrm{Contr}_k(p) \in P^{(m)}(0, n-2)$ at $1 \leq k < n$ by

   $\mathrm{Contr}_k(p) := \Big[\Big($ $^{(m)}\Big)^{\otimes(k-1)} \otimes r^* \otimes \Big($ $^{(m)}\Big)^{\otimes(n-k-1)}\Big] \cdot p.$

3. For $p \in P^{(m)}(0, n)$ define the *reflection* $\mathrm{Refl}(p) \in P^{(m)}(0, n)$ by

   $$\mathrm{Refl}(p) := \mathrm{Rot}_\downarrow^n(p^*),$$

   where $\mathrm{Rot}_\downarrow^n$ denote the n-fold rotation.

4. For $p \in P^{(m)}(0,n)$ define the *cyclic rotation* $\mathrm{Rot}_{\circlearrowleft}(p) \in P^{(m)}(0,n)$ by

$$\mathrm{Rot}_{\circlearrowleft}(p) := \left[\left(|^{(m)} \right)^{\otimes n} \otimes r^* \right] \cdot \left[|^{(m)} \otimes p \otimes |^{(m)} \right] \cdot r.$$

Remark 6.24. In the following we might omit parentheses when applying $\mathrm{Rot}_{\downarrow}$, $\mathrm{Rot}_{\uparrow}$, Contr_k, Refl, $\mathrm{Rot}_{\circlearrowleft}$ or other functions.

Using the previous operations, we can now define rotated pre-categories and corresponding homomorphisms between them.

Definition 6.25 (Rotated pre-category)**.**

1. Let $r \in P^{(m)}$ be a spatial partition which can rotate points. An *(r-)rotated pre-category* of spatial partitions is a set $C \subseteq P^{(m)}$ containing r such that every $p \in C$ has only lower points and C is closed under tensor products, contractions, reflections and cyclic rotations.

2. If C is a rotated pre-category, then we denote with $C(n) := C \cap P(0,n)$ the set of all spatial partitions with n points.

3. If $p_1, \ldots, p_k \in P^{(m)}$ are spatial partitions with only lower points, then denote with $\mathrm{cl}_r^{\mathrm{rot}} \{p_1, \ldots, p_k\}$ the smallest r-rotated pre-category containing $p_1, \ldots, p_k$.

Remark 6.26. Let C be an r-rotated pre-category and let σ be the permutation corresponding to r (see Remark 6.22). Then the operations on C can be described as follows:

1. Let $p, q \in C$. Then $p \otimes q$ is the usual tensor product of spatial partitions.

2. Let $p \in C(n)$ with $n \geq 2$. Then $\mathrm{Contr}_k(p)$ is obtained by first connecting the points (k, i) and $(k+1, \sigma(i))$ for $1 \leq i \leq m$ and then removing these points.

3. Let $p \in C$. Then $\mathrm{Refl}(p)$ is obtained by reversing the points horizontally and permuting all levels according to σ.

4. Let $p \in C$. Then $\mathrm{Rot}_{\circlearrowleft}(p)$ is obtained by cyclic shifting the points to the right. In particular, $\mathrm{Rot}_{\circlearrowleft}$ is independent of the partition r.

Remark 6.27. In order to show that a set C is an r-rotated pre-category it is sufficient to consider only contractions at $k = 1$, because for $p \in C(n)$ we have

$$\mathrm{Contr}_k\, p = \mathrm{Rot}_{\circlearrowleft}^{(k-1)} \,\mathrm{Contr}_1 \,\mathrm{Rot}_{\circlearrowleft}^{(n-k+1)}\, p.$$

Definition 6.28 (Homomorphism of rotated pre-categories)**.** Let $C_1 \subseteq P^{(m)}$ be an r_1-rotated pre-category and let $C_2 \subseteq P^{(m)}$ be an r_2-rotated pre-category. A mapping $f\colon C_1 \to C_2$ is a *homomorphism of rotated pre-categories* if

1. $f(p) \in C_2(n)$ for all $n \in N_0$ and $p \in C_1(n)$,

2. $f(p_1 \otimes p_2) = f(p_1) \otimes f(p_2)$ for all $p_1, p_2 \in C_1$,

3. $f(\mathrm{Contr}_k\, p) = \mathrm{Contr}_k\, f(p)$ for $p \in C_1(n)$ with $n \geq 2$ and $1 \leq k < n$,

4. $f(\operatorname{Refl} p) = \operatorname{Refl} f(p)$ for all $p \in C_1$,

5. $f(\operatorname{Rot}_{\circlearrowleft} p) = \operatorname{Rot}_{\circlearrowleft} f(p)$ for all $p \in C_1$.

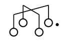
If f is bijective, then f is called isomorphism and the inverse f^{-1} is again a homomorphism. Further, two rotated pre-categories C_1 and C_2 are called isomorphic and we write $C_1 \cong C_2$, if there exists an isomorphism $f\colon C_1 \to C_2$.

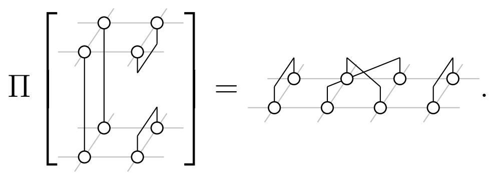
Next we show that each pre-category C which can rotate points gives rise to a rotated pre-category C^{rot}.

Definition 6.29.

1. Let $r \in P^{(m)}$ be a spatial partition which can rotate points. Then define $\Pi\colon P^{(m)} \to P^{(m)}$ by $\Pi(p) = \operatorname{Rot}_{\downarrow}^{k}(p)$ for $p \in P^{(m)}(k,l)$. Here, $\operatorname{Rot}_{\downarrow}^{k}$ denotes the k-fold rotation with respect to r.

2. Let $C \subseteq P^{(m)}$ be a pre-category of spatial partitions which can r-rotate points. Then denote with $C^{\text{rot}} := \Pi(C)$ the image of C under Π with respect to r.

Example 6.30. The mapping Π uses the partition r to rotate all upper points to the bottom and permutes the levels. Consider for example $r =$ . Then



Proposition 6.31. *Let $C \subseteq P^{(m)}$ be a pre-category of spatial partitions which can r-rotate points. Then C^{rot} is an r-rotated pre-category.*

Proof. Let $C \subseteq P^{(m)}$ be a pre-category of spatial partitions which can r-rotate points. Define the set of lower points

$$C^{\text{low}} := \bigcup_{n \in \mathbb{N}_0} C(0,n) \subseteq C.$$

Then $C^{\text{rot}} \subseteq C^{\text{low}}$ by the definition of Π. Additionally, $\Pi(p) = p$ for all $p \in C^{\text{low}}$ such that $C^{\text{low}} \subseteq C^{\text{rot}}$. Hence, C^{rot} contains exactly the partitions in C with only lower points. Because C is a pre-category containing and r, C is closed under tensor products, involutions, cyclic shifts and contractions of partitions on lower points. Further, these operations yield again partitions with only lower points. Therefore, $C^{\text{low}} = C^{\text{rot}}$ is closed under the r-rotated pre-category operations. □

Next we want to show that every rotated pre-category can be written as C^{rot} for some pre-category C. In order to do this, we first show that the operations of pre-categories can be expressed in terms of rotations and the operations of rotated pre-categories.

