UNIVERSITÄT DES SAARLANDES

UNITARY EASY GROUPS

Bachelor's Thesis

submitted by

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Sworn declaration

I declare under oath that I have prepared the paper at hand independently and without the help of others and that I have not used any other sources and recourses than the ones stated. Parts that have been taken literally or correspondingly from published or unpublished texts or other sources have been labeled as such. This paper has not been presented to any examination board in the same or similar form before.

Saarbrücken, ________________ ____________________

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Introduction

This thesis deals with a group theoretical verification of the characterisation of *unitary easy groups* — groups that were classified by Tarrago and Weber in 2016 [20].

The term “easy group” was coined by Banica and Speicher in [5] and describes (in the context of this work) closed subgroups of the orthogonal group O_n, whose associated symmetric tensor categories with duals are generated by linear maps that originate in combinatorial considerations, namely partitions of sets.

As it turned out, one can even assign a symmetric tensor category with duals to a closed subgroup of the unitary group U_n. In this context, the underlying combinatorial considerations are partitions of two-coloured sets and unitary easy groups are again such, where their associated tensor categories are generated by certain linear maps stemming from partitions of coloured sets. The easy groups classified by Banica and Speicher [5] correspond to unitary easy groups, where the underlying sets are non-coloured (i.e., all elements of the underlying sets have the same colour).

The characterisation of (unitary) easy groups in terms of symmetric tensor categories with duals that originate in orthogonal respectively unitary representations of groups makes a Tannaka-Krein-approach viable to recover a group from a given category of partitions. This is the approach chosen in [5] and [20]. An elementary group theoretical verification of the results is still missing, which will be addressed in this thesis.

To this end, we deal with unitary representations of easy groups respectively unitary easy groups and use the fact that the correspondence to a certain category of partitions gives rise to certain relations on the generators of a certain commutative universal C^*-algebras which translates to relations in terms of polynomials in the entries of matrices that belong to those (unitary) easy groups.

Due to the description of easy groups in terms of (objects associated to) categories of partitions, it is, in the spirit of the generalisation of the notions “compact group” and “compact matrix group” as discussed in chapter 2 of this thesis, possible to drop certain partitions from those categories of partitions to end up with so-called *easy quantum groups* respectively *unitary easy quantum groups*.

Additionally, we take a brief look at (unitary) easy groups as Lie groups, which they are, since they are closed subgroups of the orthogonal group O_n respectively the unitary group U_n.

An open question in this field is how to find examples of non-easy groups, that is, matrix groups G with $\mathfrak{S}_n \subseteq G \subseteq O_n$ respectively $\mathfrak{S}_n \subseteq G \subseteq U_n$ that are not easy.

In the first chapter, we recap some basic facts about matrix groups and show that $\mathrm{Gl}_n(\mathbb{K})$ is a smooth manifold, where $\mathbb{K}$ denotes either the field of real or complex numbers.

In the second chapter, we familiarise with the notions of compact quantum group and compact matrix quantum group. To form a better understanding for comultiplications, basic theory of algebras, coalgebras, bialgebras and Hopf algebras is developed.

The third chapter deals with easy groups and their relationship to so-called categories of partitions.

In chapter four, we give a group theoretical verification that the unitary easy groups found in [20] are indeed the groups given there.

The appendix is meant to give a brief overview over — and hopefully some kind of intuition for — the concepts and notions that occur throughout the thesis. The appendix is meant to make this thesis more or less self-contained and thus hopefully accessible for students of other fields of study, even though some kind of capability of mathematical thought on the readers part will be necessary and knowledge of the content of basic courses in linear algebra and analysis certainly helpful.

Finally, I want to thank my friends and family for their ongoing support throughout my studies and helpful input during the development of this thesis, namely I want to thank my parents for their incredible support in matters outside of my studies and Eileen Oberringer, Michael Brill, Daniel Krämer, Steven Klein and Christian Steinhart for reading through the text. In addition, I want to thank my supervisor and teacher Moritz Weber for his great lectures and the guidance through the topic of my thesis. Last, but not least, I want to thank Prof. Dr. Ernst-Ulrich Gekeler and Prof. Dr. Gabriela Weitze-Schmithüsen for their instructive lectures and their time and input on questions regarding this thesis.

A few remarks on notation

Apart from the following agreements on notation, we will avoid fixed notation for certain parts of the thesis (e.g., “For this section let V be a k-vector space...”) to improve readability.

Throughout the thesis, $\mathbb{N} := \{1, 2, 3, \dots\}$ denotes the natural numbers, $\mathbb{N}_0$ denotes the non-negative integers, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ denotes the real numbers and $\mathbb{C}$ denotes the complex numbers. The symbol $\mathbb{K}$ stands for either the real– or the complex numbers. Given a natural number n, $\mathbb{N}_n$ denotes the set of natural numbers at most as large as n, i.e., $\mathbb{N}_n := \{1, 2, \dots, n\}$. Given a complex number $z = z_1 + \mathrm{i}z_2$, we denote by z^* the complex conjugate of z, i.e., $z^* = z_1 - \mathrm{i}z_2$.

Given two sets A and B, we denote by $A - B := \{x \mid x \in A \wedge x \notin B\}$ the set difference of A and B to avoid confusion with coset notation. If $A \subseteq B$, we call $A^{\mathsf{c}} := B - A$ the relative complement of A in B. By B^A, we denote the set of maps from A to B, i.e., $B^A = \{f \colon A \to B\}$. If B is at least a group, we put $B^{(A)} := \{f \colon A \to B \mid f(a) \neq e_B \text{ only for finitely many } a \in A\}$.

For a non-negative integer n, we denote $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. For a natural number n, by $\mathfrak{S}_n := \mathrm{Sym}(\mathbb{N}_n)$ we denote the *symmetric group on n letters*.

Given an index set I and a family of sets $(A_i)_{i \in I}$, we denote by $\prod_{i \in I} A_i$ the cartesian product of the sets A_i, i.e., $\prod_{i \in I} A_i = \{(a_i)_{i \in I} \mid a_i \in A_i\}$.

For natural numbers n and k, we put

$$I_n^k := \{\boldsymbol{i} = (i_1, \dots, i_k) \mid i_1, \dots, i_k \in \mathbb{N}_n\},$$

Given integers i, j, we define

$$\delta_{i,j} := \begin{cases} 1, & \text{if } i = j, \\ 0 & \text{else,} \end{cases}$$

the so called *Kronecker Delta*. This definition extends to multi-indices standing to reason.

Given a natural number n and a field k, we denote by $e_i := (\delta_{i,j})^t_{1 \leq j \leq n}$ the i-th vector of the canonical basis of k^n. The set k^n turns into a k-vector space with componentwise addition and scalar multiplication. We will understand elements of k^n as column vectors. Vectors $\xi \in k^n$ will be expressed as

$$\xi = (\xi^1, \dots, \xi^n)^t = \sum_{i=1}^n \xi^i e_i$$

whenever convenient, functionals $\varphi \in (k^n)^* := \mathrm{Hom}(k^n, k)$ will be expressed as $\varphi = \sum_{i=1}^n \varphi_i \varepsilon^i$, where $\{\varepsilon^1, \dots, \varepsilon^n\} \subseteq (k^n)^*$ denotes the dual basis to

$\{e_1, \ldots, e_n\} \subseteq k^n$; accordingly, we agree on numbering row indices of matrices (that originate from linear maps) in superscript and column indices of matrices in subscript.

Given a linear map $\varphi\colon V \to W$ between Hilbert spaces over $\mathbb{K}$, we denote by $\varphi^\dagger$ the adjoint map of φ. Given a matrix $A \in \mathrm{Gl}_n(\mathbb{K})$, we write $A^\dagger := (A^*)^t$, where we mean componentwise complex conjugation.

We reserve the brackets "$\langle \cdot, \cdot \rangle$" for dual pairings and use the brackets "$(\cdot|\cdot)$" for inner products.

In a topological space $(X, \mathfrak{T})$, we denote by $\mathrm{cl}_\mathfrak{T}(A)$ the closure of $A \subseteq X$ and by $\mathrm{Int}_\mathfrak{T}(A)$ the interior of $A \subseteq X$, both with respect to $\mathfrak{T}$. If there is no confusion to be feared about the topology in question, we simply write $\mathrm{cl}(A)$ or $\mathrm{Int}(A)$ respectively.

If not explicitly stated otherwise, we assume an euclidean or unitary space to be equipped with the standard inner product and the norm induced by it. If euclidean or unitary spaces make an appearance as a topological space, we assume the topology induced by the standard inner product, if not stated otherwise explicitly.

Chapter I.

Matrix groups

The set $\mathrm{Gl}_n(k) := M_n(k)^\times$ forms a group with respect to matrix multiplication (see Example A.1.2 and Example A.3.6), subgroups of $\mathrm{Gl}_n(k)$ are called *matrix groups*. If $k = \mathbb{K}$ is the field of real or complex numbers, the group $\mathrm{Gl}_n(\mathbb{K})$ even carries a lot more structure; it then also is a topological group and a differentiable manifold in a compatible way, meaning the law of composition and the inverse map $i\colon \mathrm{Gl}_n(\mathbb{K}) \to \mathrm{Gl}_n(\mathbb{K})$, $A \mapsto A^{-1}$ are smooth. Such groups are called *Lie groups*.

It is a crucial result in the theory of Lie groups that closed subgroups of Lie groups are themselves Lie groups with the relative topology, as a consequence every closed subgroup of $\mathrm{Gl}_n(\mathbb{K})$ is itself a Lie group; such groups are called *matrix Lie groups*.

The main goals of this chapter are convincing ourselves that $\mathrm{Gl}_n(\mathbb{K})$ indeed is a Lie group, seeing that the orthogonal and unitary groups are compact and how subgroups of the orthogonal group O_n respectively the unitary group U_n can be represented on the Hilbert spaces $T^m(\mathbb{K}^n)$ for natural numbers n and m in a natural way.

For this chapter, the sections 1, 2 and 4 of Appendix A and the sections 1, 2 and 3 of Appendix B should make the content understandable.

1. The algebra of square matrices

As discussed in Remark A.2.5, in the situation of finite-dimensional vector spaces, we can identify $\mathrm{End}_k(V)$ with $M_n(k)$, where $n = \dim(V)$; under this identification, invertible linear maps correspond to matrices with non-zero determinant. The set $M_n(k)$ turns into a unital algebra over k, i.e., a k-vector space together with a k-bilinear multiplication and a neutral element with respect to this k-bilinear multiplication, in the following way:

Lemma I.1.1: *Let n be a natural number and let k be a field. Then, $M_n(k)$ turns into an algebra over k with addition, componentwise multiplication and multiplication of matrices as laws of composition. The neutral element with respect to addition of matrices is the zero-matrix, the neutral element with respect to multiplication of matrices is the identity $I_n = \mathrm{diag}(1, \ldots, 1)$.*

For $k = \mathbb{K}$, we can make $M_n(\mathbb{K})$ into a normed vector space over $\mathbb{K}$. As it turns out, this space is isometrically isomorphic to $\mathbb{K}^{n \cdot n}$ and thus a Banach space over $\mathbb{K}$, since convergence in $\mathbb{K}^m$ is precisely componentwise convergence and $\mathbb{K}$ with the usual absolute value is a Banach space. But even more holds true: $M_n(\mathbb{K})$ together with the operator norm is a unital Banach algebra, i.e., a Banach space over $\mathbb{K}$ that is at the same time a unital algebra over $\mathbb{K}$ such that the norm is submultiplicative.

Lemma I.1.2 (Operator norm): *Let n be a natural number. The map*

$$\|\cdot\|_{\mathrm{op}} \colon M_n(\mathbb{K}) \longrightarrow \mathbb{K}, \qquad A \longmapsto \sup\{\|Ax\| \mid \|x\| = 1\}$$

is a norm on $M_n(\mathbb{K})$, the so-called operator norm. *The operator norm is submultiplicative, i.e., for $A, B \in M_n(\mathbb{K})$ it holds $\|AB\|_{\mathrm{op}} \leq \|A\|_{\mathrm{op}} \|B\|_{\mathrm{op}}$.*

Proof: First, we show that $\|\cdot\|_{\mathrm{op}}$ is indeed a norm.

(i) Suppose $A = 0$. Then $Ax = 0$ for all $x \in \mathbb{K}^n$, in particular $\|Ax\| = 0$ for all $x \in \mathbb{K}^n$ and thus $\|A\|_{\mathrm{op}} = 0$.

Suppose now $\|A\|_{\mathrm{op}} = 0$. Then $\|Ax\| = 0$ for all $x \in \mathbb{K}^n$ with $\|x\| = 1$, hence for $x \in \mathbb{K}^n - \{0\}$, it holds $\|Ax\| = \|x\| \|A(x/\|x\|)\| = 0$. Because $\|\cdot\|$ is a norm on $\mathbb{K}^n$, this means $Ax = 0$ for all $x \in \mathbb{K}^n$ and thus $A = 0$.

(ii) Let $A \in M_n(\mathbb{K})$ and $\alpha \in \mathbb{K}$. Then we have

$$\begin{aligned}\sup\{\|(\alpha A)x\| \mid \|x\| = 1\} &= \sup\{|\alpha| \|Ax\| \mid \|x\| = 1\} \\ &= |\alpha| \sup\{\|Ax\| \mid \|x\| = 1\},\end{aligned}$$

i.e., $\|\alpha A\|_{\mathrm{op}} = |\alpha| \|A\|_{\mathrm{op}}$.

(iii) Let $A, B \in M_n(\mathbb{K})$. It holds

$$\begin{aligned}\sup\{\|(A+B)x\| \mid \|x\| = 1\} &\leq \sup\{\|Ax\| + \|Bx\| \mid \|x\| = 1\} \\ &= \sup\{\|Ax\| \mid \|x\| = 1\} + \sup\{\|Bx\| \mid \|x\| = 1\}.\end{aligned}$$

For the submultiplicativity, let $A, B \in M_n(\mathbb{K})$. Then we can calculate

$$\begin{aligned}\sup\{\|ABx\| \mid \|x\| = 1\} &= \sup\{\|Bx\| \|A(Bx/\|Bx\|)\| \mid \|x\| = 1\} \\ &= \|Bx\| \sup\{\|Ax\| \mid \|x\| = 1\} \\ &\leq \sup\{\|Ax\| \mid \|x\| = 1\} \sup\{\|Bx\| \mid \|x\| = 1\}. \quad \square\end{aligned}$$

This, together with the preceding remark, shows that $M_n(\mathbb{K})$ turns into a unital Banach algebra over $\mathbb{K}$ with the operator norm.

Lemma I.1.3: *The matrix multiplication*

$$\cdot\colon M_n(\mathbb{K}) \times M_n(\mathbb{K}) \longrightarrow M_n(\mathbb{K}), \qquad (A, B) \longmapsto AB$$

is continuous with respect to the topology induced by $\|\cdot\|_{\mathrm{op}}$, where we equip $M_n(\mathbb{K}) \times M_n(\mathbb{K})$ with the product topology.

Proof: We have to check that for sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ in $M_n(\mathbb{K})$ with $A_n \to A$ and $B_n \to B$ it holds $A_nB_n \to AB$. We have

$$\begin{aligned} \|A_nB_n - AB\|_{\mathrm{op}} &= \|A_nB_n - A_nB - AB + A_nB\|_{\mathrm{op}} \\ &= \|A_n(B_n - B) + (A_n - A)B\|_{\mathrm{op}} \\ &\leq \|A_n\|_{\mathrm{op}}\|B_n - B\|_{\mathrm{op}} + \|A_n - A\|_{\mathrm{op}}\|B\|_{\mathrm{op}} \end{aligned}$$

which gets arbitrarily small, since convergent sequences are bounded. □

Lemma I.1.4: *Let $A \in M_n(\mathbb{K})$ with $\|A\|_{\mathrm{op}} < C$. Then for $1 \leq i, j \leq n$, we have $|A^i_j| < C$.*

Proof: Since $\|e_i\| = 1$, we have for $1 \leq i \leq n$ that $\|Ae_i\| \leq \|A\|_{\mathrm{op}} < C$. Now

$$|A^i_j| \leq \left(\sum_{i=1}^{n} |A^i_j|^2\right)^{1/2} = \|Ae_j\| < C$$

which we wanted to show. □

We read from Lemma I.1.4 that if a sequence $(A_n)_{n\in\mathbb{N}}$ in $M_n(\mathbb{K})$ converges to $A \in M_n(\mathbb{K})$, it holds $(A_n)^i_j \to A^i_j$ for $1 \leq i, j \leq n$.
A speciality in the case of finite-dimensional $\mathbb{K}$-vector spaces is that the converse is also true. This is, however, terribly wrong in general.

Lemma I.1.5: *Let n be a natural number and let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $M_n(\mathbb{K})$ such that $(A_n)^i_j \to A$ for $1 \leq i, j \leq n$. Then $\|A_n - A\|_{\mathrm{op}} \to 0$.*

For a proof of this assertion, check Remark B.4.7.

Example I.1.6 (Vector spaces as smooth manifolds): Let V be an n-dimensional $\mathbb{R}$-vector space equipped with any norm (giving V a topology) and an ordered basis $B = (b_1, \ldots, b_n)$. In a normed vector space, any non-empty open subset is the space itself, because by normalisation, every open ball fits a basis of said normed vector space.

The coordinate map

$$D_B \colon V \longrightarrow \mathbb{R}^n, \qquad \sum_{i=1}^n v^i b_i \longmapsto \sum_{i=1}^n v^i e_i$$

is continuous, hence (V, D_B) is a chart for V. Choosing a different basis B' in V yields a different chart $(V, D_{B'})$ and the change of charts is just the change of basis-matrix, i.e., the changes of charts are diffeomorphisms as invertible linear maps.

Remark I.1.7: As an n^2-dimensional $\mathbb{R}$-vector space, $M_n(\mathbb{R})$ is a smooth n^2-dimensional manifold with the chart $(M_n(\mathbb{R}), D_B)$. For $M_n(\mathbb{C})$, we consider its realification, which is a $2n^2$-dimensional $\mathbb{R}$-vector space, and thus make $M_n(\mathbb{C})$ a smooth $2n^2$-dimensional manifold.

2. The general linear group

Inside the Banach algebra $M_n(\mathbb{K})$ sits the general linear group $\mathrm{Gl}_n(\mathbb{K})$, consisting of the matrices with non-zero determinant. These are in a one-to-one correspondence with invertible endomorphisms of $\mathbb{K}^n$.

Definition I.2.1 (Matrix group): Let n be a natural number and let k be a field. Then $\mathrm{Gl}_n(k) = \{A \in M_n(k) \mid \det(A) \neq 0\}$ forms a group with the multiplication of matrices as law of composition. A subgroup $H \subseteq \mathrm{Gl}_n(k)$ is called a *matrix group*.

Remark I.2.2: Important for the following topological considerations is the fact that $\det \colon M_n(\mathbb{K}) \to \mathbb{K}$ is continuous, since for a matrix $A = (a^i_j)_{1 \leq i,j \leq n}$,

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) a^1_{\sigma(1)} \cdots a^n_{\sigma(n)}$$

is a polynomial in the entries of A.

Lemma I.2.3: *Let n be a natural number. The general linear group* $\mathrm{Gl}_n(\mathbb{K})$ *together with multiplication of matrices and the operator norm is a topological group, i.e., multiplication of matrices and inversion of matrices are continuous with respect to the operator norm.*

Proof: The continuity of multiplication of matrices with respect to the operator norm was already shown in Lemma I.1.3. Given a matrix $A = (a^i_j)_{1\leq i,j\leq n}$, the adjugate matrix $A^\# =: B = (b^i_j)_{1\leq i,j\leq n}$ has the entries

$$b^i_j = (-1)^{i+j} \det(A^{\boldsymbol{j}}_{\boldsymbol{i}}) \tag{I.1}$$

where $A^{\boldsymbol{j}}_{\boldsymbol{i}}$ denotes the $(n-1) \times (n-1)$ matrix resulting from cancelling the j-th row and the i-th column of A.

For an invertible matrix $A \in \mathrm{Gl}_n(\mathbb{K})$ and its adjugate matrix $A^\#$ it holds $AA^\# = A^\# A = \det(A)I_n$, see section 6.4 in [12].

Given a convergent sequence $(A_m)_{m\in\mathbb{N}}$ of matrices $A_m = ([a_m]^i_j)_{1\leq i,j\leq n}$ in $\mathrm{Gl}_n(\mathbb{K})$ with limit A, the component sequence $([a_m]^i_j)_{m\in\mathbb{N}}$ converges to the entry a^i_j of A and thus, due to Eq. (I.1) and the continuity of the determinant, we have componentwise convergence of the adjugates $A^\#_m$ to the adjugate $A^\#$. Componentwise convergence implies convergence in operator norm, see Lemma I.1.5. □

Proposition I.2.4: *Let n be a natural number. Then* $\mathrm{Gl}_n(\mathbb{R})$ *is a smooth manifold.*

Proof: Since $\det\colon M_n(\mathbb{R}) \to \mathbb{R}$ is continuous, $\mathbb{R} - \{0\} \subseteq \mathbb{R}$ is open and $\mathrm{Gl}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ is an open subset of the smooth n^2-dimensional manifold $M_n(\mathbb{R})$. This renders $\mathrm{Gl}_n(\mathbb{R})$ an n^2-dimensional smooth manifold itself. The chart for $\mathrm{Gl}_n(\mathbb{R})$ is the restriction of the chart for $M_n(\mathbb{R})$ to $\mathrm{Gl}_n(\mathbb{R})$. □

Proposition I.2.5: *Let n be a natural number. Then* $\mathrm{Gl}_n(\mathbb{R})$ *is a* Lie group*, i.e.,* $\mathrm{Gl}_n(\mathbb{R})$ *is a topological group and a smooth manifold at the same time and the laws of composition are smooth with respect to the topology on* $\mathrm{Gl}_n(\mathbb{R})$*.*

Proof: We already established that $\mathrm{Gl}_n(\mathbb{R})$ is both a topological group and a smooth manifold. What remains to show is that multiplication of matrices $\mu\colon \mathrm{Gl}_n(\mathbb{R}) \times \mathrm{Gl}_n(\mathbb{R}) \to \mathrm{Gl}_n(\mathbb{R})$ and inversion $i\colon \mathrm{Gl}_n(\mathbb{R}) \to \mathrm{Gl}_n(\mathbb{R})$ are smooth maps.

Note that $\mathrm{Gl}_n(\mathbb{R}) \times \mathrm{Gl}_n(\mathbb{R})$ is a smooth manifold together with the chart $(D_B \times D_B, \mathrm{Gl}_n(\mathbb{R}) \times \mathrm{Gl}_n(\mathbb{R}))$.

A map $f\colon M \to N$ between an m-dimensional smooth manifold M and an n-dimensional smooth manifold N is called smooth in a point p in the open set $U \subseteq M$, if for a chart (ψ, W) of N around p the *local coordinate representation*

$$f_{\varphi,\psi} := \psi \circ f \circ \varphi^{-1}\colon \varphi(U) \longrightarrow \psi(W)$$

is smooth in the point $\varphi(p) \in \mathbb{R}^m$, i.e., the situation is captured in the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ \varphi(U) & \xrightarrow[f_{\varphi,\psi}]{} & \psi(W) \end{array}$$

This notion of smoothness does not depend on the chosen charts because the changes of charts are smooth, the fact that we have the chain rule and the Inverse Function Theorem.

We agree on the ordered basis $B = (E^1_1, \dots, E^1_n, E^2_1, \dots, E^2_n, \dots, E^n_1, \dots, E^n_n)$ for $M_n(\mathbb{R})$. For μ and i, we have, at any point of the domains, the local chart representations

$$\mu_{D_B,D_B} \colon \mathbb{R}^{n \cdot n} \times \mathbb{R}^{n \cdot n} \longrightarrow \mathbb{R}^{n \cdot n},$$
$$\Big((a^1_1, \dots, a^n_n), (c^1_n, \dots, c^n_n)\Big) \longmapsto \Big(\sum_{k=1}^{n} a^1_k c^k_1, \dots, \sum_{k=1}^{n} a^n_k c^k_n\Big)$$

and

$$i_{D_B,D_B} \colon \mathbb{R}^{n \cdot n} \longrightarrow \mathbb{R}^{n \cdot n}, \qquad (a^1_1, \dots, a^n_n) \longmapsto \frac{1}{\det((a^i_j)_{1 \le i,j \le n})}(b^1_1, \dots, b^n_n),$$

where $B = (b^i_j)_{1 \le i,j \le n}$ is the adjugate matrix to $(a^i_j)_{1 \le i,j \le n}$.

The determinant $\det\colon \mathrm{Gl}_n(\mathbb{R}) \to \mathbb{R}$, interpreted as map between the smooth manifolds $\mathrm{Gl}_n(\mathbb{R})$ and $\mathbb{R}$, is smooth, since for a matrix $A \in \mathrm{Gl}_n(\mathbb{R})$ it holds $\det(A) = \det_{D_B,\mathrm{id}}(a^1_1, \dots, a^n_n)$ and $\det(A)$ is a polynomial in the entries of A.

Hence, we recognise both local coordinate representations as smooth maps between subsets of euclidean spaces. □

The following important result goes back to Cartan and is also known as Closed Subgroup Theorem in english literature. It can for example be found in [11] III.2.33. Satz

Proposition I.2.6 (Closed Subgroup Theorem): *Let G be a Lie group and let H be a subgroup of G. Then H is a Lie group with respect to the induced topology if and only if H is closed in G.*

Remark I.2.7: Since all the definitions up to this point were made for topological groups that are at the same time real smooth manifolds, we have to think about $\mathrm{Gl}_n(\mathbb{C})$ for a second. For a natural number n, we embed $\mathrm{Gl}_n(\mathbb{C})$ into $\mathrm{Gl}_{2n}(\mathbb{R})$ by sending an invertible matrix $A = (a^i_j)_{1\le i\le n}$ to the block matrix $(\rho(a^i_j))_{1\le i\le n} \in \mathrm{Gl}_{2n}(\mathbb{R})$, where ρ is the homomorphism of rings

$$\rho\colon \mathbb{C} \longrightarrow M_2(\mathbb{R}), \qquad a + \mathrm{i}b \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

An invertible matrix in $\mathrm{Gl}_n(\mathbb{C})$ gets sent to an invertible matrix in $M_{2n}(\mathbb{R})$, because any matrix in $\mathrm{Gl}_n(\mathbb{C})$ is conjugated to a matrix in Jordan normal form without zero-entries on the diagonal and the determinant of a block matrix is the product of the determinants of the diagonal blocks.

Since convergence is precisely componentwise convergence, this subgroup of $\mathrm{Gl}_{2n}(\mathbb{R})$ is furthermore closed. Hence, using the Closed Subgroup Theorem, we can view $\mathrm{Gl}_n(\mathbb{C})$ as a Lie group.

3. Compactness of the unitary groups

Recall that for a natural number n, the sets

$$O_n := \{A \in \mathrm{Gl}_n(\mathbb{R}) \mid AA^t = A^tA = I_n\},$$
$$U_n := \{A \in \mathrm{Gl}_n(\mathbb{C}) \mid AA^\dagger = A^\dagger A = I_n\}$$

are called the *orthogonal* and the *unitary group*. These are indeed groups with the multiplication of matrices and it can be shown that they consist precisely of the structure preserving maps of the Hilbert spaces $\mathbb{K}^n$, i.e., linear isometries. For further details, see Corollary A.4.15.

Lemma I.3.1: *The map*

$$f\colon M_n(\mathbb{K}) \longrightarrow M_n(\mathbb{K}), \qquad A \longmapsto A^\dagger,$$

sending a matrix A to its hermitian transpose, is continuous with respect to the topology induced by $\|\cdot\|_{\mathrm{op}}$.

Proof: Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $M_n(\mathbb{K})$ with limit $A \in M_n(\mathbb{K})$. Then $(A_n^\dagger)^i_j = [(A_n)^j_i]^* \to [A^j_i]^* = (A^\dagger)^i_j$. □

Proposition I.3.2: *Let n be a natural number. Then, the groups $O_n \subseteq M_n(\mathbb{R})$ and $U_n \subseteq M_n(\mathbb{C})$ are compact and hence, in particular, matrix Lie groups.*

Proof: We only give the proof for U_n, for O_n the same arguments go through.

The map $f\colon M_n(\mathbb{K}) \to M_n(\mathbb{K})$, $A \mapsto A^\dagger A$ is continuous, since the matrix multiplication and the map mapping A to its hermitian transpose are continuous. We have $U_n = f^{-1}(\{I_n\})$, i.e., U_n is closed.

