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Master thesis

Quantum Graphs and Bigalois Extensions

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Contents

Introduction

In this thesis we will study compact quantum groups, quantum graphs, quantum automorphism groups and quantum isomorphisms. These are generalisations of compact groups, graphs and their automorphisms and isomorphisms.
Compact quantum groups can be seen as a non-commutative analog of compact groups. For a compact group G, we can look at the unital C^*-algebra of continuous functions $C(G)$. Using the multiplication of G we can now define a map $\Delta : C(G) \to C(G) \otimes C(G)$, called comultiplication, which fulfils certain properties. A compact quantum group is now by definition a unital C^*-algebra A together with such a comultiplication. Using that a unital C^*-algebra is commutative if and only if it is isomorphic to $C(X)$ for some compact space X, we can show that the C^*-algebra A is commutative if and only if it is isomorphic to $C(G)$ for some compact group G. This illustrates the term "non-commutative analog".
Similarly, quantum graphs are non-commutative analogs of graphs. To every graph with n vertices we can associate the set $(\mathbb{C}^n, \psi_n, A)$, where $\psi_n(x) = \frac{1}{n}\sum_{i=1}^n x_i$ and A is the adjacency matrix. A quantum graph is now a finite-dimensional, not necessarily commutative C^*-algebra $\mathcal{O}(X)$ together with a faithful state ψ_X and a linear map $A_X : \mathcal{O}(X) \to \mathcal{O}(X)$, which has to fulfil certain properties. Here we can also show that the C^*-algebra is commutative if and only if we are in the classical situation, i.e. the set is $(\mathbb{C}^n, \psi_n, A)$. Hence the class of classical graphs is just the set of quantum graphs with commutative C^*-algebra.
Furthermore, we will consider quantum automorphism groups of quantum sets and quantum graphs. For a classical graph this is the quantum analog of the space of graph automorphisms. We will also look at the linking algebra of two quantum graphs. This is an object with which we can define a quantum isomorphism notion for two quantum graphs. The main theorem of this thesis then states that if we have a quantum graph whose quantum automorphism group is "equivalent" (we will explain this notion of equivalence in more detail) to another compact quantum group, then there exists another quantum graph with this compact quantum group

as quantum automorphism group which is quantum isomorphic to our first quantum graph. This is a result from [BCE+20] (Theorem 4.11).

The main source of this thesis is [BCE+20]. Its structure is as follows. In the first chapter we review some basics of the theory of C^*-algebras (based on [WLV21]) such as the Gelfand-Naimark-Theorem and the construction of universal C^*-algebras. There we also introduce the notion of compact quantum groups and Hopf algebras. In the second chapter we define quantum graphs, look at some examples and clarify where the definition comes from. We then introduce quantum automorphism groups of classical graphs, quantum sets and quantum graphs and take a closer look at some examples of them. In Chapter 3 we first define the notion of a bigalois extension and then introduce quantum isomorphisms and the linking algebra of two quantum graphs. We also clarify the connection between bigalois extensions and quantum isomorphisms of quantum graphs. In the fourth chapter we discuss the representation theory of compact quantum groups (following [NT13]), define the notion of monoidal equivalence and look at some examples of monoidally equivalent compact quantum groups. In Chapter 5, we define the linking algebra also for compact quantum groups (based on [BEHY22] and [BRV05]) and prove the main theorem of this thesis (Theorem 5.4). At the end, we take a closer look at the linking algebra of two explicit compact quantum groups.

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Chapter 1

Preliminaries

1.1 Notations

First of all, we want to fix some notations. We denote the identity map with id and the characteristic function with χ. The symbol $\otimes$ denotes the tensor product of Hilbert spaces, the algebraic tensor product of (*-)algebras and the minimal tensor product of C^*-algebras. Moreover, we identify $1 \otimes x$ with x where $1 \in \mathbb{C}$. If X, Y, Z are vector spaces and $T : X \otimes Y \to X \otimes Y$ is a linear map, then e.g. $T_{13} : X \otimes Z \otimes Y \to X \otimes Z \otimes Y$ is the linear map which acts as T on X and Y and as the identity on Z.
To distinguish between the involution and the adjoint we denote the adjoint of a linear operator A with $A^\dagger$. For a vector space B, the symbol B^* denotes the dual space of B. Furthermore, our inner product is linear in the first variable. If $\xi \in H$ is an element of a Hilbert space, we denote with $\xi^\star$ the linear map $\xi^\star : H \to \mathbb{C},\ h \mapsto \langle h, \xi \rangle$. If we have two Hilbert spaces H_1 and H_2 with orthonormal bases $\{e_j\}_j$ and $\{f_i\}_i$ respectively, then we denote with $b_{ij} \in B(H_1, H_2)$ the map with $b_{ij}(e_k) = \delta_{kj} f_i$. The symbol $B(H_1, H_2)$ denotes the set of bounded linear maps from H_1 to H_2. Moreover, if we talk about $\mathbb{C}^n$ we denote with e_i the vector with $(e_i)_j = \delta_{ij}$ and if we talk about $M_n(\mathbb{C})$ we denote with e_{ij} the matrix with $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ if it is not defined otherwise.

1.2 C^*-Algebras

Now, we want to recall some basics about *-algebras and C^*-algebras. We will review the definition of these algebras, some important theorems and the construction of universal *- and C^*-algebras. The source for this section is [WLV21].

Definition 1.1. Let A be a $\mathbb{C}$-algebra. We call an antilinear map $\cdot^* : A \to A$ an *involution* if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in A$. A **-algebra* is an algebra equipped with an involution. A *C^*-algebra* is a normed and complete *-algebra where the norm is submultiplicative, i.e. $\|xy\| \leqslant \|x\|\|y\|$, and it satisfies the C^*-identity, i.e. $\|x^*x\| = \|x\|^2$.
We call a *-algebra *unital* if it contains a unit with respect to the multiplication. For two unital *-algebras A and B, a linear and multiplicative map $\varphi : A \to B$ is called a **-homomorphism* if $\varphi(x^*) = \varphi(x)^*$ and *unital* if $\varphi(1) = 1$.

For better understanding, let us look at a few examples of C^*-algebras.

Example 1.2. *i)* Let X be a compact Hausdorff space. Then $C(X)$ with the infinity norm $\|\cdot\|_\infty$, pointwise addition and multiplication and involution defined by $f^* := \overline{f}$ is a unital C^*-algebra. Indeed, $(C(X), \|\cdot\|_\infty)$ is complete and

$$\|fg\|_\infty \leqslant \|f\|_\infty \|g\|_\infty, \ \overline{\overline{f}} = f, \ \overline{fg} = \overline{f}\,\overline{g} = \overline{g}\,\overline{f}, \ \|\overline{f}f\|_\infty = \||f|^2\|_\infty = \|f\|_\infty^2$$

is true for all $f, g \in C(X)$.

ii) For $n \in \mathbb{N}$, the space $M_n(\mathbb{C})$ with the operator norm is a unital C^*-algebra, where we have the usual matrix multiplication and the involution is defined as $A^* := \overline{A}^T$, i.e. $(A^*)_{ij} := \overline{A_{ji}}$.

iii) Let H be a Hilbert space. Then $B(H)$ with the operator norm and composition as multiplication is a unital C^*-algebra with involution $T^* := T^\dagger$. In fact, even any C^*-algebra is isomorphic to a norm closed *-subalgebra of B(H), for some H. Observe that $B(H)$ is just $M_n(\mathbb{C})$ if H is finite-dimensional.

One of the fundamental theorems of C^*-algebras is the Gelfand-Naimark-Theorem. It states that a C^*-algebra is commutative if and only if it is isomorphic to the space of continuous functions on some compact space. We will need this theorem for showing that the quantum objects we will introduce are non-commutative analogs of the classical objects. A proof of this theorem is given in [[WLV21], Theorem 3.23].

Theorem 1.3 (Gelfand-Naimark, 1943)**.** *Let A be a unital C^*-algebra. Then*

$$A \text{ is commutative} \iff \exists X \text{ compact: } A \cong C(X).$$

The space X is then given by

$$Spec(A) = \{\varphi : A \to \mathbb{C} \mid \varphi \text{ is a homomorphism with } \varphi \neq 0\}.$$

Another important theorem is Wedderburn's Theorem, which says that every finite-dimensional C^*-algebra is isomorphic to the direct sum of some matrix spaces. This is proven in [[WLV21], Proposition 8.5].

Theorem 1.4 (Wedderburn's Theorem)**.** *Let A be a finite-dimensional C^*-algebra. Then there exist $m \in \mathbb{N}$ and $N_1, \dots, N_m \in \mathbb{N}$ such that*

$$A \cong \bigoplus_{i=1}^{m} M_{N_i}(\mathbb{C}).$$

Remark 1.5. Actually, Wedderburn's Theorem (or the Artin-Wedderburn Theorem) is known in the more general setting of semisimple rings. A corollary of this theorem is that any finite-dimensional $\mathbb{C}$-algebra is isomorphic to $\oplus_{i=1}^{m} M_{N_i}(\mathbb{C})$ for some $m, N_1, \dots, N_m \in \mathbb{N}$, so here we doesn't even need the C^*-structure.

Next, we want to recall the construction of universal *- and C^*-algebras since we need this to define the quantum automorphism groups and linking algebras later. We first define universal *-algebras.

Definition 1.6. Let $E = \{x_i \mid i \in I\}$ be a set of elements where I is some index set. We denote with $P(E)$ the *-algebra whose elements are polynomials $\sum_{k=1}^{N} \alpha_k y_k$ with $N \in \mathbb{N}$, $\alpha_k \in \mathbb{C}$ and $y_k \in \{x_{i_1}^{\epsilon_1} \cdots x_{i_m}^{\epsilon_m} \mid i_1, \dots, i_m \in I, \epsilon_1, \dots, \epsilon_m \in \{1, *\}\}$. The multiplication is defined as

$$(x_{i_1}^{\epsilon_1} \cdots x_{i_m}^{\epsilon_m}) \cdot (x_{j_1}^{\tilde{\epsilon_1}} \cdots x_{j_n}^{\tilde{\epsilon_n}}) := x_{i_1}^{\epsilon_1} \cdots x_{i_m}^{\epsilon_m} x_{j_1}^{\tilde{\epsilon_1}} \cdots x_{j_n}^{\tilde{\epsilon_n}}$$

and the involution via

$$(\alpha x_{i_1}^{\epsilon_1} \cdots x_{i_m}^{\epsilon_m})^* := \overline{\alpha} x_{i_m}^{\overline{\epsilon_m}} \cdots x_{i_1}^{\overline{\epsilon_1}}$$

where $\alpha \in \mathbb{C}$ and $\overline{\epsilon_k} := \begin{cases} 1, & \text{if } \epsilon_k = * \\ *, & \text{if } \epsilon_k = 1. \end{cases}$

Let $R \subseteq P(E)$ be a set of polynomials and $J(R)$ be the two-sided *-ideal generated by R. Then the *universal *-algebra* with *generators* E and *relations* R is defined as

$$\mathcal{O}(E \mid R) := {P(E)}\big/{J(R)}.$$

An important property of universal *-algebras is the so called universal property: If we have a *-algebra A with elements $\{z_i \mid i \in I\}$ which satisfy the relations R (i.e. all polynomials in R are zero, when we replace each x_i by z_i), then there is a unique *-homomorphism $\varphi : \mathcal{O}(E \mid R) \to A$ with $\varphi(x_i) = z_i$.

In order to define universal C^*-algebras, we need the notion of a C^*-seminorm.

Definition 1.7. Let A be a *-algebra. A *C^*-seminorm* on A is a map $p : A \to [0, \infty)$ such that for all $x, y \in A$ and $\lambda \in \mathbb{C}$ we have

i) $p(\lambda x) = |\lambda| p(x)$ and $p(x + y) \leqslant p(x) + p(y)$

ii) $p(xy) \leqslant p(x)p(y)$

iii) $p(x^*x) = p(x)^2$.

Hence the only thing missing for a C^*-norm is the positive definiteness.
With this definition we can now also define universal C^*-algebras.

Definition 1.8. Let E be a set of generators and $R \subseteq P(E)$ be relations. We put

$$\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } \mathcal{O}(E \mid R)\}$$

for $x \in \mathcal{O}(E \mid R)$. If $\|x\| < \infty$ for all $x \in \mathcal{O}(E \mid R)$, then $\| \cdot \|$ is a C^*-seminorm and one can check that $\{x \in \mathcal{O}(E \mid R) \mid \|x\| = 0\}$ is a two-sided *-ideal. If $\|x\| < \infty$ for all $x \in \mathcal{O}(E \mid R)$, we can define the *universal C^*-algebra* as

$$C^*(E \mid R) := \overline{\mathcal{O}(E \mid R) \big/ \{x \in \mathcal{O}(E \mid R) \mid \|x\| = 0\}}^{\|\cdot\|}.$$

For better understanding, since this construction is not very illustrative, let us look at some example of universal C^*-algebras.

Example 1.9. *i*) The universal C^*-algebra with one generator $E = \{x\}$ and the relation $R = \{xx^*x - x\}$ exists. We write $C^*(x \mid xx^*x = x)$ for this C^*-algebra. Indeed, we get for every C^*-seminorm p and $x \in \mathcal{O}(E \mid R)$

$$p(x)^2 = p(x^*x) = p(x^*xx^*x) = p(x^*x)^2 = p(x)^4$$

and therefore $p(x) \in \{0, 1\}$, which implies

$$\|x\| = \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } \mathcal{O}(E \mid R)\} < \infty.$$

ii) Let $N \geqslant 2$. Then the universal C^*-algebra

$$C^*(e_{ij}, i, j = 1, \ldots, N \mid e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il} \text{ for all } i, j, k, l)$$

is isomorphic to $M_n(\mathbb{C})$.

iii) The universal C^*-algebra

$$C^*(e_{ij}, i, j \in \mathbb{N} \mid e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il} \text{ for all } i, j, k, l)$$

is isomorphic to $\mathcal{K}(H)$, the algebra of compact operators on a separable Hilbert space H.

For a proof of *ii)* see [[WLV21], Proposition 6.11] and for *iii)* see [[WLV21], Proposition 6.13].
We can think of a universal C^*-algebra as a subset of the corresponding universal *-algebra by identifying an equivalence class with some representative of the class. This is even a dense subset, as the following lemma shows.

Lemma 1.10. *Let E be a set of generators and $R \subseteq P(E)$ be relations. Then $C^*(E \mid R)$ is a dense subset of $\mathcal{O}(E \mid R)$.*

Proof. Let $x \in \mathcal{O}(E \mid R)$ and for $n \in \mathbb{N}$ let $[x_n]$ be the equivalence class of x in $\mathcal{O}(E \mid R)/\{x \in \mathcal{O}(E \mid R) \mid \|x\| = 0\}$. Then $\|x_n - x\| = 0$ for all $n \in \mathbb{N}$ and therefore $x_n \to x$ $(n \to \infty)$. Hence $\mathcal{O}(E \mid R)/\{x \in \mathcal{O}(E \mid R) \mid \|x\| = 0\}$ is dense in $\mathcal{O}(E \mid R)$. Since $\mathcal{O}(E \mid R)/\{x \in \mathcal{O}(E \mid R) \mid \|x\| = 0\} \subseteq C^*(E \mid R)$, we get that $C^*(E \mid R)$ is also dense in $\mathcal{O}(E \mid R)$. □

Remark 1.11. Note that it is possible to have $C^*(E \mid R) \subsetneq \mathcal{O}(E \mid R)$. For example, look at the universal algebras generated by $E = \{u_{ij} \mid 1 \leqslant i, j \leqslant n\}$ and

$$R = \{u_{ij} - u_{ij}^*, \sum_{k=1}^{n} u_{ik} - 1, \sum_{k=1}^{n} u_{kj} - 1, u_{ij}u_{ik} - \delta_{jk}u_{ij}, u_{ji}u_{ki} - \delta_{jk}u_{ji} \mid 1 \leqslant i, j \leqslant n\}.$$

Then

$$C^*(E \mid R) = C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{kj} = 1)$$

since by [[Sch20], Remark 1.1.9] the relations $u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1$ already imply $u_{ij}u_{ik} = \delta_{jk}u_{ij}$ and $u_{ji}u_{ki} = \delta_{jk}u_{ji}$ in a C^*-algebra. In a *-algebra this is not true, hence in the universal *-algebra we still need all relations from R.

1.3 Compact Quantum Groups

Next we introduce the notion of compact quantum groups. These object can be defined in different ways. We will use the definition of Woronowicz [[Wor98], Definition 1.1]. He was also the one who first introduced the notion of compact quantum groups in 1987. With this definition one can easily see how it links to classical compact groups. In this section, we will also introduce Hopf algebras. These are objects that are strongly related to compact quantum groups.
The main source for this section is [BCE+20].

Definition 1.12. A *compact quantum group* (CQG) is a pair (A, Δ) where A is a unital C^*-algebra and $\Delta : A \to A \otimes A$ is a unital *-homomorphism with the following properties

1) $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ (*coassociativity*)

2) $(A \otimes 1)\Delta(A)$ and $(1 \otimes A)\Delta(A)$ are linearly dense in $A \otimes A$.
(*cancellation property*)

The map Δ is called *comultiplication*.

The first questions that might come to mind after reading this definition is: Where does this definition come from? Why does it make sense? The following remark shows that the C^*-algebra A of a CQG is commutative if and only if $A \cong C(G)$ for a compact group G, i.e. a topological group which is compact and a Hausdorff space. Thereby, we need 1) for the associativity of the multiplication and 2) for the cancellation property of the group and vice versa. Therefore the names of the properties 1) and 2) make indeed sense.

The following remark is similar to [[Web17], Remark 2.5] and [[Gro20], Proposition 2.1.2].