Proposition 6.32. *Let $r \in P^{(m)}$ be a partition which can rotate points. Further, let $p \in P^{(m)}(k_1,l_1)$ and $q \in P^{(m)}(k_2,l_2)$. Then*

1. $\Pi(p \otimes q) = \operatorname{Rot}_{\circlearrowleft}^{k_2}\left(\Pi p \otimes \operatorname{Rot}_{\circlearrowleft}^{l_2} \Pi q\right)$, *2.* $\Pi\, p^* = \operatorname{Refl} \Pi\, p$,

3. $\Pi(p \cdot q) = \operatorname{Contr}_{k_1+1} \ldots \operatorname{Contr}_{k_1+l_2} (\Pi\, q \otimes \Pi\, p)$ *if p and q are composable.*

Proof. We first consider tensor products and involutions. In order to prove the corresponding statements we only have to keep track of the points because these two operations do not change blocks. Therefore, we can visualize the partitions p and q without blocks as

$$p = \begin{array}{cccc} & p_1^m \cdots p_k^m & & \\ \iddots & & \iddots & \\ p_1^1 \cdots p_k^1 & & & \\ & p_{k+1}^m \cdots p_{k+l}^m & & \\ \iddots & & \iddots & \\ p_{k+1}^1 \cdots p_{k+l}^1 & & & \end{array} \qquad \text{and} \qquad q = \begin{array}{cccc} & q_1^m \cdots q_k^m & & \\ \iddots & & \iddots & \\ q_1^1 \cdots q_k^1 & & & \\ & q_{k+1}^m \cdots q_{k+l}^m & & \\ \iddots & & \iddots & \\ q_{k+1}^1 \cdots q_{k+l}^1 & & & \end{array}.$$

Here k, l either refer to k_1, l_1 or k_2, l_2 depending on the partition p or q. Denote with σ the permutation corresponding to r (see Remark 6.22). Then

$$\Pi\, p = \begin{array}{c} p_k^{\sigma(m)} \cdots p_1^{\sigma(m)}\, p_{k+1}^m \cdots p_{k+l}^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \\ p_k^{\sigma(1)} \cdots p_1^{\sigma(1)}\, p_{k+1}^1 \cdots p_{k+l}^1 \end{array} \qquad \text{and} \qquad \Pi\, q = \begin{array}{c} q_k^{\sigma(m)} \cdots q_1^{\sigma(m)}\, q_{k+1}^m \cdots q_{k+l}^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \\ q_k^{\sigma(1)} \cdots q_1^{\sigma(1)}\, q_{k+1}^1 \cdots q_{k+l}^1 \end{array}.$$

Further, we obtain

$$\begin{aligned} &\Pi \left[\begin{array}{c} p_1^m \cdots p_k^m \; q_1^m \cdots q_k^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \\ p_1^1 \cdots p_k^1 \; q_1^1 \cdots q_k^1 \\ p_{k+1}^m \cdots p_{k+l}^m \; q_{k+1}^m \cdots q_{k+l}^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \\ p_{k+1}^1 \cdots p_{k+l}^1 \; q_{k+1}^1 \cdots q_{k+l}^1 \end{array} \right] \\ &= \begin{array}{c} q_k^{\sigma(m)} \cdots q_1^{\sigma(m)}\, p_k^{\sigma(m)} \cdots p_1^{\sigma(m)}\, p_{k+1}^m \cdots p_{k+l}^m\, q_{k+1}^m \cdots q_{k+l}^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \quad \iddots \quad \iddots \quad \iddots \quad \iddots \\ q_k^{\sigma(1)} \cdots q_1^{\sigma(1)}\, p_k^{\sigma(1)} \cdots p_1^{\sigma(1)}\, p_{k+1}^1 \cdots p_{k+l}^1\, q_{k+1}^1 \cdots q_{k+l}^1 \end{array} \\ &= \operatorname{Rot}_{\circlearrowleft}^{k_2} \left[\begin{array}{c} p_k^{\sigma(m)} \cdots p_1^{\sigma(m)}\, p_{k+1}^m \cdots p_{k+l}^m\, q_{k+1}^m \cdots q_{k+l}^m\, q_k^{\sigma(m)} \cdots q_1^{\sigma(m)} \\ \iddots \quad \iddots \quad \iddots \quad \iddots \quad \iddots \quad \iddots \quad \iddots \quad \iddots \\ p_k^{\sigma(1)} \cdots p_1^{\sigma(1)}\, p_{k+1}^1 \cdots p_{k+l}^1\, q_{k+1}^1 \cdots q_{k+l}^1\, q_k^{\sigma(1)} \cdots q_1^{\sigma(1)} \end{array} \right] \\ &= \operatorname{Rot}_{\circlearrowleft}^{k_2} \left[\begin{array}{c} p_k^{\sigma(m)} \cdots p_1^{\sigma(m)}\, p_{k+1}^m \cdots p_{k+l}^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \\ p_k^{\sigma(1)} \cdots p_1^{\sigma(1)}\, p_{k+1}^1 \cdots p_{k+l}^1 \end{array} \otimes \operatorname{Rot}_{\circlearrowleft}^{l_2} \left[\begin{array}{c} q_k^{\sigma(m)} \cdots q_1^{\sigma(m)}\, q_{k+1}^m \cdots q_{k+l}^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \\ q_k^{\sigma(1)} \cdots q_1^{\sigma(1)}\, q_{k+1}^1 \cdots q_{k+l}^1 \end{array} \right] \right], \end{aligned}$$

which shows $\Pi(p \otimes q) = \operatorname{Rot}_{\circlearrowleft}^{k_2} \big(\Pi\, p \otimes \operatorname{Rot}_{\circlearrowleft}^{l_2} \Pi\, q\big)$. Similarly,

$$\Pi \left[\begin{array}{c} p_{k+1}^m \cdots p_{k+l}^m \\ \iddots \quad \iddots \\ p_{k+1}^1 \cdots p_{k+l}^1 \\ p_1^m \cdots p_k^m \\ \iddots \quad \iddots \\ p_1^1 \cdots p_k^1 \end{array} \right] = \begin{array}{c} p_{k+l}^{\sigma(m)} \cdots p_{k+1}^{\sigma(m)}\, p_1^m \cdots p_k^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \\ p_{k+l}^{\sigma(1)} \cdots p_{k+1}^{\sigma(1)}\, p_1^1 \cdots p_k^1 \end{array} = \operatorname{Refl} \left[\begin{array}{c} p_k^{\sigma(m)} \cdots p_1^{\sigma(m)}\, p_{k+1}^m \cdots p_{k+l}^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \\ p_k^{\sigma(1)} \cdots p_1^{\sigma(1)}\, p_{k+1}^1 \cdots p_{k+l}^1 \end{array} \right]$$