For $A \in U_n$ and $x \in \mathbb{K}^n$ we have $\|Ax\| = \|x\|$, i.e., $\|A\|_{\mathrm{op}} = 1$ and thus, U_n is bounded. Hence, by Heine-Borel, U_n is compact. ☐

4. Unitary representations

Reminder I.4.1: Let n be a natural number and let $H = \mathbb{C}^n$ be equipped with the canonical basis $\{e_1, \dots, e_n\}$. For a multi-index $\boldsymbol{i} = (i_1, \dots, i_m) \in I_n^m$, we put $e_{\boldsymbol{i}} := e_{i_1} \otimes \cdots \otimes e_{i_m}$. The set $\{e_{\boldsymbol{i}} \mid \boldsymbol{i} \in I_n^m\}$ forms a basis of the algebraic tensor product $T^m(H) := \bigotimes_{i=1}^m H$. As a finite-dimensional $\mathbb{K}$-vector space, $T^m(H)$ is a Hilbert space, too. The inner product from Remark B.5.1 in this case reads

$$(\xi|\eta)_{T^m(H)} = \Bigg(\sum_{\boldsymbol{i}\in I_n^m} \xi^{\boldsymbol{i}} e_{\boldsymbol{i}} \Bigg| \sum_{\boldsymbol{j}\in I_n^m} \eta^{\boldsymbol{j}} e_{\boldsymbol{j}} \Bigg)_{T^m(H)} = \sum_{\boldsymbol{i}\in I_n^m} \xi^{\boldsymbol{i}} (\eta^{\boldsymbol{i}})^*.$$

Definition I.4.2 (Unitary representation): Let H be a Hilbert space over $\mathbb{K}$, let $B(H)$ denote the set of bounded linear operators on H and $U(H)$ denote the subset of unitary operators on H. Furthermore let G be a locally compact group. If $\pi\colon G \to U(H)$ is a weakly continuous homomorphism, π is called a *unitary representation of G on H.*

The groups we consider meet the requirements for unitary representations. To see this, note that open and closed subsets of locally compact Hausdorff spaces are locally compact with respect to the subspace topology. Since $\mathrm{Gl}_n(\mathbb{C})$ is a smooth manifold, U_n is, in particular, locally compact.

Proposition I.4.3: *Let n be a natural number, let $H = \mathbb{C}^n$ and let $G \subseteq U_n$ be an open or closed subgroup. Then, for every non-negative integer m, the map*

$$\pi(G,m)\colon G \longrightarrow B(T^m(H)), \qquad g \longmapsto T^m(g) := g^{\otimes m}$$

is a unitary representation of G on $B(T^m(H))$.

Proof: For the unitarity, let $g \in G$. Since $G \subseteq U_n$, we have $(g\xi|g\eta) = (\xi|\eta)$ for all $\xi, \eta \in H$. For $\xi = \sum_{\boldsymbol{i} \in I_n^m} \xi^{\boldsymbol{i}} e_{\boldsymbol{i}}$ and $\eta = \sum_{\boldsymbol{j} \in I_n^m} \eta^{\boldsymbol{j}} e_{\boldsymbol{j}}$ in $T^m(H)$, we have

$$\begin{aligned}
(T^m(g)(\xi)|T^m(g)(\eta))_{T^m(H)} &= \Big(T^m(g)\Big(\sum_{\boldsymbol{i} \in I_n^m} \xi^{\boldsymbol{i}} e_{\boldsymbol{i}}\Big)\Big|T^m(g)\Big(\sum_{\boldsymbol{j} \in I_n^m} \eta^{\boldsymbol{j}} e_{\boldsymbol{j}}\Big)\Big)_{T^m(H)} \\
&= \sum_{\boldsymbol{i} \in I_n^m} \sum_{\boldsymbol{j} \in I_n^m} \xi^{\boldsymbol{i}} (\eta^{\boldsymbol{j}})^* (T^m(g)(e_{\boldsymbol{i}})|T^m(g)(e_{\boldsymbol{j}}))_{T^m(H)} \\
&= \sum_{\boldsymbol{i} \in I_n^m} \sum_{\boldsymbol{j} \in I_n^m} \xi^{\boldsymbol{i}} (\eta^{\boldsymbol{j}})^* \prod_{k=1}^m (g(e_{i_k})|g(e_{j_k})) \\
&= \sum_{\boldsymbol{i} \in I_n^m} \sum_{\boldsymbol{j} \in I_n^m} \xi^{\boldsymbol{i}} (\eta^{\boldsymbol{j}})^* \prod_{k=1}^m (e_{i_k}|e_{j_k}) \\
&= \sum_{\boldsymbol{i} \in I_n^m} \sum_{\boldsymbol{j} \in I_n^m} \xi^{\boldsymbol{i}} (\eta^{\boldsymbol{j}})^* (e_{\boldsymbol{i}}|e_{\boldsymbol{j}})_{T^m(H)} = (\xi|\eta)_{T^m(H)},
\end{aligned}$$

i.e., in fact $\pi(G,m)\colon G \to U(T^m(H))$.

To see that π is a homomorphism, let g and $h \in G$. Then

$$\begin{aligned}
\pi(G,m)(gh)(\xi) &= T^m(gh)\Big(\sum_{\boldsymbol{i} \in I_n^m} \xi^{\boldsymbol{i}} e_{\boldsymbol{i}}\Big) \\
&= \sum_{\boldsymbol{i} \in I_n^m} \xi^{\boldsymbol{i}} T^m(gh)(e_{\boldsymbol{i}}) \\
&= \sum_{\boldsymbol{i} \in I_n^m} \xi^{\boldsymbol{i}} (gh)(e_{i_1}) \otimes \cdots \otimes (gh)(e_{i_m}) \\
&= \sum_{\boldsymbol{i} \in I_n^m} \xi^{\boldsymbol{i}} T^m(g)(he_{i_1} \otimes \cdots \otimes he_{i_m}) \\
&= \sum_{\boldsymbol{i} \in I_n^m} T^m(g)\Big(T^m(h)(\xi^{\boldsymbol{i}} e_{\boldsymbol{i}})\Big) = \pi(G,m)(g)\pi(G,m)(h)(\xi).
\end{aligned}$$

Finally, for the continuity of $\pi(G,m)$ with respect to the weak operator topology on $B(T^m(H))$, let $(g_n)_{n \in \mathbb{N}}$ be a sequence in G with limit g, i.e., $(g_n\xi|\eta) \to (g\xi|\eta)$ for all $\xi, \eta \in H$, and let $\varepsilon > 0$ be given. We have to see that $(\pi(G,m)(g_n)\xi|\eta)_{T^m(H)} \to (\pi(G,m)(g)\xi|\eta)_{T^m(H)}$ for all $\xi, \eta \in T^m(H)$.

For $\xi = \sum_{\boldsymbol{i} \in I_n^m} \xi^{\boldsymbol{i}} e_{\boldsymbol{i}}$ and $\eta = \sum_{\boldsymbol{j} \in I_n^m} \eta^{\boldsymbol{j}} e_{\boldsymbol{j}}$ in $T^m(H)$ and some natural number n, we can expand

$$\begin{aligned}
&|(T^m(g_n)\xi|\eta)_{T^m(H)} - (T^m(g)\xi|\eta)_{T^m(H)}| \\
&\qquad = \Big|\sum_{\boldsymbol{i} \in I_n^m} \sum_{\boldsymbol{j} \in I_n^m} \xi^{\boldsymbol{i}} (\eta^{\boldsymbol{j}})^* \prod_{k=1}^m \Big((g_n - g)e_{i_k}\Big|e_{j_k}\Big)\Big|.
\end{aligned}$$

Put $\alpha := \sum_{\boldsymbol{i}\in I_n^m}\sum_{\boldsymbol{j}\in I_n^m}|\xi^{\boldsymbol{i}}(\eta^{\boldsymbol{j}})^*|$.

As by assumption we have $((g_n - g)e_{i_k}|e_{j_k}) \to 0$ for each $1 \leq k \leq m$, we find natural numbers N_k such that $|((g_{n_k} - g)e_{i_k}|e_{j_k})| < (\varepsilon/\alpha)^{1/m}$ for all $n_k \geq N_k$. For $N := \max\{N_k \mid 1 \leq k \leq m\}$ we thus, using the triangular inequality for $|\cdot|$, find that

$$\Big|\sum_{\boldsymbol{i}\in I_n^m}\sum_{\boldsymbol{j}\in I_n^m}\xi^{\boldsymbol{i}}(\eta^{\boldsymbol{j}})^*\prod_{k=1}^m\Big((g_n - g)e_{i_k}\Big|e_{j_k}\Big)\Big| < \alpha\prod_{k=1}^m\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{m}} = \varepsilon$$

for all $n \geq N$ which we wanted to show. $\square$

Chapter II.

Compact matrix quantum groups

In 1987, Woronowicz [24] introduced the notion of a “compact matrix pseudo group”, later renamed to *compact matrix quantum group*. This notion generalises the notion of a compact matrix group. The starting point for this notion is the algebra of continuous functions on a compact matrix group, which can be supplied with a rich structure. Dropping “the commutativity condition” yields a new object, which is not a group anymore: a so-called compact matrix quantum group.

Later, in 1995, this approach was carried out [25] on the notion of a compact group as well, leading new objects called *compact quantum groups* that generalise compact matrix quantum groups.

This chapter is dedicated to familiarising with those two notions, and to do so, we take a small detour to coalgebras, bialgebras and Hopf algebras; objects that appear in many branches of modern pure mathematics. For the most part, the presentation follows [18].

For this chapter, section 3 of Appendix A and the sections 3, 4, 5, 6 of Appendix A should be helpful.

1. Coalgebras

Reminder II.1.1 (Algebra): Let k be a field and $(A, +, \cdot)$ be a k-vector space. If there is a bilinear multiplication

$$\bullet \colon A \times A \longrightarrow A,$$

such that $(A, +, \bullet)$ forms a unital ring, the tuple $(A, +, \cdot, \bullet)$ is called an *algebra over k* or *k-algebra*.

The bilinearity of "•" from Reminder II.1.1 is precisely the requirement that "•" is compatible with the addition and scalar multiplication on A in the sense that we have distributivity from the left and right.

If, on the one hand, we start with a unital algebra $(A, +, \cdot, \bullet)$, by the universal property of the tensor product $A \otimes_k A$, there is one and only one linear map $m\colon A \otimes_k A \to A$ that renders commutative the following diagram:

$$\begin{array}{ccc} A \times A & \xrightarrow{\tau} & A \otimes_k A \\ & \searrow^{\bullet} & \downarrow m \\ & & A \end{array}$$

Given an element $a \in A$, it is easy to check that the map

$$\eta_a\colon k \longrightarrow A, \qquad \lambda \longmapsto \lambda a$$

is linear. Furthermore, the map η_a allows the recovery of a, as $\eta_a(1) = a$. If we define $\eta := \eta_1$, we can express that A is a unital ring via the commutative diagrams

$$\begin{array}{ccc} A \otimes_k A & & \\ \eta\otimes\mathrm{id}\uparrow & \searrow^{m} & \\ k \otimes_k A & \cong & A \end{array} \qquad\qquad \begin{array}{ccc} A \otimes_k A & & \\ \mathrm{id}\otimes\eta\uparrow & \searrow^{m} & \\ A \otimes_k k & \cong & A \end{array}$$

Finally, we can capture the associativity of $m\colon A \otimes_k A \to A$ by the commutative diagram

$$\begin{array}{ccccc} & & A \otimes_k A \otimes_k A & & \\ & \swarrow^{m\otimes\mathrm{id}} & & \searrow^{\mathrm{id}\otimes m} & \\ A \otimes_k A & & & & A \otimes_k A \\ & \searrow_{m} & & \swarrow_{m} & \\ & & A & & \end{array}$$

If on the other hand we have a k-vector space A with linear maps $\eta\colon k \to A$, $m\colon A \otimes_k A \to A$ that render the above diagrams commutative, one can check that in fact A is a k-algebra. Thus, in this spirit, we also write (A, m_A, η_A) for the k-Algebra $(A, +, \cdot, \bullet)$.

"Reversing the arrows" in the above commutative diagrams is what we do to define a coalgebra over k.

Definition II.1.2 (Coalgebra): Let $(C, +, \cdot)$ be a vector space over the field k. If there are linear maps $\Delta\colon C \to C \otimes_k C$ and $\varepsilon\colon C \to k$ that satisfy

(i) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,

(ii) $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$

we call the tuple $(C, +, \cdot, \Delta, \varepsilon)$ a *coalgebra over* k. We will often just write (C, Δ, ε), when addition and scalar multiplication are clear from context. The map Δ is called *comultiplication*, the map ε is called the *counit*.

Coalgebras are sometimes refered to as “cogebras”, see [6], III.§11.1.

Example II.1.3 (of Coalgebras): Let k be a field.

(i) The field k has a coalgebra structure determined by $\Delta_k(1) := 1 \otimes 1$, $\varepsilon_k(1) := 1$. This coalgebra structure is unique.

(ii) Let $\varnothing \neq S$ be a set. The k-vector space $kS := \{\sum_{s \in S} \lambda(s)s \mid \lambda \in k^{(S)}\}$ is made into a coalgebra over k by the maps

$$\Delta\colon kS \longrightarrow kS \otimes_k kS, \qquad \sum_{s \in S} \lambda(s)s \longmapsto \sum_{s \in S} \lambda(s)s \otimes s,$$
$$\varepsilon\colon kS \longrightarrow k, \qquad \sum_{s \in S} \lambda(s)s \longmapsto 1.$$

This coalgebra is called the *set coalgebra*.

(iii) Consider the k-vector space $V := k^{n \times n}$ equipped with the canonical basis $\{E^i_j \mid 1 \leq i, j \leq n\}$, where $E_{i,j} := (\delta_{i,k}\delta_{j,\ell})_{1 \leq k, \ell \leq n}$. With the maps Δ, ε defined via linear extension of

$$\Delta(E^i_j) := \sum_{k=1}^{n} E^i_k \otimes E^k_j, \qquad \varepsilon(E^i_j) := \delta_{i,j},$$

V turns into a coalgebra over k, the so-called *matrix coalgebra*. We denote $M^c_n(k) := (V, \Delta, \varepsilon)$.

The comultiplication and counit as given above are the most natural, i.e., the dual maps to matrix multiplication and the map sending $\lambda \in K$ to λI_n. Thus, (V, Δ, ε) is precisely the dual coalgebra of $M_n(k)$, see Proposition II.1.5. For the assertion, see [18] Example 2.1.15, for a proof of this, check [14], Example 2.25.

As with associative laws of composition, coassociativity implies general coassociativity. This requires, just as for associative laws of composition, a rather tricky proof; for example one can show this via induction over two parameters. Essentially, one has to show that for any natural number n, the maps

$$\Delta^{(n,k)} := \mathrm{id}^{\otimes(k-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes(n-k-1)} \qquad (k \in \mathbb{N}_n)$$

coincide. We will denote this map with Δ^n.

When working with coalgebras, a handy notation for the comultiplication is in use: Given a coalgebra (C, Δ, ε) over k and $c \in C$, we can express $\Delta(c) \in C \otimes_k C$ as a finite sum $\Delta(c) = \sum_i c_{i(1)} \otimes c_{i(2)}$ with some $c_{i(1)}, c_{i(2)} \in C$. Often, the summation and the enumerating indices of $c_{(1)}$, $c_{(2)}$ are omitted, so $\Delta(c) = c_{(1)} \otimes c_{(2)}$. This notation is the so-called *Sweedler-notation*. In the Sweedler-notation, coassociativity and counit axioms read

$$c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)} \otimes c_{(3)},$$
$$\varepsilon(c_{(1)})c_{(2)} = c = c_{(1)}\varepsilon(c_{(2)}).$$

To comfortably use the Sweedler-notation for simplifying calculations, one has to think about the following assertions:

(i) For all natural numbers $n \geq 2$, it holds $\Delta^n = (\Delta \otimes \mathrm{id}) \circ \Delta^{n-1}$,

(ii) For any natural number n, any $i \in \mathbb{N}_{n-1}$ and $m \in \mathbb{N}_{n-i} \cup \{0\}$ it holds

$$\Delta^n = (\mathrm{id}^{\otimes m} \otimes \Delta^i \otimes \mathrm{id}^{\otimes(n-i-m)}) \circ \Delta^{n-i},$$

see [18], Chapter 2, Section 1. For a detailed proof, see [14], Lemma 2.12.

To find the correct notion for a structure preserving map between coalgebras, we proceed similarly as before and express the properties of homomorphisms between algebras in commutative diagrams: For k-algebras (A, m_A, η_A) and (B, m_B, η_B), a linear map $f\colon A \to B$ which renders commutative the diagrams

$$\begin{array}{ccc} A \otimes_k A & \xrightarrow{f \otimes f} & B \otimes_k B \\ \downarrow{\scriptstyle m_A} & & \downarrow{\scriptstyle m_B} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} & k & \\ {\scriptstyle \eta_A}\swarrow & & \searrow{\scriptstyle \eta_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

is called a *homomorphism of algebras*.

Definition II.1.4 (Coalgebra map): Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras over the field k. A linear map $f\colon C \to D$ with the properties

(i) $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$,

(ii) $\varepsilon_D \circ f = \varepsilon_C$

is called a *coalgebra map* or *homomorphism of coalgebras.*

In Sweedler-notation, the properties of a coalgebra map read

$$f(c)_{(1)} \otimes f(c)_{(2)} = f(c_{(1)}) \otimes f(c_{(2)}), \qquad \varepsilon_D(f(c)) = \varepsilon_C(c).$$

That the chosen definitions are what we wanted them to be can be seen by the following proposition:

Proposition II.1.5: *Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra over the field k. Then the maps $\Delta\colon C \to C \otimes_k C$ and $\varepsilon\colon C \to k$ define dual maps $m\colon C^* \otimes C^* \to C^*$ and $\eta\colon k \to C^*$ that evaluate*

$$m(\phi, \psi)(c) = \phi(c_{(1)})\psi(c_{(2)}), \qquad \eta(\lambda)(c) = \lambda\varepsilon(c)$$

for $\phi, \psi \in C^*$, $c \in C$ and $\lambda \in k$. The coassociativity of Δ and the counit axioms for ε are precisely what turns (C^*, m, η) into an algebra over k.

Conversely, if (A, m, η) is a finite-dimensional algebra over k, (A^*, m^*, η^*) is a coalgebra over k.

The proof of this important statement is elementary and merely requires going through the diagrams.

The alert reader will have noticed that actually, the dual map of Δ should be $\Delta^*\colon (C \otimes_k C)^* \to C^*$ — here it is crucial that we can regard $(C \otimes_k C)^*$ as a subspace of $C^* \otimes_k C^*$ (see Remark A.3.11) and we mean the restriction $m := \Delta^*|_{\iota(C \otimes_k C)^*}$. This fact causes problems for infinite-dimensional algebras whose dual space in general cannot be supplied with a coalgebra structure so easily. Here one has to restrict to a special subspace of the dual to make ends meet. For finite-dimensional algebras A, it holds $A^* \otimes_k A^* \cong (A \otimes_k A)^*$ so that we can go the other way (making identifications), too.

2. Tensor products of algebras and coalgebras

In this section, we see how to give tensor products of algebras and coalgebras a natural algebra respectively coalgebra structure. This natural structure will be important for the definition of a bialgebra.

Let (A, m_A, η_A) and (B, m_B, η_B) be algebras over k. As a k-vector space, the tensor product $A \otimes_k B$ makes sense and turns into a k-algebra itself with the multiplication

$$\begin{aligned}\bullet\colon A \otimes_k B \times A \otimes_k B &\longrightarrow A \otimes_k B,\\ \Big(\sum_{i=1}^{n} a_i \otimes b_i, \sum_{j=1}^{m} a'_j \otimes b'_j\Big) &\longmapsto \sum_{i=1}^{n}\sum_{j=1}^{m} m_A(a_i, a'_j) \otimes m_B(b_i, b'_j),\end{aligned}$$

or equivalently, with the composition $\bullet$ of the following maps:

$$A \otimes_k B \otimes_k A \otimes_k B \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} A \otimes_k A \otimes_k B \otimes_k B \xrightarrow{m_A\otimes m_B} A \otimes_k B. \tag{II.1}$$

Here τ denotes the so-called *twist map* $\tau\colon A \otimes_k B \to B \otimes_k A$, defined by linear extension of $a \otimes b \mapsto b \otimes a$ for $a \in A$ and $b \in B$, which is an isomorphism of vector spaces over k.

If we start with coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ over k, we can equip the vector space tensor product $C \otimes_k D$ with a natural coalgebra structure via the comultiplication $\Delta_{C\otimes D}$ defined as the composition

$$C \otimes_k D \xrightarrow{\Delta_C\otimes\Delta_D} C \otimes_k C \otimes_k D \otimes_k D \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} C \otimes_k D \otimes_k C \otimes_k D \tag{II.2}$$

and the counit $\varepsilon_{C\otimes D} := \varepsilon_C \otimes \varepsilon_D\colon k \otimes_k k \cong k \to C \otimes_k D$.

Let $\varphi := \mathrm{id} \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id} \otimes \mathrm{id}$ and $\psi := \mathrm{id} \otimes \tau \otimes \tau \otimes \mathrm{id}$. When proving that $\Delta_{C\otimes D}$ is coassociative, we just need the isomorphism of vector spaces

$$\varphi \circ \psi\colon C \otimes_k D \otimes_k C \otimes_k D \otimes_k C \otimes_k D \longrightarrow C \otimes_k C \otimes_k C \otimes_k D \otimes_k D \otimes_k D$$

to identify

$$\begin{aligned}&c_{(1)} \otimes d_{(1)} \otimes c_{(2)(1)} \otimes d_{(2)(1)} \otimes c_{(2)(2)} \otimes d_{(2)(2)}\\ &\qquad = c_{(1)(1)} \otimes d_{(1)(1)} \otimes c_{(1)(2)} \otimes d_{(1)(2)} \otimes c_{(2)} \otimes d_{(2)},\end{aligned}$$

then the coassociativity of $\Delta_{C\otimes D}$ follows from the coassociativity of Δ_C respectively Δ_D. Everything else is straight forward computation.

Also straight forward is checking that if $f\colon C \to C'$ and $g\colon D \to D'$ are coalgebra maps, then $f \otimes g\colon C \otimes_k D \to C' \otimes D'$ (where both vector spaces are equipped with the natural coalgebra structures) is a coalgebra map, too.

3. Bialgebras and Hopf algebras

Lemma II.3.1: *Let A be a vector space over the field k, let $m_A \colon A \otimes_k A \to A$ and $\eta_A \colon k \to A$ be k-linear maps that make A an algebra over k and let $\Delta_A \colon A \to A \otimes_k A$ and $\varepsilon_A \colon A \to k$ be k-linear maps that make A a coalgebra over k. Furthermore, k be equipped with its unique coalgebra structure and $A \otimes_k A$ be equipped with the natural algebra and coalgebra structures. Then, the following are equivalent:*

(i) *m_A and η_A are coalgebra maps,*

(ii) *Δ_A and ε_A are algebra maps.*

Proof: We only show "(i) ⇒ (ii)", the other implication is proven similarly. Suppose m_A and η_A are coalgebra maps, i.e.,

$$(\Delta_A \circ m_A) = (m_A \otimes m_A) \circ \Delta_{A \otimes A}, \qquad \varepsilon_A \circ m_A = \varepsilon_{A \otimes A},$$
$$\Delta_A \circ \eta_A = (\eta_A \otimes \eta_A) \circ \Delta_k, \qquad \varepsilon_A \circ \eta_A = \varepsilon_k.$$

Plugging in suitable elements yields that this means for all $a, b \in A$ that $\Delta_A(ab) = \Delta_A(a)\Delta_A(b)$, $\Delta_A(\alpha 1_A) = \alpha 1_A \otimes 1_A$ and $\varepsilon_A(ab) = \varepsilon_A(a)\varepsilon_A(b)$, $\varepsilon_A(\alpha 1_A \otimes 1_A) = \alpha$, i.e., what we wanted to show. □

Definition II.3.2 (Bialgebra): Let A be a vector space over the field k with maps m, η, Δ and ε such that (A, m, η) is a algebra over k and (A, Δ, ε) is a coalgebra over k. If either of the statements of Lemma II.3.1 holds true, then $(A, m, \eta, \Delta, \varepsilon)$ is called a *bialgebra over k*.

Unsurprisingly, a linear map $f \colon (C, m_C, \eta_C, \Delta_C, \varepsilon_C) \to (D, m_D, \eta_D, \Delta_D, \varepsilon_D)$ between bialgebras is called a *homomorphism of bialgebras*, if f is both an algebra and a bialgebra map. If a bialgebra map $f \colon C \to D$ is invertible as a linear map, f is called an *isomorphism of bialgebras*.

Remark II.3.3 (Convolution algebra): Let (A, η, m) be an algebra over k and let (C, Δ, ε) be a coalgebra over the field k. Then, $\mathrm{Hom}_k(C, A)$ turns into an algebra over k with the product

$$f \star g := m \circ (f \otimes g) \circ \Delta,$$

where $f, g \in \mathrm{Hom}_k(C, A)$, and has the unit $\eta \circ \varepsilon$. In Sweedler-notation, the product reads $(f \star g)(c) = f(c_{(1)})g(c_{(1)})$.

Note that if $A = k$, then $\mathrm{Hom}_k(C, A) = C^*$ is the dual algebra of C and if $k = C$, then $\mathrm{Hom}_k(C, A) \cong A$.

If $f\colon C \to D$ is a coalgebra map and $\varphi\colon A \to B$ is an algebra map, the maps

$$f^*\colon \operatorname{Hom}_k(D,A) \longrightarrow \operatorname{Hom}_k(C,A), \qquad \psi \longmapsto \psi \circ f,$$
$$\varphi_*\colon \operatorname{Hom}_k(C,A) \longrightarrow \operatorname{Hom}_k(C,B), \qquad \psi \longmapsto \varphi \circ \psi$$

are algebra maps.

Definition II.3.4 (Hopf algebra): Let $(A, m, \eta, \Delta, \varepsilon)$ be a bialgebra over the field k. If id_A has a convolution inverse S in the convolution algebra $\operatorname{End}_k(A)$, i.e., $S \star \mathrm{id}_A = \mathrm{id}_A \star S = \eta \circ \varepsilon$, then $(A, m, \eta, \Delta, \varepsilon, S)$ is called a *Hopf algebra over k*. The map S is called *antipode*.

Remark II.3.5: Let $(A, m, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra over the field k. Because S is the convolution inverse of id_A, for all $a \in A$ it holds

$$S(a_{(1)})a_{(2)} = \varepsilon(a)1_A = a_{(1)}S(a_{(2)}).$$

Since $\operatorname{End}_k(A)$ together with the involution product is an algebra, the antipode is unique.

The antipode is an antialgebra map and an anticoalgebra map, i.e., for all $a, b \in A$ it holds

(i) $S(ab) = S(b)S(a)$, $S(1) = 1$,

(ii) $S \otimes S \circ \Delta(a) = \tau \circ \Delta \circ S(h)$, $(\varepsilon \circ S)(a) = \varepsilon(a)$.

A proof of this can be found in [16] Proposition 1.3.1.

Example II.3.6 (Function algebra of finite group): Let G be a finite group, let e_G denote its neutral element and let k be a field. On the vector space $k^G = \{\varphi\colon G \to k\}$, we may introduce an algebra structure via the point-wise laws of composition, i.e., for $\varphi, \psi \in k^G$, $\alpha \in k$ and $g \in G$, we define

$$(\varphi+\psi)(g) := \varphi(g)+\psi(g), \qquad (\alpha\varphi)(g) := \alpha\varphi(g), \qquad (\varphi\cdot\psi)(g) := \varphi(p)\cdot\psi(g).$$

Furthermore, we can introduce a coalgebra structure by virtue of

$$\Delta(\varphi)(g,h) := \varphi(gh), \qquad \varepsilon(\varphi) := \varphi(e_G).$$

Those two structures are compatible, i.e., $(k^G, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra. Finally, $S(\varphi)(g) := \varphi(g^{-1})$ declares an antipode on k^G, rendering k^G a Hopf algebra over k.

Denoting by δ_g the map defined via $\delta_g(h) := \delta_{g,h}$, we obtain a basis $\{\delta_g \mid g \in G\}$ of k^G. Using this basis, we can express the aforementioned maps in coordinates:

$$\Delta(\delta_g) = \sum_{\substack{(h_1,h_2)\in G^2\\ h_1h_2=g}} \delta_{h_1} \otimes \delta_{h_2}, \qquad \varepsilon(\delta_g) = \delta_{g,e_G}, \qquad S(\delta_g) = \delta_{g^{-1}}.$$

This construction uses the identification $k^{G\times G} \cong k^G \otimes k^G$, which fails when G is not finite.

4. Compact matrix quantum groups

Example II.4.1: Let $G \subseteq U_n$ be a closed subgroup. Then G is compact since U_n is compact (see Proposition I.3.2). Hence, the algebra of continuous functions $C(G) := \{\varphi\colon G \to \mathbb{C} \text{ continuous}\}$ endowed with the supremum norm and the involution defined via

$$\begin{aligned} {}^*\colon C(G) &\longrightarrow C(G), \\ &(\varphi\colon G \to \mathbb{C},\ g \mapsto \varphi(g)) \longmapsto (\varphi^*\colon G \to \mathbb{C},\ g \mapsto \varphi(g)^*) \end{aligned}$$

is in fact a C^*-algebra. That $(C(G), \|\cdot\|_\infty)$ is a Banach algebra is a standard result shown in a regular course on analysis, for a proof see for example [22], Satz 1.23. The rest is easily checked.