Remark 1.13. *i*) Let G be a compact group and let $C(G)$ be the continuous functions on G. We define

$$\Delta_G : C(G) \to C(G) \otimes C(G)$$

by

$$\Delta_G(f)(g,h) := f(gh),$$

where we used $C(G) \otimes C(G) \cong C(G \times G)$ with $(f_1 \otimes f_2)(g,h) = f_1(g)f_2(h)$ [Bla06], Theorem II.9.4.4]. Then $(C(G), \Delta_G)$ is a compact quantum group. Indeed, $C(G)$ is a unital C^*-algebra and Δ_G is a unital *-homomorphism. Moreover, the coassociativity follows from the associativity of G: For $f \in C(G)$ let $\Delta_G(f) = f_1 \otimes f_2$, i.e. $f_1(g)f_2(h) = f(gh)$, then

$$\begin{aligned}(\Delta_G \otimes id)(\Delta_G(f))(g,h,k) &= (\Delta_G(f_1) \otimes f_2)(g,h,k) = f_1(gh)f_2(k) \\ &= f(ghk) = f_1(g)f_2(hk) = (f_1 \otimes \Delta_G(f_2))(g,h,k) \\ &= (id \otimes \Delta_G)(\Delta_G(f))(g,h,k).\end{aligned}$$

The case $\Delta_G(f) = \sum_{i,j} f_i \otimes f_j$ follows analogously.
Furthermore, the space $(C(G) \otimes 1)\Delta_G(C(G))$ is spanned by functions of the form $(g,h) \mapsto f_1(g)f_2(gh)$ with $f_1, f_2 \in C(G)$. Therefore it is a unital *-subalgebra. Using the cancellation property of G (i.e. $gt = gs$ and $tg = sg$ both imply $t = s$), one can show that this set separates the points, hence we get with the Stone-Weierstrass Theorem that it is dense in $C(G) \otimes C(G)$. Similarly one can show the density of

the set $(1 \otimes C(G))\Delta_G(C(G))$.

ii) If (A, Δ) is a CQG with some commutative C^*-algebra A, then the Gelfand-Naimark Theorem (Theorem 1.3) tells us that $A \cong C(G)$ with $G = \mathrm{Spec}(A)$, which is a compact space. Moreover, the comultiplication $\Delta : A \to A \otimes A$ induces a group law $m : G \times G \to G$ by

$$m : \mathrm{Spec}(A) \times \mathrm{Spec}(A) \to \mathrm{Spec}(A), \quad (\varphi_1, \varphi_2) \mapsto (\varphi_1, \varphi_2) \circ \Delta,$$

which is associative since the coassociativity holds. Here we used

$$\begin{aligned}\mathrm{Spec}(A \otimes A) &\cong \mathrm{Spec}(C(G) \otimes C(G)) \cong \mathrm{Spec}(C(G \times G)) \\ &\cong G \times G = \mathrm{Spec}(A) \times \mathrm{Spec}(A)\end{aligned}$$

with $(\varphi_1, \varphi_2)(a \otimes b) = \varphi_1(a)\varphi_2(b)$.
Therefore G is a compact semi-group. From the linear density of $(C(G) \otimes 1)\Delta(C(G))$ and $(1 \otimes C(G))\Delta(C(G))$ now follows that G has the cancellation property and this implies together with the compactness of G that G is indeed a group. Moreover, one can check $\Delta = \Delta_G$ (where Δ_G is defined as in *i)*).

Hence we get for every compact quantum group (A, Δ):

$$A \text{ is commutative } \iff (A, \Delta) = (C(G), \Delta_G) \text{ for some compact group } G.$$

So we see that CQGs generalise compact groups as a non-commutative analog.

At this point we want to introduce two CQGs which we will need later when talking about quantum automorphism groups.

Definition 1.14. *i)* [Wan98] The quantum permutation group S_n^+ is the CQG $(\mathcal{O}(S_n^+), \Delta)$ defined by

$$\mathcal{O}(S_n^+) = C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1) \text{ and } \Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj},$$

where $1 \leqslant i, j \leqslant n$.

ii) [Gro21a] The projective version PO_n^+ of the orthogonal quantum group is the compact quantum group $(\mathcal{O}(PO_n^+), \Delta)$ where $\mathcal{O}(PO_n^+)$ is the universal C^*-algebra generated by elements v_{kl}^{ij} for $1 \leqslant l, k, i, j \leqslant n$ with

$$v_{kl}^{ij*} = v_{lk}^{ji}$$

$$\sum_{k,l=1}^{n} v_{lk}^{ji} v_{kl}^{rs} = \delta_{ir}\delta_{js} = \sum_{k,l=1}^{n} v_{ij}^{kl} v_{sr}^{lk}$$
$$\sum_{q=1}^{n} v_{kq}^{ij} v_{ql}^{rs} = \delta_{jr} v_{kl}^{is}$$

and

$$\Delta(v_{kl}^{ij}) = \sum_{r,s=1}^{n} v_{kl}^{rs} \otimes v_{rs}^{ij}.$$

Here we will also introduce an isomorphism notion for CQG, as we need it in a later theorem, by using the notion of a quantum subgroup. Moreover, we want to state the definition of Hopf algebras. These algebras are important for working with CQGs since we can associate to every CQG a corresponding Hopf algebra.

Definition 1.15. Let G_1, G_2 be two CQGs. Then G_1 is a *quantum subgroup* of G_2, written $G_1 \subseteq G_2$, if there exists a surjective *-homomorphism

$$\sigma : \mathcal{O}(G_2) \to \mathcal{O}(G_1) \text{ with } (\sigma \otimes \sigma)\Delta_2 = \Delta_1\sigma,$$

where Δ_i are the corresponding comultiplications.
If $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$, we write $G_1 \cong G_2$ and call G_1 and G_2 *isomorphic.*

Definition 1.16. A *Hopf algebra* is a quadruple (A, Δ, S, ϵ) where A is a unital algebra with multiplication map $m : A \otimes A \to A$ and

$$\Delta : A \to A \otimes A \quad (\textit{comultiplication}),$$

$$S : A \to A^{op} \quad (\textit{antipode}),$$

$$\epsilon : A \to \mathbb{C} \quad (\textit{counit})$$

are unital algebra homomorphisms satisfying

1) $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$

2) $m(id \otimes S)\Delta = \epsilon(\cdot)1 = m(S \otimes id)\Delta$

3) $(\epsilon \otimes id)\Delta = (id \otimes \epsilon)\Delta = id.$

A *Hopf *-algebra* is a Hopf algebra where A is a *-algebra and Δ and ϵ are *-homomorphisms.

Remark 1.17. *i*) With A^{op} we denote the algebra A with the opposite multiplication, i.e. $a \cdot_{A^{op}} b = b \cdot_A a$. Therefore we have in the above definition $S(ab) = S(b)S(a)$ for all $a, b \in A$.

ii) To every CQG $G = (A, \Delta)$ we can associate a Hopf *-algebra which is dense in A. We will write $\mathcal{O}(G)$ for both, the corresponding Hopf *-algebra and the corresponding C^*-algebra A. This abuse of notation will simplify the writing. We will later look at the theorem which states the existence of the corresponding Hopf *-algebra since we need the notion of representations for this. This will be Theorem 4.3.

At the end of this section, let as look at an easy example of a Hopf *-algebra.

Example 1.18. The CQG from Remark 1.13 *i*) is also a Hopf *-algebra together with

$$S : C(G) \to C(G)^{op}, \quad Sf(t) := f(t^{-1})$$

and

$$\epsilon : C(G) \to \mathbb{C}, \quad \epsilon(f) := f(e),$$

where $e \in G$ is the neutral element.

Chapter 2

Quantum Graphs and Quantum Automorphism Groups

Now we also want to introduce quantum graphs and quantum automorphism groups. Quantum graphs are quantum analogs of graphs and quantum automorphism groups of graphs are quantum analogs of graph automorphisms, i.e. we can show that the commutative analog is a classical graph respectively the space of graph automorphisms.
The main source for this chapter is again [BCE$^+$20].

2.1 Quantum Graphs

As Matsuda already described in [Mat21], quantum graphs were first introduced by Duan, Severini and Winter in [DSW13]. They were called non-commutative graphs. Since every reflexive undirected classical graph is only a reflexive symmetric relation, Weaver formulated quantum graphs as reflexive symmetric quantum relations on a von Neumann algebra in [Wea21]. Quantum relations were introduced by Kuperberg and Weaver in [KW12]. In [BCE$^+$20] quantum graphs were then formulated (similarly to the definition in [MRV18]) as finite quantum sets with an adjacency matrix. That is the definition we will use.

In this section we start with the definition of a quantum set and a δ-form since we need this to introduce the notion of quantum graphs. We will look at some examples of quantum sets and quantum graphs and show that a quantum graph is indeed the non-commutative analog of a graph.

Definition 2.1. A *(finite, measured) quantum set* is a pair $X = (\mathcal{O}(X), \psi_X)$, where $\mathcal{O}(X)$ is a finite-dimensional C^*-algebra and $\psi_X : \mathcal{O}(X) \to \mathbb{C}$ is a faithful state, i.e. a linear unital functional with $\psi_X(x^*x) \geqslant 0$ and $\psi_X(x^*x) = 0$ only if $x = 0$. With $|X|$ we denote the dimension of $\mathcal{O}(X)$.

Remark 2.2. *i*) By Wedderburn's Theorem (Theorem 1.4) there exist $m \in \mathbb{N}$ and $N_1, \dots, N_m \in \mathbb{N}$ such that $\mathcal{O}(X) \cong \bigoplus_{i=1}^m M_{N_i}(\mathbb{C})$. This implies in particular that $\mathcal{O}(X)$ is unital.

ii) By defining $\langle a, b\rangle := \psi_X(b^*a)$, we get a scalar product on $\mathcal{O}(X)$ since ψ_X is faithful, i.e. $\psi(x^*x) = 0$ implies $x = 0$. Then $(\mathcal{O}(X), \langle\cdot,\cdot\rangle)$ is a Hilbert space because

$$\psi_X(x^*x) \leqslant \|x^*x\| = \|x\|^2,$$

so $\mathcal{O}(X)$ is also complete with respect to the norm induced by the scalar product. To distinguish between the Hilbert and C^*-structures on $\mathcal{O}(X)$, we denote the Hilbert space with $L^2(X)$. Note that

$$\langle ab, c\rangle = \psi(c^*ab) = \psi((a^*c)^*b) = \langle b, a^*c\rangle$$

is true for all $a, b, c \in L^2(X)$.
With $m_X : \mathcal{O}(X)\otimes\mathcal{O}(X) \to \mathcal{O}(X)$ we denote the multiplication, i.e. $m_X(a \otimes b) = ab$. Since $m_X \in B(L^2(X) \otimes L^2(X), L^2(X))$, we can form the adjoint $m_X^\dagger$.

iii) Let $\eta_X : \mathbb{C} \to \mathcal{O}(X)$ be the unit map, i.e. $\eta_X(\alpha) = \alpha 1$. Then the adjoint of η_X is ψ_X since

$$\langle b, \eta_X(\alpha)\rangle = \langle b, \alpha 1\rangle = \psi_X(\overline{\alpha}1b) = \psi_X(b)\overline{\alpha} = \langle \psi_X(b), \alpha\rangle_{\mathbb{C}}.$$

Definition 2.3. Let $\delta > 0$. We call a state $\psi_X : \mathcal{O}(X) \to \mathbb{C}$ a *δ-form* if

$$m_X m_X^\dagger = \delta^2 id.$$

It should be remarked here that the formula in this definition is really a statement about ψ_X although it is not clear at first sight. This is the case since the adjoint of m_X is formed with respect to the scalar product $\langle a, b\rangle = \psi_X(b^*a)$.
Since the above definition is not very illustrative, let us look at a few examples of quantum sets where the corresponding state is a δ-form.

Example 2.4. *i)* Let $[n] := \{1, \ldots, n\}$ and $C([n])$ be the C^*-algebra of (continuous) functions on $[n]$ with the pointwise multiplication m. We define

$$\psi : C([n]) \to \mathbb{C}, \quad \psi(f) := \frac{1}{n}\sum_{i=1}^{n} f(i).$$

Then $(C([n]), \psi)$ is a quantum set. Furthermore, we have

$$\begin{aligned}\langle \chi_{\{j\}} \otimes \chi_{\{k\}}, n(\chi_{\{i\}} \otimes \chi_{\{i\}})\rangle &= n\langle \chi_{\{j\}}, \chi_{\{i\}}\rangle\langle \chi_{\{k\}}, \chi_{\{i\}}\rangle = n\psi(\chi_{\{i\}}\chi_{\{j\}})\psi(\chi_{\{i\}}\chi_{\{k\}})\\ &= n\delta_{ij}\delta_{ik}\psi(\chi_{\{i\}})^2 = n\delta_{ij}\delta_{ik}\frac{1}{n^2} = \delta_{ij}\delta_{ik}\frac{1}{n} = \delta_{jk}\delta_{ij}\frac{1}{n}\\ &= \delta_{jk}\delta_{ij}\psi(\chi_{\{i\}}) = \delta_{jk}\psi(\chi_{\{i\}}\chi_{\{j\}}) = \langle \delta_{jk}\chi_{\{j\}}, \chi_{\{i\}}\rangle\\ &= \langle m(\chi_{\{j\}} \otimes \chi_{\{k\}}), \chi_{\{i\}}\rangle\end{aligned}$$

and hence

$$m^\dagger(f) = m^\dagger(\sum_{i=1}^{n} f(i)\chi_{\{i\}}) = \sum_{i=1}^{n} f(i)m^\dagger(\chi_{\{i\}}) = n\sum_{i=1}^{n} f(i)(\chi_{\{i\}} \otimes \chi_{\{i\}}).$$

This implies that ψ is a $\sqrt{n}$-form since

$$m(m^\dagger(f)) = m(n\sum_{i=1}^{n} f(i)(\chi_{\{i\}}\otimes\chi_{\{i\}})) = n\sum_{i=1}^{n} f(i)m(\chi_{\{i\}}\otimes\chi_{\{i\}}) = n\sum_{i=1}^{n} f(i)\chi_{\{i\}} = nf.$$

Moreover, we get the well-known scalar product $\langle f, g\rangle = \frac{1}{n}\sum_{i=1}^{n} \overline{g(i)}f(i)$.

ii) Since $C([n]) \cong \mathbb{C}^n$, we also have the quantum set $(\mathbb{C}^n, \psi)$ with $\sqrt{n}$-form

$$\psi : \mathbb{C}^n \to \mathbb{C}, \quad \psi((x_i)_{i=1,\ldots,n}) := \frac{1}{n}\sum_{i=1}^{n} x_i.$$

This map we will denote from now on with ψ_n. We have the pointwise multiplication m and

$$m^\dagger((x_i)_{i=1,\ldots,n}) = n\sum_{i=1}^{n} x_i(e_i \otimes e_i).$$

iii) The matrix algebra $M_n(\mathbb{C})$ together with the normalized trace

$$tr : M_n(\mathbb{C}) \to \mathbb{C}, \quad tr((a_{ij})_{ij}) := \frac{1}{n}\sum_{i=1}^{n} a_{ii}$$

is a quantum set and tr is an n-form.
Indeed, one can check that $m^\dagger(a) = n\sum_{k,i,j=1}^{n} a_{ij}(e_{ik} \otimes e_{kj})$ and hence

$$\begin{aligned}m(m^\dagger(a)) &= m(n\sum_{k,i,j=1}^{n} a_{ij}(e_{ik} \otimes e_{kj})) = n\sum_{k,i,j=1}^{n} a_{ij}e_{ik}e_{kj}\\ &= n\sum_{k,i,j=1}^{n} a_{ij}e_{ij} = n^2\sum_{i,j=1}^{n} a_{ij}e_{ij} = n^2 a.\end{aligned}$$

iv) Let $n(i) \in \mathbb{N}$ and $Q_i \in M_{n(i)}(\mathbb{C})$ be positive and invertible matrices with $\sum_{i=1}^{s} tr(Q_i) = 1$ and $tr(Q_i^{-1}) = \delta^2$ for $1 \leqslant i \leqslant s$. Then the C^*-algebra $\bigoplus_{i=1}^{s} M_{n(i)}(\mathbb{C})$ together with $\psi((x_1, \dots, x_s)) := \sum_{i=1}^{s} tr(Q_i x_i)$ is a quantum set and ψ is a δ-form. In fact, one can show that every quantum set is of this form.

Now we can finally define quantum graphs. For this we first recall the definition of a classical (finite) graph without multiple edges. A classical graph X without multiple edges consists of a finite vertex set $V(X)$ and edge set $E(X) \subseteq V(X) \times V(X)$. A graph is called reflexive if $(v, v) \in E(X)$ for all $v \in V(X)$ and undirected if $(v, w) \in E(X) \implies (w, v) \in E(X)$ for all $w, v \in V(X)$. W.l.o.g we can always assume $V(X) = \{1, \dots, n\}$ for some $n \in \mathbb{N}$. The adjacency matrix $A = (a_{ij})_{ij} \in M_n(\mathbb{C})$ is defined by

$$a_{ij} := \begin{cases} 1, & \text{if } (i,j) \in E(X) \\ 0, & \text{if } (i,j) \notin E(X). \end{cases}$$

Note that a matrix is an adjacency matrix for some undirected, reflexive graphs without multiple edges if and only if $a_{ij}^2 = a_{ij}, a_{ij} = a_{ji}$ and $a_{ii} = 1$ for $i, j = 1, \dots, n$.

The following definition is taken from [BEVW20] since its a bit more general than the definition of a quantum graph given in [BCE$^+$20].

Definition 2.5. Let X be a quantum set with a δ-form ψ_X. A self-adjoint linear map $A_X : L^2(X) \to L^2(X)$ is called a *quantum adjacency matrix* if

$$m_X(A_X \otimes A_X)m_X^\dagger = \delta^2 A_X. \tag{1}$$

In the following we only want to look at undirected and reflexive quantum graphs. Therefore we require two additional conditions:

$$(id \otimes \eta_X^\dagger m_X)(id \otimes A_X \otimes id)(m_X^\dagger \eta_X \otimes id) = A_X \tag{2}$$

$$m_X(A_X \otimes id)m_X^\dagger = \delta^2 id \tag{3}$$

Then (1) guarantees that the quantum graph does not have multiple edges, (2) states that it is undirected and (3) that it is reflexive. See also Proposition 2.7. We call $X = (\mathcal{O}(X), \psi_X, A_X)$ a *quantum graph*.

It is not clear yet whether a quantum adjacency matrix really needs to be self-adjoint, so there are also definitions of quantum graphs where this is not required. Since we mostly refer to [BCE$^+$20] and it is required there, it is included in our definition.

Convention 2.6. From now on we always consider classical undirected, reflexive graphs without multiple edges.

Definition 2.5 looks quite complicated and not very intuitive. In order to understand the definition better, we now prove similarly to Remark 1.13 that a quantum graph is a classical graph if and only if the corresponding C^*-algebra is commutative. This also clarifies why the equations $(1), (2)$ and (3) guarantee the corresponding properties.

Proposition 2.7. *i) If X is a classical graph with n vertices and adjacency matrix $A_X = (a_{ij})_{ij}$, then A_X is a quantum adjacency matrix and for the conditions* $(1), (2)$ *and* (3) *in Definition 2.5 we have*

$$(1) \iff a_{ij}^2 = a_{ij}, \quad (2) \iff a_{ij} = a_{ji} \quad \textit{and} \quad (3) \iff a_{ii} = 1,$$

where the equalities have to be true for all $i, j \in \{1, \dots, n\}$.
ii) Let $(\mathcal{O}(X), \psi_X, A_X)$ be a quantum graph where $\mathcal{O}(X)$ is a commutative C^*-algebra. Then X is a classical graph.