which implies $\Pi\, p^* = \operatorname{Refl} \Pi\, p$. Now assume $k_1 = l_2$ such that p and q are composable. Then

$$\Pi \left[\begin{array}{c} p_1^m \cdots p_k^m \\ \iddots \quad \iddots \\ p_1^1 \cdots p_k^1 \\ p_{k+1}^m \cdots p_{k+l}^m \\ \iddots \quad \iddots \\ p_{k+1}^1 \cdots p_{k+l}^1 \end{array} \cdot \begin{array}{c} q_1^m \cdots q_k^m \\ \iddots \quad \iddots \\ q_1^1 \cdots q_k^1 \\ q_{k+1}^m \cdots q_{k+l}^m \\ \iddots \quad \iddots \\ q_{k+1}^1 \cdots q_{k+l}^1 \end{array} \right] = \Pi \left[\begin{array}{c} q_1^m \cdots q_k^m \\ \iddots \quad \iddots \\ q_1^1 \cdots q_k^1 \\ p_{k+1}^m \cdots p_{k+l}^m \\ \iddots \quad \iddots \\ p_{k+1}^1 \cdots p_{k+l}^1 \end{array} \right] = \begin{array}{c} q_k^{\sigma(m)} \cdots q_1^{\sigma(m)}\, p_{k+1}^m \cdots p_{k+l}^m \\ \iddots \quad \iddots \quad \iddots \quad \iddots \\ q_k^{\sigma(1)} \cdots q_1^{\sigma(1)}\, p_{k+1}^1 \cdots p_{k+l}^1 \end{array},$$

where the points p_i^j and q_{k+i}^j with $1 \leq i \leq l_2$ and $1 \leq i \leq m$ were joined and removed. This is the same as contracting the points q_{k+i}^j and $p_i^{\sigma(j)}$ using r for $1 \leq j \leq m$ and $i = l_2, \ldots, 1$ in

$$\begin{array}{ccccccccccccccc}
 & q_k^{\sigma(m)} & \cdots & q_1^{\sigma(m)} & q_{k+1}^m & \cdots & q_{k+l}^m & p_k^{\sigma(m)} & \cdots & p_1^{\sigma(m)} & p_{k+1}^m & \cdots & p_{k+l}^m \\
 & \iddots & & \iddots & \iddots & & \iddots & \iddots & & \iddots & \iddots & & \iddots \\
q_k^{\sigma(1)} & \cdots & q_1^{\sigma(1)} & q_{k+1}^1 & \cdots & q_{k+l}^1 & p_k^{\sigma(1)} & \cdots & p_1^{\sigma(1)} & p_{k+1}^1 & \cdots & p_{k+l}^1 &
\end{array}$$

Hence, $\Pi(p \cdot q) = \mathrm{Contr}_{k_1+1} \ldots \mathrm{Contr}_{k_1+l_2} (\Pi\, q \otimes \Pi\, p)$. ☐

Proposition 6.33. *Let $D \subseteq P^{(m)}$ be an r-rotated pre-category. Then there exists a pre-category $C \subseteq P^{(m)}$ which can r-rotate points such that $C^{rot} = D$.*

Proof. Define $C := \Pi^{-1}(D)$ as the pre-image of D. Since D is closed under the rotated pre-category operators, C is closed under tensor products, involutions and compositions by Proposition 6.32. Further, $\Pi({\substack{\circ\\|\\\circ}}^{(m)}) = r$ and $\Pi(r) = r$ such that C contains ${\substack{\circ\\|\\\circ}}^{(m)}$ and r. Therefore, C is a pre-category which can r-rotate points and we have $C^{\mathrm{rot}} = \Pi(C) = D$ by definition. ☐

The previous proposition now allows us to prove the following result about pre-categories which are defined by generators.

Proposition 6.34. *Let $C = \mathrm{cl}\,\{p_1, \ldots, p_n\}$ be a pre-category which can r-rotate points. Then $C^{rot} = \mathrm{cl}_r^{rot}\,\{\Pi\, p_1, \ldots, \Pi\, p_n\}$.*

Proof. By Proposition 6.31, C^{rot} is an r-rotated pre-category which contains $\Pi\, p_1$, $\ldots$, $\Pi\, p_n$ by definition. Hence, $\mathrm{cl}_r^{\mathrm{rot}}\,\{\Pi\, p_1, \ldots, \Pi\, p_n\} \subseteq C^{\mathrm{rot}}$. On the other hand, the proof of Proposition 6.34 shows that the pre-image $D := \Pi^{-1}(\mathrm{cl}_r^{\mathrm{rot}}\,\{\Pi\, p_1, \ldots, \Pi\, p_n\})$ is a pre-category which can r-rotate points and contains $p_1, \ldots, p_n$. Therefore, $C = \mathrm{cl}\,\{p_1, \ldots, p_n\} \subseteq D$, which implies

$$C^{\mathrm{rot}} = \Pi(C) \subseteq \Pi(D) = \mathrm{cl}_r^{\mathrm{rot}}\,\{\Pi\, p_1, \ldots, \Pi\, p_n\}\,.$$

☐

Finally we show that homomorphisms and isomorphisms of rotated pre-categories can be lifted to pre-categories.

Proposition 6.35. *Let $C_1 \subseteq P^{(m_1)}$ and $C_2 \subseteq P^{(m_2)}$ be pre-categories of spatial partitions which can r_1-rotate / r_2-rotate points. Then any homomorphism $f\colon C_1^{rot} \to C_2^{rot}$ of rotated pre-categories can be lifted to a homomorphism $\widetilde{f}\colon C_1 \to C_2$ of pre-categories.*

Proof. Define $\widetilde{f}\colon C_1 \to C_2$ by $\widetilde{f}(p) = \mathrm{Rot}_\uparrow^k f(\Pi\, p)$ for $p \in C_1(k,l)$. Then $\widetilde{f}(p) \in C_2(k,l)$ for $p \in C_1(k,l)$ since f is a homomorphism of rotated pre-categories and preserves the number of lower points. It remains to check the pre-category operations. Note that for all $p \in C_1$ with at least n points holds that

$$\Pi\, \widetilde{f}(p) = \Pi\, \mathrm{Rot}_\uparrow^n f(\Pi\, p) = f(\Pi\, p).$$