The coordinate functions $u^i_j\colon G \to \mathbb{C},\ A \mapsto A^i_j$ belong to the algebra of continuous functions $C(G)$, since convergence in G is precisely componentwise convergence. Furthermore, by a Stone-Weierstraß-argument, they generate $C(G)$.

Using the identification $C(G) \otimes_{\mathbb{C}} C(G) \cong C(G \times G)$, by virtue of

$$C(G) \otimes C(G) \longrightarrow C(G \times G), \qquad f \otimes g \longmapsto \big((s,t) \mapsto f(s)g(t)\big),$$

we declare a comultiplication Δ on $C(G)$ via

$$\Delta\colon C(G) \longrightarrow C(G \times G), \qquad \varphi \longmapsto (\Delta(\varphi)\colon G \times G \to \mathbb{C},\ (A,B) \mapsto \varphi(AB)).$$

As the coordinate functions u^i_j generate $C(G)$, it is sufficient to check what Δ does on these: For $(i,j) \in \{1,\dots,n\}^2$ and matrices $A, B \in G$, it holds

$$\Delta(u^i_j)(A,B) = u^i_j(AB) = \sum_{k=1}^n A^i_k B^k_j,$$

hence $\Delta(u^i_j) = \sum_{k=1}^n u^i_k u^k_j$, which corresponds to $\sum_{k=1}^n u^i_k \otimes u^k_j$. Note that this is precisely the dualised matrix multiplication of Example II.1.3.

Writing the coordinate functions into a matrix $U = (u^i_j) \in M_n(C(G))$, we obtain an invertible matrix in $M_n(C(G))$, because $U^* U^t = (UU^\dagger)^* = I_n$. To see this, note that $(U^* U^t)^i_j = \sum_{k=1}^n (u^i_k)^* u^j_k$ and, since $G \subseteq U_n$, for any $A \in G$ it holds

$$\Big(\sum_{k=1}^n (u^i_k)^* u^j_k\Big)(A) = \sum_{k=1}^n (A^i_k)^* A^j_k = \delta_{i,j} = (1(A))^i_j,$$

where 1 denotes the function $1\colon G \to M_n(C(G))$, $A \mapsto \mathrm{diag}(1, \dots, 1)$.

This is the principal example that led to the notion of a *compact matrix quantum group*. Abstracting from this example and "dropping the commutativity condition", we come up with the following definition:

Definition II.4.2 (Compact matrix quantum group): Let $n \in \mathbb{N}$ and let A be the C^*-algebra generated by n^2 elements u^i_j, where $1 \le i, j \le n$. If the matrices $U = (u^i_j)_{1\le i,j\le n}$ and $\bar{U} = (u^{i\,*}_j)_{1\le i,j\le n}$ are invertible in $M_n(A)$ and the map

$$\Delta\colon A \longrightarrow A \otimes_{\min} A, \qquad u^i_j \longmapsto \sum_{k=1}^n u^i_k \otimes u^k_j$$

is a *-homomorphism, the tuple (A, U) is called a *compact matrix quantum group*.

This definition is due to Woronowicz, the founder of the theory of compact quantum groups. At first, he called such objects "compact matrix pseudo groups", see [24] later, the term compact matrix quantum group has been agreed upon.

Abstracting even further, we carry out this process on compact groups, too. This leads to the notion of a *compact quantum group*, see [25]

Definition II.4.3 (Compact quantum group): Let A be a separable unital C^*-algebra and $\Delta\colon A \to A \otimes_{\min} A$ be a unital *-homomorphism. If it holds

(i) The homomorphism Δ is coassociative, i.e., $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,

(ii) The sets

$$\{((b \otimes 1_A) \circ \Delta)(c) \mid b, c \in A\}, \qquad \{((1_A \otimes b) \circ \Delta)(c) \mid b, c \in A\}$$

are linearly dense in $A \otimes_{\min} A$,

then (A, Δ) is called a *compact quantum group*.

The notion of a compact quantum group indeed generalises that of a compact matrix quantum group, see [21], Proposition 6.1.4.

The "compactness" in the name *compact quantum group* is due to the development of the notion as a generalisation of a compact topological group. Due to the famous Gelfand-Naimark theorem, we can go the other way as well, more precisely: If we start with a commutative unital C^*-algebra and a comultiplication $\Delta\colon A \to A \otimes_{\min} A$, we can recover a compact group from it in the following way:

Proposition II.4.4: *Let A be a commutative unital C^*-algebra together with a comultiplication $\Delta\colon A \to A \otimes_{\min} A$ rendering the pair (A, Δ) a compact quantum group. Then, A is isomorphic to the C^*-algebra of complex-valued, continuous functions $C(G)$ over some compact Hausdorff topological group G. The comultiplication, seen as a mapping $\Delta\colon C(G) \to C(G \times G)$, is, for $f \in C(G)$ and $g, h \in G$, given by $\Delta(f)(g, h) = f(gh)$. The correspondence of compact quantum groups (A, Δ) and compact groups G is unique up to equivalence.*

This result can be found in [21], Proposition 5.1.4. A detailed proof of this assertion can be found in [13], Proposition 2.1.2.

Chapter III.

Easy groups

This chapter is dedicated to the central objects for this thesis: (unitary) easy groups. Those are compact matrix groups $G \subseteq \mathrm{Gl}_n(\mathbb{K})$ with $\mathfrak{S}_n \subseteq G \subseteq O_n$ respectively $\mathfrak{S}_n \subseteq G \subseteq U_n$, whose associated symmetric tensor categories with duals are generated by linear maps associated to partitions of sets. The orthogonal case was first treated by Banica and Speicher in 2009, see [5]; the unitary case was treated by Tarrago and Weber in 2016, see [20].

The orthogonal easy groups and unitary easy groups are completely classified, this was done in the papers cited above. An open question in this field is, how to classify non-easy groups, that is, compact matrix groups $G \subseteq \mathrm{Gl}_n(\mathbb{K})$ with $\mathfrak{S}_n \subseteq G \subseteq O_n$ respectively $\mathfrak{S}_n \subseteq G \subseteq U_n$ which are not easy.

The sections 3 and 4 of Appendix A and sections 3, 5 and 6 of Appendix B should provide the means necessary to understand the content of this chapter.

1. Schur-Weyl duality

Let n be natural number and let $H := \mathbb{C}^n$. For any non-negative integer m, the group $U_n \subseteq M_n(\mathbb{C})$ has a natural unitary representation $\pi_m : U_n \to B(T^m(H))$ via

$$\pi_m : U_n \longrightarrow B(T^m(H)), \qquad g \longmapsto g^{\otimes m}$$

and the symmetric group $\mathfrak{S}_m$ operates on $T^m(H)$ via linear extension of

$$\rho_m : \mathfrak{S}_m \longrightarrow B(T^m(H)), \qquad \rho(\sigma)(e_{\boldsymbol{i}}) := e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(m)}} =: e_{\sigma(\boldsymbol{i})}.$$

These two actions are quite compatible in the following sense: It holds $\rho_m(\sigma) \circ \pi_m(g) = \pi_m(g) \circ \rho_m(\sigma)$ for all $\sigma \in \mathfrak{S}_m$ and $g \in U_n$, since for

$e_{\boldsymbol{i}} \in T^m(H)$ we find

$$\begin{aligned}(\rho_m(\sigma) \circ \pi_m(g))(e_{\boldsymbol{i}}) &= \rho_m(\sigma)(ge_{i_1} \otimes \cdots \otimes ge_{i_m}) \\ &= ge_{i_{\sigma(1)}} \otimes \cdots \otimes ge_{i_{\sigma(m)}} \\ &= \pi_m(g)(e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(m)}}) = (\pi_m(g) \circ \rho_m(\sigma))(e_{\boldsymbol{i}}).\end{aligned}$$

The statement of Schur-Weyl duality is that this is essentially everything that commutes with ρ respectively π. More precisely, it holds

$$\operatorname{Lin}(\rho_m(\mathfrak{S}_m)) = \operatorname{Lin}(\pi_m(U_n))', \qquad \operatorname{Lin}(\rho_m(\mathfrak{S}_m))' = \operatorname{Lin}(\pi_m(U_n)),$$

where M' denotes the so-called *commutant of* $M \subseteq B(T^m(H))$, which is defined as $M' := \{T \in B(T^m(H)) \mid TA = AT \text{ for all } A \in M\}$.

The above calculations show the inclusions $\operatorname{Lin}(\rho_m(\mathfrak{S}_m)) \subseteq \operatorname{Lin}(\pi_m(U_n))'$ and $\operatorname{Lin}(\pi_m(U_n)) \subseteq \operatorname{Lin}(\rho_m(\mathfrak{S}_m))'$, the other inclusions are by far out of reach in the context of this thesis and need involved arguments from representation theory. A proof can be found in [17], Chapter 9, Section 1.

The statement of classical Schur-Weyl duality can also be expressed in the terminology developed in the following pages, see [20], Section 7.

For clarity, for the next few lines, we make a distinction between $\mathfrak{S}_n$ and its fundamental representation $\rho(\mathfrak{S}_n)$. For the rest of this thesis, this distinction does not play a role.

If, instead of $\mathfrak{S}_n$, we consider its fundamental representation $\rho(\mathfrak{S}_n)$, we get a different commutant, because the operation of the permutation matrices on $\mathbb{C}^n$ is different to that of $\mathfrak{S}_n$ on $\mathbb{C}^n$ (check Example IV.2.2(ii) for the definition of the fundamental representation of $\mathfrak{S}_n$). Then, we have

$$\operatorname{Lin}(\rho(\mathfrak{S}_n))' = \mathbb{C}[P_m(n)],$$

where $P_m(n)$ is the so-called *partition monoid*, $\mathbb{C}[P_m(n)]$ is called *partition algebra* and m is an indeterminate. This is in keeping with Theorem III.4.8. Without going into too much detail, the idea is representation of partitions of sets with $2n$ elements by suitable pictures, that is, diagrams with an upper line of n points and a lower line of n points and lines joining those points that belong to the same block of the partition.

As a generalisation of this, we will in the following do the same with partitions of sets with $k + \ell$ elements and apply the same tools there.

2. Categories of partitions

In the spirit of the previous section, we want to assign to a partition of a set with $k + \ell$ points a pictorial representation with an upper row of k points, a lower row of ℓ points and connecting lines joining elements of the same subset of the partition.

Definition III.2.1 (Sets of partitions): Let k and ℓ be non-negative integers, let $\{\circ, \bullet\}$ be the set of the two colours black and white, let M' be the set $M' = \{1, \dots, k, 1', \dots, \ell'\}$ and let $M := M' \times \{\circ, \bullet\}$. The colours black and white are called *inverse to each other*. Consider a subset $S \subseteq M$ with $p_{M'}(S) = M'$ and $\#(S) = k + \ell$, where $p_{M'} \colon M \to M'$ denotes the projection onto M.

A *partition of* S is a decomposition into pairwise disjoint non-empty subsets of S, the so-called *blocks of the partition*. By $P_S^{\circ\bullet}(k, \ell)$ we denote the set of partitions of S.

Let $p \in P_S^{\circ\bullet}(k, \ell)$ be a partition of S. Draw the points of S in two rows: an upper row with k coloured points and a lower row with ℓ coloured points. We associate to p a pictorial representation by joining the points that belong to the same block with lines. If the lines can be drawn such that no two different lines cross, the partition is said to be *non-crossing*.

By $P^{\circ\bullet}(k, \ell) := \bigcup_{S \subseteq M} P^{\circ\bullet}(k, \ell)$ we denote the *set of coloured partitions of coloured sets S with $k + \ell$ points*, where $S \subseteq M$ fulfils the requirements listed above, and by $P^{\circ\bullet} := \bigcup_{k \in \mathbb{N}_0} \bigcup_{\ell \in \mathbb{N}_0} P^{\circ\bullet}(k, \ell)$, we denote the *set of coloured partitions*.

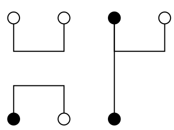
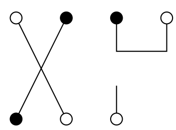
Furthermore, by $NC^{\circ\bullet}(k, \ell) \subseteq P^{\circ\bullet}(k, \ell)$, we denote the *set of non-crossing partitions of coloured sets with $k + \ell$ points* and by $NC^{\circ\bullet} \subseteq P^{\circ\bullet}$, we denote the set of *non-crossing partitions*.

Remark III.2.2: If in the situation of Definition III.2.1 we take $S \subseteq M$ with $p_{\{\circ,\bullet\}}(S) = \{\circ\}$ or $p_{\{\circ,\bullet\}}(S) = \{\bullet\}$, we end up with the same notion of partition as given in [5], Definition 1.5. Such partitions are called *non-coloured*.

Example III.2.3: Let $k = 4$, $\ell = 3$ and consider the sets

$$S = \{(1, \circ), (2, \circ), (3, \bullet), (4, \circ), (1', \bullet), (2', \circ), (3', \bullet)\},$$
$$S' = \{(1, \circ), (2, \bullet), (3, \bullet), (4, \circ), (1', \bullet), (2', \circ), (3', \circ)\}.$$

Let $p = \{(1, \circ), (2, \circ)\}$, $\{(1', \bullet), (2', \circ)\}$, $\{(3, \bullet), (4\circ), (3', \bullet)\} \in P_S^{\circ\bullet}(4, 3)$ and $p' = \{(1, \circ), (2', \circ)\}$, $\{(1', \bullet), (2, \bullet)\}$, $\{(3, \bullet), (4, \circ)\}$, $\{(3', \circ)\} \in P_{S'}^{\circ\bullet}(4, 3)$. To these partitions, the pictures

$p =$  and $p' =$ 

are associated. From those pictures, we can immediately tell that $p \in NC^{\circ\bullet}$ and that $p' \in P^{\circ\bullet} - NC^{\circ\bullet}$.

Definition III.2.4 (Operations on the set of partitions): Let k, ℓ, m and n be non-negative integers.

(i) Let $p \in P^{\circ\bullet}(k,\ell)$ and $q \in P^{\circ\bullet}(m,n)$. Horizontally concatenating p and q gives rise to a partition $p \otimes q \in P^{\circ\bullet}(k+m,\ell+n)$, the *tensor product of* p *and* q.

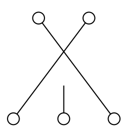
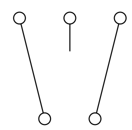
(ii) Let $q \in P^{\circ\bullet}(k,\ell)$ and $p \in P^{\circ\bullet}(\ell,m)$. If the ℓ lower points of q and the ℓ upper points of p are identically coloured, the vertical concatenation (writing p under q and removing the ℓ middle points and eventual loops) is called the *composition* $pq \in P^{\circ\bullet}(k,m)$ of p and q.

(iii) Let $p \in P^{\circ\bullet}(k,\ell)$. Reflecting p at the horizontal axis gives rise to a partition $p^* \in P^{\circ\bullet}(\ell,k)$, the *involution of* p.

(iv) Let $p \in P^{\circ\bullet}(k,\ell)$. Reflecting p at the vertical axis and inverting the colours gives rise to a partition $\tilde{p} \in P^{\circ\bullet}(\ell,k)$, the *verticolour reflection* of p.

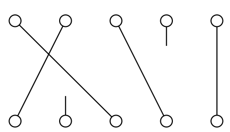
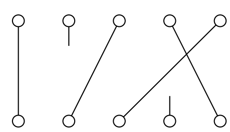
(v) Given a partition p, shifting the leftmost- or rightmost point on the upper- respectively lower line to the lower- respectively upper line and inverting this points colour yields a new partition, a *rotated version of* p.

Example III.2.5 (Composition of partitions): Let $k=2$, $\ell=3$ and consider the non-coloured partitions $p = \{1,5\}$, $\{4\}$, $\{2,4\} \in P(3,2)$ and $q = \{1,4\}$, $\{2\}$, $\{3,5\} \in P(2,3)$. Then we have the pictorial representations

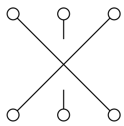
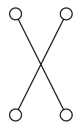
$p =$  and $q =$ 

We will use those partitions to illustrate the operations on partitions.

(i) Tensor products: The tensor products $p \otimes q$ respectively $q \otimes p$ are

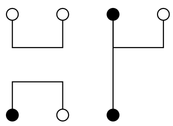
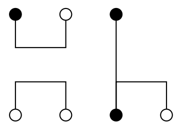
$p \otimes q =$  and $q \otimes p =$ 

(ii) Composition of partitions: We find

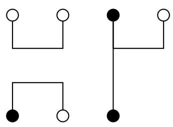
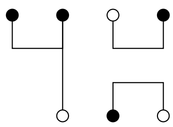
$pq =$  and $qp =$ 

Note the removal of the "isolated block on the center line" in the composition qp.

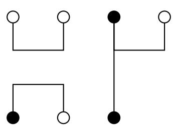
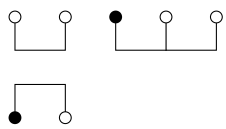
(iii) Involution: For the partition p from Example III.2.3, the involution looks like this:

$p =$  and $p^* =$ 

(iv) Verticolour reflection: For the same partition p, verticolour reflection yields the picture

$p =$  and $\tilde{p} =$ 

(v) Rotation: If we take the partition p from Example III.2.3 and rotate the rightmost point of the lower line to the upper line, we get a new partition p'. In pictures:

$p =$  and $p' =$ 

The following definition stems from [20], Section 3.3.

Definition III.2.6 (Category of partitions): For non-negative integers k, ℓ, let $C(k,\ell) \subseteq P^{\circ\bullet}(k,\ell)$ be subsets. If their union $C = \bigcup_{k\in\mathbb{N}_0}\bigcup_{\ell\in\mathbb{N}_0} C(k,\ell)$ fulfils

(i) C is closed under the tensor product,

(ii) C is closed under composition,

(iii) C is closed under involution,

(iv) C contains ,  $\in P^{\circ\bullet}(0,2)$ and ,  $\in P^{\circ\bullet}(1,1)$,

C is called a *category of partitions.*

We write $C = \langle p_1, \dots, p_k \rangle$, if C is the smallest category of partitions that contains $p_1, \dots, p_k \in P^{\circ\bullet}$ and call C the *category generated by* $p_1, \dots, p_k$. The partitions  and  are usually omitted because they are always contained by definition.

Remark III.2.7: Categories of partitions can be shown to be stable under rotation and verticolour reflection, see [20], Section 3.3.

Definition III.2.8: Let $C \subseteq P^{\circ\bullet}$ be a category of partitions. If  $\otimes$  $\in C$, then C is called *globally coloured.* Otherwise, C is called locally coloured.

The globally coloured case has been completely classified by Daniel Gromada in 2018, see [9]. It "behaves similarly to the non-coloured case", which can also be seen later in Chapter 4.

3. Partition C^*-algebras

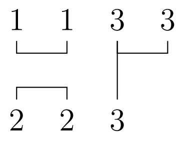
Definition III.3.1 (Adapted multi-indices to a partition): Let k and ℓ be non-negative integers and let $p \in P^{\circ\bullet}(k, \ell)$ be a partition. For multi-indices $\boldsymbol{i} = (i_1, \dots, i_k) \in I_n^k$ and $\boldsymbol{j} = (j_1, \dots, j_\ell) \in I_n^\ell$, we attach the indices $i_1, \dots, i_k$ to the k upper points of p, the indices $j_1, \dots, j_\ell$ to the ℓ lower points of p standing to reason and define

$$\delta_p(\boldsymbol{i}, \boldsymbol{j}) := \begin{cases} 1, & \text{if lines connecting blocks only join equal numbers,} \\ 0, & \text{else.} \end{cases}$$

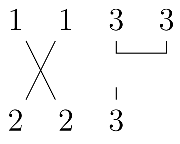
If $\delta_p(\boldsymbol{i}, \boldsymbol{j}) = 1$, we call the pair $(\boldsymbol{i}, \boldsymbol{j})$ *adapted to the partition* p.

Note that in the situation of Definition III.3.1, the colourings of the points of the underlying sets play no role.

Example III.3.2: Consider the partitions from Example III.2.3 and the multi-indices $\boldsymbol{i} = (1, 1, 3, 3) \in I_3^4$, $\boldsymbol{j} = (2, 2, 3) \in I_3^3$. Attaching the indices to the pictorial representations yields the pictures



and



Clearly, $\delta_p(\boldsymbol{i}, \boldsymbol{j}) = 1$ and $\delta_{p'}(\boldsymbol{i}, \boldsymbol{j}) = 0$.

Using the notion of adapted multi-indices to a partition, we can describe what it should mean that generators of a universal C^*-algebra "fulfil the relations of partition":

Definition III.3.3: Let n be a natural number, let u^i_j, where $1 \leq i, j \leq n$, be n^2 elements and let $C(u^i_j | 1 \leq i, j \leq n)$ be the C^*-algebra generated by the elements u^i_j.

Let furthermore $p \in P^{\circ\bullet}(k, \ell)$ be a partition, $\boldsymbol{r} = (r_1, \ldots, r_k) \in \{\circ, \bullet\}^k$ be the upper colour pattern of p and $\boldsymbol{s} = (s_1, \ldots, s_\ell) \in \{\circ, \bullet\}^\ell$ be the lower colour pattern of p. Put $(u^i_j)^\bullet := (u^i_j)^*$, $(u^i_j)^\circ := u^i_j$.

The generators u^i_j *fulfil the relation* $R(p)$, if for all multi-indices $\boldsymbol{\beta} \in I^\ell_n$ and $\boldsymbol{i} \in I^k_n$ it holds

$$\sum_{\boldsymbol{\alpha} \in I^k_n} \delta_p(\boldsymbol{\alpha}, \boldsymbol{\beta}) (u^{\alpha_1}_{i_1})^{r_1} \cdots (u^{\alpha_k}_{i_k})^{r_k} = \sum_{\boldsymbol{j} \in I^\ell_n} \delta_p(\boldsymbol{i}, \boldsymbol{j}) (u^{\beta_1}_{j_1})^{s_1} \cdots (u^{\beta_\ell}_{j_\ell})^{s_\ell}.$$

If $k = 0$, the left-hand side of the above equation is $\delta_p(\varnothing, \boldsymbol{\beta})$, if $\ell = 0$, the right-hand side of the above equation is $\delta_p(\boldsymbol{i}, \varnothing)$.

For a translation guide between pictorial representations of coloured partitions and the corresponding relations, see [20], end of Section 4.

Definition III.3.4 (Easy quantum group): Let n be a natural number, let G be a compact matrix quantum group and let $C(G)$ be the universal unital C^*-algebra generated by the elements u^i_j, where $1 \leq i, j \leq n$, such that U and $\bar{U}$ are unitary.

If there is a set of partitions $C_0 \subseteq P^{\circ\bullet}$ such that

$$C(G) = C^*(u^i_j, 1 \leq i, j \leq n \mid R(p)),$$

G is called *easy*. If, in addition, all u^i_j are self-adjoint, then G is called *orthogonal easy*.

4. Tensor categories associated to matrix groups

As established in Proposition I.4.3, a compact subgroup of O_n respectively U_n can, for each natural number m, be represented on $T^m(\mathbb{C}^n)$. Associated to these representations are so-called intertwiners, and suitable collections of those form vector spaces. This enables associating a symmetric tensor category with duals to G. By Tannaka-Krein duality, such symmetric tensor categories with duals allow for the reconstruction of a closed subgroup of O_n respectively U_n.

We first carefully go through the orthogonal case and then treat the unitary case, which only requires small modifications.

For Intertwiner spaces, we follow the definition given in [5], Definition 1.1.

Definition III.4.1 (Intertwiner spaces): Let n be a natural number, let k and ℓ be non-negative integers and let $G \subseteq O_n$ be a compact group. We denote

$$\mathrm{Mor}_G(k,\ell) := \{\varphi \in \mathrm{Hom}(T^k(\mathbb{C}^n), T^\ell(\mathbb{C}^n)) \mid \varphi \circ T^k(g) = T^\ell(g) \circ \varphi \,\forall\, g \in G\}.$$

For brevity, we denote $\mathrm{Mor}(k,\ell) := \mathrm{Hom}(T^k(\mathbb{C}^n), T^\ell(\mathbb{C}^n))$. By Mor_G we denote the collection of intertwiner spaces $\mathrm{Mor}_G(k,\ell)$, where k,ℓ are non-negative integers.

Proposition III.4.2: *Let n be a natural number and let $G \subseteq O_n$ be a compact group. Then, the collection Mor_G of vector spaces $\mathrm{Mor}_G(k,\ell)$ forms a symmetric tensor category with duals in the sense that it has the following properties:*

(i) *If $\varphi, \varphi' \in \mathrm{Mor}_G$, then $\varphi \otimes \varphi' \in \mathrm{Mor}_G$,*

(ii) *If $\varphi, \varphi' \in \mathrm{Mor}_G$ are composeable, then their composition belongs to Mor_G,*

(iii) *If $\varphi \in \mathrm{Mor}_G$, then $\varphi^* \in \mathrm{Mor}_G$,*

(iv) *The identity $\mathrm{id}_{\mathbb{C}^n}$ belongs to $\mathrm{Mor}_G(2,2)$,*

(v) *The twist map τ defined by $\tau(\xi \otimes \eta) = \eta \otimes \xi$ belongs to $\mathrm{Mor}_G(1,1)$,*

(vi) *The map $\xi = \sum_{i=1}^n e_i \otimes e_i$ belongs to $\mathrm{Mor}_G(0,2)$.*

This result is taken from Banica and Speicher, see [5], Proposition 1.2.

Proof: In the following, let k, ℓ, m and n be natural numbers.

(i) Let $\varphi \in \mathrm{Mor}_G(k,\ell)$ and $\psi \in \mathrm{Mor}_G(m,n)$. Then for their tensor product it holds $\varphi \otimes \psi \in \mathrm{Mor}_G(k+m, \ell+n)$, since for all $g \in G$ we have

$$\begin{aligned}(\varphi \otimes \psi) \circ T^{k+m}(g) &= \varphi \circ T^k(g) \otimes \psi \circ T^m(g)\\ &= T^\ell(g) \circ \varphi \otimes T^n(g) \circ \psi = T^{\ell+n}(g) \circ (\varphi \otimes \psi).\end{aligned}$$

(ii) Let $\varphi \in \mathrm{Mor}_G(k,\ell)$ and $\psi \in \mathrm{Mor}_G(n,m)$. Then $\psi\varphi \in \mathrm{Mor}_G(k,m)$, as for all $g \in G$ it is

$$(\psi \circ \varphi) \circ T^k(g) = \psi \circ T^\ell(g) \circ \varphi = T^m(g) \circ (\psi \circ \varphi).$$

(iii) For $\varphi \in \mathrm{Mor}_G(k,\ell)$, we show $\varphi^\dagger \in \mathrm{Mor}_G(\ell,k)$. Then by Definition A.4.19, also $\varphi^* \in \mathrm{Mor}_G$. It holds

$$\begin{aligned}(\varphi \circ T^k(g))^\dagger = T^k(g)^\dagger \circ \varphi^\dagger &= T^k(g^\dagger) \circ \varphi^\dagger \\ &= \varphi^\dagger \circ T^\ell(g^\dagger) = \varphi^\dagger \circ T^\ell(g)^\dagger = (T^\ell(g) \circ \varphi)^\dagger,\end{aligned}$$

what we wanted to see.

(iv) Clearly, for all $g \in G$ and $\xi \in \mathbb{C}^n$, we have $\mathrm{id}_{\mathbb{C}^n} g\xi = g\xi = g\,\mathrm{id}_{\mathbb{C}^n} \xi$.

(v) For $g \in G$ and $e_i \otimes e_j$ it holds

$$\begin{aligned}\tau \circ T^2(g)(e_i \otimes e_j) &= \tau(g(e_i) \otimes g(e_j)) \\ &= g(e_j) \otimes g(e_i) = T^2(g)(e_j \otimes e_i) = T^2(g)(\tau(e_i \otimes e_j)).\end{aligned}$$

(vi) Let $g \in G$. By the orthogonality of g, we know that the rows of g form an orthonormal basis of $\mathbb{R}^n$ (see Corollary A.4.15). Hence,

$$\begin{aligned}T^2(g)(\xi) &= \sum_{i=1}^n ge_i \otimes ge_i \\ &= \sum_{i=1}^n \Big(\sum_{j=1}^n g_i^j e_j\Big) \otimes \Big(\sum_{k=1}^n g_i^k e_k\Big) \\ &= \sum_{j=1}^n \sum_{k=1}^n \Big(\sum_{i=1}^n g_i^j g_i^k\Big) e_j \otimes e_k \\ &= \sum_{j=1}^n \sum_{k=1}^n (gg^t)_k^j e_j \otimes e_k = \sum_{j=1}^n \sum_{k=1}^n \delta_{j,k} e_j \otimes e_k = \sum_{i=1}^n e_i \otimes e_i = \xi,\end{aligned}$$

which concludes the proof. □

What makes this construction interesting, is the following important assertion (see Theorem 1.3 and Theorem 1.4 in [5]):

Theorem III.4.3: *The construction $G \mapsto \mathrm{Mor}_G$ induces a one-to-one correspondence between compact subgroups $G \subseteq O_n$ and symmetric tensor categories with duals $C_x \subseteq C$.*

For homogenous groups, that is, groups G with $\mathfrak{S}_n \subseteq G \subseteq O_n$, this assignment induces a one-to-one correspondence between homogenous groups and subcategories of $\mathrm{Mor}_{\mathfrak{S}_n}$.