Proof. *i)* For a classical graph with n vertices, we always consider the quantum set $(\mathbb{C}^n, \psi_n)$ from Example 2.4 *ii*). We look at the three equalities from the definition above.
(1)

$$\begin{aligned} m_X((A_X \otimes A_X)(m_X^\dagger(e_k))) &= m_X((A_X \otimes A_X)(ne_k \otimes e_k)) = n \cdot m_X(A_X e_k \otimes A_X e_k) \\ &= n(a_{1k}^2, \dots, a_{nk}^2)^T \overset{!}{=} n(a_{1k}, \dots, a_{nk})^T = nA_X e_k \end{aligned}$$

(2)

$$\begin{aligned} &(id \otimes \eta_X^\dagger m_X)(id \otimes A_X \otimes id)(m_X^\dagger \eta_X \otimes id)(1 \otimes e_k) \\ &= (id \otimes \eta_X^\dagger m_X)(id \otimes A_X \otimes id)(m_X^\dagger(\sum_{i=1}^n e_i) \otimes e_k) \\ &= (id \otimes \eta_X^\dagger m_X)(id \otimes A_X \otimes id)(n \sum_{i=1}^n e_i \otimes e_i \otimes e_k) \\ &= (id \otimes \eta_X^\dagger m_X)(n \sum_{i=1}^n e_i \otimes A_X e_i \otimes e_k) \\ &= n \sum_{i=1}^n e_i \otimes \eta_X^\dagger(a_{ki} e_k) \overset{\eta_X^\dagger = \psi_X}{=} n \sum_{i=1}^n e_i \otimes a_{ki} \frac{1}{n} \\ &= \sum_{i=1}^n a_{ki} e_i = A_X^T e_k \overset{!}{=} A_X e_k \end{aligned}$$

(3)

$$\begin{aligned} m_X(A_X \otimes id)m_X^\dagger(e_k) &= m_X(A_X \otimes id)(ne_k \otimes e_k) = n \cdot m_X(A_X e_k \otimes e_k) \\ &= na_{kk}e_k \stackrel{!}{=} ne_k \end{aligned}$$

Since the above equalities have to be true for all $k \in \{1, \ldots, n\}$, we get

$$(1) \iff a_{ij}^2 = a_{ij}, \quad (2) \iff a_{ij} = a_{ji} \quad \text{and} \quad (3) \iff a_{ii} = 1.$$

Hence we can see that A_X is a quantum adjacency matrix because A_X is a symmetric matrix with entries in $\{0, 1\}$ and $a_{ii} = 1$.

ii) Let $(\mathcal{O}(X), \psi_X, A_X)$ be a quantum graph with $|X| = n$ and where $\mathcal{O}(X)$ is a commutative C^*-algebra. Then the Gelfand-Naimark Theorem tells us that there exists a compact space X such that $\mathcal{O}(X) \cong C(X)$. Since $\mathcal{O}(X)$ is n-dimensional we get that the space X must have n elements, hence we can assume $X = \{1, \ldots, n\}$. Moreover, the Riesz Representation Theorem tells us that ψ_X is of the form

$$\psi_X(f) = \int_X f d\mu \quad (f \in C(X))$$

for some unique Borel probability measure $\mu : \{1, \ldots, n\} \to [0, 1]$.
We then get

$$m_X^\dagger(\chi_{\{i\}}) = \begin{cases} \frac{1}{\mu(\{i\})}\chi_{\{i\}} \otimes \chi_{\{i\}}, & \text{if } \mu(\{i\}) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

because

$$\begin{aligned} \langle \frac{1}{\mu(\{i\})}\chi_{\{i\}} \otimes \chi_{\{i\}}, \chi_{\{j\}} \otimes \chi_{\{k\}} \rangle &= \frac{1}{\mu(\{i\})} \int_X \chi_{\{i\}}\chi_{\{j\}} d\mu \int_X \chi_{\{i\}}\chi_{\{k\}} d\mu \\ &= \frac{1}{\mu(\{i\})} \delta_{ij}\mu(\{i\})\delta_{ik}\mu(\{i\}) = \delta_{ij}\delta_{ik}\mu(\{i\}) \\ &= \delta_{ik} \int_X \chi_{\{j\}}\chi_{\{i\}} d\mu = \int_X \chi_{\{j\}}\chi_{\{k\}}\chi_{\{i\}} d\mu \\ &= \langle \chi_{\{i\}}, \chi_{\{j\}}\chi_{\{k\}} \rangle \end{aligned}$$

if $\mu(\{i\}) \neq 0$ and $\langle \chi_{\{i\}}, \chi_{\{j\}}\chi_{\{k\}} \rangle = \delta_{ij}\delta_{jk}\mu(\{j\}) = 0$ if $\mu(\{i\}) = 0$.
Since ψ_X is a δ-form we get $\delta^2\chi_{\{i\}} = m_X(m_X^\dagger(\chi_{\{i\}}))$ for all $i \in \{1, \ldots, n\}$. Hence $\mu(\{i\}) \neq 0$ and $\delta^2\chi_{\{i\}} = \frac{1}{\mu(\{i\})}\chi_{\{i\}}$, which implies $\mu(\{i\}) = \frac{1}{\delta^2}$ for all $i \in \{1, \ldots, n\}$. Using

$$1 = \mu(\{1, \ldots, n\}) = \sum_{i=1}^n \mu(\{i\}) = \frac{n}{\delta^2},$$

we get $\delta^2 = n$ and $\mu(\{i\}) = \frac{1}{n}$. Therefore $\psi_X(f) = \frac{1}{n}\sum_{i=1}^n f(i)$, so we get the quantum set from Example 2.4 *ii*) (or actually part *i*), but the two spaces are isomorphic anyway). Now part *i*) implies that for $A_X = (a_{ij})_{ij}$ we get $a_{ij}^2 = a_{ij}$, $a_{ij} = a_{ji}$ and $a_{ii} = 1$. Thus $(\mathcal{O}(X), \psi_X, A_X)$ is just a classical graph. □

So here we also get the equivalence for every quantum graph $X = (\mathcal{O}(X), \psi_X, A_X)$:

$$\mathcal{O}(X) \text{ is commutative } \iff (\mathcal{O}(X), \psi_X, A_X) = (\mathbb{C}^n, \psi_n, A_X) \text{ with } n = |X|.$$

Therefore the notion of a quantum graph is also just a generalisation of a classical graph as a non-commutative analog.
We now want to look at an example of a quantum graph with a non-commutative C^*-algebra. This example is taken from [[Gro21b], Example 3.13].

Example 2.8. Let $(M_2(\mathbb{C}), tr)$ be the quantum set from Example 2.4 *iii*) with $n = 2$. We define

$$A : M_2(\mathbb{C}) \to M_2(\mathbb{C}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $(M_2(\mathbb{C}), tr, A)$ is a quantum graph.
Indeed, let $f : \{1,2\} \to \{1,2\}$, $f(1) := 2$, $f(2) := 1$, then $A(e_{ij}) = e_{f(i)f(j)} + e_{ij}$. Since

$$\begin{aligned}\langle e_{f(i)f(j)}, e_{kl}\rangle &= tr(e_{kl}^* e_{f(i)f(j)}) = tr(e_{lk} e_{f(i)f(j)}) = \delta_{kf(i)} tr(e_{lf(j)}) = \frac{1}{2}\delta_{kf(i)}\delta_{lf(j)} \\ &= \frac{1}{2}\delta_{f(k)i}\delta_{f(l)j} = \delta_{f(k)i} tr(e_{f(l)j}) = tr(e_{f(l)f(k)} e_{ij}) = tr(e_{f(k)f(l)}^* e_{ij}) = \langle e_{ij}, e_{f(k)f(l)}\rangle,\end{aligned}$$

we get

$$\begin{aligned}\langle A(e_{ij}), e_{kl}\rangle &= \langle e_{f(i)f(j)} + e_{ij}, e_{kl}\rangle = \langle e_{f(i)f(j)}, e_{kl}\rangle + \langle e_{ij}, e_{kl}\rangle \\ &= \langle e_{ij}, e_{f(k)f(l)}\rangle + \langle e_{ij}, e_{kl}\rangle = \langle e_{ij}, e_{f(k)f(l)} + e_{kl}\rangle = \langle e_{ij}, A(e_{kl})\rangle,\end{aligned}$$

so we know that A is self-adjoint. Moreover,

$$\begin{aligned}m(A \otimes A)m^\dagger(e_{ij}) &= m(2\sum_{k=1}^{2} A(e_{ik}) \otimes A(e_{kj})) \\ &= m(2\sum_{k=1}^{2}(e_{f(i)f(k)} + e_{ik}) \otimes (e_{f(k)f(j)} + e_{kj})) \\ &= 2\sum_{k=1}^{2} e_{f(i)f(j)} + e_{ij} = 4(e_{f(i)f(j)} + e_{ij}) = 2^2 A(e_{ij}),\end{aligned}$$

where the second equation is true since $f(k) \neq k$ for $k = 1, 2$. Furthermore,

$$
\begin{aligned}
&(id \otimes \eta^\dagger m)(id \otimes A \otimes id)(m^\dagger \eta \otimes id)(1 \otimes e_{ij}) \\
&\qquad = (id \otimes \eta^\dagger m)(id \otimes A \otimes id)(m^\dagger(\sum_{l=1}^{2} e_{ll}) \otimes e_{ij}) \\
&\qquad = (id \otimes \eta^\dagger m)(id \otimes A \otimes id)(2 \sum_{k,l=1}^{2} e_{lk} \otimes e_{kl} \otimes e_{ij}) \\
&\qquad = (id \otimes \eta^\dagger m)(2 \sum_{k,l=1}^{2} e_{lk} \otimes (e_{f(k)f(l)} + e_{kl}) \otimes e_{ij}) \\
&\qquad \overset{\eta^\dagger = tr}{=} 2 \sum_{k,l=1}^{2} e_{lk} \otimes tr(e_{f(k)f(l)} e_{ij} + e_{kl} e_{ij}) \\
&\qquad = \sum_{k,l=1}^{2} e_{lk} \otimes (\delta_{f(l)i} \delta_{f(k)j} + \delta_{li} \delta_{kj}) \\
&\qquad = e_{f(i)f(j)} + e_{ij} = A(e_{ij})
\end{aligned}
$$

and

$$
\begin{aligned}
m(A \otimes id)m^\dagger(e_{ij}) &= m(2 \sum_{k=1}^{2} A(e_{ik}) \otimes e_{kj}) = m(2 \sum_{k=1}^{2} (e_{f(i)f(k)} + e_{ik}) \otimes e_{kj}) \\
&= 2 \sum_{k=1}^{2} e_{ik} e_{kj} = 2^2 e_{ij}.
\end{aligned}
$$

Hence the three conditions from Definition 2.5 are fulfilled.

However, since with this example you do not really have a picture in mind, we want to look at two further examples of quantum graphs which have classical analogs. To prove that these examples are indeed quantum graph, we first need the following lemma. The idea for this lemma is taken from [[Bra12], Remark 3.6].

Lemma 2.9. *Let $X = (\mathcal{O}(X), \psi_X)$ be a quantum set. Then*

$$(\psi_X \otimes id) m_X^\dagger = (id \otimes \psi_X) m_X^\dagger = id$$

and

$$m_X^\dagger m_X = (m_X \otimes id)(id \otimes m_X^\dagger).$$

Proof. We have $m_X(\eta_X \otimes id) = m_X(id \otimes \eta_X) = id$, so forming the adjoint and using $\eta_X^\dagger = \psi_X$ yields

$$(\psi_X \otimes id) m_X^\dagger = (id \otimes \psi_X) m_X^\dagger = id.$$

Moreover,

$$\begin{aligned}\langle (m_X \otimes id)(a \otimes b \otimes c), d \otimes e\rangle &= \langle ab \otimes c, d \otimes e\rangle = \langle ab, d\rangle\langle c, e\rangle \\ &= \langle b, a^* d\rangle\langle c, e\rangle = \langle b \otimes c, a^* d \otimes e\rangle\end{aligned}$$

implies

$$\begin{aligned}\langle (m_X \otimes id)(id \otimes m_X^\dagger)(g \otimes h), d \otimes e\rangle &= \langle (m_X \otimes id)(g \otimes m_X^\dagger(h)), d \otimes e\rangle \\ &= \langle m_X^\dagger(h), g^* d \otimes e\rangle \\ &= \langle h, g^* de\rangle = \langle gh, de\rangle \\ &= \langle m_X(g \otimes h), m_X(d \otimes e)\rangle \\ &= \langle m_X^\dagger m_X(g \otimes h), d \otimes e\rangle\end{aligned}$$

and hence $(m_X \otimes id)(id \otimes m_X^\dagger)(g \otimes h) = m_X^\dagger m_X(g \otimes h)$ for all $g, h \in \mathcal{O}(X)$. □

Proposition 2.10. *Let $X = (\mathcal{O}(X), \psi_X)$ be a quantum set with a δ-form ψ_X.*
i) The set X together with $A_X := \delta^2\psi_X(\cdot)1$ is a quantum graph, called complete quantum graph. If X is the quantum set $(\mathbb{C}^n, \psi_n)$, then we have the complete classical graph.
ii) The set X together with $A_X := id$ is a quantum graph, called trivial quantum graph. If X is the quantum set $(\mathbb{C}^n, \psi_n)$, then we have the trivial classical graph.

Proof. *i)* First of all, A_X is self-adjoint because $A_X = \delta^2\psi_X(\cdot)1 = \delta^2\langle \cdot, 1\rangle 1$ implies

$$\langle A_X(a), b\rangle = \langle \delta^2\langle a, 1\rangle 1, b\rangle = \delta^2\langle a, 1\rangle\langle 1, b\rangle = \delta^2\langle a, 1\rangle\overline{\langle b, 1\rangle} = \langle a, \delta^2\langle b, 1\rangle 1\rangle = \langle a, A_X(b)\rangle.$$

We continue by checking the equalities from Definition 2.5.
(1) The equality

$$m_X(A_X \otimes A_X)(a \otimes b) = \delta^2\langle a, 1\rangle\delta^2\langle b, 1\rangle 1 = \delta^4\langle a \otimes b, 1 \otimes 1\rangle 1$$

implies

$$m_X(A_X \otimes A_X)(m_X^\dagger(a)) = \delta^4\langle m_X^\dagger(a), 1 \otimes 1\rangle 1 = \delta^4\langle a, 1\rangle 1 = \delta^2 A_X(a).$$

(2) We have

$$\begin{aligned}(id \otimes \eta_X^\dagger m_X)(id \otimes A_X \otimes id)(a \otimes b \otimes c) &= (id \otimes \psi_X m_X)(a \otimes A_X(b) \otimes c) \\ &= a\delta^2\psi_X(\psi_X(b)c) = a\delta^2\psi_X(b)\psi_X(c) \\ &= (id \otimes \psi_X)(a \otimes b)\delta^2\psi_X(c)\end{aligned}$$

and thus we get

$$\begin{aligned}(id\otimes\eta_X^\dagger m_X)(id\otimes A_X\otimes id)(m_X^\dagger\eta_X\otimes id)(1\otimes c)\\ &= (id\otimes\eta_X^\dagger m_X)(id\otimes A_X\otimes id)(m_X^\dagger\eta_X(1)\otimes c)\\ &= (id\otimes\psi_X)((m_X^\dagger\eta_X(1))\delta^2\psi_X(c) \overset{Lemma\ 2.9}{=} 1\delta^2\psi_X(c) = A_X(c).\end{aligned}$$

3) We conclude from the equality

$$m_X(A_X\otimes id)(a\otimes b) = m_X(\delta^2\psi_X(a)1\otimes b) = \delta^2\psi_X(a)b = \delta^2(\psi_X\otimes id)(a\otimes b)$$

that

$$m_X(A_X\otimes id)m_X^\dagger = \delta^2(\psi_X\otimes id)m_X^\dagger \overset{Lemma\ 2.9}{=} \delta^2 id.$$

Hence, $(\mathcal{O}(X),\psi_X,A_X)$ is a quantum graph.
If $(\mathcal{O}(X),\psi_X) = (\mathbb{C}^n,\psi_n)$, then

$$A_X(e_k) = \sqrt{n}^2\psi_n(e_k)1 = n\frac{1}{n}1 = 1 = \sum_{i=1}^n e_i$$

for all $1\leqslant k\leqslant n$. This implies $(A_X)_{ij} = 1$ for all $1\leqslant i,j\leqslant n$, hence $(\mathbb{C}^n,\psi_n,A_X)$ is the complete classical graph.

ii) Of course, the identity map is self-adjoint. We check again the three conditions from Definition 2.5 to show that X with $A_X = id$ is a quantum graph.
(1)

$$m_X(A_X\otimes A_X)m_X^\dagger = m_X(id\otimes id)m_X^\dagger = m_Xm_X^\dagger = \delta^2 id = \delta^2 A_X$$

(2) We have

$$\begin{aligned}(id\otimes\eta_X^\dagger m_X)(id\otimes A_X\otimes id) &= (id\otimes\eta_X^\dagger m_X)(id\otimes id\otimes id)\\ &= id\otimes\psi_Xm_X = (id\otimes\psi_X)(id\otimes m_X)\end{aligned}$$

and therefore

$$\begin{aligned}&(id\otimes\eta_X^\dagger m_X)(id\otimes A_X\otimes id)(m_X^\dagger\eta_X\otimes id)(1\otimes a)\\ &= (id\otimes\eta_X^\dagger m_X)(id\otimes A_X\otimes id)(m_X^\dagger\eta_X(1)\otimes a)\\ &= (id\otimes\psi_X)(id\otimes m_X)(m_X^\dagger\eta_X(1)\otimes a)\\ &= (id\otimes\psi_X)(id\otimes m_X)(m_X^\dagger\otimes id)(1\otimes a)\\ &= (id\otimes\psi_X)((m_X\otimes id)(id\otimes m_X^\dagger))^\dagger(1\otimes a)\\ &\overset{Lemma\ 2.9}{=} (id\otimes\psi_X)(m_X^\dagger m_X)^\dagger(1\otimes a)\end{aligned}$$

$$= (id \otimes \psi_X)(m_X^\dagger m_X)(1 \otimes a)$$
$$= (id \otimes \psi_X)(m_X^\dagger(a)) \overset{Lemma\ 2.9}{=} a = A_X(a).$$

(3)

$$m_X(A_X \otimes id)m_X^\dagger = m_X(id \otimes id)m_X^\dagger = m_X m_X^\dagger = \delta^2 id.$$

If $(\mathcal{O}(X), \psi_X) = (\mathbb{C}^n, \psi_n)$, then $A_X(e_k) = e_k$ for all $1 \leqslant k \leqslant n$. Hence $(A_X)_{ij} = \delta_{ij}$ and therefore $(\mathbb{C}^n, \psi_n, A_X)$ is the trivial classical graph. □

2.2 Quantum Automorphism Groups of Quantum Graphs

Next we want to introduce quantum automorphism groups of quantum sets and quantum graphs. For this we first look at classical automorphism groups and quantum automorphism groups of classical graphs to motivate the definition of quantum automorphism groups of quantum graphs. We show again that the quantum objects are the non-commutative analogs of the classical ones. At the end of this section we look at a more concrete example of a quantum automorphism group of some quantum graph and prove that the CQGs S_n^+ and PO_n^+ are quantum automorphism groups of some quantum sets.
The following two definitions, the next proposition and its proof are taken from [Sch20].