Denote the previous equation with $(*)$ and let $p \in P^{(m)}(k_1, l_1)$ and $q \in P^{(m)}(k_2, l_2)$. Then

$$\Pi\, \widetilde{f}(p \otimes q) = f\, \Pi(p \otimes q) \qquad (*)$$

$$
\begin{aligned}
&= f\,\mathrm{Rot}_{\circlearrowleft}^{k_2}\left(\Pi\, p \otimes \mathrm{Rot}_{\circlearrowleft}^{l_2}\,\Pi\, q\right) && (6.32)\\
&= \mathrm{Rot}_{\circlearrowleft}^{k_2}\left(f\,\Pi\, p \otimes \mathrm{Rot}_{\circlearrowleft}^{l_2}\, f\,\Pi\, q\right)\\
&= \mathrm{Rot}_{\circlearrowleft}^{k_2}\left(\Pi\, \widetilde{f}p \otimes \mathrm{Rot}_{\circlearrowleft}^{l_2}\,\Pi\, \widetilde{f}q\right) && (*)\\
&= \Pi(\widetilde{f}p \otimes \widetilde{f}q). && (6.32)
\end{aligned}
$$

Since $\widetilde{f}(p\otimes q)$ and $\widetilde{f}(p)\otimes\widetilde{f}(q)$ have the same number of upper and lower points, we obtain $\widetilde{f}(p\otimes q) = \widetilde{f}(p)\otimes\widetilde{f}(q)$. The same computation using $(*)$ and Proposition 6.32 and the same argument by counting the points can also be applied to the involution and the composition. Thus, $\widetilde{f}(p^*) = \widetilde{f}(p)^*$ and $\widetilde{f}(p\cdot q) = \widetilde{f}(p)\cdot\widetilde{f}(q)$ if p and q are composable. Therefore, $\widetilde{f}$ is a homomorphism of pre-categories. ☐

Proposition 6.36. *Let $C_1 \subseteq P^{(m_1)}$ and $C_2 \subseteq P^{(m_2)}$ be pre-categories of spatial partitions which can r_1-rotate / r_2-rotate points. If $C_1^{rot} \cong C_2^{rot}$ as rotated pre-categories then $C_1 \cong C_2$ as pre-categories.*

Proof. Assume there exists an isomorphism $f\colon C_1^{\mathrm{rot}} \to C_2^{\mathrm{rot}}$. Using Proposition 6.35, we obtain a homomorphism $\widetilde{f}\colon C_1 \to C_2$ given by $\widetilde{f}(p) = \mathrm{Rot}_{\uparrow}^{k} f(\Pi\, p)$ for $p \in C_1(k,l)$. Similarly, we obtain from f^{-1} a homomorphism $\widetilde{g}\colon C_2 \to C_1$ defined by $\widetilde{g}(p) = \mathrm{Rot}_{\uparrow}^{k} f^{-1}(\Pi\, p)$ for $p \in C_2(k,l)$. In the following we show that $\widetilde{f}$ and $\widetilde{g}$ are inverse such that $C_1 \cong C_2$. Let $k,l \in \mathbb{N}_0$ and $p \in C_1(k,l)$. Then

$$
\begin{aligned}
\widetilde{g}(\widetilde{f}(p)) &= \widetilde{g}(\mathrm{Rot}_{\uparrow}^{k} f(\Pi\, p))\\
&= \mathrm{Rot}_{\uparrow}^{k} f^{-1}\,\Pi\,\mathrm{Rot}_{\uparrow}^{k} f(\Pi\, p)\\
&= \mathrm{Rot}_{\uparrow}^{k} f^{-1} f(\Pi\, p)\\
&= \mathrm{Rot}_{\uparrow}^{k}\,\Pi\, p.
\end{aligned}
$$

Since p has exactly k upper points, we have $\mathrm{Rot}_{\uparrow}^{k}\,\Pi\, p = p$. This implies $\widetilde{g}(\widetilde{f}(p)) = p$. The same argument also shows that $\widetilde{f}(\widetilde{g}(p))$ for all $p \in C_2(k,l)$. ☐

7 Applications to Finite Quantum Spaces

In the following we consider a category C_B which was defined by Cébron-Weber in [CW22] and which is related to quantum automorphism groups of finite quantum spaces. Our main result is the construction of an isomorphism $C_B \cong NC_{\circ\bullet}$ in Theorem 7.19, which can be found in Section 7.3. Here, $NC_{\circ\bullet}$ denotes the category of colored non-crossing partitions. We begin in Section 7.1 with some background regarding finite quantum spaces and the definition of the category C_B. Then we introduce the rotated pre-category of non-crossing cycles in Section 7.2 and show that it is isomorphic to the pre-category of rotated non-crossing partitions. In Section 7.3, we use the results from Section 6.2 and Section 6.3 to lift this isomorphism to a homomorphism $\phi\colon NC_{\circ\bullet} \to P_{2,\circ\bullet}^{(2)}$. Finally, we show that $C_B \cong \Gamma_\phi \cong NC_{\circ\bullet}$ as categories of colored partitions.

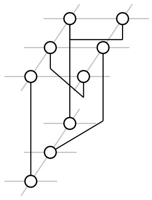
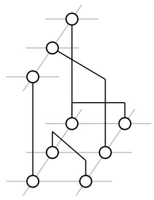
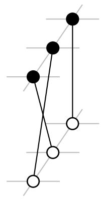
7.1 Quantum Symmetries of Finite Quantum Spaces

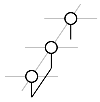
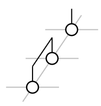
Consider the finite space $X_n = \{1, \dots, n\}$ and the C^*-algebra $C(X_n) \cong \mathbb{C}^n$ of continuous functions on X_n. In [Wan98], Wang defined the quantum automorphism group of X_n as the maximal compact quantum group acting on $C(X_n)$ and computed that it corresponds to the quantum permutation group S_n^+.
More generally, consider a finite quantum space (B, ψ), where B is a finite-dimensional C^*-algebra and ψ is a positive linear functional on B. Here B can be seen as an algebra of continuous functions on a "non-commutative space" and ψ corresponds to a measure on this space. In this setting, Wang similarly defined a quantum automorphism group of (B, ψ), which we will denote with $G^+(B, \psi)$ in the following. If ψ satisfies further properties, then $G^+(B, \psi)$ can be described as a universal C^*-algebra with generators and relations (see [Mro13]). In [CW22], Cébron-Weber found examples where these relations come from spatial partitions.

Definition 7.1 (Cébron-Weber)**.** Let $n, N \in \mathbb{N}$ and consider the finite quantum space

$$B = \bigoplus_{i=1}^{n} M_N(\mathbb{C}), \qquad \psi(x_1 \oplus \cdots \oplus x_n) = \frac{1}{nN} \sum_{i=1}^{n} \mathrm{Tr}(x_i).$$

Then $G^+(B, \psi)$ agrees with the spatial partition quantum group associated to the category of colored spatial partitions $C_B = \langle p_{1a}, p_{1b}, p_2, p_{3a}, p_{3b} \rangle$, where

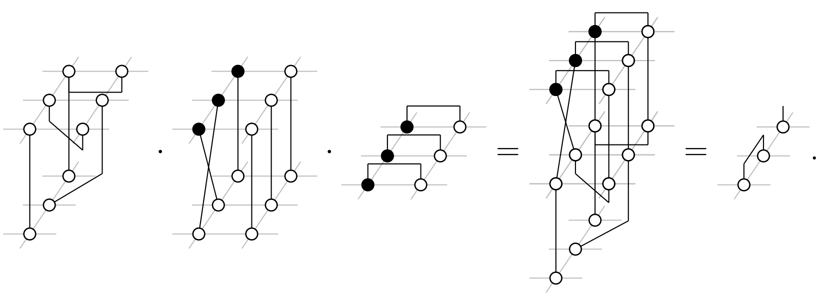
$p_{1a} =$  , $p_{1b} =$  , $p_2 =$  ,

$p_{3a} =$  , $p_{3b} =$  .