To deal with the unitary case, i.e., groups G with $\mathfrak{S}_n \subseteq G \subseteq U_n$, we have to modify the definition of an intertwiner in the following way:

Definition III.4.4: Let k and ℓ be non-negative integers, let $\mathfrak{S}_n \subseteq G \subseteq U_n$ be a compact group, let S be a coloured set with $k+\ell$ points and let $\boldsymbol{r} = (r_1, \dots, r_k) \in \{\circ, \bullet\}^k$, $\boldsymbol{s} = (s_1, \dots, s_\ell) \in \{\circ, \bullet\}^\ell$ be the colourings of the upper respectively lower row of S. Let again $g^\circ := g$, $g^\bullet := g^\dagger$.

A linear map $\varphi\colon T^k(\mathbb{C}^n) \to T^\ell(\mathbb{C}^n)$ is called *intertwiner*, if for all $g \in G$ it holds

$$\varphi \circ (g^{r_1} \otimes \dots \otimes g^{r_k}) = (g^{s_1} \otimes \dots \otimes g^{s_\ell}) \circ \varphi.$$

The set

$$\mathrm{Mor}_G^S(k,\ell) := \{\varphi \in \mathrm{Hom}_{\mathbb{C}}(T^k(\mathbb{C}^n), T^\ell(\mathbb{C}^n)) \mid \varphi \text{ is intertwiner for } G \text{ and } S\}$$

is called *intertwiner space of G with respect to the colouring of S*. Letting $M' = \{1, \dots, k, 1', \dots, \ell'\}$, $M := M' \times \{\circ, \bullet\}$ and

$$\mathcal{S} = \{S \subseteq M \mid p_{M'}(S) = M' \text{ and } \#(S) = k+\ell\},$$

we put $\mathrm{Mor}_G(k,\ell) := \bigcup_{S \subseteq \mathcal{S}} \mathrm{Mor}_G^S(k,\ell)$ and $\mathrm{Mor}_G := \bigcup_{k \in \mathbb{N}_0} \bigcup_{\ell \in \mathbb{N}_0} \mathrm{Mor}_G(k,\ell)$.

It is immediate that the collection of intertwiner spaces of a compact group G, where $\mathfrak{S}_n \subseteq G \subseteq U_n$, is a tensor category with duals and that this category contains the generators ,  and any of their rotations.

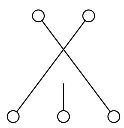
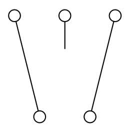
Finding groups G with $\mathfrak{S}_n \subseteq G \subseteq O_n$ respectively $\mathfrak{S}_n \subseteq G \subseteq U_n$ thus can be done by finding suitable symmetric tensor categories with duals. Finding well controllable tensor categories will be the strategy in the following.

Definition III.4.5: Let k and ℓ be natural numbers and let $p \in P^{\circ\bullet}(k,\ell)$. The map defined via linear extension of

$$T_p\colon T^k(\mathbb{C}^n) \longrightarrow T^\ell(\mathbb{C}^n), \qquad e_{\boldsymbol{i}} \longmapsto \sum_{\boldsymbol{j} \in I_n^\ell} \delta(\boldsymbol{i}, \boldsymbol{j}) e_{\boldsymbol{j}},$$

where $\boldsymbol{i} \in I_n^k$, is called *linear map associated to p*.

Example III.4.6: Consider the partitions p and q

$p =$  and $q =$ 

of Example III.2.5(i) and let $n \in \mathbb{N}$. The associated linear maps are uniquely determined by

$$T_p(e_i \otimes e_j) = \sum_{\boldsymbol{j} \in I_n^3} \delta_p((i,j), \boldsymbol{j}) e_{j_1} \otimes e_{j_2} \otimes e_{j_3} = \sum_{k=1}^n e_j \otimes e_k \otimes e_i,$$

$$T_q(e_i \otimes e_j \otimes e_\ell) = \sum_{\boldsymbol{j} \in I_n^2} \delta_q((i,j,\ell), \boldsymbol{j}) e_{j_1} \otimes e_{j_2} = e_i \otimes e_\ell.$$

For the linear maps associated to the composition pq and the composition $T_p \circ T_q$, we get

$$(T_p \circ T_q)(e_i \otimes e_j \otimes e_\ell) = T_p(e_i \otimes e_\ell) = \sum_{k=1}^{n} e_\ell \otimes e_k \otimes e_i,$$
$$(T_{pq})(e_i \otimes e_j \otimes e_\ell) = \sum_{k=1}^{n} e_\ell \otimes e_k \otimes e_i,$$

and for the linear maps associated to the composition qp and for the composition $T_q \circ T_p$, we obtain

$$(T_q \circ T_p)(e_i \otimes e_j) = T_q\Big(\sum_{k=1}^{n} e_j \otimes e_k \otimes e_i\Big) = \sum_{i=1}^{n} e_j \otimes e_i = n \cdot (e_j \otimes e_i),$$
$$T_{qp}(e_i \otimes e_j) = \sum_{\boldsymbol{j} \in I_n^2} \delta_{qp}((i,j), \boldsymbol{j}) e_{\boldsymbol{j}} = e_j \otimes e_i.$$

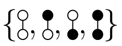
This assignment of linear maps to partitions behaves well with the operations we already defined on partitions, see [5] Proposition 1.9.

Proposition III.4.7: *Let p and q be partitions. Then, for their associated linear maps, it holds:*

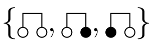
(i) $T_{p \otimes q} = T_p \otimes T_q$,

(ii) $T_{pq} = n^{-b(p,q)} T_p \circ T_q$,

(iii) $T_{p^*} = T_p^*$,

(iv) *For* $p \in$  *it holds* $T_p = \mathrm{id}_{\mathbb{C}^n}$,

(v) *For* $p =$  *it holds* $T_p = \tau$,

(vi) *For* $p \in$  *it holds* $T_p(1) = \xi$.

Here, $b(p,q)$ denotes the number of points that get erased from the middle line when forming pq, τ denotes the twist map and ξ is the map from Proposition III.4.2.

Because we already saw such a case in Example III.4.6, we have an idea, where the correction term $n^{-b(p,q)}$ comes from: Dropping the isolated block on the center line in the composition qp creates the factor n^1 in the evaluations of T_{qp}. This has to be accounted for when comparing T_{qp} and $T_q \circ T_p$.

As a result of Proposition III.4.7, we can describe the tensor category associated to the symmetric group $\mathfrak{S}_n$:

Theorem III.4.8: *Let n be a natural number and let k and ℓ be non-negative integers. Then,*

$$\mathrm{Mor}_{\mathfrak{S}_n}(k,\ell) = \mathrm{Lin}(\{T_p \mid p \in P(k,\ell)\}),$$

hence $\mathrm{Mor}_{\mathfrak{S}_n}$ *consists of the linear span of linear maps associated to partitions.*

5. Easy groups

Now, everything is in place to define what an easy group is; see [20], Definition 5.5.

Definition III.5.1 (Easy group): Let n be a natural number and let G be a closed matrix group with $\mathfrak{S}_n \subseteq G \subseteq U_n$. If there is a category of partitions $C \subseteq P^{\circ\bullet}$ such that ⌘ $\in C$ and such that for all non-negative integers k, ℓ and coloured sets S with upper colouring $\boldsymbol{r} \in \{\circ, \bullet\}^k$ and lower colouring $\boldsymbol{s} \in \{\circ, \bullet\}^\ell$ it holds that $\mathrm{Mor}_G^S(k,\ell)$ is spanned by all linear maps T_p, where $p \in C(k,\ell)$ and p has upper colouring according to $\boldsymbol{r}$ and lower colouring according to $\boldsymbol{s}$, then G is called *unitary easy.*

If C is a non-coloured category of partitions, it holds $\mathfrak{S}_n \subseteq G \subseteq O_n$ and the group G is called *orthogonal easy.*

This means: Easy groups can be recovered via Tannaka-Krein duality and thus are closed subgroups of O_n respectively U_n. Without the requirement "⌘ $\in C$", this is precisely the definition of a easy quantum group. Here "the" is justified because it is equivalent to the definition given in Definition III.3.4, see [20], Lemma 5.6.

Theorem III.5.2 (Easy groups): *Let n be a natural number. There are exactly 6 easy groups, namely:*

(i) *The symmetric group $\mathfrak{S}_n$,*

(ii) *The group $\mathfrak{S}_n' = \mathbb{Z}_2 \times \mathfrak{S}_n$,*

(iii) *The hyperoctahedral group $H_n = \mathbb{Z}_2 \wr \mathfrak{S}_n$,*

(iv) *The bistochastic group B_n,*

(v) *The group $B_n' = \mathbb{Z}_2 \times B_n$,*

(vi) *The orthogonal group O_n.*

As a generalisation of the bistochastic group B_n, consisting of orthogonal matrices with row and column sums equal to one, we denote by C_n the group of unitary matrices with row and column sums equal to one.

Theorem III.5.3 (Unitary easy groups): *Let n be a natural number. The unitary easy groups are the following:*

(i) $\mathcal{O}_{\text{grp,glob}}(k) : O_n \times \mathbb{Z}_k$*, where* $k \in 2\mathbb{N}_0$*,*

(ii) $\mathcal{O}_{\text{grp,loc}} : U_n$*,*

(iii) $\mathcal{H}_{\text{grp,glob}}(k) : (\mathbb{Z}_2 \wr \mathfrak{S}_n) \times \mathbb{Z}_k = H_n \times \mathbb{Z}_k$*, where* $k \in 2\mathbb{N}_0$*,*

(iv) $\mathcal{H}_{\text{grp,loc}}(k,d) : (\mathbb{Z}_d \wr \mathfrak{S}_n) \times \mathbb{Z}_k$*, where* $d \in \mathbb{N}_0 - \{1,2\}$ *and* $k \mid d$*,*

(v) $\mathcal{S}_{\text{grp,glob}}(k) : \mathfrak{S}_n \times \mathbb{Z}_k$*, where* $k \in \mathbb{N}_0$*,*

(vi) $\mathcal{B}_{\text{grp,glob}}(k) : B_n \times \mathbb{Z}_k$*, where* $k \in 2\mathbb{N}_0$*,*

(vii) $\mathcal{B}_{\text{grp,loc}}(k) : C_n \times \mathbb{Z}_k$*, where* $k \in \mathbb{N}_0$*.*

The orthogonal easy groups were completely classified in 2009 by Banica and Speicher, see Theorem 2.8 in [5], the unitary case was treated by Tarrago and Weber in 2016, see Theorem 7.2 in [20].

The product "$\tilde{\times}$" for the easy groups from Theorem III.5.3 is in fact the usual direct product of groups, as $\mathbb{Z}_k$ is a finite group and $0 \in G$ has finite order, hence we omit the symbol here.

For $k = 0$, we put $\mathbb{Z}_0 := \{0\}$. This means that the orthogonal easy groups reappear in the list above as special cases of unitary easy groups.

5.1. Categories corresponding to easy groups

The categories of partitions corresponding to the easy groups are

Orthogonal case	Unitary case
$\mathfrak{S}_n = \langle \overset{\uparrow}{\circ}, \sqcap\!\sqcap\!\sqcap_{\circ\circ\circ\circ}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$	$\mathcal{O}_{\text{grp,glob}}(k) = \langle \sqcap_{\circ\circ}^{\otimes \text{nest}(k/2)}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$
$\mathbb{Z}_2 \times \mathfrak{S}_n = \langle \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\circ}, \sqcap\!\sqcap\!\sqcap_{\circ\circ\circ\circ}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$	$\mathcal{O}_{\text{grp,loc}} = \langle \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$
$\mathbb{Z}_2 \wr \mathfrak{S}_n = \langle \sqcap\!\sqcap\!\sqcap_{\circ\circ\circ\circ}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$	$\mathcal{H}_{\text{grp,glob}}(k) = \langle b_k, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$
$B_n = \langle \overset{\uparrow}{\circ}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$	$\mathcal{H}_{\text{grp,loc}}(k,d) = \langle b_k, b_d \otimes \tilde{b}_d, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$
$\mathbb{Z}_2 \times B_n = \langle \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\circ}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$	$\mathcal{S}_{\text{grp,glob}}(k) = \langle \overset{\uparrow}{\circ}^{\otimes k}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$
$O_n = \langle \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$	$\mathcal{B}_{\text{grp,glob}}(k) = \langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$
	$\mathcal{B}_{\text{grp,loc}}(k) = \langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet}, \overset{\circ\circ}{\underset{\circ\circ}{\times}} \rangle$

see [5] for the result, [13], Theorem 2.6.13, for the categories corresponding to the orthogonal easy groups and [20], Theorem 7.2, for the result for the unitary easy groups.

Chapter IV.

A hands-on approach to easy groups

In this chapter, we show that the (unitary) easy groups as classified by Banica and Speicher respectively Tarrago and Weber allow a representation on $\mathbb{K}$ of suitable dimension that fits the description provided through polynomials in the matrix entries.

To be able to do this, we need to understand the occurring products of groups and recall some basic facts on group representations. As for the products involved, the wreath product is probably the one unfamiliar to the reader.

Sections 1 and 3 of Appendix A and section 3 of Appendix B should be helpful for this chapter.

1. Wreath products

The following section aims at understanding the law of composition on the groups $\mathbb{Z}_d \wr \mathfrak{S}_n$, where n and d are natural numbers. It is quite technical.

Recall that if $(G, \bullet)$ and $(H, \star)$ are groups, X is a set and $\alpha\colon H \times X \to X$ is a group action of H on X, then this group action α induces a group homomorphism $\boldsymbol{\alpha}\colon H \to \operatorname{Aut} G^X$, which gives rise to a group operation of H on G^X. The *wreath product of G and H* is the group

$$G \wr_X H := G^X \rtimes_{\boldsymbol{\alpha}} H.$$

In literature, this product sometimes is called the *unrestricted wreath product* [19], §34, Aufgabe 16, while $G^{(X)} \rtimes_{\boldsymbol{\alpha}} H$ is called *restricted wreath product* or just *wreath product*. For the groups in question in this thesis, this distinction does not matter.

For further details and a proof of the above assertion, see Appendix 1, in particular Lemma A.1.11.

The key example of wreath products that appear in this thesis is the following:

Example IV.1.1: Let n and d be natural numbers, let $G = (\mathbb{Z}_d, +)$, $X = \mathbb{N}_n$ and $H = (\mathfrak{S}_n, \circ)$. In this setting, $G^X = \{f\colon \mathbb{N}_n \to \mathbb{Z}_d\}$, which we can identify with $\prod_{i=1}^n \mathbb{Z}_d$ by identifying the map $f\colon \mathbb{N}_n \to \mathbb{Z}_d$ with the tuple $(f(1), \dots, f(n)) \in \prod_{i=1}^n \mathbb{Z}_d$. The group action $\alpha\colon \mathfrak{S}_n \times \mathbb{N}_n \to \mathbb{N}_n$ induces the maps

$$\boldsymbol{\alpha}_\sigma\colon \prod_{i=1}^n \mathbb{Z}_d \longrightarrow \prod_{i=1}^n \mathbb{Z}_d, \qquad (i_1, \dots, i_n) \longmapsto (i_{\sigma^{-1}(1)}, \dots, i_{\sigma^{-1}(n)})$$

and the group homomorphism $\boldsymbol{\alpha}\colon \mathfrak{S}_n \to \mathrm{Aut}(\prod_{i=1}^n \mathbb{Z}_d)$. The wreath product $\mathbb{Z}_d \wr_{\mathbb{N}_n} \mathfrak{S}_n = (\prod_{i=1}^n \mathbb{Z}_d) \rtimes_{\boldsymbol{\alpha}} \mathfrak{S}_n$, which as a set is $(\prod_{i=1}^n \mathbb{Z}_d) \times \mathfrak{S}_n$, has the following law of composition:

$$\begin{aligned}((i_1, \dots, i_n), \sigma) \star ((j_1, \dots, j_n), \delta) \\ = ((i_1, \dots, i_n) + (j_{\sigma^{-1}(1)}, \dots, j_{\sigma^{-1}(n)}), \sigma \circ \delta).\end{aligned}$$

2. Group representations

In this section, we collect all group representations necessary to carry out the work of the following section.

Definition IV.2.1 (Group representation): Let G be a group and V be a finite-dimensional vector space over the field k. A group homomorphism $\rho\colon G \to \mathrm{Gl}(V)$, where by $\mathrm{Gl}(V)$ we mean the invertible linear endomorphisms of V, is called a *group representation of G on V*. By abuse of language, if the homomorphism ρ is clear from context, we refer to V as the representation. If ρ is injective, the representation is called *faithful*. We call the dimension of V the *dimension of the representation ρ*.

We call

$$\ker \rho := \{g \in G \mid \rho(g) = \mathrm{id}\}$$

the *kernel of the representation ρ*. The kernel of ρ is a normal subgroup of G.

If $W \subseteq V$ is a linear subspace that is stable under ρ, i.e., $\rho(g)W \subseteq W$ for all $g \in G$, the map $\rho_W\colon G \to \mathrm{Gl}(W)$ is called a *subrepresentation of G on W*.

If there is no non-trivial subrepresentation of G on V, ρ is called *irreducible*.

Example IV.2.2 (Some representations): (i) The map

$$\rho\colon \mathbb{C} \longrightarrow M_2(\mathbb{R}), \qquad x + \mathrm{i}y \longmapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

is a ring homomorphism and the restriction $\rho' := \rho|_{\mathbb{C}^\times}$ is a representation of the multiplicative group of $\mathbb{C}$ on $\mathbb{R}^2$.

Since ρ is a ring homomorphism, its kernel $\ker\rho \subseteq \mathbb{C}$ is an ideal, i.e., $\ker\rho = \{0\}$ or $\ker\rho = \mathbb{C}$ since $\mathbb{C}$ is a field. As ρ is not the zero-map, ρ must be injective. Hence ρ' is a faithful representation.

Note that ρ is not surjective, which does not come as a surprise: $\mathrm{Gl}_2(\mathbb{R})$ is not commutative, but $\rho(\mathbb{C}) \subseteq \mathrm{Gl}_2(\mathbb{R})$ is; $\mathrm{Gl}_2(\mathbb{R})$ has non-trivial zero-divisors, but $\rho(\mathbb{C}) \subseteq \mathrm{Gl}_2(\mathbb{R})$ does not. In fact, the image of $\mathbb{C}$ under ρ is a sub ring of $\mathrm{Gl}_n(\mathbb{R})$ that is even a field.

(ii) Let n be a natural number. Then

$$\rho_n\colon \mathfrak{S}_n \longrightarrow \mathrm{Gl}_n(\mathbb{R}), \qquad \sigma \longmapsto A_{\sigma^{-1}} := \begin{cases} (A_{\sigma^{-1}})^i_j = 1, & \text{if } i = \sigma^{-1}(j), \\ (A_{\sigma^{-1}})^i_j = 0, & \text{else,} \end{cases}$$

is a representation of $\mathfrak{S}_n$, the so-called *fundamental representation*. It is a faithful representation.

(iii) Let n be a natural number. Then

$$\sigma_n\colon \mathbb{Z}_n \longrightarrow \mathbb{C}^\times, \qquad [j] \longmapsto \exp\left(\frac{2\pi\mathrm{i}j}{n}\right)$$

is an irreducible faithful representation of $\mathbb{Z}_n$. In fact, any of the n-th roots of unity $\exp(2\pi\mathrm{i}(j/n))$, $1 \leq j \leq n$, gives rise to an irreducible faithful representation of $\mathbb{Z}_n$.

Lemma IV.2.3: *Let G and H be groups, V and W be vector spaces over the field k and $\rho\colon G \to \mathrm{Gl}(V)$, $\sigma\colon H \to \mathrm{Gl}(W)$ be representations. Then*

$$\rho \otimes \sigma\colon G \times H \longrightarrow \mathrm{Gl}(V \otimes_k W), \qquad (g,h) \longmapsto \rho(g) \otimes \sigma(h)$$

is a representation of $G \times H$ on $V \otimes_k W$.

Proof: Let (g_1, h_1) and (g_2, h_2) be elements of the direct product $G \times H$. Since $\rho(g_i)$, $\sigma(h_i)$ are in particular elements of $\mathrm{End}(V)$, $\mathrm{End}(W)$, their tensor products $\rho(g_i) \otimes \sigma(h_i)$ are elements of $\operatorname{End}(V \otimes_k W)$, which is an algebra. We have

$$\begin{aligned}(\rho \otimes \sigma)(g_1 g_2, h_1 h_2) &= \rho(g_1 g_2) \otimes \sigma(h_1 h_2)\\ &= (\rho(g_1) \circ \rho(g_2)) \otimes (\sigma(h_1) \circ \sigma(h_2))\\ &= (\rho(g_1) \otimes \sigma(h_1)) \circ (\rho(g_2) \otimes \sigma(h_2)),\end{aligned}$$

i.e., $\rho \otimes \sigma$ is a group homomorphism. As mentioned in Remark A.3.10, the bijectivity of any $\rho(g)$ and any $\sigma(h)$, where $g \in G$ and $h \in H$, enforces the bijectivity of $\rho(g) \otimes \sigma(h)$, thus $\rho \otimes \sigma$ is well-defined. □

Example IV.2.4: Let $G = \mathbb{Z}_d$, $H = \mathfrak{S}_n$ and let ρ, σ be the representations of said groups from Example IV.2.2. Then

$$\sigma \otimes \rho \colon G \times H \longrightarrow \operatorname{Gl}_n(\mathbb{C}), \qquad (z, \delta) \longmapsto \sigma(z) \cdot \rho(\delta),$$

where we used the canonical identification $\mathbb{C} \otimes_{\mathbb{C}} V \cong V$ for any $\mathbb{C}$-vector space V, i.e., images of tuples from $G \times H$ are just permutation matrices multiplied by some d-th root of unity.

We already know the effect of taking the direct product of some matrix group G and $\mathbb{Z}_d$, where d is a natural number. Because we understand the law of composition of $\mathbb{Z}_d \wr \mathfrak{S}_n$, the following statement is at least plausible:

Lemma IV.2.5: *Let n and d be natural numbers. Then,*

$$(\mathbb{Z}_d \wr \mathfrak{S}_n) \cong \{A = \operatorname{diag}(\xi_1, \ldots, \xi_n)\rho(\sigma) \mid \sigma \in \mathfrak{S}_n, \xi_i \in \mathbb{S}^1, \xi_i^d = 1, 1 \le i \le d\},$$

where ρ denotes the fundamental representation of $\mathfrak{S}_n$.

For a proof of this assertion, see [4], Proposition 2.1. The proof uses group presentations, which exceeds the scope of this thesis. Lemma IV.2.5 tells us that we can identify the group $\mathbb{Z}_d \wr \mathfrak{S}_n$ with the monomial matrices, whose non-zero entries are d-th roots of unity.

3. Relations associated to easy groups

Since easy groups come about as commutative compact matrix quantum groups, the description in terms of categories in [13], Theorem 2.6.13, respectively [20] Section 4 gives rise to a description of the matrices contained in a given easy group in terms of relations on the entries of these matrices.

In the following, we will reproduce these relations (see [20], Section 4) and give a direct proof that the easy groups as equal to the groups recovered via Tannaka-Krein duality.

In the orthogonal case, the results presented here are no news. They are already contained in [5], more precisely Proposition 2.4 of this paper. In the unitary case, these are new considerations that cannot be found in existing literature.

3.1. Orthogonal easy groups

Let n be a natural number. In the orthogonal case, the occurring relations are

(R_1) "$\uparrow$": $\sum_{\ell=1}^{n} a_j^\ell = 1$,

(R_2) "$\sqcap\!\sqcap\!\sqcap$": $a_i^k a_j^k = a_k^i a_k^j = 0$, if $i \neq j$,

(R_3) "$\uparrow \otimes \uparrow$": $(\sum_{\ell=1}^{n} a_{j_1}^\ell) \cdot (\sum_{k=1}^{n} a_{j_2}^k) = 1$.

Here, indices that are no summation indices are to be understood as viable choices, i.e., elements of $\{1, \ldots, n\}$.

A matrix that fulfils (R_2) is *monomial*, that is, contains at most one non-zero entry per row and column.

As, by definition, $\sqcap$ and $|$ are contained in all categories of partitions, the matrices considered are elements of O_n. Furthermore, because the orthogonal easy groups are recovered via Tannaka-Krein duality, they are indeed groups and thus contain the transpose (the inverse) of each matrix. Hence, (R_1) and (R_3) are actually relations on row and column sums at the same time.

Using the above relations, the orthogonal easy groups can be characterised as

$$
\begin{aligned}
\mathfrak{S}_n &= \{A = (a_j^i) \in O_n \mid \textstyle\sum_{\ell=1}^{n} a_j^\ell = 1, a_i^k a_j^k = a_k^i a_k^j = 0 \text{ if } i \neq j\} \\
\mathbb{Z}_2 \times \mathfrak{S}_n &= \{A = (a_j^i) \in O_n \mid (\textstyle\sum_{\ell=1}^{n} a_{j_1}^\ell) \cdot (\textstyle\sum_{k=1}^{n} a_{j_2}^k) = 1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad a_i^k a_j^k = a_k^i a_k^j = 0 \text{ if } i \neq j\} \\
\mathbb{Z}_2 \wr \mathfrak{S}_n &= \{A = (a_j^i) \in O_n \mid a_i^k a_j^k = a_k^i a_k^j = 0 \text{ if } i \neq j\} \\
B_n &= \{A = (a_j^i) \in O_n \mid \textstyle\sum_{\ell=1}^{n} a_j^\ell = 1\} \\
\mathbb{Z}_2 \times B_n &= \{A = (a_j^i) \in O_n \mid (\textstyle\sum_{\ell=1}^{n} a_{j_1}^\ell) \cdot (\textstyle\sum_{k=1}^{n} a_{j_2}^k) = 1\}
\end{aligned}
$$

For each orthogonal easy group that is not a matrix group to begin with, we have a faithful n-dimensional representation on $\mathbb{C}$ and it is obvious that matrices in the images of these representations fulfil the given relations. Also the other inclusions are immediate. Note that for $\mathbb{Z}_2 \wr \mathfrak{S}_n$, the orthogonality forces the monomial matrices to only have non-zero entries 1 or -1 and that for $\mathbb{Z}_2 \times B_n$, the requirement that the square of each row sum equals one but also products of pairs of different row sums equal one forces all row sums to be either 1 or -1.

3.2. Unitary easy groups

Let n be a natural number. In the unitary case, the occurring relations are the following:

(R_1) "$\sqcap_{\circ\circ}^{\otimes \mathrm{nest}(k/2)}$": $a^{i_1}_{j_1} \cdots a^{i_{k/2}}_{j_{k/2}} = (a^{i_1}_{j_1})^* \cdots (a^{i_{k/2}}_{j_{k/2}})^*$,

(R_2) "$\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}$": $(a^i_j)^* a^k_\ell = a^i_j (a^k_\ell)^*$,

(R_3) "$\sqcap_{\circ\bullet\circ\bullet}$": $a^k_i (a^k_j)^* = a^i_k (a^j_k)^* = 0$, if $i \neq j$,

(R_4) "b_k": $\sum_\ell a^{i_1}_\ell \cdots a^{i_k}_\ell = \delta_{i_1,i_2} \cdots \delta_{i_{k-1},i_k}$,

(R_5) "$b_d \otimes \tilde{b}_d$": $\sum_k \delta_{i_1,i_2} \cdots \delta_{i_{d-1},i_d} a^k_{j_1} \cdots a^k_{j_d} = \sum_\ell \delta_{j_1,j_2} \cdots \delta_{j_{d-1},j_d} a^{i_1}_\ell \cdots a^{i_d}_\ell$,

(R_6) "$\uparrow_\circ \otimes \uparrow_\bullet$": $(\sum_k a^k_j) = (\sum_\ell a^i_\ell)$,

(R_7) "$\uparrow_\circ^{\otimes k}$": $(\sum_{\ell_1} a^{\ell_1}_{j_1}) \cdots (\sum_{\ell_k} a^{\ell_k}_{j_k}) = 1$

Here, indices that don't appear in the name of the partition are to be understood as viable choices, i.e., elements of $\{1, \ldots, n\}$. Again, relations on the rows of a matrix are at the same time relations on the columns of this matrix, because unitary easy groups are subgroups of U_n and thus stable under inversion; the relations themselves are stable under complex conjugation. The fact that $\sqcap_{\circ\bullet}$, $\sqcap_{\bullet\circ}$ and $\overset{\circ}{\circ}$, $\overset{\bullet}{\bullet}$ are contained in each category of partitions ensures that we the occurring matrices are unital.