Definition 2.11. Let X be a classical graph (see Convention 2.6). A *graph automorphism* is a bijection $\sigma : V(X) \to V(X)$ such that

$$(i,j) \in E(X) \iff (\sigma(i), \sigma(j)) \in E(X).$$

The set of all graph automorphisms of X together with the composition forms a group which we denote with $\mathrm{Aut}(X)$.

Remark 2.12. Every $\sigma \in \mathrm{Aut}(X)$ we can view as a matrix

$$\sigma \in M_n(\{0,1\}) \text{ where } \sigma_{ij} = \delta_{\sigma(i)j}$$

and n is the number of vertices of X. Then $(i,j) \in E(X) \iff (\sigma(i), \sigma(j)) \in E(X)$ is equivalent to $A_X \sigma = \sigma A_X$. Moreover, we have for all $1 \leqslant i, j \leqslant n$ the equalities $\sigma_{ij}^* = \sigma_{ij} = \sigma_{ij}^2$ and $\sum_{k=1}^n \sigma_{ik} = \sum_{k=1}^n \sigma_{kj} = 1$ since σ is bijective. Hence σ is a permutation matrix (i.e. a matrix in $M_n(\{0,1\})$ with exactly one non-zero entry in

every row and column) which commutes with A_X.
Conversely, if we have a permutation matrix $(\sigma_{ij})_{ij}$ which commutes with A_X. Then

$$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}, \quad \sigma(i) := j \text{ if } \sigma_{ij} = 1$$

is a graph automorphism. Therefore we can identify $\mathrm{Aut}(X)$ with all permutation matrices that commute with A_X.

This motivates the following definition.

Definition 2.13. Let X be a classical graph with n vertices and adjacency matrix A_X. We define $\mathcal{C}(G_X)$ as the universal C^*-algebra with generators $\{u_{ij}\}_{i,j=1,\dots,n}$ and relations $u_{ij}^* = u_{ij} = u_{ij}^2$, $\sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1$ for all $i, j = 1, \dots, n$ and $uA_X = A_X u$ (with $u = (u_{ij})_{ij}$).
One can show that $G_X = (\mathcal{C}(G_X), \Delta)$ with $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ is a CQG [[Sch20], Lemma 2.1.2]. We call G_X the *quantum automorphism group* of X.

Now we want to show that the quantum automorphism group is just the non-commutative analog of the classical graph automorphism group. So we show that if we add the relations $u_{ij}u_{kl} = u_{kl}u_{ij}$ to the universal C^*-algebra $\mathcal{C}(G_X)$ we get $C(\mathrm{Aut}(X))$, the continuous functions on $\mathrm{Aut}(X)$.

Proposition 2.14. *Let X be a classical graph and*

$$A := C^*(u_{ij} \mid u_{ij}^* = u_{ij} = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1, uA_X = A_X u, u_{ij}u_{kl} = u_{kl}u_{ij}),$$

then $A \cong C(Aut(X))$.

Proof. First note that A is also a CQG with Δ from Definition 2.13. This can be shown similarly to [[Sch20], Lemma 2.1.2]. Because A is commutative we get again $A \cong C(\mathrm{Spec}(A))$ by Gelfand-Naimark. From Remark 1.13 *ii*) we know that

$$m : \mathrm{Spec}(A) \times \mathrm{Spec}(A) \to \mathrm{Spec}(A), \quad (\varphi_1, \varphi_2) \mapsto (\varphi_1, \varphi_2) \circ \Delta$$

turns $\mathrm{Spec}(A)$ into a compact group.
Let $\sigma \in \mathrm{Aut}(X)$. Then we get by Remark 2.12 and since $\sigma_{ij} \in \{0, 1\}$ (so the elements commute) that the matrix elements σ_{ij} satisfy the relations from the universal C^*-algebra A. Hence there exists a unique *-homomorphism $\varphi_\sigma : A \to \mathbb{C}$ with $\varphi_\sigma(u_{ij}) = \sigma_{ij}$. Moreover,

$$\varphi_\sigma(1) = \varphi_\sigma(\sum_k u_{ik}) = \sum_k \sigma_{ik} = 1.$$

This implies $\varphi_\sigma \neq 0$ and therefore $\varphi_\sigma \in \text{Spec}(A)$. Thus we can define the map

$$\Phi : \text{Aut}(X) \to \text{Spec}(A), \quad \sigma \mapsto \varphi_\sigma.$$

This map is obviously injective since $\varphi_\sigma = \varphi_{\tilde{\sigma}}$ implies $\sigma_{ij} = \varphi_\sigma(u_{ij}) = \varphi_{\tilde{\sigma}}(u_{ij}) = \tilde{\sigma}_{ij}$ for all i, j. For $\varphi \in \text{Spec}(A)$ we define $\sigma_{ij} := \varphi(u_{ij})$. Then $(\sigma_{ij})_{ij}$ is a permutation matrix which commutes with A_X since the elements u_{ij} satisfy the corresponding relations and φ is a unital *-homomorphism. Hence $\sigma \in \text{Aut}(X)$ and $\varphi_\sigma(u_{ij}) = \sigma_{ij} = \varphi(u_{ij})$, so $\varphi_\sigma = \varphi$. Therefore, Φ is also surjective.
Now it is left to show that Φ is also a group homomorphism. This follows from

$$\begin{aligned}\Phi(\sigma \circ \tilde{\sigma})(u_{ij}) = (\sigma \circ \tilde{\sigma})_{ij} = \sum_k \sigma_{ik}\tilde{\sigma}_{kj} = \sum_k \varphi_\sigma(u_{ik})\varphi_{\tilde{\sigma}}(u_{kj}) = \sum_k (\varphi_\sigma, \varphi_{\tilde{\sigma}})(u_{ik} \otimes u_{kj}) \\ = (\varphi_\sigma, \varphi_{\tilde{\sigma}})(\Delta(u_{ij})) = m(\varphi_\sigma, \varphi_{\tilde{\sigma}})(u_{ij}).\end{aligned}$$

Moreover, the map Φ is continuous since $\text{Aut}(X)$ is finite and therefore $\text{Spec}(A) = \{\varphi_\sigma \mid \sigma \in \text{Aut}(X)\}$ is also finite. Hence $\text{Spec}(A) \cong \text{Aut}(X)$ as compact groups. This also implies $C(\text{Spec}(A)) \cong C(\text{Aut}(X))$ as compact groups via the group isomorphism

$$C(\text{Spec}(A)) \to C(\text{Aut}X)), \quad g \mapsto g \circ \Phi.$$

Finally, $A \cong C(\text{Aut}(X))$. □

Remark 2.15. In the following chapters we will define some certain universal *-algebras and denote them with the letter $\mathcal{O}$. All this universal *-algebras can also be defined as universal C^*-algebras. Since they are all generated by the elements of a unitary matrix, one can check that they exist. We will not list the definition for the universal C^*-algebras again, but simply denote the corresponding universal C^*-algebra with the letter $\mathcal{C}$. Observe, that Lemma 1.10 implies that the universal C^*-algebra is always dense in the universal *-algebra.
In [BCE+20] only the universal *-algebras are considered, but we also look at the universal C^*-algebras to see the connection of the quantum automorphism groups of quantum graphs and classical graphs.

So now we also want to define quantum automorphism groups of quantum sets and quantum graphs. To motivate the definition we first note the following fact:
If X is a classical graph and u_{ij} the generating elements from $\mathcal{C}(G_X)$, then $(u_{ij})_{ij}$ is a unitary matrix and the map

$$\rho_X : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathcal{C}(G_X), \quad \rho_X(e_i) := \sum_{j=1}^n e_j \otimes u_{ji}$$

is a unital *-homomorphism satisfying

$$\rho_X(A_X\cdot) = (A_X \otimes id)\rho_X.$$

This leads us to the following definition.

Definition 2.16. *i*) Let $X = (\mathcal{O}(X), \psi_X)$ be a quantum set with $n = |X|$ and fix an orthonormal basis $\{e_i\}_{i=1}^n$ for $L^2(X)$.
We define $\mathcal{C}(\mathrm{Aut}^+(X))$ as the universal C^*-algebra generated by the coefficients u_{ij} of a unitary matrix $u = (u_{ij})_{ij} \in M_n(\mathcal{C}(\mathrm{Aut}^+(X)))$ that satisfies the relations which make the map

$$\rho_X : \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{C}(\mathrm{Aut}^+(X)), \quad \rho_X(e_i) := \sum_{j=1}^n e_j \otimes u_{ji}$$

a unital *-homomorphism.

ii) Let $X = (\mathcal{O}(X), \psi_X, A_X)$ be a quantum graph with $n = |X|$ and fix an orthonormal basis $\{e_i\}_{i=1}^n$ for $L^2(X)$.
We define $\mathcal{O}(G_X)$ to be the universal *-algebra generated by the entries of a unitary matrix $u = (u_{ij})_{i,j=1}^n \in M_n(\mathcal{O}(G_X))$ that fulfils the relations which make the map

$$\rho_X : \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G_X), \quad \rho_X(e_i) := \sum_{j=1}^n e_j \otimes u_{ji}$$

a unital *-homomorphism satisfying the A_X-covariance condition, i.e.

$$\rho_X(A_X\cdot) = (A_X \otimes id)\rho_X.$$

The notation $\mathcal{O}(G_X)$ implies that this *-algebra comes from a CQG G_X. We will later see that this is the case for both *-algebras, so they are associated to some CQGs $\mathrm{Aut}^+(X)$ and G_X. We call $\mathrm{Aut}^+(X)$ and G_X the *quantum automorphism group* of X and u the *fundamental representation* of the respective CQG.

For better understanding of the definition, let us see what the properties of ρ_X mean for the fundamental representation u.

Lemma 2.17. *Let X be a quantum graph with $|X| = n$. If we view u as a linear map*

$$u : L^2(X) \otimes \mathcal{O}(G_X) \to L^2(X) \otimes \mathcal{O}(G_X), \quad u(\xi \otimes a) := \sum_{i,j=1}^n b_{ij}(\xi) \otimes u_{ij}a.$$

Then

i) $u(\cdot \otimes 1) = \rho_X$ *and* $u(\xi \otimes a) = u(\xi \otimes 1)(1 \otimes a)$ *for all* $\xi \in L^2(X)$, $a \in \mathcal{O}(G_X)$,

ii) u *is unital,*

iii) $u(\xi^* \otimes 1) = u(\xi \otimes 1)^*$ *for all* $\xi \in L^2(X)$,

iv) $u(\xi_1 \otimes 1)u(\xi_2 \otimes 1) = u(\xi_1\xi_2 \otimes 1)$ *for all* $\xi_1, \xi_2 \in L^2(X)$,

v) $u(A_X \otimes id) = (A_X \otimes id)u$.

Viewing u *and* A_X *as matrices, the last point is equivalent to*

$$\sum_{j=1}^{n} e_j \otimes (uA_X)_{jk} = \sum_{j=1}^{n} e_j \otimes (A_X u)_{jk}. \quad (1 \leqslant k \leqslant n)$$

Proof. $i)$ We have

$$u(e_k \otimes 1) = \sum_{i,j=1}^{n} b_{ij}(e_k) \otimes u_{ij} = \sum_{i,j=1}^{n} \delta_{jk} e_i \otimes u_{ij} = \sum_{i=1}^{n} e_i \otimes u_{ik} = \rho_X(e_k),$$

hence $u(\cdot \otimes 1) = \rho_X$. Moreover,

$$u(\xi \otimes a) = \sum_{i,j=1}^{n} b_{ij}(\xi) \otimes u_{ij} a = (\sum_{i,j=1}^{n} b_{ij}(\xi) \otimes u_{ij})(1 \otimes a) = u(\xi \otimes 1)(1 \otimes a)$$

for all $\xi \in L^2(X)$ and $a \in \mathcal{O}(G_X)$.
Using that ρ_X is a unital *-homomorphism we get the corresponding properties of u as a linear map.

ii) $u(1 \otimes 1) = \rho_X(1) = 1 \otimes 1$

iii) $u(\xi^* \otimes 1) = \rho_X(\xi^*) = \rho_X(\xi)^* = u(\xi \otimes 1)^*$

iv) $u(\xi_1 \otimes 1)u(\xi_2 \otimes 1) = \rho_X(\xi_1)\rho_X(\xi_2) = \rho_X(\xi_1\xi_2) = u(\xi_1\xi_2 \otimes 1)$

v)
$$\begin{aligned} u(A_X\xi \otimes a) &= u(A_X\xi \otimes 1)(1 \otimes a) = \rho_X(A_X\xi)(1 \otimes a) = (A_X \otimes id)\rho_X(\xi)(1 \otimes a) \\ &= (A_X \otimes id)u(\xi \otimes 1)(1 \otimes a) = (A_X \otimes id)u(\xi \otimes a). \end{aligned}$$

If we view A_X as a matrix $(a_{ij})_{ij} \in M_n(\mathbb{C})$, then the equality $v)$ implies

$$\begin{aligned} \sum_{j=1}^{n} e_j \otimes (uA_X)_{jk} &= \sum_{j=1}^{n} e_j \otimes \sum_{i=1}^{n} u_{ji} a_{ik} = \sum_{i=1}^{n} a_{ik} \sum_{j=1}^{n} e_j \otimes u_{ji} = \sum_{i=1}^{n} a_{ik} u(e_i \otimes 1) \\ &= u(\sum_{i=1}^{n} a_{ik} e_i \otimes 1) = u(A_X e_k \otimes 1) \overset{v)}{=} (A_X \otimes id)u(e_k \otimes 1) \\ &= \sum_{i=1}^{n} A_X e_i \otimes u_{ik} = \sum_{i,j=1}^{n} a_{ji} e_j \otimes u_{ik} = \sum_{j=1}^{n} e_j \otimes \sum_{i=1}^{n} a_{ji} u_{ik} \\ &= \sum_{j=1}^{n} e_j \otimes (A_X u)_{jk}. \end{aligned}$$

and since

$$\begin{aligned} u(A_X e_k \otimes 1)(1 \otimes a) &= u(A_X e_k \otimes a) \\ \text{and } (A_X \otimes id)u(e_k \otimes 1)(1 \otimes a) &= (A_X \otimes id)u(e_k \otimes a) \end{aligned}$$

we get the equivalence. □

Remark 2.18. In some sources the generating matrix of the quantum automorphism group does not have to be a unitary, but ρ_X has to satisfy the ψ_X-invariance condition $(\psi_X \otimes id)\rho_X = \psi_X(\cdot)1$. These two statements are equivalent. We just show one direction of this equivalence since the other is way more complicated to show. If u is a unitary, i.e. $\sum_{k=1}^n u_{ki}^* u_{kj} = \sum_{k=1}^n u_{ik} u_{jk}^* = \delta_{ij}$, then

$$\begin{aligned} (\psi_X \otimes id)(\rho_X(e_i^* e_k)) &= (\psi_X \otimes id)(\rho_X(e_i)^* \rho(e_k)) \\ &= (\psi_X \otimes id)(\sum_{j=1}^n e_j^* \otimes u_{ji}^*)(\sum_{l=1}^n e_l \otimes u_{lk})) \\ &= \sum_{j,l=1}^n \psi_X(e_j^* e_l) \otimes u_{ji}^* u_{lk} \\ &= \sum_{j,l=1}^n \delta_{jl} \otimes u_{ji}^* u_{lk} = \sum_{j=1}^n u_{ji}^* u_{jk} \\ &= \delta_{ik} = \psi_X(e_i^* e_k)1 \end{aligned}$$

and this implies $(\psi_X \otimes id)\rho_X = \psi_X(\cdot)1$ since with $1 = \sum_{i=1}^n \alpha_i e_i$ we get

$$\begin{aligned} (\psi_X \otimes id)(\rho_X(e_k)) = (\psi_X \otimes id)(\rho_X(1^* e_k)) &= (\psi_X \otimes id)(\rho_X(\sum_{i=1}^n \overline{\alpha_i} e_i^* e_k)) \\ &= \sum_{i=1}^n \overline{\alpha_i}(\psi_X \otimes id)(\rho_X(e_i^* e_k)) = \sum_{i=1}^n \overline{\alpha_i} \psi_X(e_i^* e_k)1 \\ &= \psi_X(\sum_{i=1}^n \overline{\alpha_i} e_i^* e_k)1 = \psi_X(e_k)1. \end{aligned}$$

To obtain the CQGs $\mathrm{Aut}^+(X)$ and G_X from the *-algebras $\mathcal{C}(\mathrm{Aut}^+(X))$ and $\mathcal{O}(G_X)$ we need (among other requirements) that these *-algebras are Hopf *-algebras. This result is proved in the following proposition.

Proposition 2.19. *The *-algebras $\mathcal{C}(Aut^+(X))$ and $\mathcal{O}(G_X)$ admit a Hopf *-algebra structure defined by*

$$\Delta(u_{ij}) := \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad S(u_{ij}) := u_{ji}^*, \quad \epsilon(u_{ij}) := \delta_{ij} \quad (1 \leqslant i, j \leqslant n).$$

Proof. Both *-algebras are unital since u is a unitary matrix. First of all, one can check that the defined (*-)homomorphism exist, so e.g. the matrix $(\sum_{k=1}^{n} u_{ik} \otimes u_{kj})_{ij}$ fulfills the same properties as $(u_{ij})_{ij}$. Moreover,

$$\begin{aligned}\Delta(1) &= \Delta(\sum_{j=1}^{n} u_{ij}u_{ij}^*) = \sum_{j=1}^{n} \Delta(u_{ij})\Delta(u_{ij})^* = \sum_{j=1}^{n}(\sum_{k=1}^{n} u_{ik} \otimes u_{kj})(\sum_{l=1}^{n} u_{il}^* \otimes u_{lj}^*) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} u_{ik}u_{il}^* \otimes u_{kj}u_{lj}^* = \sum_{k=1}^{n}\sum_{l=1}^{n} u_{ik}u_{il}^* \otimes \delta_{kl} = \sum_{k=1}^{n} u_{ik}u_{ik}^* \otimes 1 = 1 \otimes 1\end{aligned}$$

and analogously one can show that S and ϵ are unital.
We continue by checking the conditions $1) - 3)$ from Definition 1.16.
1)

$$\begin{aligned}(id \otimes \Delta)\Delta(u_{ij}) &= \sum_{k=1}^{n} u_{ik} \otimes \Delta(u_{kj}) = \sum_{k=1}^{n}\sum_{l=1}^{n} u_{ik} \otimes u_{kl} \otimes u_{lj} \\ &= \sum_{l=1}^{n} \Delta(u_{il}) \otimes u_{lj} = (\Delta \otimes id)\Delta(u_{ij})\end{aligned}$$

2)

$$\begin{aligned}m(id \otimes S)(\Delta(u_{ij})) &= m(\sum_{k=1}^{n} u_{ik} \otimes S(u_{kj})) = m(\sum_{k=1}^{n} u_{ik} \otimes u_{jk}^*) \\ &= \sum_{k=1}^{n} u_{ik}u_{jk}^* = \delta_{ij} = \epsilon(u_{ij})1\end{aligned}$$

3)

$$(\epsilon \otimes id)(\Delta(u_{ij})) = \sum_{k=1}^{n} \epsilon(u_{ik}) \otimes u_{kj} = \sum_{k=1}^{n} \delta_{ik} \otimes u_{kj} = u_{ij}$$

and in the same way one can show $(id \otimes \epsilon)\Delta = id$. □

Now we want to check whether the notation $\mathcal{C}(G_X)$ is fine, i.e. whether Definition 2.13 and 2.16 are compatible.