We refer to [CW22] for more details and the exact relations corresponding to the partitions in C_B. Our goal is now to construct an isomorphism $C_B \cong NC_{\circ\bullet}$ and to prove Theorem 7.19 in Section 7.3. But first, we simplify the generators of C_B.

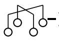
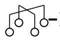
Proposition 7.2. *$C_B = \langle p_{1b}, p_2 \rangle$ as category of colored spatial partitions.*

Proof. We have $p_{1a} = p_{1b}^*$, $p_{3a} = p_{3b}^*$ and $p_{3b} = p_{1a} \cdot (p_2 \otimes$ $^{(3)}) \cdot$ $^{(3)}$, where the latter can be visualized as

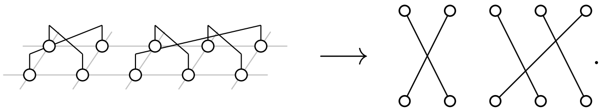


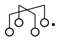
□

7.2 Non-Crossing Cycles

In order to prove Theorem 7.19 in Section 7.3, we need the isomorphism $NCC \cong NC^{\text{rot}}$ between non-crossing cycles NCC and rotated non-crossing partitions NC^{rot}. In the following we begin with the definition of the set $NCC \subseteq P_2^{(2)}$ of non-crossing cycles and show that NCC is a -rotated pre-category. But before we come to the definition of NCC, we first give some remarks regarding -rotated partitions.

Remark 7.3. In the following we will only consider rotated partitions on two levels. Therefore, we will visualize such partitions from the top as 2D partitions. For example



Remark 7.4. Consider the partition $r =$ . Then r swaps both levels when rotating points. Recall from Remark 6.26 that the r-rotated category operations can be described as follows:

1. Let $p \in P_2^{(2)}(0, n_1)$ and $q \in P_2^{(2)}(0, n_2)$. Then $p \otimes q$ is the usual tensor product of partitions.
2. Let $p \in P_2^{(2)}(0, n)$ with $n \geq 2$. Then $\text{Contr}_k(p)$ is obtained by connecting the point $(k, 1)$ with $(k+1, 2)$ and the point $(k, 2)$ with $(k+1, 1)$. Then these points are removed and their blocks are merged.
3. Let $p \in P_2^{(2)}(0, n)$. Then $\text{Refl}(p)$ is obtained by reversing the points horizontally and then swapping both levels.
4. Let $p \in P_2^{(2)}(0, n)$. Then $\text{Rot}_{\circlearrowright}(p)$ is obtained by cyclic shifting the points to the right.

Next we introduce cycles and the set NCC of non-crossing cycles.

Definition 7.5 (Cycle)**.** Let $p \in P_2^{(2)}(0, l)$ and $1 \leq i_1 < \cdots < i_n \leq l$. We say $i_1, \ldots, i_n$ form a *cycle* in p if

1. $(i_k, 2)$ and $(i_{k+1}, 1)$ are a block for $1 \leq k < n$,
2. $(i_1, 1)$ and $(i_n, 2)$ are a block.

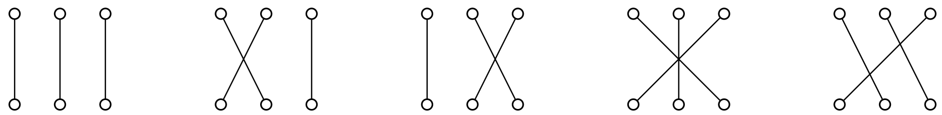
Further, denote with n the length of the cycle.

Definition 7.6 (Non-crossing cycles)**.**

1. Let $i_1, \ldots, i_n$ and $j_1, \ldots, j_m$ be two cycles in the same partition. We say the cycles *cross* if there exist indices k_1, k_2, k_3, k_4 such that $i_{k_1} < j_{k_2} < i_{k_3} < j_{k_4}$ or $j_{k_1} < i_{k_2} < j_{k_3} < i_{k_4}$.
2. Let $p \in P_2^{(2)}(0, l)$ be a partition. We say p is a union of cycles if we can partition the points $\{1, \ldots, l\}$ into subsets such that the points in each subset form a cycle in p.
3. Define the set of non-crossing cycles

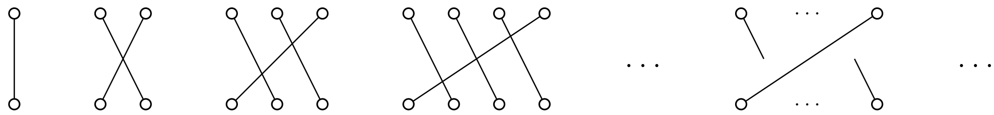
$$NCC := \{p \in P_2^{(2)} \mid p \text{ is the union of pairwise non-crossing cycles}\}.$$

Example 7.7. For example, the set $NCC(3)$ of non-crossing cycles on 3 lower and 3 upper points is given by the following partitions:



Next we show that the set NCC is a -rotated pre-category. This is done by first considering the class of simple cycles and showing that it is closed under reflections, cyclic shifts and contractions.

Definition 7.8 (Simple cycles)**.** Since the definition of a cycle completely describes its block structure, there exists exactly one cycle of length n on n points. These cycles are called *simple cycles* and are given by the following partitions:

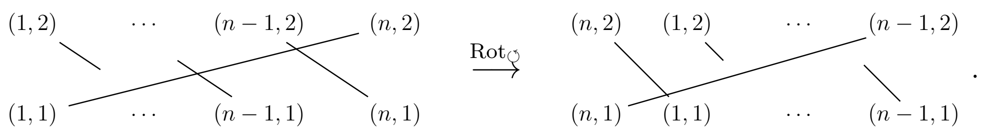


Note that the empty partition is also a simple cycle.

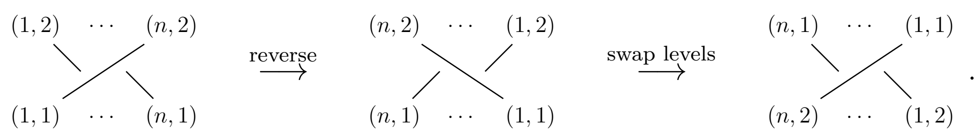
Proposition 7.9. *The set of simple cycles is closed under reflections, cyclic shifts and contractions with respect to* .