In terms of the above relations, we have the characterisations

$$\begin{aligned}
\mathcal{O}_{\mathrm{grp,loc}}(k) &= \{A = (a^i_j) \in U_n \mid (R_1), (R_2)\}, \\
\mathcal{H}_{\mathrm{grp,glob}}(k) &= \{A = (a^i_j) \in U_n \mid (R_2), (R_3), (R_4)\}, \\
\mathcal{H}_{\mathrm{grp,loc}}(k,d) &= \{A = (a^i_j) \in U_n \mid (R_3), (R_4), (R_5)\}, \\
\mathcal{S}_{\mathrm{grp,glob}}(k) &= \{A = (a^i_j) \in U_n \mid (R_2), (R_3), (R_6), (R_7)\}, \\
\mathcal{B}_{\mathrm{grp,glob}}(k) &= \{A = (a^i_j) \in U_n \mid (R_2), (R_6), (R_7)\}, \\
\mathcal{B}_{\mathrm{grp,loc}}(k) &= \{A = (a^i_j) \in U_n \mid (R_6), (R_7)\}.
\end{aligned}$$

Proposition IV.3.1: *Let n be a natural number and let $A = (a^i_j) \in U_n$ such that for all $i, j, k, \ell \in \{1, \ldots, n\}$ it holds $(a^i_j)^* a^k_\ell = a^i_j (a^k_\ell)^*$. Then, there are $\zeta \in \mathbb{S}^1$ and $B \in O_n$ such that $A = \zeta B$.*

The proof is due to "user1551" on math.stackexchange, see [1].

Proof: Since $AA^* = A^*A = I_n$, there are some indices i and j such that $a^i_j \neq 0$. Put $\zeta := (a^i_j)^*/|a^i_j|$. The property $a^i_j(a^k_\ell)^* = (a^i_j)^* a^k_\ell$ enforces $\operatorname{Re}(a^i_j)\operatorname{Im}(a^k_\ell) = \operatorname{Re}(a^k_\ell)\operatorname{Im}(a^i_j)$, thus, for all $k, \ell \in \{1, \dots, n\}$,

$$\zeta a^k_\ell = \frac{\operatorname{Re}(a^i_j)\operatorname{Re}(a^k_\ell) + \operatorname{Im}(a^i_j)\operatorname{Im}(a^k_\ell) + \mathrm{i}[\operatorname{Re}(a^i_j)\operatorname{Im}(a^k_\ell) - \operatorname{Re}(a^k_\ell)\operatorname{Im}(a^i_j)]}{\operatorname{Re}(a^i_j)^2 + \operatorname{Im}(a^i_j)^2}$$

is in fact a real number. This means that $B := \zeta A$ is a real matrix. It holds

$$BB^t = BB^\dagger = (\zeta A)(\zeta A)^\dagger = (\zeta A)(\zeta^* A^\dagger) = I_n,$$

thus B is orthogonal, $A = \zeta^* B$ and $\zeta \in \mathbb{S}^1$. □

Note that matrices in unitary easy groups stemming from globally coloured partitions, each matrix is just a orthogonal matrix multiplied by some complex number of absolute value 1.

Theorem IV.3.2: *Let n be a natural number. Then, it holds:*

(i) $O_n \times \mathbb{Z}_k = \mathcal{O}_{\text{grp,glob}}(k)$, *where* $k \in 2\mathbb{N}_0$,

(ii) $(\mathbb{Z}_2 \wr \mathfrak{S}_n) \times \mathbb{Z}_k = \mathcal{H}_{\text{grp,loc}}(k)$, *where* $k \in \mathbb{N}_0$,

(iii) $(\mathbb{Z}_d \wr \mathfrak{S}_n) \times \mathbb{Z}_k = \mathcal{H}_{\text{grp,loc}}(k, d)$, *where* $d \in \mathbb{N}_0 - \{1, 2\}$ *and* $k \mid d$,

(iv) $\mathfrak{S}_n \times \mathbb{Z}_k = \mathcal{S}_{\text{grp,glob}}(k)$, *where* $k \in \mathbb{N}_0$,

(v) $B_n \times \mathbb{Z}_k = \mathcal{B}_{\text{grp,glob}}(k)$, *where* $k \in 2\mathbb{N}_0$,

(vi) $C_n \times \mathbb{Z}_k = \mathcal{B}_{\text{grp,loc}}(k)$, *where* $k \in \mathbb{N}_0$.

Proof: Throughout the proof, we use the names from Example IV.2.2 for the representations.

(i) "$\subseteq$": Let $A \in O_n \times \mathbb{Z}_k$, i.e., there are $B \in O_n$ and $\zeta \in \mathbb{S}^1$ with $\zeta^k = 1$ such that $A = \zeta B$. For $i, j, k, \ell \in \{1, \dots, n\}$ it holds

$$a^i_j(a^k_\ell)^* = \zeta b^i_j \zeta^* b^k_\ell = \zeta^* b^i_j \zeta b^k_\ell = (a^i_j)^* a^k_\ell,$$

furthermore, for indices $i_1, \dots, i_{k/2}, j_1, \dots, j_{k/2} \in \{1, \dots, n\}$ it holds

$$\begin{aligned} a^{i_1}_{j_1} \cdots a^{i_{k/2}}_{j_{k/2}} = (\zeta b^{i_1}_{j_1}) \cdots (\zeta b^{i_{k/2}}_{j_{k/2}}) &= \zeta^{k/2} b^{i_1}_{j_1} \cdots b^{i_{k/2}}_{j_{k/2}} \\ &= (\zeta^*)^{k/2} b^{i_1}_{j_1} \cdots b^{i_{k/2}}_{j_{k/2}} = (a^{i_1}_{j_1})^* \cdots (a^{i_{k/2}}_{j_{k/2}})^*, \end{aligned}$$

i.e., $O_n \times \mathbb{Z}_k \subseteq \mathcal{O}_{\text{grp,glob}}(k)$.

"$\supseteq$": Since the elements of $\mathcal{O}_{\text{grp,loc}}(k)$ satisfy (R_1), we know by Proposition IV.3.1 that, for each $A \in \mathcal{O}_{\text{grp,loc}}(k)$, there is some complex number $\zeta \in \mathbb{S}^1$ such that $\zeta^* A \in O_n$. Now, (R_2) ensures that ζ is a k-th root of unity.

(ii) It is clear that the matrices in the image of our representation fulfil the relations given.

Let now $A \in \mathcal{H}_{\mathrm{grp,glob}}(k)$. By Proposition IV.3.1, we know that $A = \zeta B$ with $\zeta \in \mathbb{S}^1$ and $B \in O_n$. The relation (R_3) yields that A is monomial and (R_4) ensures that every non-zero entry is a k-th root of unity. This shows the inclusion.

(iii) Again, it is obvious that the matrices in the image of our representation fulfil the relations.

A matrix $A \in \mathcal{H}_{\mathrm{grp,loc}}(k, d)$ is certainly monomial, every non-zero entry of A is a k-th root of unity and for all $i, j, k, \ell \in \{1, \ldots, n\}$ we have $(a^i_j)^d = (a^k_\ell)^d$, thus every non-zero entry is a product of some m-th root of unity, where $m \mid d$.

(iv) That the matrices in the image of the representation $\rho_n \otimes \sigma_n$ fulfil the given relations is obvious.

For a matrix A in $\mathcal{S}_{\mathrm{grp,loc}}(k)$, the relation (R_3) forces the matrix to be monomial. Furthermore, (R_2) ensures that A is the product of some monomial orthogonal matrix and a complex number $\zeta \in \mathbb{S}^1$. Now (R_6) and (R_7) yield that all row and column sums are equal and k-th roots of unity, i.e., A is a permutation matrix multiplied by a k-th root of unity.

(v) That the matrices in the image of our representation meet the relations is clear.

Let now $A \in \mathcal{B}_{\mathrm{grp,glob}}(k)$ be given. The relation (R_2) ensures that A is a product ζB with $\zeta \in \mathbb{S}^1$ and $B \in O_n$. Furthermore all row and column sums of A are the same and equal a k-th root of unity, our ζ. Thus, multiplying A with ζ^* leaves us with a real orthogonal matrix whose row and column sums are all the same and equal to one, i.e., B is a bistochastic matrix.

(vi) This is clear. □

4. Easy groups as Lie groups

Because (unitary) easy groups are closed subgroups of the orthogonal group O_n respectively the unitary group U_n, they are in particular Lie groups. A crucial data for a Lie group is its Lie algebra, that is, its tangent space at the neutral element. The idea to determine the Lie algebras of the easy groups originated in the following consideration: Given a natural number n, we have the inclusions

$$\begin{array}{ccccc} B_n & \subseteq & \mathbb{Z}_2 \times B_n & \subseteq & O_n \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\ \mathfrak{S}_n & \subseteq & \mathbb{Z}_2 \times \mathfrak{S}_n & \subseteq & \mathbb{Z}_2 \wr \mathfrak{S}_n \end{array}$$

of orthogonal easy groups. Since all inclusions are inclusions of closed subgroups in Lie groups, all inclusions are inclusions of sub Lie groups in Lie groups. This also means that the Lie algebras of one of those subgroups is sub Lie algebras of the Lie algebra of the respective containing group. Via the exponential map, it is possible to lift a sub Lie algebra of the Lie algebra of a matrix Lie group to a (sub) Lie group of said matrix Lie group, thus, the above considerations could have led to other homogenous groups. Unfortunately, the Lie algebras of the finite easy groups are not particularly interesting.

Lemma IV.4.1: *Let $G \subseteq \mathrm{Gl}_n(\mathbb{K})$ be a finite subgroup. Then, the induced topology on G is the discrete topology and the Lie algebra $\mathrm{Lie}(G)$ of G is the zero space.*

Proof: Since the topology on $\mathrm{Gl}_n(\mathbb{K})$ is Hausdorff, we find suitably small neighbourhoods for each point of G that don't contain any other point, i.e., for every element $g \in G$, the singleton $\{g\}$ is contained in the induced topology. It is well known that continuous maps into the discrete topology are locally constant and since differentiable maps are in particular continuous, any path to G at the neutral element is locally constant. As the tangent space of G has the characterisation Theorem B.3.10, it must be the zero space. □

Since matrix Lie groups are smooth embedded submanifolds of $M_n(\mathbb{K})$ and not $\mathbb{K}^n$, it is worthwhile to think about the characterisation of the tangent space. A proof that the above characterisation holds can be found in [10], Corollary 3.46.

Lemma IV.4.2: *Let G and H be Lie groups with Lie algebras $\mathrm{Lie}(G)$ and $\mathrm{Lie}(H)$.*

(i) *The product Lie group $G \times H$ has the Lie algebra $\mathrm{Lie}(G) \oplus \mathrm{Lie}(H)$.*

(ii) *If $K \subseteq G$ is normal and closed, then G/K is a Lie group with Lie algebra $\mathrm{Lie}(G/K) \cong \mathrm{Lie}(G)/\mathrm{Lie}(H)$.*

For the assertion in (i), see [7] III.§3.8 and for the assertion in (ii), see [11], III.3.12 Satz.

Let n be a natural number. Recall that the Lie algebras to the classical Lie groups O_n and U_n are

$$\mathfrak{so}(n) := \{A \in \mathrm{Gl}_n(\mathbb{R}) \mid -A = A^t\},$$
$$\mathfrak{u}(n) := \{A \in \mathrm{Gl}_n(\mathbb{C}) \mid -A = A^\dagger\}.$$

Note that the special orthogonal group and the orthogonal group have the same Lie algebra, since the special orthogonal group is the connected component of the identity in the orthogonal group.

Remark IV.4.3: The same argument that shows $B_n \cong O_{n-1}$ also shows $C_n \cong U_{n-1}$.

Proposition IV.4.4 (Lie algebras of easy groups): *Let n be a natural number. In the orthogonal case, the Lie algebras of the easy groups are*

(i) $\mathrm{Lie}(\mathfrak{S}_n) = \{0\}$,
(ii) $\mathrm{Lie}(\mathfrak{S}'_n) = \{0\}$,
(iii) $\mathrm{Lie}(H_n) = \{0\}$,
(iv) $\mathrm{Lie}(B_n) \cong \mathfrak{so}(n-1)$,
(v) $\mathrm{Lie}(B'_n) \cong \mathfrak{so}(n-1)$,
(vi) $\mathrm{Lie}(O_n) = \mathfrak{so}(n)$.

In the unitary case, the Lie algebras of the easy groups are

(i) $\mathrm{Lie}(O_n \times \mathbb{Z}_k) = \mathfrak{so}(n)$, *where* $k \in 2\mathbb{N}_0$,
(ii) $\mathrm{Lie}(U_n) = \mathfrak{u}(n)$,
(iii) $\mathrm{Lie}(H_n \times \mathbb{Z}_k) = \{0\}$, *where* $k \in 2\mathbb{N}_0$,
(iv) $\mathrm{Lie}((\mathbb{Z}_d \wr \mathfrak{S}_n) \times \mathbb{Z}_k) = \{0\}$, *where* $d \in \mathbb{N}_0 - \{1, 2\}$ *and* $k \in d\mathbb{N}$,
(v) $\mathrm{Lie}(\mathfrak{S}_n \times \mathbb{Z}_k) = \{0\}$,
(vi) $\mathrm{Lie}(B_n \times \mathbb{Z}_k) \cong \mathfrak{so}(n-1)$, *where* $k \in 2\mathbb{N}_0$,
(vii) $\mathrm{Lie}(C_n \times \mathbb{Z}_k) \cong \mathfrak{u}(n-1)$, *where* $k \in \mathbb{N}_0$.

Appendix

Appendix A.

Essentials from abstract algebra

In this chapter, we give a short overview over basic concepts from abstract algebras such as groups, group actions on sets, vector spaces and linear maps, transformation matrices, dual spaces, tensor products of vector spaces and linear maps, euclidean or unitary spaces, isometries, that is, structure preserving maps of euclidean or unitary spaces, the Riesz Representation Theorem and the adjoint linear map.

1. Groups, group actions and products

In this section, we recap the concepts of groups, group homomorphisms and group actions of groups on sets. The given examples are taken from the content of the thesis.

Definition A.1.1 (Group): Let $\varnothing \neq G$ be a set and let $\circ\colon G \times G \to G$ be a law of composition on G, where we write $g \circ h := \circ(g, h)$ or briefly $gh := g \circ h$. If the axioms

(i) There is $e \in G$ with $e \circ g = g = g \circ e$ for all $g \in G$,

(ii) For each $g \in G$ there is $h \in G$ with $g \circ h = e = h \circ g$,

(iii) For all $f, g, h \in G$ it holds $f \circ (g \circ h) = (f \circ g) \circ h$,

are satisfied, the tuple $(G, \circ)$ is called a *group*. If no confusion concerning the law of composition is to be feared, we briefly write G for the group $(G, \circ)$. The element e is called *the neutral element of* G. For a given $g \in G$, the element h from (ii) is called the *inverse* to g, often denoted by g^{-1}. If in addition it holds

(iv) For all $g, h \in G$ it holds $gh = hg$, the group is called *abelian*.

Example A.1.2: (i) Let X be a set. Then

$$\mathrm{Sym}(X) := \{f\colon X \longrightarrow X \mid f \text{ is bijective}\}$$

turns into a group with composition of maps. The identity map is the neutral element with respect to this law of composition, associativity is a well-known property of composition of maps and the usual inverse maps are the inverses with respect to composition of maps.

(ii) Let n be a natural number. Given a field k, denote by $M_n(k) := k^{n\times n}$ the set of square matrices with entries in k. The subset

$$\mathrm{Gl}_n(k) := \{A \in M_n(k) \mid \det(A) \neq 0\}$$

of invertible matrices turns into a group with the matrix multiplication: Given matrices $A = (a^i_j)$, $B = (b^i_j)$, the product matrix $AB := A \cdot B$ is defined entrywise as

$$(AB)^i_j := \sum_{k=1}^{n} a^i_k b^k_j.$$

The neutral element with respect to this law of composition is the identity matrix $I_n := (\delta_{i,j})_{1\leq i,j\leq n}$, associativity of the matrix multiplication can be checked in a straight foreward calculation and for a given invertible matrix A, the inverse can (but rarely should) be calculated via $A^{-1} := (\det A)^{-1}A^{\#}$, where $A^{\#}$ denotes the so called adjugate matrix of A, see [12] Section 6.4.

Definition A.1.3 (Subgroup): Let $(G, \circ)$ be a group and let $H \subseteq G$ be subset. If it holds

(i) $H \neq \varnothing$,

(ii) $HH^{-1} = \{h' \circ h^{-1} \mid h', h \in H\} \subseteq H$,

then $(H, \circ|_{H\times H})$ is called a *subgroup* (with the restricted law of composition $\circ|_{H\times H}$). H then is a group in its own right.

As for any algebraic structure, structure preserving maps are of interest:

Definition A.1.4 (Group homomorphism): Let $(G, \circ)$ and $(H, \bullet)$ be groups and let $\varphi\colon G \to H$ be a map. If for any $a, b \in G$ it holds

$$\varphi(a \circ b) = \varphi(a) \bullet \varphi(b),$$

then φ is called *homomorphism of groups*. By writing $\varphi\colon (G, \circ) \to (H, \bullet)$ we indicate that φ is a structure preserving map.

A group homomorphism $\varphi\colon G \to H$ is called *group isomorphism*, if it is invertible (i.e., bijective) as a map. The inverse map φ^{-1} then is a homomorphism of groups as well.

Example A.1.5: Let $(G, \circ)$ be a group. Then

$$\mathrm{Aut}(G) := \{\varphi\colon (G, \circ) \longrightarrow (G, \circ) \mid \varphi \text{ is bijective}\}$$

is a group with composition of maps.

Definition A.1.6 (Group action): Let X be a set and let $(G, \circ)$ be a group. A mapping $\alpha\colon G \times X \to X$ satisfying the axioms

(i) For all $x \in X$ it holds $\alpha(e, x) = x$,

(ii) For all $g, h \in G$ and $x \in X$ it holds $\alpha(g \circ h, x) = \alpha(g, \alpha(h, x))$,

is called a *group action* (or more precisely a *group left action*) on X. We often abbreviate $g.x := \alpha(g, x)$ to improve readability.

Example A.1.7: Let n be a natural number.

(i) The symmetric group $\mathfrak{S}_n := \mathrm{Sym}(\mathbb{N}_n)$ operates canonically on the set $\mathbb{N}_n$ via $\sigma.i := \sigma(i)$: Obviously, $\mathrm{id}(i) = i$ for all $i \in \mathbb{N}_n$ and given two permutations σ and δ, we have

$$\sigma.(\delta.i) = \sigma.\delta(i) = \sigma(\delta(i)) = (\sigma \circ \delta).i.$$

In the same way, $\mathfrak{S}_n$ acts on an arbitrary set with n elements $X = \{x_1, \dots, x_n\}$ via $\sigma.x_i := x_{\sigma(i)}$.

(ii) Given a field k, the group $\mathrm{Gl}_n(k)$ operates on k^n via $A.\xi := A \cdot \xi$ for $A \in \mathrm{Gl}_n(k)$ and $\xi \in k^n$, since for all vectors $\xi \in k^n$ and matrices $A, B \in \mathrm{Gl}_n(k)$ we have $I_n.\xi = I_n \cdot \xi = \xi$ and $A.(B.\xi) = (A \cdot B).\xi$, where "$\cdot$" means usual matrix multiplication.

(iii) Let V be an n-dimensional k-vector space, $B = (b_1, \dots, b_n)$ be an ordered basis of V and $\xi = \sum_{i=1}^n \xi^i b_i$ be an arbitrary element of V. By defining

$$\sigma.\xi := \sum_{i=1}^n \xi^i b_{\sigma(i)},$$

the action of $\mathfrak{S}_n$ on B extends to an action on V.

Lemma A.1.8: *Let G be a group and let X be a set. A group homomorphism $\pi\colon G \to \mathrm{Sym}(X)$ gives rise to a group action of G on X and vice versa.*

Proof: If, on the one hand, G acts on X and we fix $g \in G$, then $\pi_g\colon X \to X$, $x \mapsto g.x$ is bijective with inverse map $\pi_{g^{-1}}$ and the map

$$\pi\colon G \longrightarrow \mathrm{Sym}(X), \qquad g \longmapsto \pi_g$$

is a group homomorphism. If, on the other hand, $\pi\colon G \to \mathrm{Sym}(X)$, $g \mapsto \pi_g$ is a group homomorphism,

$$G \times X \longrightarrow X, \qquad (g, x) \longmapsto \pi_g(x) =: g.x$$

defines a group action of G on X. □

Definition A.1.9 (Direct product): Let I be an index set and let $(G_i, \circ_i)_{i\in I}$ be a family of groups. The cartesian product $\prod_{i\in I} G_i$ equipped with the law of composition

$$(g_i)_{i\in I} \circ (h_i)_{i\in I} := (g_i \circ_i h_i)_{i\in I}$$

is itself a group, the *direct product of the family* $(G_i, \circ_i)_{i\in I}$.

Definition A.1.10 (Semidirect product): Let $(G, \circ)$ and $(H, \bullet)$ be groups and let $\varphi\colon H \to \operatorname{Aut} G$ be a group homomorphism. The cartesian product $G \times H$ together with the law of composition

$$(g_1, h_1) \star (g_2, h_2) := (g_1 \circ \varphi(h_1)(g_2), h_1 \bullet h_2)$$

is again a group, the *semidirect product of G and H with respect to φ*, denoted $G \rtimes_\varphi H$.

The group structure of $G \rtimes_\varphi H$ depends decisively on the choice of φ. Choosing the homomorphism $\varphi\colon H \to \operatorname{Aut} G$, $h \mapsto \mathrm{id}_G$ for all $h \in H$ gives back the direct product, so the semidirect product can be seen as a generalisation of the direct product of groups.

Lemma A.1.11: *Let $(G, \bullet)$, $(H, \star)$ be groups, let X be a set and let $\alpha\colon H \times X \to X$ be a group action of H on X. Then α induces a group homomorphism $\boldsymbol{\alpha}\colon H \to \operatorname{Aut} G^X$, which gives rise to a group operation of H on G^X.*

Proof: First of all, we note that G^X becomes a group with the pointwise law of composition (i.e., for f and $g \in G^X$, we define the map $f \bullet g \in G^X$ via $(f \bullet g)(x) := f(x) \bullet g(x)$) and we write $(G^X, \bullet)$ for this group, but $\operatorname{Aut}(G^X)$

becomes a group via composition of maps as law of composition and we write $(\mathrm{Aut}(G^X), \circ)$ for that group. For $h \in H$ we define the map

$$\boldsymbol{\alpha}_h\colon G^X \longrightarrow G^X, \qquad \boldsymbol{\alpha}_h(f)(x) := f(\alpha(h^{-1}, x)).$$

The maps $\boldsymbol{\alpha}_h$ are group homomorphisms, since for $f, g \in G^X$ and for all $x \in X$ it holds

$$\begin{aligned}\boldsymbol{\alpha}_h(f \bullet g)(x) &= (f \bullet g)(\alpha(h^{-1}, x)) \\ &= f(\alpha(h^{-1}, x)) \bullet g(\alpha(h^{-1}, x)) = \boldsymbol{\alpha}_h(f)(x) \bullet \boldsymbol{\alpha}_h(g)(x).\end{aligned}$$

If f and g are functions from X to G with $\boldsymbol{\alpha}_h(f) = \boldsymbol{\alpha}_h(g)$, for all $x \in X$ we have

$$\boldsymbol{\alpha}_h(f)(x) = f(\alpha(h^{-1}, x)) = g(\alpha(h^{-1}, x)) = \boldsymbol{\alpha}_h(g)(x);$$

in particular for $h = e_H$ we have $f(x) = g(x)$ for all $x \in X$, i.e., the maps $\boldsymbol{\alpha}_h$ are injective. Given $h \in H$ and a map $g\colon X \to G$, put $f\colon X \to G$, $x \mapsto g(\alpha(h, x))$. Then we have

$$\boldsymbol{\alpha}_h(f)(x) = f(\alpha(h^{-1}, x)) = g(\alpha(h, \alpha(h^{-1}, x)) = g(x),$$

i.e., $g = \boldsymbol{\alpha}_h(f)$ and we established the surjectivity of the maps $\boldsymbol{\alpha}_h$. The maps $\boldsymbol{\alpha}_h$ hence induce a map

$$\boldsymbol{\alpha}\colon H \longrightarrow \mathrm{Aut}(G^X),$$

which induces a group action of H on G^X, since we have

(i) $(\boldsymbol{\alpha}(e_H))(f)(x) = \boldsymbol{\alpha}_{e_H}(f)(x) = f(x)$, i.e., $(\boldsymbol{\alpha}(e_H))(f) = f$,

(ii) For $h_1, h_2 \in H$ it holds

$$\begin{aligned}(\boldsymbol{\alpha}(h_1 \star h_2))(f)(x) &= f(\alpha(h_2^{-1} \star h_1^{-1}, x)) \\ &= f(\alpha(h_2^{-1}, \alpha(h_1^{-1}, x)) \\ &= \boldsymbol{\alpha}(h_1)(f(\alpha(h_2^{-1}, \cdot))(x) = (\boldsymbol{\alpha}(h_1))(\boldsymbol{\alpha}(h_2)(f))(x),\end{aligned}$$

where $f(\alpha(h_2^{-1}, \cdot))$ means the map $X \to G$, $x \mapsto f(\alpha(h_2^{-1}, x))$. From the above calculation we read off $\boldsymbol{\alpha}(h_1 \star h_2) = \boldsymbol{\alpha}(h_1) \circ \boldsymbol{\alpha}(h_2)$, thus, in fact $\boldsymbol{\alpha}\colon H \to \mathrm{Aut}(G^X)$ is a group homomorphism which concludes the proof. $\square$

Definition A.1.12 (Wreath product): Let G and H be groups, let X be a set and let α be a group action of H on X. Then

$$G \wr_X H := G^X \rtimes_{\boldsymbol{\alpha}} H$$

is called the *wreath product of G and H with respect to the group action of H on X*, where $\boldsymbol{\alpha}$ means the group homomorphism from Lemma A.1.11.

From a mathematical standpoint, the wreath product plays a vital role in the classification of group extensions. In 1953, Krasner and Kaloujnine proved the universal embedding theorem that states that any group extension of a group H by a group A is isomorphic to a subgroup of the regular wreath product $A \wr_H H$, where H operates on itself by left-multiplication, see [15]. It also arises naturally in some combinatorial problems.

2. Vector spaces, linear maps and matrices

This section gives a very brief overview over basic facts on vector spaces and linear maps, the associated structure preserving maps. Many elementary facts from this section are heavily used in the thesis, e.g., that one can associate a transformation matrix to a given linear map between finite-dimensional vector spaces, that linear maps are uniquely determined by their values on a basis of the domain and the homomorphism theorem.

For the sake of completeness, we recap the definitions of ring and field.

Definition A.2.1 (Ring): Let R be a set and let $+\colon R\times R \to R$, $\cdot\colon R\times R \to R$ be two laws of composition. If it holds

(i) $(R,+)$ is an abelian group (with neutral element 0_R),

(ii) The law of composition "$\cdot$" is associative,

(iii) There is an element $0_R \neq 1_R \in R$ such that for all $r \in R$ it holds

$$1_R \cdot r = r \cdot 1_R = r,$$

(iv) For all $m, r, s \in R$ it holds

$$m \cdot (r+s) = m \cdot r + m \cdot s, \qquad (r+s) \cdot m = r \cdot m + s \cdot m,$$

the tuple $(R,+,\cdot)$ is called a *unital ring*. If no confusion is to be feared, we just call R an unital ring. If in addition it holds

(v) For all $r, s \in R$ it holds $r \cdot s = s \cdot r$,

R is called a *commutative ring*. If in addition it holds

(vi) For all $r \in R$ there is $s \in R$ such that $r \cdot s = s \cdot r = 1_R$,

$(R, +, \cdot)$ is called a *field*. We agree that "$\cdot$" has priority over "$+$", e.g., we write $m + s \cdot r := m + (s \cdot r)$. The inverse of $r \in R$ with respect to "$+$" is denoted $-r$, the inverse of $r \in R$ with respect to "$\cdot$" is denoted r^{-1}.

Let $(S, +, \cdot)$ be another ring and $\varphi\colon R \to S$ be a map. If it holds

$$\varphi(r+s) = \varphi(r) + \varphi(s), \qquad \varphi(r \cdot s) = \varphi(r) \cdot \varphi(s), \qquad \varphi(1) = 1,$$

we call φ a (unital) *ring homomorphism*. Ring homomorphisms are field homomorphisms, too.