Proposition 2.20. *If X is a classical graph, then $\mathcal{C}(G_X)$ from Definition 2.16 is the same as the quantum automorphism group from Definition 2.13. Moreover, we get $Aut^+(\mathbb{C}^n, \psi_n) = S_n^+$.*

Proof. If we look at a classical graph with n vertices, we consider again the quantum set $(\mathbb{C}^n, \psi_n)$. Let $(e_j)_i = \sqrt{n}\delta_{ij}$. Then $\{e_i\}_i$ is an orthonormal basis for $\mathbb{C}^n$ since

$$\langle e_i, e_j \rangle = \psi_n(e_j^* e_i) = \psi_n(\delta_{ij}\sqrt{n}e_i) = \delta_{ij}\frac{1}{n}\sqrt{n}^2 = \delta_{ij}.$$

Note that $\sum_{j=1}^n e_j \otimes x_j = \sum_{j=1}^n e_j \otimes y_j \iff x_i = y_i$ for all $1 \leqslant i \leqslant n$ since the equality of the sums implies

$$\sqrt{n} e_i \otimes x_i = (e_i \otimes 1)(\sum_{j=1}^n e_j \otimes x_j) = (e_i \otimes 1)(\sum_{j=1}^n e_j \otimes y_j) = \sqrt{n} e_i \otimes y_i$$

and therefore $e_i \otimes (x_i - y_i) = 0 \iff x_i = y_i$.
We have

$$\sum_{j=1}^n e_j \otimes u_{ji} = \rho_X(e_i) = \rho_X(e_i^*) = \rho_X(e_i)^* = \sum_{j=1}^n e_j \otimes u_{ji}^*,$$

which is equivalent to $u_{ji} = u_{ji}^*$ for all i, j. Moreover,

$$\begin{aligned}\sqrt{n} \sum_{j=1}^n e_j \otimes \delta_{ik} u_{ji} = \sqrt{n} \delta_{ik} \sum_{j=1}^n e_j \otimes u_{ji} &= \sqrt{n} \delta_{ik} \rho_X(e_i) = \rho_X(e_i e_k) \\ &= \rho_X(e_i)\rho_X(e_k) = (\sum_{j=1}^n e_j \otimes u_{ji})(\sum_{l=1}^n e_l \otimes u_{lk}) \\ &= \sum_{j,l=1}^n e_j e_l \otimes u_{ji} u_{lk} = \sqrt{n} \sum_{j=1}^n e_j \otimes u_{ji} u_{jk},\end{aligned}$$

which is equivalent to $\delta_{ik} u_{ji} = u_{ji} u_{jk}$ for all i, j. In particular $u_{ij} = u_{ij}^2$. Since u is a unitary we have

$$\sum_{k=1}^n u_{ki}^* u_{kj} = \sum_{k=1}^n u_{ik} u_{jk}^* = \delta_{ij}$$

and therefore we get

$$\sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1.$$

The relations

$$u_{ij} = u_{ij}^* = u_{ij}^2 \text{ and } \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1$$

already imply $\delta_{ik} u_{ji} = u_{ji} u_{jk}$ in a C^*-algebra [[Sch20], Remark 1.1.9] and they imply $\rho_X(1) = 1 \otimes 1$ since

$$\rho_X(1) = \rho_X(\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i) = \frac{1}{\sqrt{n}} \sum_{i,j=1}^n e_j \otimes u_{ji} = \frac{1}{\sqrt{n}} \sum_{j=1}^n e_j \otimes \sum_{i=1}^n u_{ji} = 1 \otimes 1.$$

Thus we get that u satisfying the relations which make ρ_X a unital *-homomorphism is equivalent to

$$u_{ij} = u_{ij}^* = u_{ij}^2 \text{ and } \sum_{k=1}^n u_{kj} = \sum_{k=1}^n u_{ik} = 1.$$

Moreover, the A_X-covariance condition is equivalent to $uA_X = A_Xu$ because from Lemma 2.17 we get $\sum_{j=1}^n e_j \otimes (uA_X)_{jk} = \sum_{j=1}^n e_j \otimes (A_Xu)_{jk}$ for all $1 \leqslant k \leqslant n$. Hence we have

$$\mathcal{C}(G_X) = C^*(u_{ij} \mid u_{ij}^* = u_{ij} = u_{ij}^2, \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1, uA_X = A_Xu).$$

Therefore both definitions coincide for classical graphs.
This proof also shows $\mathrm{Aut}^+(\mathbb{C}^n, \psi_n) = S_n^+$ since in the definition of quantum automorphism groups of quantum graphs, the only additional condition is the A_X-covariance condition compared to the definition of the quantum automorphism groups of quantum sets. Hence we get

$$\mathcal{C}(\mathrm{Aut}^+(\mathbb{C}^n, \psi_n)) = C^*(u_{ij} \mid u_{ij}^* = u_{ij} = u_{ij}^2, \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1) = \mathcal{O}(S_n^+)$$

and therefore $\mathrm{Aut}^+(\mathbb{C}^n, \psi_n) = S_n^+$ since the comultiplications are also the same. □

The last point in this chapter is a more concrete example of a quantum automorphism group of a quantum graph. The following proposition and its proof is similar to [[Wan98], Theorem 4.1].

Proposition 2.21. *Let $X = (M_n(\mathbb{C}), tr, A)$ be the quantum set from Example 2.4 iii) with some quantum adjacency matrix A. Then $\mathcal{O}(G_X)$ is the universal *-algebra generated by elements v_{kl}^{ij} with $1 \leqslant i, j, k, l \leqslant n$ which satisfy the following relations*

$$v_{kl}^{ij*} = v_{lk}^{ji} \tag{2.1}$$

$$\sum_{k,l=1}^n v_{lk}^{ji} v_{kl}^{rs} = \delta_{ir}\delta_{js} = \sum_{k,l=1}^n v_{ij}^{kl} v_{sr}^{lk} \tag{2.2}$$

$$\sum_{q=1}^n v_{kq}^{ij} v_{ql}^{rs} = \delta_{jr} v_{kl}^{is} \tag{2.3}$$

$$(v_{kl}^{ij})_{klij} A = A (v_{kl}^{ij})_{klij}. \tag{2.4}$$

Thus, we also have $Aut^+(M_n(\mathbb{C}), tr) = PO_n^+$.

Proof. The proof is quite similar to the one of Proposition 2.20.
Let $(e_{ij})_{kl} = \sqrt{n}\delta_{ik}\delta_{jl}$. Then $\{e_{ij}\}_{ij}$ is an orthonormal basis for $M_n(\mathbb{C})$ since

$$\langle e_{ij}, e_{kl} \rangle = tr(e_{kl}^* e_{ij}) = tr(e_{lk}e_{ij}) = \delta_{ki}\sqrt{n} tr(e_{lj}) = \delta_{ki}\sqrt{n}\delta_{lj}\frac{1}{n}\sqrt{n} = \delta_{ki}\delta_{lj}.$$

One can show that

$$\sum_{k,l=1}^n e_{kl} \otimes x_{kl} = \sum_{k,l=1}^n e_{kl} \otimes y_{kl} \Longleftrightarrow x_{ij} = y_{ij} \text{ for all } 1 \leqslant i, j \leqslant n.$$

We have $\rho_X(e_{ij}) = \sum_{k,l} e_{kl} \otimes v_{kl}^{ij}$. Therefore we get

$$\sum_{k,l=1}^{n} e_{kl} \otimes v_{kl}^{ji} = \rho_X(e_{ji}) = \rho_X(e_{ij})^* = \sum_{k,l=1}^{n} e_{kl}^* \otimes v_{kl}^{ij*} = \sum_{k,l=1}^{n} e_{lk} \otimes v_{kl}^{ij*},$$

which is equivalent to (2.1). Furthermore, $(v_{ij}^{kl})_{ijkl}$ has to be a unitary matrix. This implies

$$\delta_{ir}\delta_{js} = \sum_{k,l=1}^{n} (v^*)_{ij}^{kl} v_{kl}^{rs} = \sum_{k,l=1}^{n} v_{kl}^{ij*} v_{kl}^{rs} \overset{(2.1)}{=} \sum_{k,l=1}^{n} v_{lk}^{ji} v_{kl}^{rs}$$

and

$$\delta_{ir}\delta_{js} = \sum_{k,l=1}^{n} v_{ij}^{kl} (v^*)_{kl}^{rs} = \sum_{k,l=1}^{n} v_{ij}^{kl} v_{rs}^{kl*} \overset{(2.1)}{=} \sum_{k,l=1}^{n} v_{ij}^{kl} v_{sr}^{lk}$$

which is equivalent to (2.2). Moreover,

$$\begin{aligned}
\sqrt{n} \sum_{k,l=1}^{n} e_{kl} \otimes \delta_{jr} v_{kl}^{is} &= \rho_X(\delta_{jr}\sqrt{n} e_{is}) = \rho_X(e_{ij} e_{rs}) = \rho_X(e_{ij})\rho_X(e_{rs}) \\
&= (\sum_{k,l=1}^{n} e_{kl} \otimes v_{kl}^{ij})(\sum_{p,q=1}^{n} e_{pq} \otimes v_{pq}^{rs}) = \sum_{k,l,p,q=1}^{n} e_{kl} e_{pq} \otimes v_{kl}^{ij} v_{pq}^{rs} \\
&= \sqrt{n} \sum_{k,l,p,q=1}^{n} e_{kq} \delta_{lp} \otimes v_{kl}^{ij} v_{pq}^{rs} = \sqrt{n} \sum_{k,l,q=1}^{n} e_{kq} \otimes v_{kl}^{ij} v_{lq}^{rs} \\
&= \sqrt{n} \sum_{k,q=1}^{n} e_{kq} \otimes \sum_{l=1}^{n} v_{kl}^{ij} v_{lq}^{rs},
\end{aligned}$$

and this is equivalent to (2.3). By Proposition 2.19 we know that $\mathcal{O}(G_X)$ is a Hopf *-algebra with $S(v_{kl}^{ij}) = v_{ij}^{kl*} \overset{(2.1)}{=} v_{ji}^{lk}$. Applying S to both sides of (2.3) leads us to

$$\delta_{jr} v_{si}^{lk} = S(\delta_{jr} v_{kl}^{is}) = S(\sum_{q=1}^{n} v_{kq}^{ij} v_{ql}^{rs}) = \sum_{q=1}^{n} v_{sr}^{lq} v_{ji}^{qk}. \tag{$\star$}$$

With these relations we also get that ρ_X is unitary because

$$\begin{aligned}
\rho_X(1) &= \rho_X(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_{ii}) = \frac{1}{\sqrt{n}} \sum_{k,l,i=1}^{n} e_{kl} \otimes v_{kl}^{ii} = \frac{1}{\sqrt{n}} \sum_{k,l=1}^{n} e_{kl} \otimes \sum_{i=1}^{n} v_{kl}^{ii} \\
&\overset{(\star)}{=} \frac{1}{\sqrt{n}} \sum_{k,l=1}^{n} e_{kl} \otimes \sum_{i,j=1}^{n} v_{kp}^{ij} v_{pl}^{ji} \overset{(2.2)}{=} \frac{1}{\sqrt{n}} \sum_{k,l=1}^{n} e_{kl} \otimes \delta_{kl} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e_{kk} \otimes 1 = 1 \otimes 1.
\end{aligned}$$

Furthermore, the A-covariance condition is equivalent (2.4) because from Lemma 2.17 we get that it is equivalent to $\sum_{k,l} e_{kl} \otimes (vA)_{klij} = \sum_{k,l} e_{kl} \otimes (Av)_{klij}$ for all $1 \leqslant i, j \leqslant n$ where $v = (v_{kl}^{ij})_{klij}$.

Similarly as in the proof of Proposition 2.20 this also implies that $\mathcal{C}(\mathrm{Aut}^+(M_n(\mathbb{C}), tr))$ is the universal C^*-algebra generated by elements v_{kl}^{ij} which satisfy the relations (2.1)-(2.3) and hence $\mathrm{Aut}^+(M_n(\mathbb{C}), tr) = PO_n^+$ since the comultiplications are also the same by Proposition 2.19. □

Chapter 3

Linking Algebras of Quantum Graphs

In this chapter we want to define a quantum isomorphism notion for quantum graphs. For this we need the definition of some linking algebra of two quantum graphs. We also look at the definition of bigalois extensions, which is a quantum analog of a torsor in the context of group actions, and show that this linking algebra is a bigalois extension if it is non-zero.
The source for this chapter is again mainly [BCE+20].

3.1 Bigalois Extensions

In this section, we will introduce bigalois extensions. We show that a bigalois extension is the quantum analog of a torsor and that every Hopf *-algebra itself is a bigalois extension. First, we need the notion of a *-comodule algebra and a Galois extension. Throughout this section, let (A, Δ, S, ϵ) be a Hopf *-algebra.

Definition 3.1. A *left A *-comodule algebra* is a unital *-algebra Z equipped with a unital *-homomorphism $\alpha : Z \to A \otimes Z$ which satisfies

1) $(id \otimes \alpha)\alpha = (\Delta \otimes id)\alpha$

2) $(\epsilon \otimes id)\alpha = id.$

Similarly, a *right A *-comodule algebra* is a unital *-algebra Z equipped with a unital *-homomorphism $\beta : Z \to Z \otimes A$ which satisfies

1) $(\beta \otimes id)\beta = (id \otimes \Delta)\beta$

2) $(id \otimes \epsilon)\beta = id$.

Definition 3.2. A left A *-comodule algebra (Z, α) is called a *left A Galois extension* if the linear map

$$\kappa_l : Z \otimes Z \to A \otimes Z, \quad \kappa_l(x \otimes y) = \alpha(x)(1 \otimes y)$$

is bijective. Similarly, a right A *-comodule algebra (Z, β) is called a *right A Galois extension* if the linear map

$$\kappa_r : Z \otimes Z \to Z \otimes A, \quad \kappa_r(x \otimes y) = (x \otimes 1)\beta(y)$$

is bijective.

The following example illustrates what a Galois extension means in the context of finite groups.

Example 3.3. If G is a finite group and $G \curvearrowright X$ is an action of G on a finite space X, then we call X a left G-torsor if the action is free and transitive, i.e. $t \cdot x = x$ implies $t = e$ and for all $x, y \in X$ there exists a $g \in G$ with $g \cdot x = y$. This is equivalent to the fact that the map

$$\Phi : G \times X \to X \times X, \quad (g, x) \mapsto (g \cdot x, x)$$

is bijective. We set $\mathcal{O}(X) = C(X)$ and $\mathcal{O}(G) = C(G)$. Then $\mathcal{O}(X)$ is a left $\mathcal{O}(G)$ *-comodule algebra with the map

$$\alpha : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X) \cong C(G \times X), \quad \alpha(f)(g, x) := f(g \cdot x).$$

Moreover, X is a left G-torsor if and only if the map

$$\kappa_l : \mathcal{O}(X) \otimes \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X), \quad \kappa_l(f_1 \otimes f_2) := (f_1 \otimes f_2) \circ \Phi$$

is bijective and that is the case if and only if $\mathcal{O}(X)$ is a left $\mathcal{O}(G)$ Galois extension since

$$(f_1 \otimes f_2)(\Phi(g, x)) = (f_1 \otimes f_2)(g \cdot x, x) = f_1(g \cdot x) f_2(x) = (\alpha(f_1)(1 \otimes f_2))(g, x).$$

Therefore we can regard a Galois extension as a quantum analogue of a torsor in the context of group actions.

Now we finally state the definition of a bigalois extension. In short, it merges the terms of a left and right Galois extension.

Definition 3.4. Let A and B be Hopf *-algebras. A unital *-algebra Z is called an *A-B-bigalois extension* if

1) (Z, α) is a left A Galois extension

2) (Z, β) is a right B Galois extension

3) Z is an A-B-bicomodule algebra, i.e. $(id \otimes \beta)\alpha = (\alpha \otimes id)\beta$.

In fact, every Hopf *-algebra is itself a bigalois extension. This result is formulated in the following proposition. In its proof, the idea for the inverse maps of κ_l and κ_r is taken from [[Sch04], Lemma 4.4.1], but the explicit calculation is new.

Proposition 3.5. *Let (A, Δ, S, ϵ) be a Hopf *-algebra. Then A itself with the map Δ is an A-A-bigalois extension.*

Proof. We know $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$ and $(\epsilon \otimes id)\Delta = (id \otimes \epsilon)\Delta = id$ from the definition of Hopf algebras, hence (A, Δ) is a left and right A *-comodule algebra and an A-A-bicomodule algebra.
To prove that the maps κ_l and κ_r from Definition 3.2 are bijective, we use Sweedler's sumless notation. That is, we write $\Delta(b) = b_{(1)} \otimes b_{(2)}$ for $b \in A$, even if $\Delta(b)$ is not an elementary tensor. So we omit the possible summation sign. This is not a problem because all occurring functions are linear.
The inverse maps are given by

$$\kappa_r^{-1}(a \otimes b) = aS(b_{(1)}) \otimes b_{(2)} \text{ and } \kappa_l^{-1}(a \otimes b) = a_{(1)} \otimes S(a_{(2)})b.$$

We check this by using the equations 1), 2), 3) from Definition 1.16 which imply

1) $b_{(1)} \otimes b_{(2)(1)} \otimes b_{(2)(2)} = b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)}$

2) $b_{(1)}S(b_{(2)}) = \epsilon(b)1 = S(b_{(1)})b_{(2)}$

3) $\epsilon(b_{(1)})b_{(2)} = b_{(1)}\epsilon(b_{(2)}) = b$

for all $b \in A$. Since

$$\kappa_r(a \otimes b) = (a \otimes 1)\Delta(b) = ab_{(1)} \otimes b_{(2)},$$

we get

$$\begin{aligned}\kappa_r^{-1}(\kappa_r(a \otimes b)) = \kappa_r^{-1}(ab_{(1)} \otimes b_{(2)}) &= ab_{(1)}S(b_{(2)(1)}) \otimes b_{(2)(2)} \\ &= (a \otimes 1)(b_{(1)}S(b_{(2)(1)}) \otimes b_{(2)(2)})\end{aligned}$$

$$
\begin{aligned}
&= (a \otimes 1)(m \otimes id)(id \otimes S \otimes id)(b_{(1)} \otimes b_{(2)(1)} \otimes b_{(2)(2)}) \\
&\overset{1)}{=} (a \otimes 1)(m \otimes id)(id \otimes S \otimes id)(b_{(1)(1)} \otimes b_{(1)(2)} \otimes b_{(2)}) \\
&= (a \otimes 1)(b_{(1)(1)} S(b_{(1)(2)}) \otimes b_{(2)}) \\
&\overset{2)}{=} (a \otimes 1)(\epsilon(b_{(1)}) 1 \otimes b_{(2)}) \\
&= (a \otimes 1)(1 \otimes \epsilon(b_{(1)}) b_{(2)}) \\
&\overset{3)}{=} (a \otimes 1)(1 \otimes b) = a \otimes b
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\kappa_r(\kappa_r^{-1}(a \otimes b)) &= \kappa_r(aS(b_{(1)}) \otimes b_{(2)}) = aS(b_{(1)}) b_{(2)(1)} \otimes b_{(2)(2)} \\
&\overset{1)}{=} aS(b_{(1)(1)}) b_{(1)(2)} \otimes b_{(2)} \overset{2)}{=} a\epsilon(b_{(1)}) \otimes b_{(2)} \overset{3)}{=} a \otimes b.
\end{aligned}
$$

In the same way, one can check $\kappa_l^{-1}(a \otimes b) = a_{(1)} \otimes S(a_{(2)})b$. □

3.2 Quantum Isomorphisms of Quantum Graphs

Now we want to introduce the linking algebra of two quantum graphs X and Y to define an isomorphism notion of quantum graphs. We will show later that the linking algebra is a $\mathcal{O}(G_Y)$-$\mathcal{O}(G_X)$-bigalois extension if it is non-zero. But first, we want to look at isomorphisms and quantum isomorphisms of classical graphs to motivate our definition. This is similar to the definition of graph automorphisms and the quantum automorphism group of a classical graph.