Proof.

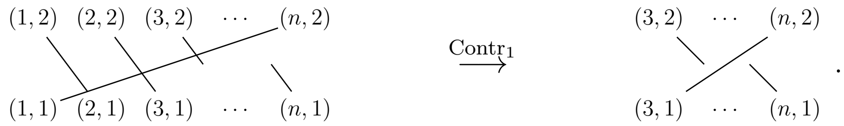
1. Cyclic shifts: One directly checks that $\mathrm{Rot}_{\circlearrowleft}(\,)=$ and $\mathrm{Rot}_{\circlearrowleft}(\,)=$. For $n \geq 3$, we obtain



2. Reflections: The base case is given by $\mathrm{Refl}(\,)=$. For $n \geq 2$, we have



3. Contractions: Since simple cycles are closed under cyclic shifts, it is sufficient to consider only Contr_1 (see Remark 6.27). One checks that $\mathrm{Contr}_1(\,)$ is the empty partition and $\mathrm{Contr}_1(\,)=$. In the case $n \geq 4$, we contract the points $(1,1)$ and $(2,2)$ such that $(3,1)$ and $(n,2)$ are connected. Further, the points $(1,2)$ and $(2,1)$ are contracted such that we obtain



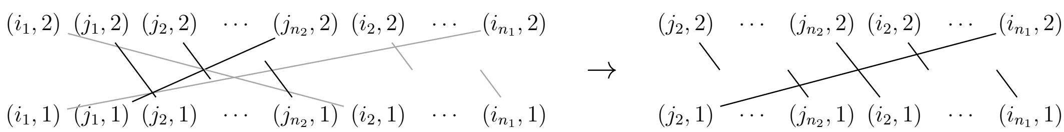
□

Proposition 7.10. *The set NCC is a -rotated pre-category of spatial partitions.*

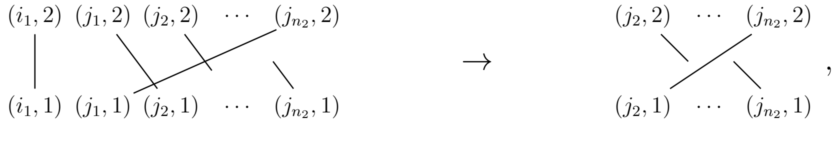
Proof. We have to show that NCC is closed under tensor products, cyclic shifts, reflections and contractions. Let $p, q \in NCC$. Then $p \otimes q \in NCC$ since the tensor product causes no crossings and $p \otimes q$ consists of the union of the cycles in p and q. Next we consider cyclic shifts and reflections. These operations reorder the points but do not change blocks and do not cause crossings. Therefore, we have to show that each individual cycle stays a cycle. But this follows directly from Proposition 7.9, where it was shown for simple cycles.
It remains to show that NCC is closed under contractions. By Remark 6.27, it is sufficient to consider the case $k = 1$. Let $p \in NCC(n)$ with $n \geq 2$ and observe that Contr_1 only affects the cycles containing the first two lower and upper points. We distinguish two cases:

1. The first points belong to the same cycle. Then Proposition 7.9 implies that after contracting these points the remaining points still form a cycle. Further, no crossings are produced when removing the first points such that $\mathrm{Contr}_1(p)$ is still a union of non-crossing cycles.

2. The first points belong to different cycles of length n_1 and n_2 with points $i_1, \dots, i_{n_1}$ and $j_1, \dots, j_{n_2}$. In the general case, we have
$$i_1 < j_1 < \cdots < j_{n_2} < i_2 < \cdots < i_{n_1}$$
since the two cycles do not cross. Then $(j_1, 1)$ and $(i_1, 2)$ are contracted such that $(j_{n_2}, 2)$ and $(i_2, 1)$ will be connected. Similarly, $(i_1, 1)$ and $(j_1, 2)$ are contracted such that $(j_2, 1)$ and $(i_{n_1}, 2)$ will be connected. Hence, both cycles get merged into a larger cycle. In the following we visualize this contraction in the case $n_1, n_2 \geq 3$. Note that the gray lines correspond to the first cycle and are colored differently for a better visualization.



It remains to check the edge cases. In the case $n_1 = n_2 = 1$ both cycles get removed and in the cases $n_1 = 1, n_2 > 2$ and $n_1 > 2, n_2 = 1$, we can visualize the contraction as



Note that this works similarly in the cases $n_1 = 1, n_2 = 2$ and $n_1 = 2, n_2 = 1$. Hence, in each case $\mathrm{Contr}_1(p)$ is again a union of non-crossing cycles.

□

In the following we introduce non-crossing partitions and show that rotated non-crossing partitions NC^{rot} are isomorphic to non-crossing cycles NCC.

Definition 7.11. Define the pre-categories of non-crossing partitions

$$NC = \{p \in P \mid \text{no blocks cross when drawing } p\},$$
$$NC_{\circ\bullet} = \{p \in P_{\circ\bullet} \mid \text{no blocks cross when drawing } p\}.$$

One checks that NC and $NC_{\circ\bullet}$ are indeed pre-categories of partitions, see also Example 2.9. Further, denote with NC^{rot} the $\sqcap_{\circ\circ}$-rotated pre-category obtained from NC.

Proposition 7.12. *The pre-categories of non-crossing partitions can be written in terms of generators as*

$$NC_{\circ\bullet} = \mathrm{cl}\left\{|^{\circ}_{\circ}, |^{\bullet}_{\bullet}, \sqcap_{\bullet\circ}, \sqcap_{\circ\bullet}, |^{\bullet}_{\circ}, \vdash^{\circ}_{\circ\circ}\right\}, \quad NC = \mathrm{cl}\left\{|^{\circ}_{\circ}, \sqcap_{\circ\circ}, \vdash^{\circ}_{\circ\circ}\right\}, \quad NC^{rot} = \mathrm{cl}^{rot}_{\sqcap_{\circ\circ}}\{\sqcap_{\circ\circ}, \sqcap\!\sqcap_{\circ\circ\circ}\}.$$

In particular, we have $NC = S_{|^{\bullet}_{\circ}}(NC_{\circ\bullet})$ in the notation of Section 6.2.