A principal example of a ring is the so called *polynomial ring*. The following standard construction tries to formalise the concept of an "indeterminate".

Example A.2.2 (Polynomial ring): Let k be a field. The set $k[X] := k^{(\mathbb{N})}$ turns into a ring with the laws of composition

$$(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} := (a_n + b_n)_{n\in\mathbb{N}},$$
$$(a_n)_{n\in\mathbb{N}} \cdot (b_n)_{n\in\mathbb{N}} := (c_n)_{n\in\mathbb{N}} \quad \text{where} \quad c_n := \sum_{j=0}^{n} a_j b_{n-j}.$$

We call $aX^0 := (a, 0, 0, \dots)$ the *constant polynomial with value* a and we call $X := (\delta_{1,j})_{j\in\mathbb{N}}$ the *indeterminate polynomial*; note that by induction, one can show that $X^n = (\delta_{n,j})_{j\in\mathbb{N}}$. Thus, we can embed k into $k[X]$ via

$$\iota\colon k \longrightarrow k[X], \qquad a \longmapsto aX^0$$

and write any element $(a_n)_{n\in\mathbb{N}}$ of $k[X]$ as $(a_n)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} a_n X^n$. Note that since $(a_n)_{n\in\mathbb{N}} \in k[X]$, there is some natural number N_0 such that $a_n = 0$ for all $n \geq N$, hence this sum is indeed finite. The elements of $k[X]$ are called *polynomials*.

Reminder A.2.3: For the rest of this remark, let k be any field.

(i) Let $(V, +)$ be an abelian group with an exterior operation $\cdot\colon k \times V \to V$. We call the tuple $(V, +, \cdot)$ a k-vector space or vector space over k, if for all $x, y \in V$ and $\lambda, \mu \in k$ it holds $1 \cdot x = x$, $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$, $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$ and $\lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x$. The map $\cdot\colon k \times V \to V$ is then called scalar multiplication. We will most of the time omit the "$\cdot$" and instead abbreviate $\lambda v := \lambda \cdot v$. Furthermore, if no confusion is to be feared, we just call V a k-vector space.

(ii) A subset $\varnothing \neq W \subseteq V$ is called a linear subspace, if for all $v, w \in W$ and $\lambda, \mu \in k$ it holds $\lambda v + \mu w \in W$.

(iii) Let W be a subset of V. We put

$$k^{(W)} := \{f \colon W \longrightarrow k \mid f(w) \neq 0 \text{ only for finitely many } w \in W\}.$$

For $\lambda \in k^{(W)}$, we call $\sum_{w \in W} \lambda(w)w$ a linear combination of W. The set

$$\mathrm{Lin}(W) := \Big\{ \sum_{w \in W} \lambda(w)w : \lambda \in k^{(W)} \Big\}$$

is called the linear span of W. The linear span of W is the smallest linear subspace of V that contains the set W. It holds

$$\mathrm{Lin}(W) = \bigcap \{U \subseteq V \mid U \text{ is a linear subspace of } V \text{ with } W \subseteq U\}.$$

(iv) A subset $W \subseteq V$ is called linearly independent, if for all $\lambda \in k^{(W)}$ it holds: If $\sum_{w \in W} \lambda(w)w = 0$, then $\lambda \equiv 0$. Linear independency is precisely the requirement that linear combinations are unique, i.e., if $v \in W$ and $v = \sum_{w \in W} \lambda(w)w = \sum_{w \in W} \mu(w)w$ for $\lambda, \mu \in k^{(W)}$, then $\lambda = \mu$, since

$$0 = \sum_{w \in W} \lambda(w)w - \sum_{w \in W} \mu(w)w = \sum_{w \in W} (\lambda(w) - \mu(w))w.$$

If W is an infinite set, W is linearly independent if and only if every finite subset of W is linearly independent.

(v) If $W \subseteq V$ is linearly independent and $\mathrm{Lin}(W) = V$, then W is called a basis of V. The cardinality of W is called the dimension of V, where "the" is justified, as every basis of V has the same cardinality. Believing Zorns lemma, every k-vector space has a basis.

(vi) If V is a k-vector space and $W_1, \ldots, W_n \subseteq V$ are linear subspaces, $\sum_{i=1}^n W_i := \{\sum_{i=1}^n w_i \mid w_i \in W_i\}$ is a linear subspace of V. The sum $\sum_{i=1}^n W_i$ is called *direct*, if it holds: If $\sum_{i=1}^n w_i = 0$, then $w_i = 0$ for $1 \leq i \leq n$. We then write $\bigoplus_{i=1}^n W_i$ for the direct sum of the W_i. An immediate consequence of the definition is that $W_i \cap W_j = \{0\}$ for $i \neq j$. This direct sum sometimes is also called *inner direct sum*, because the W_i are linear subspaces of V and thus the whole process takes place inside some vector space.

For any k-vector spaces $(W_i)_{i \in I}$, the *exterior direct sum* just is the subspace

$$\bigoplus_{i \in I} W_i = \{(w_i)_{i \in I} \mid \text{Only finitely many } w_i \text{ are non-zero}\} \subseteq \prod_{i \in I} W_i.$$

(vii) A map $\varphi\colon V \to W$ between k-vector spaces is called a k-vector space homomorphism or linear map if for all $\lambda, \mu \in k$ and $x, y \in V$ it holds $\varphi(\lambda x + \mu y) = \lambda\varphi(x) + \mu\varphi(y)$. The subsets $\ker\varphi := \{v \in V \mid \varphi(v) = 0\} \subseteq V$ and $\varphi(V) = \{\varphi(v) \mid v \in V\} \subseteq W$ are linear subspaces of V respectively W. If φ is bijective as a map, its inverse map is a linear map as well; φ is then called a k-vector space isomorphism.

(viii) Linear maps and linear independency interact in the following way: If $\varphi\colon V \to W$ is a linear map and $U \subseteq \varphi(V)$ is linearly independent, then $\varphi^{-1}(U) \subseteq V$ is linearly independent; if φ is injective and $U' \subseteq V$ is linearly independent, then $\varphi(U')$ is linearly independent; φ is surjective if and only if $\varphi(V)$ contains a basis of W and finally φ is bijective if and only if for any basis $B \subseteq V$ is holds that $\varphi(B) \subseteq W$ is a basis.

(ix) If $\varphi\colon V \to W$ is a linear map and $B \subseteq V$ is a basis, φ is already uniquely determined by $\varphi|_B\colon B \to W$. Furthermore if $\psi\colon B \to W$ is a map, there is one and only one linear map $\varphi\colon V \to W$ with $\varphi|_B = \psi$. This allows us to define linear maps by prescribing values on a basis of the domain.

(x) The set

$$\mathrm{Hom}_k(V, W) := \{\varphi\colon (V, +, \cdot) \longrightarrow (W, +, \cdot)\} \subseteq W^V$$

becomes a k-vector space itself with the pointwise laws of composition, i.e., $f + g\colon V \to W$ is defined pointwise via $(f + g)(v) := f(v) + g(v)$, likewise for $\alpha f\colon V \to W$, where $\alpha \in k$.

For later use, we state the important theorem from linear algebra, that any linearly independent set can be extended to a basis.

Theorem A.2.4: *Let k be a field and let V be a k-vector space. If $M \subseteq V$ is a linearly independent set, there is a basis $B \subseteq V$ with $M \subseteq B$.*

Again, the general proof needs Zorns lemma. For finite-dimensional vector spaces, the Steinitz exchange lemma does the trick.

Remark A.2.5 (Matrices and linear maps): When talking about finite-dimensional vector spaces, we are in the extraordinary situation that we can identify the homomorphisms between two vector spaces with suitably sized matrices. This can be seen in the following way:

(i) Given a matrix $A = (a^i_j) \in M_{n\times m}(k)$, the map $\varphi_A\colon k^m \to k^n$ defined by $\varphi_A(v) := Av$ is a linear map. Applying A to e_i gives the vector $(a^1_i, \dots, a^n_i)^t$, i.e., the i-th column of A.

(ii) Let $\psi\colon k^n \to k^m$ be a linear map. Then, motivated by the previous thoughts, for the matrix $A := (\psi(e_1), \dots, \psi(e_n)) \in M_{m\times n}(k)$ it holds $\psi = \varphi_A$, since we have $\varphi_A(e_i) = Ae_i = \psi(e_i)$.

(iii) If V and W are k-vector spaces with $\dim V = n$, $\dim W = m$ and ordered bases $B = (b_1, \dots, b_n) \subseteq V$, $(c_1, \dots, c_m) \subseteq W$, then the maps

$$D_B\colon V \longrightarrow k^n, \qquad \sum_{i=1}^{n} v^i b_i \longmapsto \sum_{i=1}^{n} v^i e_i,$$

$$D_C\colon W \longrightarrow k^m, \qquad \sum_{i=1}^{m} w^i c_i \longmapsto \sum_{i=1}^{m} w^i e_i,$$

are k-vector space isomorphisms. For a linear map $\varphi\colon V \to W$, we express the vectors $\varphi(b_j)$, $1 \le j \le n$, in terms of the basis C, i.e., $\varphi(b_j) = \sum_{i=1}^{m} \alpha^i_j c_i$, and thus get a matrix $A = (\alpha^i_j) \in k^{m\times n}$. This matrix renders commutative the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle D_B}\big\downarrow & & \big\downarrow{\scriptstyle D_C} \\ k^n & \xrightarrow[v\mapsto Av]{} & k^m \end{array}$$

The matrix $D_{C,B}(\varphi) := A$ is called the transformation matrix of φ with respect to the bases B and C.

This means we can identify $\mathrm{Hom}_k(k^n, k^m) \cong M_{m\times n}(k)$. Under this identification, composition of linear maps corresponds to multiplication of matrices. Note that this identification is not canonical, since we need to fix bases in k^n and k^m. In fact, it will break for infinite-dimensional vector spaces.

Given a set X, a relation R on X is a subset $R \subseteq X \times X$. The relation R is called reflexive, if $(x, x) \in R$ for all $x \in X$; it is called symmetric, if it holds: If $(x, y) \in R$, then $(y, x) \in R$; it is called antisymmetric, if it holds: If $(x, y) \in R$ and $(y, x) \in R$, then $x = y$ and finally, it is called transitive, if it holds: If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

A reflexive, symmetric, transitive relation is called equivalence relation, a reflexive, antisymmetric, transitive relation is called a partial order.

Let R be an equivalence relation on X. If $(x, y) \in R$, we write $x \sim y$. By $[x] := \{y \in X \mid x \sim y\}$ we denote the equivalence class of $x \in X$. By $X/{\sim}$ we denote the set of equivalence classes of the relation R. An equivalence relation on the set X provides a partition of X, i.e., a decomposition of X into disjoint subsets of X, namely the equivalence classes of the relation R.

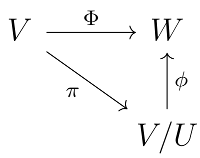
If X has enough structure and we take the right equivalence relation, we can make $X/{\sim}$ into an interesting object itself.

Example A.2.6: Let k be a field, let V be a k-vector space and let $U \subseteq V$ be a linear subspace. For $x, y \in V$, $x \sim y :\Leftrightarrow x - y \in U$ declares an equivalence relation on V. We denote by $V/U := V/\!\sim$ the set of equivalence classes with respect to this equivalence relation. Now, we can equip V/U with a k-vector space structure via the well-defined maps

$$+\colon V/U \times V/U \longrightarrow V/U, \qquad [v]_\sim + [w]_\sim := [v+w]_\sim,$$
$$\cdot\colon k \times V/U \longrightarrow V/U, \qquad \alpha[v]_\sim := [\alpha v]_\sim.$$

The map $\pi\colon V \to V/U$, $v \mapsto [v]_\sim$ is called the canoncial projection. The k-vector space V/U is called quotient vector space (of V by U), it's neutral element is $[0]_\sim = U$.

Proposition A.2.7 (Homomorphism theorem): *Let k be a field, let V and W be k-vector spaces, let $U \subseteq V$ be a linear subspace and let $\Phi\colon V \to W$ be a linear map. If $U \subseteq \ker\Phi$, there is one and only one linear map $\phi\colon V/U \to W$ rendering commutative the diagram*



If $U = \ker\Phi$, the map ϕ is injective.

Proposition A.2.7 can be understood like this: Whenever we have a linear map $\Phi\colon V \to W$, we have an injective linear map $\phi\colon V/\ker\Phi \to W$. To this end, we can define linear maps $\phi\colon V/\ker\Phi \to W$ without having to worry about well-definedness, i.e., without checking that the prescription for our map does not depend on the chosen representative of $[x]_\sim$, which we usually would have to, since we otherwise directly declared a map on equivalence classes. Given a vector space V/U, the work usually amounts to finding a suitable linear map $\varphi\colon V \to W$ with $\ker\Phi = U$.

In practice, Proposition A.2.7 is used to get rid of “inessential data” or to force certain properties.

Example A.2.8: Let k be a field, let V be a finite-dimensional k-vector space with basis $B = (b_1, \ldots, b_n)$, let U be the subspace generated by $b_1, \ldots, b_{i-1}$ for some fixed $i \in \mathbb{N}_n$ and let $\varphi\colon V \to V$ be declared by

$$\varphi(b_j) := \begin{cases} 0, & \text{if } 1 \le j < i, \\ b_j, & \text{otherwise.} \end{cases}$$

Then $\ker\varphi = U$ and by Proposition A.2.7, there is an injective linear map $\phi\colon V/U \to V$ with $\phi(V/U) = \varphi(V)$, i.e., $V/U \cong \varphi(V)$. The subset $\{b_i, \dots, b_n\} \subseteq B$ is linearly independent, thus

$$\dim\varphi(V) = \dim V - \dim U = \dim V/U. \tag{A.1}$$

This means that for a linear map $\varphi\colon V \to W$ between finite-dimensional k-vector spaces, it holds $\dim\varphi(V) = \dim V - \dim\ker\varphi$. Thus, if it holds $\dim V = \dim W$ and $\dim\ker\varphi = 0$, then $\dim\varphi(V) = \dim W$ and hence φ is surjective.

Since we can canonically identify V and $V/\{0\}$, Proposition A.2.7 also gives that $\varphi\colon V \to W$ is injective if and only if $\ker\varphi = \{0\}$.

3. Multilinear algebra

In this section, we deal with dual spaces and multilinear maps, i.e., maps from direct products of vector spaces that are linear in each argument, such as the determinant; one of the most important multilinear maps. Furthermore, we introduce the tensor products of finitely many vector spaces and linear maps.

Definition A.3.1 (Dual vector space): Let k be a field and let V be a k-vector space. Then $V^* := \mathrm{Hom}_k(V, k)$ is called the *dual vector space* to V. Elements of V are called functionals.

If V is finite-dimensional with basis $B = (b_1, \dots, b_n)$, denote by B^* the set of linear maps $\beta^i\colon V \to k$ with $\beta^i(b_j) = \delta_{i,j}$. This set B^* then is a basis of V^*, the *dual basis* of B.

Note that those maps $\beta^i\colon V \to k$ exist in the first place: because of Reminder A.2.3 (viii), the requirement $\beta^i(b_j) = \delta_{i,j}$ allows us to prescribe them on a basis, which extends uniquely. Furthermore, β^i depends on *all* vectors $b_1, \dots, b_n$.

Remark A.3.2: Let V be a k-vector space and let V^* be its dual space. For a given $v \in V$, the map $\iota_v\colon V^* \to k$, $\varphi \mapsto \varphi(v)$ is linear. This gives rise to a map

$$\iota\colon V \longrightarrow V^{**}, \qquad v \longmapsto \iota_v$$

which is again linear but injective in addition (where in the infinite-dimensional case, we need Zorns lemma to prove this), i.e., we can naturally embed V into V^{**}. If V is finite-dimensional, then ι even is an isomorphism of k-vector spaces, since $\dim V = \dim V^* = \dim V^{**}$.

Definition A.3.3 (Dual pair): Let V be a k-vector space and let V^* be its dual vector space. Then the map

$$\langle \cdot, \cdot \rangle \colon V^* \times V \longrightarrow k, \qquad (\varphi, v) \longmapsto \varphi(v),$$

is bilinear. This map is a *dual pairing* of V and V^*. The tuple $(V, V^*, \langle \cdot, \cdot \rangle)$ is called a *dual pair*.

Definition A.3.4 (Dual map): Let k be a field, let V and W be a k-vector spaces and let $f \colon V \to W$ be a linear map. Then

$$f^* \colon W^* \longrightarrow V^*, \qquad \varphi \longmapsto \varphi \circ f$$

is called the *dual map* or *transpose* of f.

In terms of the dual pairing, for all $\varphi \in W^*$ and $v \in V$ it holds

$$\langle \varphi, f(v) \rangle_{W^* \times W} = \langle f^*(\varphi), y \rangle_{V^* \times V}.$$

The map $f^{**} \colon V^{**} \to W^{**}$ naturally extends f, i.e., the follwoing diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ {\scriptstyle \iota_V}\big\downarrow & & \big\downarrow{\scriptstyle \iota_W} \\ V^{**} & \xrightarrow[f^{**}]{} & W^{**} \end{array}$$

Definition A.3.5: Let n be a natural number, let k be a field, let $V_1, \ldots, V_n$ and U be k-vector spaces and let $\beta \colon \prod_{i=1}^n V_i \to U$ be a map. The map β is called *n-times multilinear*, if for fixed i and fixed $v_j \in V_j$, where $1 \leq j \leq n$ and $i \neq j$, the maps

$$\beta_i \colon V_i \longrightarrow U, \qquad v \longmapsto \beta(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_n)$$

are linear. If $U = k$, then β is called an *n-times multilinear form*. By $\mathfrak{m}(V_1, \ldots, V_n; U)$ we denote the set of n-times multilinear maps from $\prod_{i=1}^n V_i$ to U. For $n = 2$, we speak of bilinear maps or bilinear forms respectively.

Example A.3.6 (Determinant): Let n be a natural number, let k be a field with $\operatorname{char}(k) \neq 2$, and let V be a k-vector space of dimension n with basis $\{b_1, \ldots, b_n\}$. A set $\{v_1, \ldots, v_n\} \subseteq V$ of vectors $v_i = \sum_{j=1}^n v_i^j b_i$ is a basis of V if and only if

$$\det(\{v_1, \ldots, v_n\}) := \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) v_1^{\sigma(1)} \cdots v_n^{\sigma(n)} \neq 0.$$

Even though it doesn't look like it, $\det\colon \prod_{i=1}^{n} V \to k$ is indeed an n-times mulitlinear form. One way of deriving this formula is by starting with an n-times multilinear form $D\colon \prod_{i=1}^{n} V \to k$ that vanishes, if two arguments coincide, and fulfills $D(b_1, \ldots, b_n) = 1$ and then expand $D(v_1, \ldots, v_n)$ multilinearly.

Identifying a matrix $A \in M_n(k)$ with the set $\{A_{\boldsymbol{1}}, \ldots, A_{\boldsymbol{n}}\}$ of its column vectors allows the definition of a map

$$\det\colon M_n(k) \longrightarrow k, \qquad A \longmapsto \det(\{A_{\boldsymbol{1}}, \ldots, A_{\boldsymbol{n}}\}).$$

Because of Reminder A.2.3 (viii), $\det(A)$ is a decisive number for the matrix A: If $\det(A)$ is non-zero, the linear map $v \mapsto Av$ is invertible since then $\{A_{\boldsymbol{1}}, \ldots, A_{\boldsymbol{n}}\} = \{Ae_1, \ldots, Ae_n\}$ forms a basis of k^n. If $\det(A) = 0$, A can't be injective due to the linear independency of $\{e_1, \ldots, e_n\}$.

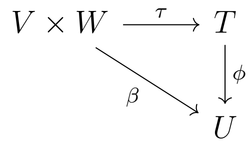
Given a matrix A, we denote by $A^{\boldsymbol{i}}_{\boldsymbol{j}}$ the matrix that results from cancelling the i-th row and the j-th column of A.

Given a linear map $\varphi\colon V \to V$, the number $\det(\{\varphi(b_1), \ldots, \varphi(b_n)\})$ tells us, if φ is a vector space isomorphism. As it turns out, this number doesn't depend on the basis chosen in V.

Let k be a field, let $V_1, \ldots, V_n$ be finite-dimensional k-vector spaces and let U be a k-vector space. Furthermore, let $B_i = \{b^i_j \mid 1 \leq j \leq n_i\} \subseteq V_i$ for $1 \leq i \leq n$ be bases. As for linear maps, a map $\beta \in \mathfrak{m}(V_1, \ldots, V_n; U)$ is uniquely determined by its values on n-tuples $(b^1_{j_1}, \ldots, b^n_{j_n})$ with $j_i \in \mathbb{N}_{n_i}$.

Now everything is in place to talk about tensor products of vector spaces.

Proposition A.3.7 (Tensor product of $\boldsymbol{k}$-vector spaces): *Let k be a field and let V and W be k-vector spaces. A pair (T, τ) consisting of a k-vector space T and a bilinear map $\tau\colon V \times W \to T$ is called* tensor product of V and W, *if it holds: For any bilinear map $\beta\colon V \times W \to U$, there is one and only one linear map $\phi\colon T \to U$ rendering commutative the diagram*



This property of the tensor product (T, τ) is called universal property *or* universal mapping property*.*

A short train of thought yields that if V and W are k-vector spaces and have a tensor product (T, τ), it is unique up to unique isomorphism — this is granted by the universal mapping property. It is thus justified, to talk of *the* tensor product of V and W.

We now have to give a reason why such a tensor product should exist in general. One way of answering this question is to construct a tensor product of two given vector spaces explicitly.

The following proof, which I find quite instructive, stems from my teacher Gabriela Weitze-Schmithüsen.

Proof (of Proposition A.3.7): Let V and W be k-vector spaces. If we put $T' := k^{(V\times W)}$ (which can be understood as the vector space of formal linear combinations of *any* elements of $V \times W$) and $\tau' \colon V \times W,\ (v, w) \longmapsto f_{(v,w)}$, where

$$f_{(v,w)} \colon V \times W \longrightarrow k, \qquad f_{(v,w)}(x, y) := \begin{cases} 1, & \text{if } x = v \text{ and } y = w, \\ 0, & \text{else,} \end{cases}$$

we already are in the situation that for any bilinear map $\beta \colon V \times W \to U$, we have one and only one linear map $\Phi \colon k^{(V\times W)} \to U$ such that $\Phi \circ \tau' = \beta$, since $\{f_{(v,w)} \mid v \in V, w \in W\}$ is a basis for $k^{(V\times W)}$. However, τ' is far from bilinear, which we want to forcibly correct in the following. Denote by D the linear subspace of $k^{(V\times W)}$ spanned by the elements

$$\begin{aligned}\{f_{(\alpha v_1+v_2,\beta w_1+w_2)} - \alpha\beta f_{(v_1,w_1)} - \alpha f_{(v_1,w_2)} - \beta f_{(v_2,w_1)} - f_{(v_2,w_2)} :\\ f \in k^{(V\times W)}, v_1, v_2 \in V, w_1, w_2 \in W, \alpha, \beta \in k\}\end{aligned} \qquad \text{(A.2)}$$

and put $T := T'/D$. Essentially, dividing out D makes every element of the form Eq. (A.2) a representant of the zero class, i.e., for every element of said form it now holds

$$[f_{(\alpha v_1+v_2,\beta w_1+w_2)}] = [\alpha\beta f_{(v_1,w_1)}] + [\alpha f_{(v_1,w_2)}] + [\beta f_{(v_2,w_1)}] + [f_{(v_2,w_2)}],$$

rendering $\tau := \pi \circ \tau'$ bilinear. Now one can check using Proposition A.2.7 that for any bilinear map $\beta \colon V \times W \to U$, the map $\phi = \pi \circ \Phi$ renders commutative the diagram

$$\begin{array}{ccccc} V \times W & \xrightarrow{\tau'} & k^{(V\times W)} & \xrightarrow{\pi} & k^{(V\times W)}/D \\ & \searrow^{\beta} & \downarrow{\Phi} & \swarrow_{\phi} & \\ & & U & & \end{array}$$

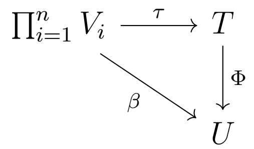
i.e., (T, τ) has the universal mapping property and thus is a tensor product of V and W. □

To work with tensor products, the construction is not essential. It even is probably best to not think of the tensor product as a set but in terms of the property mentioned in Proposition A.3.7 when making first contact with it. As for notation, the tensor product of V and W is denoted by $V \otimes_k W$ and the map τ is "forgotten". The image of $(v, w) \in V \times W$ under τ is denoted by $v \otimes w \in V \otimes_k W$. One can show that if $B = \{b_i\}_{i \in I} \subseteq V$ and $C = \{c_i\}_{i \in I} \subseteq W$ are bases of V respectively W, then $\{b_i \otimes c_i \mid i \in I\} \subseteq V \otimes_k W$ is a basis. Thus, if V and W are finite-dimensional, it holds $\dim V \otimes_k W = \dim V \cdot \dim W$.

It is crucial to remember that any element t of $V \otimes_k W$ has a (highly non-unique) representation $t = \sum_{i=1}^n v_i \otimes w_i$ with some integer n, $v_i \in V$ and $w_i \in W$ for $1 \leq i \leq n$, but that in general one won't find $v \in V$ and $w \in W$ such that $t = v \otimes w$. Elements t of $V \otimes_k W$ for which there are $v \in V$ and $w \in W$ with $t = v \otimes w$ are called *pure tensors*.

Having seen there is a tensor product of two k-vector spaces, we know the analogue statement for finitely many k-vector spaces as well.

Corollary A.3.8: *Let n be a natural number and let $V_1, \dots, V_n$ and U be k-vector spaces. There is a tuple (T, τ) consisting of a k-vector space T and an n-times multilinear map $\tau\colon \prod_{i=1}^n V_i \to T$ such that it holds: For any n-times multilinear map $\beta\colon \prod_{i=1}^n V_i \to U$, there is one and only one linear map $\Phi\colon T \to U$ rendering commutative the diagram*



A tensor product (T, τ) of $V_1, \dots, V_n$ is uniquely determined up to unique isomorphism and therefore called the tensor product of $V_1, \dots, V_n$, denoted by $\bigotimes_{i=1}^n V_i$. In the special case that $V_1 = \cdots = V_n = V$, we also write $T^n(V) := \bigotimes_{i=1}^n V$.

Note that it holds $\Phi(\bigotimes_{i=1}^n V_i) = \mathrm{Lin}(\beta(\prod_{i=1}^n V_i))$ for the map Φ from Corollary A.3.8, thus it can be understood as an extension of β.

Remembering the construction of the tensor product, we can (almost immediately) read off the following calculation rules for tensors: For vectors $v_1, \dots, v_n, v \in V$ and $\alpha \in k$ it holds $v_1 \otimes \cdots \otimes v_n = 0$ if and only if $v_i = 0$ for some $i \in \mathbb{N}_n$, $\alpha(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes \alpha v_i \otimes \cdots \otimes v_n$ for all $i \in \mathbb{N}_n$, and

$$\begin{aligned} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i + v \otimes v_{i+1} \otimes \cdots \otimes v_n \\ &= v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \\ &\quad + v_1 \otimes \cdots \otimes v_{i-1} \otimes v \otimes v_{i+1} \otimes \cdots \otimes v_n. \end{aligned}$$

The last two properties are obvious when remembering that we defined $v_1 \otimes \cdots \otimes v_n := \tau(v_1, \ldots, v_n)$. The first one is a bit more tricky: If one of the vectors is zero, then so is the tensor $v_1 \otimes \cdots \otimes v_n$. If we assume $v_i \neq 0$ for $i \in \mathbb{N}_n$, there were bases B_i of V_i with $v_i \in B_i$ and thus linear maps $\beta_i \colon V_i \to k$ with $\beta_i(v_i) = 1$. The map $\beta\colon \prod_{i=1}^n V_i \to k$, $(w_1, \ldots, w_n) \mapsto \prod_{i=1}^n \beta_i(w_i)$ were n-times multilinear with $\beta(v_1, \ldots, v_n) = 1$ and by the universal property of the tensor product, there was a linear map $\Phi\colon \bigotimes_{i=1}^n V_i \to k$ with $\Phi \circ \tau = \beta$, i.e., $\Phi(v_1 \otimes \cdots \otimes v_n) = 1$, thus $v_1 \otimes \cdots \otimes v_n \neq 0$.

What follows is a motivation, what tensor products are really good for.

Remark A.3.9: Using the universal mapping property from Proposition A.3.7, we can immediately see that for k-vector spaces V, W and U, we have the canonical identification $\mathfrak{m}(V, W; U) \cong \operatorname{Hom}_k(V \otimes_k W, U)$.