Definition 3.6. Let X and Y be to classical graphs. Then X and Y are *isomorphic*, written $X \cong Y$, if there is a bijection $\sigma : V(X) \to V(Y)$ such that

$$(i, j) \in E(X) \iff (\sigma(i), \sigma(j)) \in E(Y).$$

As one can already imagine, similarly to Remark 2.12, one can show that we can identify graph isomorphisms with all permutation matrices σ which fulfill $\sigma A_X = A_Y \sigma$. This motivates again the quantum isomorphism notion for two classical graphs.

Definition 3.7. Let X and Y be two classical graphs. Then we call X and Y *quantum isomorphic*, written $X \cong_q Y$, if there exists a unital C^*-algebra with elements p_{ij} for $i \in V(X)$ and $j \in V(Y)$ such that

$$p_{ij}^* = p_{ij}^2 = p_{ij}, \ \sum_i p_{ij} = \sum_j p_{ij} = 1 \text{ and } A_Y p = p A_X.$$

Remark 3.8. *i*) Clearly, $X \cong Y$ implies $X \cong_q Y$ since the matrix elements σ_{ij} of the bijection between $V(X)$ and $V(Y)$ satisfy the relations from Definition 3.7. The other direction is not true in general, so there are quantum isomorphic graphs which are not isomorphic.

ii) For two classical graphs there are also other quantum isomorphism notions with additional requirements for the C^*-algebra, see for example Chapter 4 of [LMR20]. By Theorem 4.4 of [LMR20], some of them are even equivalent.

Next we want to define a quantum isomorphism notion for quantum graphs. For this we need the linking algebra. In [BCE$^+$20] this linking algebra was usually introduced to extend the definition of the graph isomorphism game *-algebra $\mathcal{A}(Iso(X,Y))$ to include quantum graphs. This *-algebra characterises whether the graph isomorphism game has a perfect A^*-strategy. More details can be found in Chapter 2 of [BCE$^+$20].

Similar to the part about quantum automorphism groups, we get that if $X \cong_q Y$, then $(p_{ij})_{ij}$ is a unitary matrix and the map $\rho_{Y,X}(e_j) := \sum_i e_i \otimes p_{ij}$ is a unital *-homomorphism satisfying $\rho_{Y,X}(A_X \cdot) = (A_Y \otimes id)\rho_{Y,X}$. This motivates the definition of the linking algebra.

Definition 3.9. Let $X = (\mathcal{O}(X), \psi_X, A_X)$ and $Y = (\mathcal{O}(Y), \psi_Y, A_Y)$ be quantum graphs with $|X| = n$ and $|Y| = m$ and let $\{e_j\}_{j=1,\dots,n}$ and $\{f_i\}_{i=1,\dots,m}$ be orthonormal bases for $L^2(X)$ and $L^2(Y)$. We define the *linking algebra* of X and Y as the universal *-algebra $\mathcal{O}(G_Y, G_X)$ generated by the entries of a unitary matrix

$$p = (p_{ij})_{ij} \in B(L^2(X), L^2(Y)) \otimes \mathcal{O}(G_Y, G_X)$$

that satisfies the relations which make the map

$$\rho_{Y,X} : \mathcal{O}(X) \to \mathcal{O}(Y) \otimes \mathcal{O}(G_Y, G_X), \quad \rho_{Y,X}(e_j) := \sum_{i=1}^{m} f_i \otimes p_{ij}$$

a unital *-homomorphism with

$$\rho_{Y,X}(A_X \cdot) = (A_Y \otimes id)\rho_{Y,X}.$$

If $\mathcal{O}(G_Y, G_X) \neq 0$, we call X *algebraically quantum isomorphic* to Y and write $X \cong_{A^*} Y$.

In [BCE$^+$20] there were also other quantum isomorphism notions introduced such as the notion of a C^*-algebraically quantum isomorphism, written $X \cong_{C^*} Y$. That

is, the linking algebra admits a non-zero C^*-representation. Moreover, the quantum isomorphism notion $X \cong_{qc} Y$, which is, the linking algebra admits a tracial state. It even holds that $X \cong_{A^*} Y \Longleftrightarrow X \cong_{C^*} Y$ [[BCE+20], Corollary 4.8].
Since we will only use the term algebraically quantum isomorphic in this thesis, we will drop the term "algebraically" and just write quantum isomorphic.

Remark 3.10. *i*) If we have two finite-dimensional Hilbert spaces H_1 and H_2 and an algebra A, then we can view an element in $B(H_1, H_2) \otimes A$ as an element in $M_{\dim(H_2)\times\dim(H_1)}(A)$ by identifying the element $b_{ij} \otimes a$ with the matrix $(\delta_{ij}a)_{ij}$. If $u \in B(H_1, H_2) \otimes A$, then $u = \sum_{i,j} b_{ij} \otimes u_{ij}$ for some $u_{ij} \in A$, so we identify u with the matrix $(u_{ij})_{ij}$.

ii) If $X = Y$, then we get $\mathcal{O}(G_Y, G_X) = \mathcal{O}(G_X)$ and $\rho_{Y,X} = \rho_X$. This follows directly from the definition.

iii) If X and Y are just classical graphs, then the properties of $\rho_{Y,X}$ are in a C^*-algebra equivalent to

$$p_{ij}^* = p_{ij}^2 = p_{ij}, \ \sum_i p_{ij} = \sum_j p_{ij} = 1 \text{ and } A_Y p = pA_X.$$

Hence

$$\mathcal{C}(G_Y, G_X) = C^*(p_{ij} \mid p_{ij}^* = p_{ij}^2 = p_{ij}, \sum_i p_{ij} = \sum_j p_{ij} = 1, A_Y p = pA_X),$$

where $1 \leqslant i \leqslant |Y|$ and $1 \leqslant j \leqslant |X|$. This can be proven similarly to Proposition 2.20. Therefore we get for classical graphs:

$$\mathcal{C}(G_Y, G_X) \neq 0 \Longleftrightarrow X \cong_q Y.$$

iv) If $X = (M_n(\mathbb{C}), tr, A_X)$ and $Y = (M_m(\mathbb{C}), tr, A_Y)$ with some adjacency matrices A_X and A_Y, then one can show similar to Proposition 2.21 that $\mathcal{O}(G_Y, G_X)$ is generated by elements v_{kl}^{ij} with $1 \leqslant i,j \leqslant n$ and $1 \leqslant k,l \leqslant m$ which satisfy (2.1)-(2.3) from Proposition 2.21 and $(v_{kl}^{ij})_{klij} A_X = A_Y (v_{kl}^{ij})_{klij}$.

As already mentioned we now want to show that the linking algebra is a bigalois extension if it is non-zero. This means that a quantum isomorphism between two quantum graphs X and Y is nothing other than a $\mathcal{O}(G_Y)$-$\mathcal{O}(G_X)$-bigalois extension. Note that the other direction is not true in general, i.e.

$$\text{"there exists a } \mathcal{O}(G_Y)\text{-}\mathcal{O}(G_X)\text{-bigalois extension} \not\Rightarrow X \cong_{A^*} Y\text{"}.$$

Theorem 3.11. *Let X, Y be quantum graphs. If $\mathcal{O}(G_Y, G_X)$ is non-zero, then $\mathcal{O}(G_Y, G_X)$ is a $\mathcal{O}(G_Y)$-$\mathcal{O}(G_X)$-bigalois extension.*

Proof. Let $\mathcal{O}(G_Y, G_X) \neq 0$ and p, u and v be the matrices of generators of $\mathcal{O}(G_Y, G_X)$, $\mathcal{O}(G_Y)$ and $\mathcal{O}(G_X)$ respectively. Recall that a matrix $(a_{ij})_{ij}$ is unitary if and only if $\sum_i a_{ik}^* a_{il} = \delta_{kl} = \sum_j a_{kj} a_{lj}^*$.
1) One can check that the matrix $(\sum_k u_{ik} \otimes p_{kj})_{ij}$ fulfills the same properties as $(p_{ij})_{ij}$. Therefore, by the universal property, we get a *-homomorphism

$$\alpha : \mathcal{O}(G_Y, G_X) \to \mathcal{O}(G_Y) \otimes \mathcal{O}(G_Y, G_X) \text{ with } \alpha(p_{ij}) = \sum_k u_{ik} \otimes p_{kj}.$$

Then α is unital since

$$\begin{aligned}
\alpha(1) &= \alpha(\sum_i p_{ij}^* p_{ij}) = \sum_i \alpha(p_{ij})^* \alpha(p_{ij}) = \sum_i (\sum_k u_{ik} \otimes p_{kj})^* (\sum_l u_{il} \otimes p_{lj}) \\
&= \sum_i \sum_k \sum_l u_{ik}^* u_{il} \otimes p_{kj}^* p_{lj} = \sum_k \sum_l \delta_{kl} \otimes p_{kj}^* p_{lj} \\
&= \sum_k 1 \otimes p_{kj}^* p_{kj} = 1 \otimes 1.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
(id \otimes \alpha)(\alpha(p_{ij})) &= (id \otimes \alpha)(\sum_k u_{ik} \otimes p_{kj}) = \sum_k \sum_l u_{ik} \otimes u_{kl} \otimes p_{lj} \\
&= (\Delta \otimes id)(\sum_l u_{il} \otimes p_{lj}) = (\Delta \otimes id)(\alpha(p_{ij})),
\end{aligned}$$

hence $(id \otimes \alpha)\alpha = (\Delta \otimes id)\alpha$ and

$$(\epsilon \otimes id)(\alpha(p_{ij})) = (\epsilon \otimes id)(\sum_k u_{ik} \otimes p_{kj}) = \sum_k \delta_{ik} p_{kj} = p_{ij},$$

hence $(\epsilon \otimes id)\alpha = id$. Therefore $(\mathcal{O}(G_Y, G_X), \alpha)$ is a left $\mathcal{O}(G_Y)$ *-comodule.
We define the map

$$\eta_l : \mathcal{O}(G_Y) \otimes \mathcal{O}(G_Y, G_X) \to \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_Y, G_X)$$

by

$$\eta_l := (id \otimes m)(\gamma \otimes id)$$

with

$$\gamma : \mathcal{O}(G_Y) \to \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_Y, G_X), \quad \gamma(u_{ij}) := \sum_t p_{it} \otimes p_{jt}^*$$

and m is the multiplication of $\mathcal{O}(G_Y, G_X)$. Then η_l is the inverse of

$$\kappa_l : \mathcal{O}(G_Y, G_X) \otimes \mathcal{O}(G_Y, G_X) \to \mathcal{O}(G_Y) \otimes \mathcal{O}(G_Y, G_X), \quad \kappa_l(x \otimes y) = \alpha(x)(1 \otimes y),$$

since

$$\begin{aligned}
\eta_l(\kappa_l(p_{ij}\otimes p_{kl})) &= (id\otimes m)(\gamma\otimes id)(\alpha(p_{ij})(1\otimes p_{kl}))\\
&= (id\otimes m)(\gamma\otimes id)(\sum_s u_{is}\otimes p_{sj}p_{kl})\\
&= (id\otimes m)(\sum_s\sum_t p_{it}\otimes p_{st}^*\otimes p_{sj}p_{kl})\\
&= \sum_s\sum_t p_{it}\otimes p_{st}^*p_{sj}p_{kl}\\
&= \sum_t p_{it}\otimes \delta_{tj}p_{kl} = p_{ij}\otimes p_{kl}
\end{aligned}$$

and

$$\begin{aligned}
\kappa_l(\eta_l(u_{ij}\otimes p_{kl})) &= \kappa_l((id\otimes m)(\sum_t p_{it}\otimes p_{jt}^*\otimes p_{kl})) = \kappa_l(\sum_t p_{it}\otimes p_{jt}^*p_{kl})\\
&= \sum_t \alpha(p_{it})(1\otimes p_{jt}^*p_{kl}) = \sum_t(\sum_s u_{is}\otimes p_{st})(1\otimes p_{jt}^*p_{kl})\\
&= \sum_t\sum_s u_{is}\otimes p_{st}p_{jt}^*p_{kl} = \sum_s u_{is}\otimes \delta_{sj}p_{kl} = u_{ij}\otimes p_{kl}.
\end{aligned}$$

2) Similarly, one can check that $(\mathcal{O}(G_Y,G_X),\beta)$ is a right $\mathcal{O}(G_X)$ *-comodule with

$$\beta:\mathcal{O}(G_Y,G_X)\to\mathcal{O}(G_Y,G_X)\otimes\mathcal{O}(G_X),\quad \beta(p_{ij}):=\sum_k p_{ik}\otimes v_{kj}$$

and one can analogously construct an inverse of

$$\kappa_r:\mathcal{O}(G_Y,G_X)\otimes\mathcal{O}(G_Y,G_X)\to\mathcal{O}(G_Y,G_X)\otimes\mathcal{O}(G_X),\quad \kappa_r(x\otimes y)=(x\otimes 1)\beta(y).$$

3) Moreover, $\mathcal{O}(G_Y,G_X)$ is an $\mathcal{O}(G_Y)$-$\mathcal{O}(G_X)$-bicomodule since

$$\begin{aligned}
(id\otimes\beta)(\alpha(p_{ij})) &= (id\otimes\beta)(\sum_k u_{ik}\otimes p_{kj}) = \sum_k\sum_l u_{ik}\otimes p_{kl}\otimes v_{lj}\\
&= (\alpha\otimes id)(\sum_l p_{il}\otimes v_{lj}) = (\alpha\otimes id)(\beta(p_{ij})),
\end{aligned}$$

hence $(id\otimes\beta)\alpha=(\alpha\otimes id)\beta$.

□

Chapter 4

Representation Theory

In the fourth chapter we want to take a closer look at the representation theory of CQGs. This will enable us to introduce the notion of monoidal equivalence. As already noted we also need the notion of representations to construct the Hopf *-algebra which lies dense in the C^*-algebra of a CGQ. In the fifth chapter we will then make a connection between monoidal equivalence of quantum automorphism groups and quantum isomorphisms of quantum graphs.
The main sources for this chapter are [NT13] and [BCE$^+$20].

4.1 Representations of CQGs

In this section, we start with the definition of a representation and then state the theorem about the existence of the dense Hopf *-algebra. We will also define a certain tensor product for representations. Moreover, we prove that the fundamental representation of a quantum automorphism group is indeed a representation and using this we can finally prove the existence of the quantum automorphism groups.

Definition 4.1. Let A be a *-algebra with a unital *-homomorphism $\Delta : A \to A \otimes A$ and H a finite-dimensional Hilbert space. A *(finite-dimensional) representation* of (A, Δ) on H is an invertible element $v \in B(H) \otimes A$ such that

$$(id \otimes \Delta)(v) = v_{12}v_{13}. \quad \text{(see Section 1.1 for notation)}$$

A representation of G is called *unitary* if $v \in B(H) \otimes A$ is unitary.

Remark 4.2. *i*) Here we took the definition of representations from [NT13] instead of [BCE$^+$20] since in [BCE$^+$20] a representation is an element of $A \otimes B(H)$ instead of $B(H) \otimes A$. The Hilbert space as the first component of the tensor product fits better in our case because in Definition 2.16 we can view the fundamental representation u as an element of $B(L^2(X) \otimes \mathcal{O}(G_X))$ and then get $u(\cdot \otimes 1) = \rho_X$, so here the Hilbert space is also the first component.

ii) Let $\{e_i : 1 \leqslant i \leqslant n\}$ be an orthonormal basis of H. If $v \in B(H) \otimes \mathcal{O}(G)$ is a representation, we get

$$(id \otimes \Delta)(v) = (id \otimes \Delta)(\sum_{i,j=1}^{n} b_{ij} \otimes v_{ij}) = \sum_{i,j=1}^{n} b_{ij} \otimes \Delta(v_{ij}).$$

Since $v_{12}v_{13} = \sum_{i,j,k=1}^{n} b_{ij} \otimes v_{ik} \otimes v_{kj}$, we have that v is a representation if and only if

$$\Delta(v_{ij}) = \sum_{k=1}^{n} v_{ik} \otimes v_{kj}.$$

iii) There is always the trivial representation on $\mathbb{C}$ given by $1 \in \mathcal{O}(G) \cong B(\mathbb{C}) \otimes \mathcal{O}(G)$, since $\Delta(1) = 1 \otimes 1$.

Now we can look at how the associated Hopf *-algebra $\mathcal{O}(G)$ corresponding to a CQG G is constructed, as we already announced in Remark 1.17 *ii*). The *-algebra consists of the matrix coefficients of all finite-dimensional unitary representations of G. This is formulated in the following theorem.

Theorem 4.3 ([Web17], Theorem 4.10)**.** *Let $G = (A, \Delta)$ be a CQG and A_0 be the subspace of A which is spanned by the matrix coefficients of all finite-dimensional unitary representations of G. Then*

- *$A_0 \subseteq A$ is a dense *-algebra,*
- *A_0 is a Hopf *-algebra with comultiplication $\Delta|_{A_0}$.*

Now we also want to introduce the notion of intertwiners. These are special linear maps between the Hilbert spaces of two representations. For the rest of this section let G be a CQG.

Definition 4.4. Let $u \in B(H) \otimes \mathcal{O}(G)$ and $v \in B(K) \otimes \mathcal{O}(G)$ be two representations. We call a linear map $T : H \to K$ an *intertwiner* or *morphism* between u and v if

$$v(T \otimes id) = (T \otimes id)u.$$

We denote the space of all morphisms between u and v with $\mathrm{Mor}(u, v)$. Two representation u and v are called *equivalent* if there exists an invertible element in $\mathrm{Mor}(u, v)$, so we get $v = (T \otimes id)u(T^{-1} \otimes id)$ for some linear map T. Moreover, we call u *irreducible* if $\mathrm{Mor}(u, u) = \mathbb{C}id$.