Proof. Define the pre-categories from the statement

$$C_1 = \mathrm{cl}\left\{|^{\circ}_{\circ}, |^{\bullet}_{\bullet}, \sqcap_{\bullet\circ}, \sqcap_{\circ\bullet}, |^{\bullet}_{\circ}, \vdash^{\circ}_{\circ\circ}\right\}, \qquad C_2 = \mathrm{cl}\left\{|^{\circ}_{\circ}, \sqcap_{\circ\circ}, \vdash^{\circ}_{\circ\circ}\right\}.$$

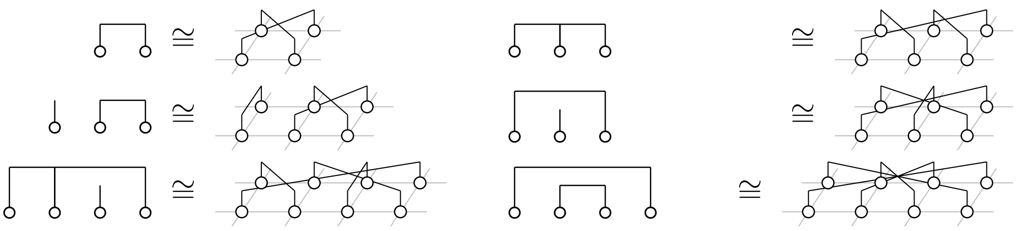
According to [Web13], the set NC is a category of partitions and is given by $\langle \downarrow_{\circ}, \sqcap\!\sqcap\!\sqcap_{\circ\circ\circ\circ}\rangle = \mathrm{cl}\left\{|^{\circ}_{\circ}, \sqcap_{\circ\circ}, \downarrow_{\circ}, \sqcap\!\sqcap\!\sqcap_{\circ\circ\circ\circ}\right\}$. Since $\downarrow_{\circ} = \sqcup^{\circ\circ}_{\circ} \cdot \sqcap_{\circ\circ}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\circ\circ\circ} = (\vdash^{\circ}_{\circ\circ} \otimes \vdash^{\circ}_{\circ\circ}) \cdot \sqcap_{\circ\circ}$, we have $\langle \downarrow_{\circ}, \sqcap\!\sqcap\!\sqcap_{\circ\circ\circ\circ}\rangle \subseteq C_2$. Conversely, $\vdash^{\circ}_{\circ\circ} = (\sqcup^{\circ\circ} \otimes \uparrow^{\circ} \otimes |^{\circ}_{\circ} \otimes |^{\circ}) \cdot (|^{\circ} \otimes \sqcap\!\sqcap\!\sqcap_{\circ\circ\circ\circ})$ such that $C_2 \subseteq \langle \downarrow_{\circ}, \sqcap\!\sqcap\!\sqcap_{\circ\circ\circ\circ}\rangle$. Thus, $NC = C_2$.
Now consider $NC_{\circ\bullet}$. Since all generators of C_1 are non-crossing partitions and include the colored base partitions, we have $C_1 \subseteq NC_{\circ\bullet}$. Further, we can use $|^{\bullet}_{\circ}$ in combination with $|^{\circ}_{\circ}$ and $|^{\bullet}_{\bullet}$ to arbitrarily color partitions. This implies $\sqcap_{\circ\circ} \in C_1$ such that $C_2 \subseteq C_1$. Since C_2 contains all white non-crossing partitions, we can arbitrarily color them and we obtain $NC_{\circ\bullet} \subseteq C_1$. Hence, $NC_{\circ\bullet} = C_1$.
Further, $|^{\bullet}_{\circ}$ can swap colors and by Proposition 6.7 we can apply $S_{|^{\bullet}_{\circ}}$ to the generators of $NC_{\circ\bullet}$ to obtain $S_{|^{\bullet}_{\circ}}(NC_{\circ\bullet}) = \mathrm{cl}\left\{|^{\circ}_{\circ}, \sqcap_{\circ\circ}, \vdash^{\circ}_{\circ\circ}\right\} = NC$. Similarly, Proposition 6.34 implies $NC^{\mathrm{rot}} = \mathrm{cl}^{\mathrm{rot}}_{\sqcap_{\circ\circ}}\{\sqcap_{\circ\circ}, \sqcap\!\sqcap_{\circ\circ\circ}\}$, where $\sqcap\!\sqcap_{\circ\circ\circ}$ is obtained by rotating $\vdash^{\circ}_{\circ\circ}$ using $\sqcap_{\circ\circ}$. □

Proposition 7.13. *The $\sqcap_{\circ\circ}$-rotated pre-category NC^{rot} is isomorphic to the $\sqcap\!\!\times\!\!\sqcap_{\circ\circ\circ\circ}$-rotated pre-category NCC.*

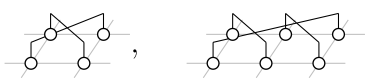
Proof. Define the mapping $f\colon NC^{\mathrm{rot}} \to NCC$ where $f(p)$ has the same number of points as p and $i_1, \dots, i_n$ form a cycle in $f(p)$ whenever the points $i_1, \dots, i_n$ form a block in p. Then f is well-defined because partitions in NC^{rot} are disjoint unions of non-crossing blocks, which get mapped to disjoint unions of non-crossing cycles. Similarly, one can map non-crossing cycles back to non-crossing blocks such that f has an inverse and is bijective.

It remains to check the category operations. Tensor products, reflections and cyclic shifts concatenate or reorder blocks and cycles in the same way such that f preserves these operators. Further, contracting non-crossing blocks with ⊓ just removes two points and merges neighbouring blocks. This is the same as contracting non-crossing cycles with ⊓ as described in the proof of Proposition 7.10. Hence, f respects contractions and is an isomorphism. □

Example 7.14. In the following we list some non-crossing partitions and their corresponding non-crossing cycles under the isomorphism from Proposition 7.13:



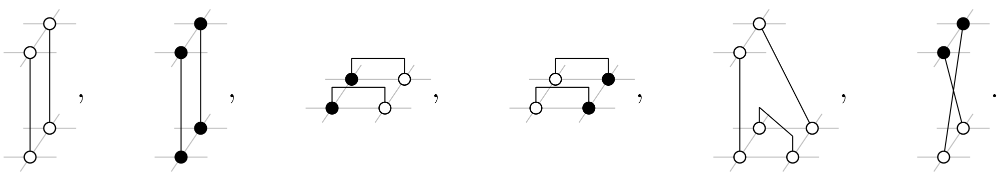
Remark 7.15. We can apply the isomorphism from Proposition 7.13 to $NC^{\mathrm{rot}} = \mathrm{cl}^{\mathrm{rot}}\{⊓, ⊓⊓\}$ from Proposition 7.12 to obtain that NCC is generated by



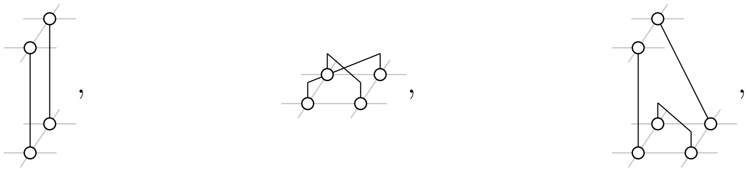
as ⊓-rotated pre-category.