On the other hand, we can establish a canonical identification between $\mathfrak{m}(V, W; U)$ and $\operatorname{Hom}_k(V, \operatorname{Hom}_k(W, U))$: If we take $\beta \in \mathfrak{m}(V, W; U)$, by definition for every $v \in V$, we get linear maps $\beta_v \colon W \to U$, $w \mapsto \beta(v, w)$, i.e, β gives rise to a linear maping $\boldsymbol{\beta}\colon V \to \operatorname{Hom}(W, U)$, $v \mapsto \beta_v$.

Given $\phi \in \operatorname{Hom}_k(V, \operatorname{Hom}_k(W, U))$, we get a bilinear map $\beta \in \mathfrak{m}(V, W; U)$ via $(v, w) \mapsto [\phi(v)](w)$. In total we thus have

$$\mathfrak{m}(V, W; U) \cong \operatorname{Hom}_k(V \otimes_k W, U) \cong \operatorname{Hom}_k(V, \operatorname{Hom}_k(W, U)).$$

Analogously, it holds

$$\begin{aligned}\mathfrak{m}(V_1, \ldots, V_n; U) &\cong \operatorname{Hom}_k(V_1 \otimes_k \cdots \otimes_k V_n, U)\\ &\cong \operatorname{Hom}_k(V_1, \operatorname{Hom}_k(V_2, \ldots, \operatorname{Hom}_k(V_n, U) \cdots)).\end{aligned}$$

This is an outstanding property of the tensor product.

The universal mapping property of the tensor product also allows for the construction of certain linear maps of tensor products from linear maps of “regular” vector spaces.

Remark A.3.10: Let V_1, V_2, W_1, W_2 be k-vector spaces and let $\varphi\colon V_1 \to W_1$ and $\psi\colon V_2 \to W_2$ be k-vector space homomorphisms. Then

$$\varphi \times \psi\colon V_1 \times V_2 \longrightarrow W_1 \times W_2, \qquad (v_1, v_2) \longmapsto (\varphi(v_1), \psi(v_2))$$

is a bilinear map. This gives us the commutative diagram

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{\tau_1} & V_1 \otimes_k V_2 \\ {\scriptstyle \varphi\times\psi}\downarrow & \searrow^{\tau_2\circ(\varphi\times\psi)} & \downarrow{\scriptstyle \exists!\,\Phi} \\ W_1 \times W_2 & \xrightarrow[\tau_2]{} & W_1 \otimes_k W_2 \end{array}$$

with $\Phi\colon V_1 \otimes_k V_2 \to W_1 \otimes_k W_2$ being linear and satisfying

$$\Phi(v_1 \otimes v_2) = \varphi(v_1) \otimes \psi(v_2).$$

This map Φ we call $\varphi \otimes \psi$. If the involved vector spaces are finite-dimensional, $\varphi \otimes \psi$ is injective (surjective) if and only if φ and ψ are injective (surjective).

Completely analogously, if we take k-vector spaces $V_1, \dots, V_n$; $W_1, \dots, W_n$ and linear maps $\varphi_i\colon V_i \to W_i$ for $1 \leq i \leq n$, we get a linear map

$$\varphi_1 \otimes \cdots \otimes \varphi_n\colon \bigotimes_{i=1}^n V_i \longrightarrow \bigotimes_{i=1}^n W_i$$
$$\text{with} \quad (\varphi_1 \otimes \cdots \otimes \varphi_n)(v_1 \otimes \cdots \otimes v_n) = \varphi_1(v_1) \otimes \cdots \otimes \varphi_n(v_n).$$

Given a map $\varphi\colon V \to W$ and a natural number m, we denote by $T^m(\varphi)$ the map declared via

$$T^m(\varphi) := \varphi \otimes \cdots \otimes \varphi\colon T^m(V) \longrightarrow T^m(W),$$
$$v_1 \otimes \cdots \otimes v_m \longmapsto \varphi(v_1) \otimes \cdots \otimes \varphi(v_n).$$

We want to finish this section with some useful facts regarding tensor products of vector spaces that involve dual spaces.

Remark A.3.11: For this remark, let k be a field.

(i) Let V and W be two k-vector spaces. Then

$$\beta\colon V^* \otimes W \longrightarrow \operatorname{Hom}_k(V, W), \qquad \varphi \otimes w \longmapsto (v \mapsto \varphi(v)w)$$

defines an injective vector space homomorphism. It is bijective if and only if V or W is finite-dimensional.

(ii) Let $V_1, \dots, V_n$ be k-vector spaces. Then

$$\iota\colon \bigotimes_{i=1}^n V_i^* \longrightarrow \Big(\bigotimes_{i=1}^n V_i\Big)^*, \qquad \iota(\varphi^1 \otimes \cdots \otimes \varphi^n)(v_1 \otimes \cdots \otimes v_n) := \prod_{i=1}^n \varphi^i(v_i)$$

defines an injective vector space homomorphism. It is an isomorphism if the vector spaces $V_1, \dots, V_n$ are finite-dimensional.

Lemma A.3.12: *Let n be a natural number and let $\varphi_i\colon V_i \to W_i$, $1 \leq i \leq n$ be linear maps between finite-dimensional k-vector spaces. Furthermore, let $\iota_V\colon \bigotimes_{i=1}^n V_i^* \to (\bigotimes_{i=1}^n V_i)^*$ and $\iota_W\colon \bigotimes_{i=1}^n W_i^* \to (\bigotimes_{i=1}^n W_i)^*$ be the isomorphisms from Remark A.3.11(ii). Then, it holds*

$$(\varphi_1 \otimes \cdots \otimes \varphi_n)^* = \iota_V^{-1} \circ (\varphi_1^* \otimes \cdots \otimes \varphi_n^*) \circ \iota_W.$$

Proof: We have to show that the square

$$\begin{array}{ccc} \bigotimes_{i=1}^n W_i^* & \xrightarrow{\varphi_1^*\otimes\cdots\otimes\varphi_n^*} & \bigotimes_{i=1}^n V_i^* \\ {\scriptstyle \iota_W}\big\downarrow & & \big\downarrow{\scriptstyle \iota_V} \\ (\bigotimes_{i=1}^n W_i)^* & \xrightarrow[(\varphi_1\otimes\cdots\otimes\varphi_n)^*]{} & (\bigotimes_{i=1}^n W_i^*)^* \end{array}$$

commutes. Let therefore $\psi^1 \otimes \cdots \otimes \psi^n \in \bigotimes_{i=1}^n W_i^*$. We want to show that

$$\begin{aligned} &(\varphi_1 \otimes \cdots \otimes \varphi_n)^* \circ (\iota_W(\psi^1 \otimes \cdots \otimes \psi^n)) \\ &\qquad\qquad = (\varphi_1^* \otimes \cdots \otimes \varphi_n^*) \circ \iota_V(\psi^1 \otimes \cdots \otimes \psi^n)). \end{aligned}$$

Let thus $v_1 \otimes \cdots \otimes v_n \in \bigotimes_{i=1}^n V_i$. Then

$$\begin{aligned} &[(\varphi_1 \otimes \cdots \otimes \varphi_n)^* \circ (\iota_W(\psi^1 \otimes \cdots \otimes \psi^n))](v_1 \otimes \cdots \otimes v_n) \\ &\quad = [(\iota_V(\psi^1 \otimes \cdots \otimes \psi^n)) \circ (\varphi_1 \otimes \varphi_n)](v_1 \otimes \cdots \otimes v_n) \\ &\quad = (\iota_V(\psi^1 \otimes \cdots \otimes \psi^n))(\varphi_1(v_1) \otimes \cdots \otimes \varphi_n(v_n)) \\ &\quad = \psi^1(\varphi_1(v_1)) \cdots \psi^n(\varphi_n(v_n)) \\ &\quad = \iota_V((\psi^1 \circ \varphi_1) \otimes \cdots \otimes (\psi^n \circ \varphi_n))(v_1 \otimes \cdots \otimes v_n) \\ &\quad = [(\iota_V \circ (\varphi_1^* \otimes \ldots \varphi_n^*))(\psi^1 \otimes \cdots \otimes \psi^n)](v_1 \otimes \cdots \otimes v_n). \end{aligned}$$

Since ι_V and ι_W are isomorphisms, this shows the claim. □

4. Euclidean and unitary spaces

This section deals with pre-Hilbert spaces, i.e., vector spaces over $\mathbb{K}$ in which we have a concept of angles between vectors and lengths of vectors, what makes them interesting for analysis as well, and in particular with finite-dimensional pre-Hilbert spaces which are called euclidean respectively unitary spaces depending on the underlying field.

Many of the facts presented here hold more generally for Hilbert spaces, which is why the presentation is made in a way that allows for an easy transfer.

The main goals of this section are the introduction of the associated structure preserving maps, a description of their transformation matrices, the concept of orthogonal systems, the Riesz Representation Theorem and as a consequence, the existence of the adjoint map.

Definition A.4.1 (Inner product): Let V be a $\mathbb{K}$-vector space and let

$$(\cdot|\cdot)\colon V \times V \longrightarrow \mathbb{K}$$

be a map. If for all $v, w \in V$ and $\alpha, \beta \in \mathbb{K}$ it holds

(i) $(v|v) \geq 0$ and $(v|v) = 0$ if and only if $v = 0$,

(ii) $(v|w) = (w|v)^*$,

(iii) For fixed $w \in V$, the map $v \mapsto (v|w)$ is linear,

the map $(\cdot|\cdot)$ is called an *inner product*. The tuple $(V, (\cdot|\cdot))$ is then called *inner product space* or *pre-Hilbert space*. If V is finite-dimensional and $\mathbb{K} = \mathbb{R}$, V is called an *euclidean space*, if V is finite-dimensional and $\mathbb{K} = \mathbb{C}$, V is called *unitary space*.

Note that $(v|w) = (w|v)^*$ ensures that $(v|v) \in \mathbb{R}$ for all $v \in V$, so that condition (i) of Definition A.4.1 can make sense. A direct consequence of the definition is that for all $v, w_1, w_2 \in V$ and $\alpha, \beta \in \mathbb{K}$ it holds

$$\begin{aligned}(v|\alpha w_1 + \beta w_2) &= (\alpha w_1 + \beta w_2|v)^* \\ &= [\alpha(w_1|v) + \beta(w_2|v)]^* \\ &= \alpha^*(w_1|v)^* + \beta^*(w_2|v)^* = \alpha^*(v|w_1) + \beta^*(v|w_2),\end{aligned}$$

i.e., for fixed $v \in V$, the map $w \mapsto (v|w)$ is antilinear or conjugate-linear.

Remark A.4.2: Let $(V, (\cdot|\cdot))$ be a pre-Hilbert space and let $v_1, v_2 \in V$ be two vectors. If it holds $(v_1|v) = (v_2|v)$ (and thus $(v|v_1) = (v|v_2)$) for all $v \in V$, then $v_1 = v_2$. Indeed, since by assumption we have $(v_1|v_1 - v_2) = (v_2|v_1 - v_2)$, it holds $0 = (v_1 - v_2|v_1 - v_2)$ and thus $v_1 = v_2$.

Example A.4.3: The vector space $\mathbb{K}^n$ becomes an euclidean respectively unitary space via the inner product

$$(\cdot|\cdot)\colon \mathbb{K}^n \times \mathbb{K}^n \longrightarrow \mathbb{K}, \qquad \Big(\sum_{i=1}^n \xi^i e_i, \sum_{j=1}^n \eta^j e_j\Big) \longmapsto \sum_{i=1}^n \xi^i (\eta^j)^*.$$

This inner product is called the *standard inner product* on $\mathbb{K}^n$.

Definition A.4.4 (Normed vector space): Let V be a $\mathbb{K}$-vector space and let

$$\|\cdot\|\colon V \times V \longrightarrow \mathbb{K}$$

be a map. If for all $v, w \in V$ and $\alpha \in \mathbb{K}$ it holds

(i) $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$,

(ii) $\|\alpha v\| = |\alpha|\|v\|$,

(iii) $\|v + w\| \leq \|v\| + \|w\|$,

the map $\|\cdot\|$ is called a *norm*. The tuple $(V, \|\cdot\|)$ is then called a *normed* $\mathbb{K}$-*vector space*. Without the second part of (i), $\|\cdot\|$ is called a *seminorm*.

Remark A.4.5 (Cauchy-Schwarz inequality): Let V be a pre-Hilbert space over $\mathbb{K}$. Then for all $v, w \in V$, it holds

$$(v|w)^2 \leq (v|v)(w|w). \tag{A.3}$$

A proof can be found in any book on functional analysis or in any good linear algebra book.

The Cauchy-Schwarz inequality is essential to see that pre-Hilbert spaces are special normed vector spaces: Let $(V, (\cdot|\cdot))$ be a pre-Hilbert space over $\mathbb{K}$. Since $(v|v) \geq 0$ for all $v \in V$,

$$\|\cdot\|: V \longrightarrow \mathbb{K}, \qquad v \longmapsto (v|v)^{1/2}$$

is well defined. Conditions (i) and (ii) from Definition A.4.4 are obviously met and with Eq. (A.3), it is easy to see that (iii) holds for $\|\cdot\|$ as well, thus $\|\cdot\|$ makes V into a normed $\mathbb{K}$-vector space.

Additionally, the Cauchy-Schwarz inequality allows for a definition of angles in pre-Hilbert spaces over $\mathbb{K}$: Since for all $v, w \in V$ it holds

$$-1 \leq \frac{(v|w)}{\|v\|\|w\|} \leq 1,$$

there is one and only one $\alpha \in [0, \pi]$ with $\cos(\alpha) = (v|w)/(\|v\|\|w\|)$. This α we call the measured angle between v and w, sometimes denoted $\measuredangle(v, w)$.

Definition A.4.6 (Orthogonality): Let $(V, (\cdot|\cdot))$ be a pre-Hilbert space over $\mathbb{K}$ and let $v, w \in V$. If $(v|w) = 0$, the vectors v and w are called *orthogonal*.

Let $S, T \subseteq V$ be subsets. If it holds: For any $s \in S$, $(s|t) = 0$ for all $t \in T$, the sets S and T are called *orthogonal*.

The set $S^\perp := \{v \in V \mid (v|s) = 0 \text{ for all } s \in S\}$ is called *orthogonal complement of* S.

Note that for any set $S \subseteq V$, the set $S^\perp \subseteq V$ is a linear subspace. We will later see that even for a linear subspace $S \subseteq V$, $S^{\perp\perp} := (S^\perp)^\perp \neq S$ in general — we only have $S \subseteq S^{\perp\perp}$.

As pre-Hilbert spaces are special cases of normed vector spaces, one might ask if one can determine, when a normed vector space $(V, \|\cdot\|)$ is in fact a pre-Hilbert space, i.e., if it's norm was induced by an inner product. Indeed, one can determine if this is the case: If for all $v, w \in V$ it holds

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2),$$

the norm is induced by an inner product. The above equation is called *parallelogram identity*.

In case the norm is induced by an inner product, one even can recover the inner product from the norm via the so-called *polarisation identity*:

$$(v|w) = \begin{cases} \frac{1}{4}(\|v + w\|^2 + \|v - w\|^2), & \text{if } \mathbb{K} = \mathbb{R}, \\ \frac{1}{4}(\|v + w\|^2 + \|v - w\|^2 + \mathrm{i}\|v - \mathrm{i}w\|^2 - \mathrm{i}\|v + \mathrm{i}w\|^2), & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$

Definition A.4.7 (Orthonormal system): Let $(V, (\cdot|\cdot))$ be a $\mathbb{K}$ pre-Hilbert space, let I be an index set and let $S = \{s_i \mid i \in I\} \subseteq V$ be a subset. If for $i, j \in I$ with $i \neq j$ it holds $(s_i|s_j) = 0$ and $(s_i|s_i) > 0$ for $i \in I$, the set S is called an *orthogonal system*. The set S is called *orthonormal system*, if for $i, j \in I$ it holds $(s_i|s_j) = \delta_{i,j}$.

Remark A.4.8: Orthogonal systems are linearly independent. Indeed, let $S = \{s_i \mid i \in I\} \subseteq V$ be an orthogonal system and $0 = \sum_{i=1}^n \alpha^i s_i$. Then it holds

$$0 = \Big(\sum_{i=1}^n \alpha^i s_i \Big| \sum_{j=1}^n \alpha^j s_j \Big) = \sum_{i=1}^n \sum_{j=1}^n \alpha^i (\alpha^j)^* (s_i|s_j) = \sum_{i=1}^n |\alpha^i|^2 (s_i|s_i),$$

i.e., $\alpha^1 = \cdots = \alpha^n = 0$.

Definition A.4.9: Let V be an euclidean respectively unitary space with $\dim V = n$. An orthogonal system $B = \{b_1, \ldots, b_n\} \subseteq V$ is called *orthogonal basis*, an orthonormal system $B = \{b_1, \ldots, b_n\}$ is called an *orthonormal basis*.

Remark A.4.10 (Fourier expansion): If $(V, (\cdot|\cdot))$ is an euclidean or unitary space with orthonormal basis $\{b_1, \ldots, b_n\}$, it holds $v = \sum_{i=1}^n (v|b_i) b_i$ for all $v \in V$. To see this, let $v = \sum_{i=1}^n \lambda^i b_i \in V$. Then we have

$$(v|b_i) = \Big(\sum_{j=1}^n \lambda^j b_j \Big| b_i \Big) = \sum_{j=1}^n \lambda^j (b_j|b_i) = \lambda^i.$$

Definition A.4.11 (Isometry): Let $(V, (\cdot|\cdot))$ be a pre-Hilbert space over $\mathbb{K}$ and let $\|\cdot\|$ be the induced norm. A linear map $\varphi\colon V \to V$ that for all $v \in V$ satisfies $\|\varphi(v)\| = \|v\|$ is called an *isometry*. If φ is in addition surjective, it is called *unitary*.

If V is finite-dimensional, isometries are also called *orthogonal* if $\mathbb{K} = \mathbb{R}$ respectively *unitary*, if $\mathbb{K} = \mathbb{C}$.

Isometries are the structure preserving maps for pre-Hilbert spaces. Note that in the setting of Definition A.4.11, the seemingly weaker condition "For all $v \in V$ it holds $\|\varphi(v)\| = \|v\|$" is equivalent to "For all $v, w \in V$ it holds $(\varphi(v)|\varphi(w)) = (v|w)$". This follows from the polarisation identity. To this end, isometries preserve the linear structure, norms as well as angles and thus all structure of inner product spaces.

Remark A.4.12: Let $(V, (\cdot|\cdot))$ be a pre-Hilbert space and let $\varphi\colon V \to V$ be an isometry. Then φ is injective. Indeed, let $v, w \in V$ with $\varphi(v) = \varphi(w)$. Then

$$0 = \|\varphi(v) - \varphi(w)\| = \|\varphi(v - w)\| = \|v - w\|,$$

i.e., $v = w$.

If V is euclidean or unitary, φ is thus automatically bijective (by Example A.2.8) and one can check that φ^{-1} is an isometry as well. For general pre-Hilbert spaces, one can give counterexamples so that the requirement for surjectivity is indeed necessary.

Example A.4.13: Denote by $(\mathbb{K}^n, (\cdot|\cdot)_{\mathbb{K}^n})$ the coordinate vector space equipped with the canonical inner product and let $A \in M_n(\mathbb{K})$ be a matrix. Denote by $A^\dagger$ defined via $(A^\dagger)^i_j := (A^j_i)^*$ the so called *hermitian transpose* or *conjugate transpose* of A and by A^* defined via $(A^*)^i_j := (A^i_j)^*$ the conjugate of A.

The linear map $\xi \mapsto A\xi$ is orthogonal respectively unitary if and only if for all $\xi = \sum_{i=1}^n \xi^i e_i$, $\eta = \sum_{j=1}^n \eta^j e_j$ it holds

$$(\xi|\eta) = \xi^t \eta^* = \xi^t A^t A^* \eta^* = (A\xi)^t (A\eta)^* = (A\xi|A\eta).$$

This is the case if and only if $A^t A^* = I_n$, which is the case if and only if $A^\dagger A = I_n$.

Lemma A.4.14: *Let $(V, (\cdot|\cdot)_V)$ be a euclidean or unitary space with orthonormal basis $B = (b_1, \dots, b_n)$ and let $\varphi\colon V \to V$ be orthogonal respectively unitary. Then, for the transformation matrix $A := D_{B,B}(\varphi)$ it holds $A^\dagger A = I_n$, i.e., the map $\xi \mapsto A\xi$ is orthogonal respectively unitary.*

Proof: Again, denote by $(\mathbb{K}^n, (\cdot|\cdot)_{\mathbb{K}^n})$ the coordinate vector space equipped with the canonical inner product. Note that since B is an orthonormal basis of V, for $v = \sum_{i=1}^n v^i b_i$ and $w = \sum_{j=1}^n w^j b_j \in V$ it holds

$$(v|w)_V = \left(\sum_{i=1}^n v^i b_i \middle| \sum_{j=1}^n w^j b_j\right)_V = \sum_{i=1}^n v^i (w^i)^* = (D_B(v)|D_B(w))_{\mathbb{K}^n},$$

i.e., D_B is an isometry. Thus, also D_B^{-1} is an isometry. Using the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V \\ {\scriptstyle D_B}\downarrow & & \downarrow{\scriptstyle D_B} \\ \mathbb{K}^n & \xrightarrow[A]{} & \mathbb{K}^n \end{array}$$

we can check that $\xi \mapsto A\xi$ is orthogonal respectively unitary. For $1 \leq i, j \leq n$ it holds

$$\begin{aligned}(Ae_i|Ae_j)_{\mathbb{K}^n} &= (D_B[\varphi D_B^{-1}(e_i)]|D_B[\varphi D_B^{-1}(e_j)])_{\mathbb{K}^n} \\ &= (D_B[\varphi(b_i)]|D_B[\varphi(b_j)])_{\mathbb{K}^n} = (\varphi(b_i)|\varphi(b_j))_V = \delta_{i,j} = (e_i|e_j)_{\mathbb{K}^n},\end{aligned}$$

which we wanted to see. □

Corollary A.4.15 (on spotting isometries): *Let $A \in M_n(\mathbb{K})$. The matrix A corresponds to a structure preserving map of euclidean respectively unitary spaces if and only if one of the following holds:*

(i) $A^\dagger A = I_n$,

(ii) *The columns of A form an orthonormal basis of $\mathbb{K}^n$,*

(iii) *The rows of A form an orthonormal basis of $\mathbb{K}^n$.*

Definition A.4.16: Let n be a natural number. The sets

$$O_n := \{A \in M_n(\mathbb{R}) \mid A^t A = I_n\} \subseteq \mathrm{Gl}_n(\mathbb{R}),$$
$$U_n := \{A \in M_n(\mathbb{C}) \mid A^\dagger A = I_n\} \subseteq \mathrm{Gl}_n(\mathbb{C})$$

are called *orthogonal group* respectively *unitary group*. Both sets are indeed groups with matrix multiplication as law of composition.

We now turn to dual spaces of euclidean or unitary spaces.

Definition A.4.17: Let $(V, (\cdot|\cdot))$ be a pre-Hilbert space over $\mathbb{K}$ and let $v \in V$. Then, the map

$$f_v \colon V \longrightarrow \mathbb{K}, \qquad w \longmapsto (w|v)$$

is linear. We call f_v the *functional associated to* v.

As it turns out, for euclidean and unitary spaces as well as Hilbert spaces over $\mathbb{K}$, all functionals on those spaces are of this form, i.e., there is an isomorphism from V to V^*. This is the famous Riesz representation theorem.

Theorem A.4.18 (Riesz representation theorem): *Let* $(V, (\cdot|\cdot))$ *be a finite-dimensional pre-Hilbert space over* $\mathbb{K}$*. Then the map*

$$j \colon V \longrightarrow V^*, \qquad v \longmapsto f_v$$

is an anti-linear isomorphism, i.e., $V \cong V^*$ *canonically.*

To see this, we only have to convince ourselves that j is injective, the rest is down to dimension. Let thus $v, w \in V$ with $f_v = f_w$, i.e., for all $u \in V$ it holds $f_v(u) = (u|v) = (u|w) = f_w(u)$. Then Remark A.4.2 ensures that $v = w$ and thus j is injective. Anti-linearity is easily checked.

The Riesz representation theorem provides the means for the definition of a very important linear map to a given linear map, the so called *adjoint map*.

Definition A.4.19 (Adjoint map): Let V and W be euclidean or unitary spaces and let $\varphi \colon V \to W$ be a linear map. Then there is one and only one linear map $\varphi^\dagger \colon W \to V$ such that for all $v \in V$ and $w \in W$ it holds

$$(\varphi(v)|w) = (v|\varphi^\dagger(w)).$$

Fix $w \in W$ and define

$$\alpha \colon V \longrightarrow \mathbb{K}, \qquad v \longmapsto (\varphi(v)|w).$$

The map α is linear, since the inner product of W is linear in the first component and φ is linear. By Theorem A.4.18, there is some vector $u_w \in V$ such that $\alpha = f_{u_w}$, i.e., $(\varphi(v)|w) = (v|u_w)$ for all $v \in V$. We now define $\varphi^\dagger(w) := u_w$. The assignment $w \mapsto \varphi^\dagger(w)$ is linear, since for all $\lambda, \mu \in \mathbb{K}$, $v \in V$ and $w_1, w_2 \in W$ we have

$$\begin{aligned}(v|\varphi^\dagger(\lambda w_1 + \mu w_2)) &= (\varphi(v)|\lambda w_1 + \mu w_2) \\ &= \lambda^*(\varphi(v)|w_1) + \mu^*(\varphi(v)|w_2) = (v, \lambda\varphi^\dagger(w_1) + \mu\varphi^\dagger(w_2)).\end{aligned}$$

Since we use Theorem A.4.18 to define $\varphi^\dagger$, the uniqueness is clear.

For a given linear map $\varphi\colon V \to W$ between euclidean or unitary spaces, we now have the maps $\varphi^*\colon W^* \to V^*$ and $\varphi^\dagger\colon W \to V$, who are linked in the following way: The diagram

$$\begin{array}{ccc} W & \xrightarrow{\varphi^\dagger} & V \\ {\scriptstyle j_W}\big\downarrow & & \big\downarrow{\scriptstyle j_V} \\ W^* & \xrightarrow[\varphi^*]{} & V^* \end{array}$$

commutes, since for all $v \in V$ and $w \in W$ we have

$$\begin{aligned} (\varphi^*(j_W(w))(v) &= (\varphi^*(f_w))(v) \\ &= f_w(\varphi(v)) \\ &= (\varphi(v)|w) = (v|\varphi^\dagger(w)) = f_{\varphi^\dagger(w)}(v) = (j_V(\varphi^\dagger(w))(v), \end{aligned}$$

i.e., we have $\varphi^\dagger = j_V^{-1} \circ \varphi^* \circ j_W$ respectively $\varphi^* = j_V \circ \varphi^\dagger \circ j_W^{-1}$.

Remark A.4.20 (Tensor product of adjoint maps): Let $\varphi_1\colon V_1 \to W_1$ and $\varphi_2\colon V_2 \to W_2$ be linear maps between finite-dimensional Hilbert spaces over $\mathbb{K}$. Then, for all $v_1 \otimes v_2 \in V_1 \otimes_{\mathbb{K}} V_2$ and $w_1 \otimes w_2 \in W_1 \otimes_{\mathbb{K}} W_2$, it holds

$$\begin{aligned} ((\varphi_1 \otimes \varphi_2)(v_1 \otimes v_2)|w_1 \otimes w_2) &= (\varphi_1(v_1) \otimes \varphi_2(v_2)|w_1 \otimes w_2) \\ &= (\varphi_1(v_1)|w_1)(\varphi_2(v_2)|w_2) \\ &= (v_1|\varphi_1^\dagger(w_1))(v_2|\varphi_2^\dagger(w_2)) \\ &= (v_1 \otimes v_2|\varphi_1^\dagger(w_1) \otimes \varphi_2^\dagger(w_2)) \\ &= (v_1 \otimes v_2|(\varphi_1^\dagger \otimes \varphi_2^\dagger(w_2)), \end{aligned}$$

i.e., $(\varphi_1 \otimes \varphi_2)^\dagger = \varphi_1^\dagger \otimes \varphi_2^\dagger$.

Appendix B.

Essentials from analysis

In this chapter, we want to recap some fundamental concepts from analysis, in particular spaces in which we can practice analysis, convergence and continuity, differentiability, submainfolds of $\mathbb{R}^n$ for some natural number n, basics on Hilbert spaces and operators on Hilbert spaces, Hilbert space tensor products, basics on C^*-algebras and some types of tensor products of C^*-algebras.

1. Convergence and continuity

In this section, we deal with the concept of convergence, a concept of approximation, and continuity, a property of maps between spaces in which we can practice analysis that behaves well with continuity.

As for the spaces, the most specific setting is the setting of pre-Hilbert spaces over $\mathbb{K}$ (refer to Definition A.4.1), which we already have seen to be a special case of the setting of normed vector spaces over $\mathbb{K}$ (refer to Definition A.4.4). Linear subspaces of pre-Hilbert spaces or normed vector spaces are themselves pre-Hilbert spaces respectively normed vector spaces with the restriction of the inner product respectively the norm to the linear subspace.