For two representations u and v we can also define a certain tensor product which we will denote with $u \oplus v$ and introduce in the next proposition. For $n \in \mathbb{N}_0$ we denote with $u^{\oplus n}$ the element $u \oplus \cdots \oplus u$, i.e. $n-1$ times the tensor product of u with itself if $n \in \mathbb{N}$ and $u^{\oplus 0} = 1$.

Proposition 4.5. *Let $u \in B(H) \otimes \mathcal{O}(G)$ and $v \in B(K) \otimes \mathcal{O}(G)$ be two representations.*
i) We define the tensor product of u and v as $u \oplus v := u_{13}v_{23}$.
Then $u \oplus v \in B(H \otimes K) \otimes \mathcal{O}(G)$ is a representation and $u \oplus 1$ is equivalent to u.
ii) If u and v are unitary, then $T \in Mor(u, v) \Longleftrightarrow T^\dagger \in Mor(v, u)$.

Proof. *i)* Note that

$$B(H) \otimes B(K) \cong M_{\dim(H)}(\mathbb{C}) \otimes M_{\dim(K)}(\mathbb{C}) \cong M_{\dim(H)\dim(K)}(\mathbb{C}) \cong B(H \otimes K).$$

Since

$$u_{13}v_{23} = \sum_{k,i,j,l} b_{ij} \otimes b_{kl} \otimes u_{ij}v_{kl}$$

we have $u \oplus v \in B(H) \otimes B(K) \otimes \mathcal{O}(G) \cong B(H \otimes K) \otimes \mathcal{O}(G)$. Moreover, the corresponding matrix of $u \oplus v$ is the Kronecker product of the corresponding matrices of u and v (see [Gro20]). Using

$$\Delta(u_{ij}v_{kl}) = \Delta(u_{ij})\Delta(v_{kl}) = (\sum_p u_{ip} \otimes u_{pj})(\sum_q v_{kq} \otimes v_{ql}) = \sum_{p,q} u_{ip}v_{kq} \otimes u_{pj}v_{ql}$$

and $(u \oplus v)_{\dim(K)(i-1)+k, \dim(K)(j-1)+l} = u_{ij}v_{kl}$, we get that $u \oplus v$ is a representation. Furthermore, we have $u \oplus 1 = \sum_{i,j} b_{ij} \otimes b_{11} \otimes u_{ij} = \sum_{i,j} b_{ij} \otimes id \otimes u_{ij} = u_{13}$. Using $H \cong H \otimes \mathbb{C}$ we get that $id : H \to H$ is an element of $\mathrm{Mor}(u, u \oplus 1)$ since

$$u_{13}(\xi \otimes 1 \otimes a) = \sum_{i,j} b_{ij}(\xi) \otimes 1 \otimes u_{ij}a = \sum_{i,j} b_{ij}(\xi) \otimes u_{ij}a = u(\xi \otimes a).$$

Therefore $u \oplus 1$ is equivalent to u.
ii) Let $T \in \mathrm{Mor}(u, v)$, then $T^\dagger : K \to H$ and

$$u(T^\dagger \otimes id)v^* = (v(T \otimes id)u)^\dagger = (T \otimes id)^\dagger = T^\dagger \otimes id,$$

since $u^* = u^\dagger$. Therefore $u(T^\dagger \otimes id) = (T^\dagger \otimes id)v$, hence $T^\dagger \in \mathrm{Mor}(v, u)$. □

The next example shows that the fundamental representation of a quantum automorphism group of a quantum graph or quantum set X is indeed a representation and that m_X, η_X and A_X are intertwiners.

Example 4.6. *i*) The fundamental representation u of a quantum automorphism group of a quantum graph or quantum set X is a representation of $(\mathcal{O}(G_X), \Delta)$ respectively $(\mathcal{C}(\mathrm{Aut}^+(X)), \Delta)$ on $L^2(X)$. For the quantum graph this is true since u is an element of $M_n(\mathcal{O}(G_X)) \cong L^2(X) \otimes \mathcal{O}(G_X)$, as a unitary it is invertible and the equality $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ is true by Proposition 2.19. The reasoning for the quantum set is the same.

ii) Let X be a quantum graph and u the fundamental representation of G_X. Then we have $m_X \in \mathrm{Mor}(u \oplus u, u), \eta_X \in \mathrm{Mor}(1, u)$ and $A_X \in \mathrm{Mor}(u, u)$. Indeed, let $\{e_i\}_i$ be an orthonormal basis of $L^2(X)$. We have $m_X : L^2(X) \otimes L^2(X) \to L^2(X)$ and

$$\begin{aligned}
&u(m_X \otimes id)(e_i \otimes e_k \otimes a) = u(e_i e_k \otimes a) = \rho_X(e_i e_k)(1 \otimes a) \\
&= \rho_X(e_i)\rho_X(e_k)(1 \otimes a) = (\sum_j e_j \otimes u_{ji})(\sum_l e_l \otimes u_{lk})(1 \otimes a) \\
&= \sum_{j,l} e_j e_l \otimes u_{ji} u_{lk} a = (m_X \otimes id)(\sum_{j,l} e_j \otimes e_l \otimes u_{ji} u_{lk} a) \\
&= (m_X \otimes id)(u_{13}(\sum_l e_i \otimes e_l \otimes u_{lk} a) = (m_X \otimes id)(u_{13}(u_{23}(e_i \otimes e_k \otimes a))) \\
&= (m_X \otimes id)(u \oplus u)(e_i \otimes e_k \otimes a))),
\end{aligned}$$

$\eta_X : \mathbb{C} \to L^2(X)$ $(B(\mathbb{C}) \cong \mathbb{C})$ and

$$\begin{aligned}
u(\eta_X \otimes id)(\alpha \otimes a) &= u(\alpha 1 \otimes a) = \rho_X(\alpha 1)(1 \otimes a) \\
&= \alpha \rho_X(1)(1 \otimes a) = \alpha 1 \otimes a = (\eta_X \otimes id)1(\alpha \otimes a),
\end{aligned}$$

$A_X : L^2(X) \to L^2(X)$ and

$$u(A_X \otimes id) = (A_X \otimes id)u$$

by Lemma 2.17 *v*).

Finally, we state the theorem which proves the existence of the quantum automorphism group of a quantum graph or a quantum set.

Theorem 4.7 ([NT13], Theorem 1.6.6)**.** *Let (A, Δ) be a Hopf *-algebra such that A is generated by the matrix coefficients of finite-dimensional unitary representations of (A, Δ), then $(A, \Delta) = (\mathcal{O}(G), \Delta)$ for some CQG G.*

Remark 4.8. The above theorem shows that the notation in Definition 2.16 makes sense and that the CQGs $\mathrm{Aut}^+(X)$ and G_X really exists since $\mathcal{C}(\mathrm{Aut}^+(X))$ and $\mathcal{O}(G_X)$ are both Hopf *-algebras by Proposition 2.19 and generated by their fundamental representation which is indeed a representation by Example 4.6 *i*).

At the end of this section we state another theorem which we will need when working with monoidal equivalence in the next sections.

Theorem 4.9 ([NT13], Theorem 1.3.7). *Every finite-dimensional representation of G is equivalent to a direct sum of irreducible representations.*

4.2 Monoidal Equivalence

In this section we introduce the notion of monoidal equivalence of two CGQs and look at some examples of monoidal equivalent CQGs. In the next chapter we will show that if a quantum automorphism group of a graph X is monoidally equivalent to another CQG, then there exists a corresponding quantum graph to this CQG which is quantum isomorphic to X.

For a CGQ G, let $\mathrm{Rep}(G)$ be the representation category, i.e. the category whose objects are equivalence classes of representations of G and whose morphisms are given by the intertwiner spaces $\mathrm{Mor}(u, v)$. With $\mathrm{Irr}(G)$ we denote the set of equivalence classes of irreducible objects in $\mathrm{Rep}(G)$. Note that from now on we also denote the equivalence class of some representation $u \in B(H) \otimes \mathcal{O}(G)$ with $u \in \mathrm{Rep}(G)$.

Definition 4.10. Let G_1 and G_2 be two CQGs. We say that G_1 and G_2 are *monoidally equivalent* and write $G_1 \sim^{mon} G_2$ if there exists a bijection

$$\varphi : \mathrm{Irr}(G_1) \to \mathrm{Irr}(G_2)$$

with $\varphi(1_{G_1}) = 1_{G_2}$ (where 1_{G_i} is the trivial representation of G_i) and for all $u_i \in \mathrm{Irr}(G_1)$ and $v_j \in \mathrm{Irr}(G_2)$ ($i = 1, \dots, n$, $j = 1, \dots, m$) there are linear isomorphisms

$$\varphi : \mathrm{Mor}(u_1 \oplus \dots u_n, v_1 \oplus \dots v_m) \to \mathrm{Mor}(\varphi(u_1) \oplus \dots \varphi(u_n), \varphi(v_1) \oplus \dots \varphi(v_m))$$

with $\varphi(id) = id$ and for all intertwiners S, T we have:

- $\varphi(S \circ T) = \varphi(S) \circ \varphi(T)$ (if $S \circ T$ is well-defined)
- $\varphi(S^\dagger) = \varphi(S)^\dagger$
- $\varphi(S \otimes T) = \varphi(S) \otimes \varphi(T)$.

Remark 4.11. By Theorem 4.9 we can extend φ to a functor $\varphi : \mathrm{Rep}(G_1) \to \mathrm{Rep}(G_2)$ since every $u \in \mathrm{Rep}(G_1)$ is equivalent to a direct sum $\oplus_i u_i$ with $u_i \in \mathrm{Irr}(G_1)$, so we can define $\varphi(u) := \oplus_i \varphi(u_i)$. This functor is in particular *essentially surjective* which means that every object in $\mathrm{Rep}(G_2)$ is of the form $\varphi(u)$ for some $u \in \mathrm{Rep}(G_1)$.

If we have two quantum sets with δ-forms, then it is quite easy to see whether their quantum automorphism groups are monoidally equivalent. This is exactly the case if the two δ are the same. This statement is formulated in the following theorem. With this we can easily follow that the two CQGs $S_{n^2}^+$ and PO_n^+ from Definition 1.14 are monoidally equivalent.

Theorem 4.12 ([DRVV10], Theorem 4.7)**.** *Let (B_i, ψ_i) be quantum sets where ψ_i is a δ_i-form $(i = 1, 2)$. Then the CQGs $Aut^+(B_1, \psi_1)$ and $Aut^+(B_2, \psi_2)$ are monoidally equivalent if and only if $\delta_1 = \delta_2$.*

Corollary 4.13. *The CQGs $S_{n^2}^+$ and PO_n^+ are monoidally equivalent.*

Proof. From Example 2.4 *ii*) and *iii*) we know that $\psi_{n^2} : \mathbb{C}^{n^2} \to \mathbb{C}$ and $tr : M_n(\mathbb{C}) \to \mathbb{C}$ are both n-forms. Therefore we get with Theorem 4.12

$$\mathrm{Aut}^+(\mathbb{C}^{n^2}, \psi_{n^2}) \sim^{mon} \mathrm{Aut}^+(M_n(\mathbb{C}), tr)$$

and then the Propositions 2.20 and 2.21 imply $S_{n^2}^+ \sim^{mon} PO_n^+$. □

Using this theorem we get another interesting monoidal equivalence.

Example 4.14. The group $SO(3)$ is defined as all orthogonal matrices in $\mathbb{R}^{3\times 3}$ with determinant 1. By [Ban99] we know that $SO(3)$ is the quantum automorphism group of $(M_2(\mathbb{C}), tr)$. This implies $SO(3) \sim_{mon} S_4^+$ since tr and ψ_4 are both 2-forms.

Chapter 5

Linking Algebras of CQGs

In this chapter we also want to define a linking algebra for two monoidally equivalent CQGs, which do not necessarily have to be quantum automorphisms groups of some quantum graphs. With the help of this new linking algebra we can finally prove the theorem which we already mentioned at the beginning of Section 4.2. It connects the monoidal equivalence of quantum automorphism groups with the quantum isomorphism of the corresponding quantum graphs. In the end we will also have a closer look at the linking algebra of $S_{n^2}^+$ and PO_n^+.
The main sources of this chapter are [BEHY22] and [BCE+20].

The existence of the linking algebra in the following definition is proved in [[BRV05], Theorem 3.9 and Proposition 3.13].

Definition 5.1. Let G_1 and G_2 be two monoidally equivalent CQGs and $\varphi : \mathrm{Rep}(G_1) \to \mathrm{Rep}(G_2)$ be the map from Remark 4.11. Then there exists a unique unital *-algebra $\hat{\mathcal{O}}(G_1, G_2)$ which is spanned by the matrix coefficients of unitary elements $X^x \in B(H_x, H_{\varphi(x)}) \otimes \hat{\mathcal{O}}(G_1, G_2)$ where $x \in \mathrm{Irr}(G_1)$. We call $\hat{\mathcal{O}}(G_1, G_2)$ the *linking algebra* of G_1 and G_2.

Remark 5.2. *i)* Since φ is defined on $\mathrm{Rep}(G_1)$, we also have unitary elements $X^x \in B(H_x, H_{\varphi(x)}) \otimes \hat{\mathcal{O}}(G_1, G_2)$ for all $x \in \mathrm{Rep}(G_1)$. Moreover, we know from the proof of [[BRV05], Theorem 3.9] that

$$(\varphi(S) \otimes id)X_{13}^y X_{23}^z = X^x(S \otimes id)$$

for all $S \in \mathrm{Mor}(y \oplus z, x)$,

$$(\varphi(T) \otimes id)X^x = X_{13}^y X_{23}^z(T \otimes id)$$

for all $T \in \mathrm{Mor}(x, y \oplus z)$ and that X^1 is the unit element of $\hat{\mathcal{O}}(G_1, G_2)$. Therefore we also get

$$(\varphi(T) \otimes id)X^x = X^y(T \otimes id)$$

for all $T \in \mathrm{Mor}(x, y)$ and

$$(X^y)^*_{13}(\varphi(T) \otimes id) = X^z_{23}(T \otimes id)$$

for all $T \in \mathrm{Mor}(1, y \oplus z)$ because $(X^y)^*_{13}X^y_{13} = id$ since X^y is a unitary element. These equalities will be useful in the proof of the next theorem.

ii) It is not mentioned in [BEHY22], but similar to Theorem 3.11 one can show that $\hat{\mathcal{O}}(G_1, G_2)$ is a $\mathcal{O}(G_1)$-$\mathcal{O}(G_2)$-bigalois extension if it is non-zero.

iii) If we have a quantum set $(\mathcal{O}(X), \psi_X)$, then $(\xi^\star \otimes id)(m_X^\dagger \eta_X)(1) = \xi^*$ (see Section 1.1 for notation). The idea that this equality holds is taken from [Rij07].
Indeed,

$$\begin{aligned}\langle(\xi^\star \otimes id)(m_X^\dagger \eta_X)(1), y\rangle &= \langle(m_X^\dagger \eta_X)(1), (\xi^\star \otimes id)^\dagger(y)\rangle \\ &= \langle(m_X^\dagger \eta)(1), \xi \otimes y\rangle = \langle 1, \xi y\rangle = \langle \xi^*, y\rangle,\end{aligned}$$

where the second equation is true since

$$\langle \xi \otimes y, x \otimes z\rangle = \langle \xi, x\rangle\langle y, z\rangle = \langle y, \langle x, \xi\rangle z\rangle = \langle y, (\xi^\star \otimes id)(x \otimes z)\rangle.$$

If two quantum graphs are quantum isomorphic, then their quantum automorphism groups are monoidally equivalent (see [BCE+20] Section 4.2). The converse is not true in general. However, we get another theorem connecting monoidal equivalence with quantum isomorphism, which we can now finally prove. It states that if a CQG is monoidally equivalent to a quantum automorphism group of some quantum graph X, then there exists a quantum graph Y with $X \cong_{A*} Y$ such that the CQG is its quantum automorphism group. For the proof of this we first need the following lemma.

Lemma 5.3. *Let H be a finite-dimensional Hilbert space and also a unital *-algebra, where the equality $\langle a^*b, c\rangle = \langle b, ac\rangle$ holds for all $a, b, c \in H$. Then H is a C^*-algebra.*

Proof. We define the map $\Phi : H \to B(H)$, $a \mapsto \phi_a$ with $\phi_a(h) = ah$. Then Φ is well defined because H is finite-dimensional, thus all linear maps are bounded and hence $\phi_a \in B(H)$. Moreover, Φ is injective since $\phi_a = \phi_b$ implies $a = \phi_a(1) = \phi_b(1) = b$. Therefore, $H \cong \Phi(H)$. The space $\Phi(H)$ is norm-closed since it is isomorphic to H

and hence finite-dimensional. Furthermore, it is a *-subalgebra of $B(H)$ because $\phi_a^\dagger = \phi_{a^*}$ since $\langle a^* h_1, h_2\rangle = \langle h_1, ah_2\rangle$. This implies that $\Phi(H)$ is a C^*-algebra, hence H is a C^*-algebra. □

The proof of the following theorem is based on the given proof in [BCE+20] (Theorem 4.11) and the proof of Theorem 3.6.5 in [Rij07].