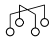
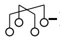
7.3 Constructing the Isomorphism

In the following we use the isomorphism $NCC \cong NC^{\mathrm{rot}}$ from the previous section to construct a homomorphism $\phi\colon NC_{\circ\bullet} \to P^{(2)}_{2,\circ\bullet}$. This homomorphism will then be used to prove the isomorphism $C_B \cong NC_{\circ\bullet}$ in Theorem 7.19. Now consider the pre-category $D_{\circ\bullet}$ which is generated by

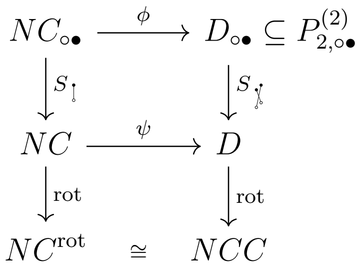


Note that $D_{\circ\bullet}$ is given by the first two levels of the category C_B from Definition 7.1. Since $q =$ ⟋ can swap colors, we can use the mapping S_q from Section 6.2 to obtain the pre-category $D := S_q(D_{\circ\bullet})$. It is generated by the spatial partitions



which are obtained by applying S_q to the generators of $D_{\circ\bullet}$. Similarly,  can rotate points such that D^{rot} is a -rotated pre-category in the sense of Section 6.3. According to Proposition 6.34, it is generated by

However, we have $D^{rot} = NCC \cong NC^{rot}$ by Remark 7.15 and Proposition 7.13. Using Proposition 6.36, we can now lift this isomorphism to an isomorphism $\psi\colon NC \to D$. Because $NC = S_q(NC_{\circ\bullet})$ with $q =$, we can apply Proposition 6.17 and lift this isomorphism again to an isomorphism $\phi\colon NC_{\circ\bullet} \to D_{\circ\bullet}$. Since $D_{\circ\bullet} \subseteq P^{(2)}_{2,\circ\bullet}$, we can view ϕ as an injective homomorphism $\phi\colon NC_{\circ\bullet} \to P^{(2)}_{2,\circ\bullet}$ with image $D_{\circ\bullet}$. This construction can be visualized in the following diagram:



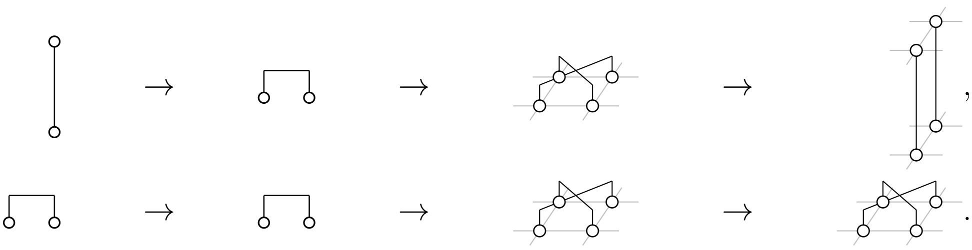
Remark 7.16. If one combines the constructions of the isomorphisms in Proposition 6.17, Proposition 6.36 and Proposition 7.13, one obtains that ϕ can be computed as follows. First consider some $p \in NC_{\circ\bullet}(k,l)$ with only white points. Then lifting the isomorphism from Proposition 7.13 as described in Proposition 6.36 yields:

1. The k upper points are rotated to the bottom.
2. The blocks are replaced with non-crossing cycles.
3. The k left-most points are rotated again to the top while swapping their levels.

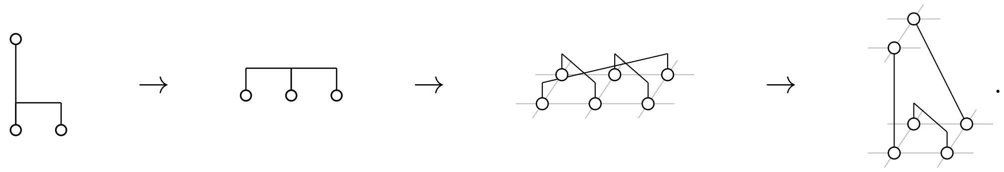
Now consider some arbitrary $p \in NC_{\circ\bullet}$. Then lifting the previous isomorphism as described in Proposition 6.17 yields:

1. All points are colored white.
2. ϕ is applied as described previously.
3. The colors are restored while swapping the levels of black points.

Example 7.17. Using the previous description of ϕ, one computes in the colorless case



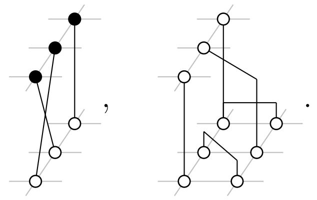
This implies for the colored case that $\phi(\cdot) = \cdot$, $\phi(\sqcap) = \sqcap^{(2)}$ $\phi(\sqcap) = \sqcap^{(2)}$ and $\phi(\cdot) = \cdot^{(2)}$. Similarly, one computes



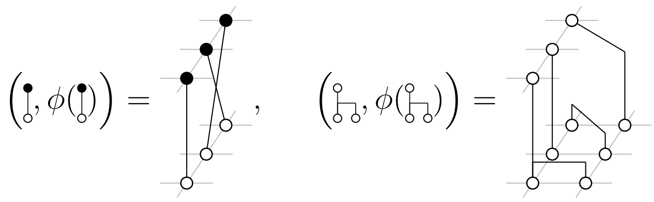
Using the explicit description of ϕ, we can prove the following lemma which directly implies the main theorem of this section.

Lemma 7.18. *The category C_B is isomorphic to the graph Γ_ϕ as categories of colored partitions.*

Proof. Recall from Proposition 7.2 that C_B is the category of colored spatial partitions generated by



In Example 7.17, we showed that ϕ maps colored base partitions to colored base partitions and we computed the image of ⟟ and ⊢. According to Proposition 7.12, we have $NC_{\circ\bullet} = \langle \cdot, \cdot \rangle$ as category of colored partitions. This implies that Γ_ϕ is generated by



as category of colored spatial partitions (see Proposition 6.9). Hence, Γ_ϕ is isomorphic to C_B by swapping level 2 and level 3 with level 1. Since swapping the levels respects the colored base partitions, the isomorphism $\Gamma_\phi \cong C_B$ is an isomorphism of categories of colored partitions. ☐

Theorem 7.19. *The category C_B is isomorphic to the category of colored non-crossing partitions $NC_{\circ\bullet}$ as categories of colored spatial partitions.*

Proof. Apply Proposition 6.9 to Γ_ϕ in Lemma 7.18 and note that the isomorphism respects the colored base partitions. ☐

Remark 7.20. In Example 7.17, we computed that ϕ maps colored base partitions to colored base partitions. Therefore, ϕ embeds $NC_{\circ\bullet}$ into $P^{(2)}_{2,\circ\bullet}$ as categories of colored spatial partitions and for any subcategory $C \subseteq NC_{\circ\bullet}$ the image $\phi(C)$ is an isomorphic subcategory of $P^{(2)}_{2,\circ\bullet}$. However, this applies not to the category NC since the colorless base partition $\sqcap$ gets twisted by ϕ. It remained open if it is possible to adjust the construction of ϕ such that the colorless base partitions are preserved and NC embeds into $P^{(2)}_2$ as categories of spatial partitions.

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