Theorem 5.4. *Let $X = (\mathcal{O}(X), \psi_X, A_X)$ be a quantum graph and G_X its quantum automorphism group. Let G be another compact quantum group that is monoidally equivalent to G_X. Then there exists a quantum graph $Y = (\mathcal{O}(Y), \psi_Y, A_Y)$ such that $G \cong G_Y$ and X is quantum isomorphic to Y, i.e. $X \cong_{A^*} Y$.*

Proof. 1. Construction of Y:
Let $\varphi : \mathrm{Rep}(G_X) \to \mathrm{Rep}(G)$ be the map from Remark 4.11 and u be the fundamental representation of G_X. We define $v := \varphi(u)$. Then $v \in B(H)\otimes\mathcal{O}(G)$ for some Hilbert space H. We set $L^2(Y) := H$ and $d_Y := \dim(H)$. Additionally we define

$$\begin{aligned} &m_Y := \varphi(m_X) \in \mathrm{Mor}(v \oplus v, v), \eta_Y := \varphi(\eta_X) \in \mathrm{Mor}(1, v),\\ &\psi_Y := \eta_Y^\dagger \in \mathrm{Mor}(v, 1) \text{ and } A_Y := \varphi(A_X) \in \mathrm{Mor}(v, v). \end{aligned}$$

Then m_Y is associative since

$$m_Y(id \otimes m_Y) = \varphi(m_X(id \otimes m_X)) = \varphi(m_X(m_X \otimes id)) = m_Y(m_Y \otimes id)$$

and η_Y is a unit map because

$$m_Y(id \otimes \eta_Y) = \varphi(m_X(id \otimes \eta_X)) = \varphi(id) = id = \varphi(m_X(\eta_X \otimes id)) = m_Y(\eta_Y \otimes id).$$

We define the map $\cdot^\# : L^2(Y) \to L^2(Y)$ by

$$\xi \mapsto \xi^\# := (\xi^\star \otimes id)(m_Y^\dagger \eta_Y)(1).$$

Then

$$\begin{aligned} \langle \xi^\#, y\rangle &= \langle (\xi^\star \otimes id)(m_Y^\dagger\eta_Y)(1), y\rangle = \langle (m_Y^\dagger\eta_Y)(1), (\xi^\star \otimes id)^\dagger y\rangle \\ &\overset{Remark\ 5.1iii)}{=} \langle (m_Y^\dagger\eta_Y)(1), \xi \otimes y\rangle = \langle 1, m_Y(\xi \otimes y)\rangle = \langle 1, \xi y\rangle \end{aligned}$$

and therefore also

$$\begin{aligned} \langle \xi^\# \otimes x, y \otimes z\rangle &= \langle \xi^\#, y\rangle\langle x, z\rangle = \langle 1, \xi y\rangle\langle x, z\rangle = \langle 1, m_Y(\xi \otimes y)\rangle\langle x, z\rangle \\ &= \langle m_Y^\dagger(1), \xi \otimes y\rangle\langle x, z\rangle = \langle m_Y^\dagger(1) \otimes x, \xi \otimes y \otimes z\rangle \end{aligned}$$

for all $\xi, x, y, z \in L^2(Y)$. Together with

$$m_Y^\dagger m_Y = \varphi(m_X^\dagger m_X) \overset{Lemma\ 2.9}{=} \varphi((m_X \otimes id)(id \otimes m_X^\dagger)) = (m_Y \otimes id)(id \otimes m_Y^\dagger)$$

this implies

$$\begin{aligned}
\langle \xi^\# x, 1\rangle &= \langle m_Y(\xi^\# \otimes x), 1\rangle = \langle \xi^\# \otimes x, m_Y^\dagger(1)\rangle \\
&= \langle m_Y^\dagger(1) \otimes x, \xi \otimes m_Y^\dagger(1)\rangle \\
&= \langle (m_Y^\dagger \otimes id)(1 \otimes x), (id \otimes m_Y^\dagger)(\xi \otimes 1)\rangle \\
&= \langle 1 \otimes x, (m_Y \otimes id)(id \otimes m_Y^\dagger)(\xi \otimes 1)\rangle \\
&= \langle 1 \otimes x, m_Y^\dagger m_Y(\xi \otimes 1)\rangle \\
&= \langle m_Y(1 \otimes x), m_Y(\xi \otimes 1)\rangle = \langle x, \xi\rangle.
\end{aligned}$$

This shows that the scalar product of $L^2(Y)$ is induced by ψ_Y since

$$\psi_Y(\xi^\# x) = \eta_Y^\dagger(\xi^\# x) = \eta_Y^\dagger(\xi^\# x)\overline{1} = \langle \eta_Y^\dagger(\xi^\# x), 1\rangle = \langle \xi^\# x, 1\rangle = \langle x, \xi\rangle.$$

With this knowledge we now get that # is an involution because the equalities

$$\begin{aligned}
\langle (\alpha\xi)^\#, y\rangle &= \langle 1, \alpha\xi y\rangle = \langle \overline{\alpha}\xi^\#, y\rangle, \\
\langle (\xi^\#)^\#, y\rangle &= \langle 1, \xi^\# y\rangle = \langle \xi, y\rangle, \\
\langle (\xi_1\xi_2)^\#, y\rangle &= \langle 1, \xi_1\xi_2 y\rangle = \langle \xi_1^\#, \xi_2 y\rangle = \langle \xi_2^\#\xi_1^\#, y\rangle
\end{aligned}$$

imply $(\alpha\xi)^\# = \overline{\alpha}\xi^\#$, $(\xi^\#)^\# = \xi$ and $(\xi_1\xi_2)^\# = \xi_2^\#\xi_1^\#$.
Therefore $L^2(Y)$ is a unital *-algebra with multiplication m_Y, unit map η_Y and involution #. In fact, $L^2(Y)$ is even a C^*-algebra because it fulfils the requirements of Lemma 5.3. We denote this C^*-algebra with $\mathcal{O}(Y)$.
We also get that ψ_Y is a faithful state since $\psi_Y(1) = \langle 1, 1\rangle = \|1\|^2 = 1$ and $\psi_Y(\xi^\#\xi) = \langle \xi, \xi\rangle = \|\xi\|$, hence $\psi_Y(\xi^\#\xi) \geqslant 0$ and $\psi_Y(\xi^\#\xi) = 0$ only if $\xi = 0$. In addition, $\psi_Y : L^2(Y) \to \mathbb{C}$ is a δ-form since

$$m_Y m_Y^\dagger = \varphi(m_X m_X^\dagger) = \varphi(\delta^2 id) = \delta^2 id.$$

Moreover, $A_Y : L^2(Y) \to L^2(Y)$ is a quantum adjacency matrix because

$$A_Y^\dagger = \varphi(A_X)^\dagger = \varphi(A_X^\dagger) = \varphi(A_X) = A_Y,$$

so A_Y is self-adjoint and (1)-(3) from Definition 2.5 are true:

(1) $m_Y(A_Y \otimes A_Y)m_Y^\dagger = \varphi(m_X(A_X \otimes A_X)m_X^\dagger) = \varphi(\delta^2 A_X) = \delta^2 A_Y$

(2) $(id \otimes \eta_Y^\dagger m_Y)(id \otimes A_Y \otimes id)(m_Y^\dagger \eta_Y \otimes id)$
$= \varphi((id \otimes \eta_X^\dagger m_X)(id \otimes A_X \otimes id)(m_X^\dagger \eta_X \otimes id)) = \varphi(A_X) = A_Y$

(3) $m_Y(A_Y \otimes id)m_Y^\dagger = \varphi(m_X(A_X \otimes id)m_X^\dagger) = \varphi(\delta^2 id) = \delta^2 id.$

Therefore $(\mathcal{O}(Y), \psi_Y, A_Y)$ is a quantum graph.

2. $G_Y \cong G$:
Let G_Y be the quantum automorphism group of Y with fundamental representation $w \in M_{d_Y}(\mathcal{O}(G_Y))$. We have that $\mathcal{O}(G)$ is generated by the entries of $\varphi(u) = v$ because φ is essentially surjective and $\mathcal{O}(G_X)$ is generated by the entries of u. Therefore, we get with Proposition 3.4.15 from [Gro20] that it is enough to show

$$\mathrm{Mor}(w^{\oplus m}, w^{\oplus n}) = \mathrm{Mor}(v^{\oplus m}, v^{\oplus n})$$

for all $m, n \in \mathbb{N}_0$ to prove that $G_Y \cong G$. The monoidal equivalence implies $\mathrm{Mor}(v^{\oplus m}, v^{\oplus n}) = \varphi(\mathrm{Mor}(u^{\oplus m}, u^{\oplus n}))$ for all $n, m \in \mathbb{N}_0$. Moreover, by some categorical reasoning, the space $\bigcup_{n,m\in\mathbb{N}} \mathrm{Mor}(u^{\oplus m}, u^{\oplus n})$ is generated by the maps $\{id, m_X, \eta_X, A_X\}$ and therefore the space $\bigcup_{n,m\in\mathbb{N}} \mathrm{Mor}(v^{\oplus m}, v^{\oplus n})$ is generated by the images

$$\{\varphi(id), \varphi(m_X), \varphi(\eta_X), \varphi(A_X)\} = \{id, m_Y, \eta_Y, A_Y\}.$$

But the set $\bigcup_{n,m\in\mathbb{N}} \mathrm{Mor}(w^{\oplus m}, w^{\oplus n})$ is also generated by the maps $\{id, m_Y, \eta_Y, A_Y\}$. Therefore we have

$$\bigcup_{n,m\in\mathbb{N}} \mathrm{Mor}(v^{\oplus m}, v^{\oplus n}) = \bigcup_{n,m\in\mathbb{N}} \mathrm{Mor}(w^{\oplus m}, w^{\oplus n})$$

and hence $\mathrm{Mor}(v^{\oplus m}, v^{\oplus n}) = \mathrm{Mor}(w^{\oplus m}, w^{\oplus n})$ for all $m, n \in \mathbb{N}_0$.

3. $X \cong_{A^*} Y$:
Since we now get the monoidal equivalence of G_X and G_Y, Remark 5.2 $i)$ implies that there exists an element $X^u \in B(L^2(X), L^2(Y)) \otimes \hat{\mathcal{O}}(G_X, G_Y)$ which satisfies

(i) $\eta_Y \otimes id = \varphi(\eta_X) \otimes id = X^u(\eta_X \otimes id)$

(ii) $(m_Y \otimes id)X_{13}^u X_{23}^u = (\varphi(m_X) \otimes id)X_{13}^u X_{23}^u = X^u(m_X \otimes id)$

(iii) $(X^u)_{13}^*(m_Y^\dagger \eta_Y \otimes id) = (X^u)_{13}^*(\varphi(m_X^\dagger \eta_X) \otimes id) = X_{23}^u(m_X^\dagger \eta_X \otimes id)$

(iv) $(A_Y \otimes id)X^u = (\varphi(A_X) \otimes id)X^u = X^u(A_X \otimes id).$

The map $\rho : \mathcal{O}(X) \to \mathcal{O}(Y) \otimes \hat{\mathcal{O}}(G_X, G_Y)$, $\rho(e_i) = \sum_j f_j \otimes X^u_{ji}$ is then a unital *-homomorphism with $\rho(A_X \cdot) = (A_Y \otimes id)\rho$.
Indeed, we have

$$\rho(1) = X^u(1 \otimes 1) = X^u((\eta_X \otimes id)(1 \otimes 1)) \overset{(i)}{=} (\eta_Y \otimes id)(1 \otimes 1) = 1 \otimes 1,$$

$$\begin{aligned}\rho(e_i e_k) &= X^u(e_i e_k \otimes 1) = X^u((m_X \otimes id)(e_i \otimes e_k \otimes 1))\\ &\overset{(ii)}{=} (m_Y \otimes id)(X^u_{13} X^u_{23}(e_i \otimes e_k \otimes 1)) = (m_Y \otimes id)(X^u_{13}(e_i \otimes \sum_j f_j \otimes X^u_{jk}))\\ &= (m_Y \otimes id)(\sum_{j,l} f_l \otimes f_j \otimes X^u_{li} X^u_{jk})) = \sum_{j,l} f_l f_j \otimes X^u_{li} X^u_{jk}\\ &= (\sum_l f_l \otimes X^u_{li})(\sum_j f_j \otimes X^u_{jk}) = \rho(e_i)\rho(e_k),\end{aligned}$$

$$\begin{aligned}\rho(e_i^*) &= X^u(e_i^* \otimes 1) \overset{Remark\ 5.2iii)}{=} X^u((e_i^\star \otimes id)(m_X^\dagger \eta_X)(1) \otimes 1)\\ &= (e_i^\star \otimes id \otimes id) X^u_{23}((m_X^\dagger \eta_X)(1) \otimes 1)\\ &= (e_i^\star \otimes id \otimes id) X^u_{23}((m_X^\dagger \eta_X \otimes id)(1 \otimes 1))\\ &\overset{(iii)}{=} (e_i^\star \otimes id \otimes id) X^{u*}_{13}(m_Y^\dagger \eta_Y \otimes id)(1 \otimes 1))\\ &= (e_i^\star \otimes id \otimes id) X^{u*}_{13}((m_Y^\dagger \eta_Y)(1) \otimes 1)\\ &= (e_i^\star \otimes id \otimes id)(\sum_{j,k}(b_{kj} \otimes id)(m_Y^\dagger \eta_Y)(1) \otimes (X^{u^*})_{kj})\\ &= (e_i^\star \otimes id \otimes id)(\sum_{j,k}(f_j^\star e_k \otimes id)(m_Y^\dagger \eta_Y)(1) \otimes (X^u_{jk})^*)\\ &= \sum_{j,k}(f_j^\star \delta_{ik} \otimes id)(m_Y^\dagger \eta_Y)(1) \otimes (X^u_{jk})^*\\ &= \sum_j (f_j^\star \otimes id)(m_Y^\dagger \eta_Y)(1) \otimes (X^u_{ji})^* = \sum_j f_j^\# \otimes (X^u_{ji})^* = \rho(e_i)^*,\end{aligned}$$

$$\begin{aligned}\rho(A_X e_i) &= X^u(A_X e_i \otimes 1) = X^u(A_X \otimes id)(e_i \otimes 1) \overset{(iv)}{=} (A_Y \otimes id) X^u(e_i \otimes 1)\\ &= (A_Y \otimes id)\rho(e_i).\end{aligned}$$

By the universal property of $\mathcal{O}(G_Y, G_X)$ we know that there is a *-homomorphism $\psi : \mathcal{O}(G_Y, G_X) \to \hat{\mathcal{O}}(G_X, G_Y)$ with $\psi(p_{ij}) = X^u_{ij}$ (where p_{ij} generates $\mathcal{O}(G_Y, G_X)$). Therefore, $\mathcal{O}(G_Y, G_X) \neq 0$, so $X \cong_{A^*} Y$ (see Definition 3.9). □

Now we want to have a closer look at the linking algebra of the CQGs $S^+_{n^2}$ and PO^+_n. For this we first state the following remark, which makes it easier to calculate the linking algebra explicitly.

Remark 5.5. Let $\mathrm{Aut}^+(B_1, \psi_1)$ and $\mathrm{Aut}^+(B_2, \psi_2)$ be two monoidally equivalent quantum automorphism groups. Then their linking algebra is given by the universal *-algebra generated by the coefficients of a unital *-homomorphism

$$\rho : B_1 \to B_2 \otimes \hat{\mathcal{O}}(\mathrm{Aut}^+(B_1, \psi_1), \mathrm{Aut}^+(B_2, \psi_2))$$

with

$$(\psi_2 \otimes id)\rho = \psi_1(\cdot)1.$$

The coefficients of ρ are defined as the set $\{(w \otimes id)\rho(x) : x \in B_1, w \in B_2^*\}$.
For example, if A, B and C are vector spaces where A and B are finite-dimensional, then the coefficients of the map $A \to B \otimes C$ with $a \mapsto \sum_{i,j} b_{ij}(a) \otimes c_{ij}$ is the set $\mathrm{span}(\{c_{ij}\}_{ij})$. Indeed, let $\{a_j\}_j$ and $\{b_i\}_i$ be a basis of A and B respectively. Then the maps $\{b_i^*\}_i$ form a basis of B^* where $b_i^*(b_k) = \delta_{ik}$. Hence, every coefficient is a linear combination of elements of the form

$$(b_k^* \otimes id)(\sum_{i,j} b_{ij}(a) \otimes c_{ij}) = \sum_{i,j} b_k^*(\alpha_j b_i) c_{ij} = \sum_{i,j} \delta_{ki} \alpha_j c_{ij} = \sum_j \alpha_j c_{kj},$$

where we assumed $a = \sum_j \alpha_j a_j$. Therefore

$$\{(w \otimes id)\rho(x) : x \in A, w \in B^*\} = \mathrm{span}(\{c_{ij}\}_{ij}).$$

By Corollary 4.13 we know that the CQGs $S_{n^2}^+$ and PO_n^+ are monoidally equivalent, hence we can look at their linking algebra using the above remark. This proposition is a new result.

Proposition 5.6. *The linking algebra of $S_{n^2}^+$ and PO_n^+ is the universal *-algebra generated by elements p_{ijk} where $1 \leqslant i, j \leqslant n$ and $1 \leqslant k \leqslant n^2$ with*

$$\sum_{k=1}^{n^2} p_{ijk} = \delta_{ij} \tag{5.1}$$

$$p_{ijk}^* = p_{jik} \tag{5.2}$$

$$\sum_{r=1}^{n} p_{irk} p_{rjl} = \delta_{kl} p_{ijk} \tag{5.3}$$

$$\sum_{i=1}^{n} p_{iik} = \frac{1}{n}. \tag{5.4}$$

Proof. By Remark 5.5 the *-algebra $\hat{\mathcal{O}}(S_{n^2}^+, PO_n^+)$ is generated by the coefficients of a unital *-homomorphism

$$\rho : \mathbb{C}^{n^2} \to M_n(\mathbb{C}) \otimes \hat{\mathcal{O}}(S_{n^2}^+, PO_n^+)$$

with

$$(tr \otimes id)\rho = \psi_{n^2}(\cdot)1.$$

We can write $\rho(e_k) = \sum_{i,j} e_{ij} \otimes p_{ijk}$ for some elements $p_{ijk} \in \hat{\mathcal{O}}(S_{n^2}^+, PO_n^+)$. The proof is now similar to the proof of Proposition 2.21.

Since ρ has to be unital we get

$$\sum_{i,j} e_{ij} \otimes \sum_k p_{ijk} = \sum_{i,j,k} e_{ij} \otimes p_{ijk} = \rho(\sum_k e_k) = \rho(1) \stackrel{!}{=} 1 \otimes 1 = \sum_{i,j} e_{ij} \otimes \delta_{ij}$$

which is equivalent to (5.1). Moreover, ρ has to be involutive. This implies

$$\sum_{i,j} e_{ji} \otimes p_{ijk}^* = \rho(e_k)^* \stackrel{!}{=} \rho(e_k^*) = \rho(e_k) = \sum_{i,j} e_{ji} \otimes p_{jik}$$

and this is equivalent to (5.2). Furthermore, we need ρ to be multiplicative. Hence

$$\begin{aligned}\sum_{i,j} e_{ij} \otimes \sum_r p_{irk} p_{rjl} &= \sum_{i,j,r} e_{ij} \otimes p_{irk} p_{rjl} = \sum_{i,j,r,s} \delta_{sr} e_{ij} \otimes p_{isk} p_{rjl} \\ &= \sum_{i,j,r,s} e_{is} e_{rj} \otimes p_{isk} p_{rjl} = (\sum_{i,s} e_{is} \otimes p_{isk})(\sum_{r,j} e_{rj} \otimes p_{rjl}) \\ &= \rho(e_k)\rho(e_l) \stackrel{!}{=} \rho(e_k e_l) = \delta_{lk}\rho(e_k) = \sum_{i,j} e_{ij} \otimes \delta_{lk} p_{ijk}\end{aligned}$$

which is equivalent (5.3). Finally, the equality $(tr \otimes id)\rho = \psi_{n^2}(\cdot)1$ is equivalent to (5.4) since

$$1 = n^2 \psi_{n^2}(e_k)1 \stackrel{!}{=} n^2 (tr \otimes id)\rho(e_k) = n^2 \sum_{i,j} tr(e_{ij}) p_{ijk} = n^2 \sum_{i,j} \delta_{ij} \frac{1}{n} p_{ijk} = n \sum_i p_{iik}.$$

□

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