Normed vector spaces over $\mathbb{K}$ are easily identified as a special case of so called metric spaces:

Definition B.1.1 (Metric space): Let X be a set and let

$$d\colon X \times X \longrightarrow \mathbb{R}^+, \qquad (x,y) \longmapsto d(x,y)$$

be a map. If for all $x, y, z \in X$ it holds

(i) $d(x,x) = 0$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$,

the tuple (X, d) is called a *metric space*, the map d is called *metric* on X.

Note that there is no special structure presupposed on the set X. If we take a normed vector space $(V, \|\cdot\|)$ over $\mathbb{K}$, we can make it into a metric space by defining

$$d \colon V \times V \longrightarrow \mathbb{R}^+, \qquad (x, y) \longmapsto \|x - y\|.$$

In metric spaces, we can give a quite intuitive introduction to the concepts of convergence and continuity.

Definition B.1.2 (Convergence, Continuity): Let (X, d) be a metric space and x be a point in X. A map $x \colon \mathbb{N} \to X$ is identified with the family $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, where $x_n := x(n)$, and is called *sequence*. The image $x(n)$ for $n \in \mathbb{N}$ is called *term of the sequence*.

The sequence $(x_n)_{n\in\mathbb{N}}$ *converges to* x, if for all errors $\varepsilon > 0$ there is some natural number N such that $d(x_n, x) < \varepsilon$ for all natural numbers $n \geq N$. The point x is then called *limit* and we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

The sequence $(x_n)_{n\in\mathbb{N}}$ in X is *convergent*, if there is some limit for this sequence.

The sequence $(x_n)_{n\in\mathbb{N}}$ is called a *Cauchy sequence*, if for all errors $\varepsilon > 0$ there is some natural number N such that $d(x_n, x_m) < \varepsilon$ for all natural numbers $n, m \geq N$.

Let (Y, d') be another metric space. A map $f \colon X \to Y$ is said to be *sequentially continuous in* x, if for any convergent sequence $(x_n)_{n\in\mathbb{N}}$ in X with limit x it holds: $f(x)$ is the limit of the sequence $(f(x_n))_{n\in\mathbb{N}}$ in Y.

If f is sequentially continuous in every point in X, f is called *sequentially continuous*.

A convergent sequence "approximates" a certain point in our metric space, since we can undercut any "error", i.e., distance to said point: For any error, we find an index such that any term with index at least as high is closer to said point. The properties of the metric grant that limits of sequences are unique, i.e., if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $x_n \to x$ and $x_n \to y$, then $x = y$. Note that convergent sequences are always Cauchy sequences.

Cauchy sequences are significant sequences when talking about a metric space: The definition looks like Cauchy sequences should always converge but in fact, they do not in general and it is a property of the metric space, if they always do. If Cauchy sequences always converge in a given metric space, the space is called *complete*. To give an example, $\mathbb{Q}$ becomes a metric space with $d(x, y) := |x - y|$, where $|\cdot|$ denotes the usual absolute value on $\mathbb{Q}$, and one can construct Cauchy sequences with rational terms that converge to irrational numbers, namely certain roots of rational numbers (look up "Babylonian method"). Since the limits are not contained in $\mathbb{Q}$, the constructed sequences do not converge in $\mathbb{Q}$, but they do in a "bigger" metric space, namely $\mathbb{R}$ (with distance $d(x, y) := |x - y|$).

Definition B.1.3 (Complete metric spaces): Let (X, d) be a metric space. If every Cauchy sequence in X converges, X is called *complete*.

If $(V, \|\cdot\|)$ is a normed vector space that is complete as metric space with the induced metric $d(x, y) := \|x - y\|$, V is called a *Banach space*.

If $(V, (\cdot|\cdot))$ is a pre-Hilbert space that is complete as metric space with the metric induced by the induced norm $\|v\| := (v|v)^{1/2}$, V is called a *Hilbert space*.

Sequentially continuous maps between metric spaces behave well with convergence. We can express the definition of sequential continuity in the following terms: If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in (X, d) with limit x, then

$$f(\lim_{n\in\mathbb{N}} x_n) = f(x) = \lim_{n\in\mathbb{N}} f(x_n),$$

i.e., taking the limit and applying f interchange.

The reader might be more acquainted with the definition of continuity due to Weierstraß:

Definition B.1.4 (Continuity): Let (X, d) and (Y, d') be metric spaces, let x be a point in X and let $f: X \to Y$ be a map. The map f is called *continuous in* x, if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d'(f(x), f(y)) < \varepsilon$.

If f is continuous in every point in X, f is called *continuous*.

Luckily, continuity due to Weierstraß and sequential continuity are equivalent concepts in metric spaces.

Lemma B.1.5: *Let (X, d) and (Y, d') be metric spaces, let $f: X \to Y$ be a map and let x be a point in X. Then f is continuous in x if and only if f is sequentially continuous in x.*

Definition B.1.6 (Metric subspace): Let (X, d) be a metric space and Y be a subset of X. Then Y turns into a metric space itself with the restriction

$$d|_{Y \times Y} \colon Y \times Y \longrightarrow \mathbb{R}^+.$$

The restriction $d|_{Y \times Y}$ is called *induced metric* and the tuple $(Y, d|_{Y \times Y})$ is called *metric subspace of* (X, d).

For the sake of completeness, we include the definition of structure preserving maps of metric spaces.

Definition B.1.7 (Isometry): Let (X, d) and (Y, d') be metric spaces and let $f \colon X \to Y$ be a map. If for all $x, y \in X$ it holds

$$d'(f(x), f(y)) = d(x, y),$$

f is called an *isometry*.

We already had an encounter with (special) isometries (see Definition A.4.11) that in fact are isometries as maps between the underlying metric spaces as well. With a similar argument to the one used in Remark A.4.12 we see that isometries are injective. Obviously, isometries are sequentially continuous.

If we have an isometry $\iota \colon X \to Y$, we can identify X with $\iota(X) \subseteq Y$ and thus regard X as a metric subspace of Y. Often we hence write $X \subseteq Y$ in this case, which is clear abuse of notation. The reader might have done so himself at some point when writing "$\mathbb{Q} \subseteq \mathbb{R}$".

With a bit of effort, we can identify metric spaces to be a special case of so called topological spaces:

Definition B.1.8 (Topological space): Let X be a set and let $\mathfrak{T}$ be a subset of the powerset of X, i.e., $\mathfrak{T} \subseteq \mathfrak{P}(X) := \{Y \subseteq X\}$. If it holds

(i) $\varnothing \in \mathfrak{T}$, $X \in \mathfrak{T}$,

(ii) For two sets $U, V \in \mathfrak{T}$ it holds $U \cap V \in \mathfrak{T}$,

(iii) For a family of subsets $(U_i)_{i \in I} \in \mathfrak{T}^I$ it holds $\bigcup_{i \in I} U_i \in \mathfrak{T}$,

$\mathfrak{T}$ is called a *topology on* X and the tuple $(X, \mathfrak{T})$ is called a *topological space*. The sets in $\mathfrak{T}$ are called *open*, the relative complements of open sets in X are called *closed*. Elements of topological spaces are often refered to as *points*.

For any $x \in X$, a set $V \subseteq X$ is called a *neighbourhood of* x, if there is an open set U with $x \in U \subseteq V$. If V is open itself, V is called an *open neighbourhood of* x. By $\mathfrak{U}(x)$ we denote the set of *neighbourhoods of* x.

A topological space $(X, \mathfrak{T})$ is called *Hausdorff topological space*, if for any two $x, y \in X$ with $x \neq y$ there are neighbourhoods $U \in \mathfrak{U}(x)$, $Y \in \mathfrak{U}(y)$ with $U \cap V = \varnothing$.

A subset $Y \subseteq X$ turns into a topological space with the topology

$$\mathfrak{T}' := \{U \mid U = U' \cap Y \text{ with } U' \in \mathfrak{T}\},$$

the so called *relative topology.* The tuple $(Y, \mathfrak{T}')$ is called *subspace of* $(X, \mathfrak{T})$.

To see this, we first need to define special neighbourhoods in metric spaces.

Definition B.1.9 (Balls): Let (X, d) be a metric space, let $x \in X$ be a point and let $r > 0$. We define the sets

$$B(x, r) := \{y \in X \mid d(x, y) < r\} \subseteq X,$$
$$\mathrm{cl}(B(x, r)) := \{y \in X \mid d(x, y) \leq r\}.$$

The set $B(x, r)$ is called the *open ball around x with radius r*, $\mathrm{cl}(B(x, r))$ is called the *closed ball around x with radius r.*

Proposition B.1.10: *Let (X, d) be a metric space. Then*

$$\mathfrak{T} := \{U \subseteq X \mid \textit{For all } u \in U \textit{ there is } \varepsilon > 0 \textit{ with } B(u, \varepsilon) \subseteq U\} \subseteq \mathfrak{P}(X)$$

is topology on X, and $(X, \mathfrak{T})$ even is a Hausdorff topological space.

Proof: First of all, we check that $\mathfrak{T}$ is a topology.

(i) It holds $\varnothing \in \mathfrak{T}$, as the statement is trivially true. We have $X \in \mathfrak{T}$, since by definition $B(x, \varepsilon) \subseteq X$ for all $x \in X$.

(ii) If $U, V \in \mathfrak{T}$, then $U \cap V \in \mathfrak{T}$ even more so.

(iii) Let $(U_i)_{i \in I} \in \mathfrak{T}^I$ and $U := \bigcup_{i \in I} U_i$. For $u \in U$, by definition there is some $i \in I$ with $u \in U_i$. As $U_i \in \mathfrak{T}$, there is some $\varepsilon > 0$ with $B(u, \varepsilon) \subseteq U_i \subseteq U$, i.e., $U \in \mathfrak{T}$.

For the Hausdorffness of $(X, \mathfrak{T})$, take points $x, y \in X$ with $x \neq y$. Then $\varepsilon := d(x, y) > 0$, $B(x, \varepsilon/2) \cap B(y, \varepsilon/2) = \varnothing$ and obviously $B(x, \varepsilon/2)$ and $B(y, \varepsilon/2)$ are non-empty open neighbourhoods of x respectively y. □

The properties "open" and "closed" can be transfered to metric spaces using the topological definition of the terms.

We are now able to give a third definition of continuity that conincides on metric spaces with the concepts defined beforehand, but is also applicable to the more general setting of topological spaces.

Definition B.1.11 (Continuity): Let $(X, \mathfrak{T}_X)$, $(Y, \mathfrak{T}_Y)$ be topological spaces, let $f\colon X \to Y$ be a map and let $x \in X$ be a point. The map f is *continuous in* x, if for all $V \in \mathfrak{U}(f(x))$ it holds $f^{-1}(V) \in \mathfrak{U}(x)$. If f is continuous in every point $x \in X$, f is called *continuous*.

Lemma B.1.12: *Let (X, d) and (Y, d') be metric spaces and let $f\colon X \to Y$ be a map. Then f is continuous as defined in Definition B.1.4 if and only if it is continuous as defined in Definition B.1.11 (with respect to the induced topologies on X and Y).*

Furthermore, for topological spaces we have the following characterisations of continuous maps:

Lemma B.1.13 (Characterisation of continuous maps): *Let $(X, \mathfrak{T}_X)$ as well as $(Y, \mathfrak{T}_Y)$ be topological spaces and let $f\colon X \to Y$ be a map. Then f is continuous if and only if for all $V \in \mathfrak{T}_V$ it holds $f^{-1}(V) \in \mathfrak{T}$.*

Equivalently, f is continuous if and only if the preimage of any closed set in Y is closed in X.

Continuous maps entered the scene as maps that behave well with convergence, and indeed, there is a generalisation of the concept of convergence that makes sense in topological spaces and that allows for an analogue statement about continuous maps.

Definition B.1.14 (Filtration): Let $\varnothing \neq \Lambda$ be a set and let "$\leqslant$" be a partial order on Λ. If for all $\lambda, \mu \in \Lambda$ there is $\nu \in \Lambda$ with $\lambda \leqslant \nu$ and $\mu \leqslant \nu$, the set Λ is called a *filtration*.

Note that any two elements in a filtration are not comparable in general.

Definition B.1.15 (Net): Let $(X, \mathfrak{T})$ be a topogical space, let x be a point in X and let $(\Lambda, \leqslant)$ be a filtration. A family $(x_\lambda)_{\lambda \in \Lambda} \in X^\Lambda$ is called a *net*. The net $(x_\lambda)_{\lambda \in \Lambda}$ *converges to* x, if for every $U \in \mathfrak{U}(x)$ there is $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geqslant \lambda_0$. In this case, we write $x_\lambda \to x$.

Nets are often called *Moore-Smith sequences* in literature. There is an equivalent concept for convergence in topological spaces (namely the concept of filters), which we do not treat here.

Choosing $(\Lambda, \leqslant) = (\mathbb{N}, \leq)$ gives back the well-known concept of sequences, i.e., we can regard nets as a generalisation of sequences.

Lemma B.1.16 (Continuity and convergence revised): *Let* $(X, \mathfrak{T})$, $(Y, \mathfrak{T}')$ *be topological spaces, let* $(\Lambda, \leqslant)$ *be a filtration, let* $f \colon X \to Y$ *be a map and let* x *be a point in* X*. Then* f *is continuous in* x *if and only if for every net* $(x_\lambda)_{\lambda\in\Lambda}$ *with* $x_\lambda \to x$ *it holds* $f(x_\lambda) \to f(x)$*.*

Nets and sequences also play a crucial role in the characterisation of specific sets that are associated to subsets of topological spaces, their so called *closure*.

Definition B.1.17 (Interior and closure): Let $(X, \mathfrak{T})$ be a topological space and let Y be a subset of X. We define

$$\operatorname{Int}(Y) := \bigcup \{U \subseteq X \mid U \text{ is open and } U \subseteq Y\},$$
$$\operatorname{cl}(Y) := \bigcap \{A \subseteq X \mid A \text{ is closed and } Y \subseteq A\}.$$

The set $\operatorname{Int}(Y)$ is called the *interior of* Y and it is the biggest open subset of X contained in Y, the set $\operatorname{cl}(Y)$ is called the *closure of* Y and it is the smallest closed set in X that contains Y.

It is easy to see that $\operatorname{Int}(\operatorname{Int}(Y)) = \operatorname{Int}(Y)$ and $\operatorname{cl}(\operatorname{cl}(Y)) = \operatorname{cl}(Y)$ for all $Y \subseteq X$, furthermore $Y \subseteq X$ is open respectively closed if and only if $Y = \operatorname{Int}(Y)$ respectively $Y = \operatorname{cl}(Y)$. It is also useful to remember that if $A, B \subseteq X$ with $A \subseteq B$, then $\operatorname{Int}(A) \subseteq \operatorname{Int}(B)$ and $\operatorname{cl}(A) \subseteq \operatorname{cl}(B)$.

Lemma B.1.18 (Characterisation of closure via convergence): *Let* (X, d) *be a metric space and let* Y *be a subset of* X*. Then we have*

$$\operatorname{cl}(Y) = \{y \in X \mid \textit{There is } (y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}} \textit{ with } y_n \to y\}.$$

Let $(X, \mathfrak{T})$ *be a topological space and* Y *be a subset of* X*. Then we have*

$$\operatorname{cl}(Y) = \{y \in X \mid \Lambda \textit{ is a filtration and there is } (y_\lambda)_{\lambda\in\Lambda} \in Y^{\Lambda} \textit{ with } y_\lambda \to y\}.$$

That we can make due with sequences in metric spaces and have to sidestep to nets in topological spaces is on grounds of the fact that metric spaces are “first countable”, while general topological spaces are not. If they are, sequences are enough — for closures as well as continuity.

Now that we have the analogies out of the way, its time to pay attention to the differences. In general topological spaces, there is no reason why nets should have unique limits. In fact, it can happen that a net converges to every point of the topological space; take for instance X to be any set and $\mathfrak{T} = \{\varnothing, X\}$. We do however have unique limits of nets in Hausdorff topological spaces:

Remark B.1.19: Let $(X, \mathfrak{T})$ be a Hausdorff topological space and let Λ be a filtration. If $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X such that $x_\lambda \to x$ and $x_\lambda \to y$ with $x, y \in X$, then $x = y$.

An outstanding topological concept is the concept of compactness.

Definition B.1.20 (Compactness): Let $(X, \mathfrak{T})$ be a topological space and let $K \subseteq X$ be a set. A family $(U_i)_{i\in I}$ with $U_i \in \mathfrak{T}$ for $i \in I$ and $K \subseteq \bigcup_{i\in I} U_i$ is called an *open cover for* K. If for every open cover $(U_i)_{i\in I}$ for K there are indices $i_1, \ldots, i_n \in I$, where n is a natural number, such that $K \subseteq \bigcup_{j=1}^{n} U_{i_j}$, K is called *compact*. The family $(U_{i_j})_{1\leq j\leq n}$ is called *finite subcover for* K.

As a generalisation of the extreme value theorem in general topological spaces we have the following theorem:

Theorem B.1.21: *Let $f\colon X \to Y$ be a continuous map between topological spaces. If $K \subseteq X$ is compact, then so is $f(X) \subseteq Y$.*

In finite-dimensional normed vector spaces, we have a useful characterisation of compact sets:

Theorem B.1.22 (Heine-Borel): *Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space over $\mathbb{K}$. A subset $K \subseteq V$ is compact if and only if K is closed and bounded, i.e., if there is $r > 0$ such that $K \subseteq B(0, r)$.*

Finally, a crucial result for the continuous functions on a compact Hausdorff topological space is the Stone-Weierstraß theorem.

Definition B.1.23 (Unital *-algebra): Let K be a compact Hausdorff topological space and let $A \subseteq C(K) = \{f\colon K \to \mathbb{C} \text{ is continuous}\}$ be a subset. If it holds

(i) If f and g are functions in A, then $fg \in A$,

(ii) If f and g are functions in A and μ and λ are complex numbers, then $\lambda f + \mu g \in A$,

(iii) If f is a function in A, then so is f^*[1].

(iv) The constant function $f\colon K \to \mathbb{C}$, $a \mapsto 1$ is in A,

A is called a *unital *-algebra.* If for all $a, b \in K$ there is $f \in A$ such that $f(a) \neq f(b)$, A is said to *separate points.*

[1]For a function $f\colon A \to \mathbb{C}$, the function f^* is the point-wise complex conjugation of f, i.e., $f^*(a) = f(a)^*$ for all $a \in K$.

With this definition, we are able to formulate the Stone-Weierstraß theorem.

Theorem B.1.24 (Stone-Weierstraß): *Let K be a compact topological Hausdorff space and let $A \subseteq C(K)$ be a unital *-algebra separating points. Then A is dense in $C(K)$. In particular: If A is closed, then $A = C(K)$.*

As a corollary, we get the classical theorem due to Weierstraß:

Corollary B.1.25: *The algebra of real valued polynomial functions P on $[0,1]$, i.e.,*

$$P := \Big\{ f\colon [0,1] \to \mathbb{R} : z \mapsto \sum_{i=0}^{n} \alpha_i z^i, n \in \mathbb{N}, \alpha_i \in \mathbb{R} \Big\},$$

is dense in $C_{\mathbb{R}}([0,1])$.

For the sake of completeness, we mention two other types of topological spaces that are often encountered in practice:

Definition B.1.26 (Topological groups and vector spaces): Let G be a group and let $\mathfrak{T}$ be a topology on G. If the maps $\circ\colon G \times G \to G$ and $i\colon G \to G$, $g \mapsto g^{-1}$ are continuous (with respect to $\mathfrak{T}$), the tuple $(G, \mathfrak{T})$ is called a *topological group.*

Let V be a $\mathbb{K}$-vector space and let $\mathfrak{T}$ be a topology on V. If the maps $+\colon V \times V \to V$ and $\cdot\colon \mathbb{K} \times V \to V$ are continuous (with respect to $\mathfrak{T}$), the tuple $(V, \mathfrak{T})$ is called *topological vector space.*

Definition B.1.27 (Locally convex vector space): Let V be a $\mathbb{K}$-vector space, let P be a family of seminorms on V and let $\mathfrak{T}$ be a topology on V. If the topology $\mathfrak{T}$ is generated by the seminorms, i.e., if a set $U \subseteq V$ is open if and only if

$$\forall\, u \in U\, \exists\, n \in \mathbb{N}\, \exists\, p_1, \dots, p_n \in P\, \exists\, \varepsilon_1, \dots, \varepsilon_n > 0 : \bigcap_{i=1}^{n} B_{p_i}(u, \varepsilon_i) \subseteq U,$$

the tuple $(V, \mathfrak{T}, P)$ is called a *locally convex vector space* (over $\mathbb{K}$). Here, $B_p(u, \varepsilon) := \{v \in V \mid p(u,v) < \varepsilon\}$ means the open ball with radius ε around u with respect to the seminorm p.

Remark B.1.28: As another exception, linear maps between finite-dimensional normed vector spaces are automatically continuous: Using Remark A.2.5, we can retreat to matrices and for a matrix $A \in M_{n \times m}(\mathbb{K})$,

$$\|A\|_{\mathrm{op}} := \sup\{\|Ax\|_{\mathbb{K}^n} \mid v \in \mathbb{K}^m, \|v\|_{\mathbb{K}^m} = 1\},$$

defines a norm on $M_{n\times m}(\mathbb{K})$, the so called operator norm, see Lemma I.1.2.

For a convergent sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{K}^m$ with $x_n \to x$ we have that

$$\|Ax_n - Ax\| = \|x_n - x\| \left\| A\left(\frac{x_n - x}{\|x_n - x\|}\right)\right\| \leq \|A\|_{\mathrm{op}}\|x_n - x\|,$$

i.e., $Ax_n \to Ax$ and A is thus sequentially continuous.

However, for linear maps (then called *linear operators*) between infinite-dimensional vector spaces, this does not hold true. Using Zorns lemma, one can construct counterexamples.

To decide if a linear operator is continuous, it is enough to know if it is continuous in one point:

Proposition B.1.29: *Let $(V, \mathfrak{T})$ and $(W, \mathfrak{T}')$ be topological vector spaces and let $T\colon V \to W$ be a linear map. Then the following are equivalent:*

(i) *T is continuous,*

(ii) *T is continuous in some point,*

(iii) *T is continuous in 0.*

Here, the two implications "(i) ⇒ (iii)" and "(iii) ⇒ (ii)" are trivial. For "(ii) ⇒ (i)" one uses the linearity of T to shift any point back to the point where T is continuous.

And in the second step, we can give a characterisation of continuous linear operators between normed spaces:

Theorem B.1.30: *Let $(V, \|\cdot\|)$ and $(W, \|\cdot\|')$ be normed vector spaces over $\mathbb{K}$ and let $T\colon V \to W$ be a linear map. Then the following are equivalent:*

(i) *T is continuous,*

(ii) *There is a constant $C > 0$ such that for all $v \in V$ it holds $\|Tv\|' \leq C\|v\|$.*

This motivates the following definition:

Definition B.1.31: Let $(V, \|\cdot\|)$ and $(W, \|\cdot\|')$ be normed vector spaces over $\mathbb{K}$ and let $T\colon V \to W$ be a continuous linear map. We call

$$\|T\| := \inf\{C \geq 0 \mid \|Tv\|' \leq C\|v\| \text{ for all } v \in V\}$$

the *operator norm* of T and $B(V, W) := \{T\colon V \to W \mid T \text{ linear, bounded}\}$ the space of linear and bounded operators from V to W. The operator norm turns $B(V, W)$ into a normed vector space over $\mathbb{K}$.

In the special case $V = W$, we write $B(V) := B(V, V)$, in the special case $W = \mathbb{K}$, we write $V' := B(V, \mathbb{K})$.

As a side note: It is due to historical development that continuous operators are called bounded, since they are not really bounded but bounded on bounded sets. If V and W are normed vector spaces and W is complete, then so is $B(V, W)$ with the operator norm.

2. Differentiablity

This section deals with the concept of differentiability. We give the definition for differentiability of functions from $\mathbb{K}$ to a normed vector space over $\mathbb{K}$, which includes the well-known notion of differentiability for functions from $\mathbb{R}$ to $\mathbb{R}$ and functions from $\mathbb{C}$ to $\mathbb{C}$ as well as the notion of differentiability for curves. In this setting, we see two equivalent characterisations of differentiability that separate, when we generalise to functions from normed vector spaces to normed vector spaces.

The main goals for this section are understanding the different concepts of differentiability that arise when generalising to maps from normed vector spaces to normed vector spaces, the chain rule and the differentiability criterion.

Definition B.2.1 (Limit for functions): Let $\varnothing \neq U \subseteq \mathbb{K}$ be a subset, let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$, let $u_0 \in U$ be a *contact point*, i.e., there is a sequence $(u_n)_{n\in\mathbb{N}}$ in U with $u_n \to u_0$, let $f\colon U \to V$ be a map and let $v \in V$ be given. If for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $0 < |u - u_0| < \delta$ it holds $\|f(u) - v\| < \varepsilon$, we write

$$\lim_{u\to u_0} f(u) = v.$$

Note that if $\lim_{u\to u_0} f(u)$ exists, f is continuous in u_0.

Definition B.2.2 (Differentiability): Let $\varnothing \neq U \subseteq \mathbb{K}$ be open, let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$, let x_0 be a point in X and let $f\colon X \to V$ be a map. If the limit

$$\left.\frac{df}{dx}\right|_{x=0} := f'(x_0) := \dot{f}(x_0) := \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, f is called *differentiable at* x_0. We call $f'(x_0) \in V$ the *derivative of* f *at* x_0. If f is differentiable in every point of X, we call f *differentiable* and call the map $f'\colon U \to V$, $x \mapsto f'(x)$ the *derivative of* f.

The above definition is probably the most famous one. As it turns out, there is another equivalent characterisation for differentiability:

Theorem B.2.3: *Let $\varnothing \neq U \subseteq \mathbb{K}$ be open, let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$, let x_0 be a point in U and let $f\colon U \to V$ be a map. The following are equivalent:*

(i) *f is differentiable in x_0,*

(ii) *There are $c \in \mathbb{K}$ and $\varphi\colon B(0,\varepsilon) \to V$ such that $\lim_{\xi\to 0} \frac{1}{\xi}\varphi(\xi) = 0$ and*

$$f(x_0 + \xi) = f(x_0) + c\xi + \varphi(\xi).$$

In this case $c = f'(x_0)$ and $\xi \mapsto c\xi$ (respectively $\xi \mapsto f(x_0) + c\xi$) is a linear approximation of f at x_0.

The property from part (ii) of Theorem B.2.3 is called *linear approximablitly*. If a function from a field to a normed vector space is differentiable in a point, the "tangent line" through that point is a good approximation for the function in a small region around that point.

What is the same for functions from a field to a normed vector space becomes different yet related concepts if we generalise differentiability to functions from normed vector spaces to normed vector spaces.

The following definition stems from [2], Definition 2.3.1.

Definition B.2.4: Let $\mathbf{E}, \mathbf{F}$ be normed vector spaces, let U be an open subset of E and let $f\colon U \subset \mathbf{E} \to \mathbf{F}$ be a given mapping. Let $u_0 \in U$. We say that f is *differentiable at* the point u_0 provided there is a bounded linear map $\mathbf{D}f(u_0)\colon E \to F$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $0 < \|u - u_0\| < \delta$, we have

$$\frac{\|f(u) - f(u_0) - \mathbf{D}f(u_0)\cdot(u - u_0)\|}{\|u - u_0\|} < \varepsilon.$$

Here we need a bounded linear map $\mathbf{D}f(u_0)\colon E \to F$, because we need to have sequential continuity for the definition to make sense.

The content of Definition B.2.4 embodies part (ii) of Theorem B.2.3. There is an analogous notion to part (i) of Theorem B.2.3, namely the notion of the *directional derivative*:

Definition B.2.5: Let $f\colon U \subset \mathbf{E} \to \mathbf{F}$ and let $u \in U$. We say f *has a derivative in the direction* $e \in \mathbf{E}$ at u if

$$\left.\frac{d}{dt} f(u + te)\right|_{t=0}$$

exists. We call this element of $\mathbf{F}$ *the directional derivative of f in the direction e in u.*

The relation between both is the following (see [2], Proposition 2.4.6):

Proposition B.2.6: *If f is differentiable in u, then the directional derivatives of f exist in u and are given by*

$$\frac{d}{dt} f(u+te)\bigg|_{t=0} = \mathbf{D}f(u)\cdot e.$$

The converse does not hold: If all directional derivatives of f exist in a given point, f need not be differentiable in that point. It is easy to give counterexamples; even in the finite-dimensional case, this is wrong.

As for nomenclature: A function whose directional derivatives in a point exist is called *Gâteaux-differentiable* in said point, a function that is differentiable in a point in the sense of Definition B.2.4 is called *Fréchet-differentiable* in said point.

We will only deal with maps between finite-dimensional normed vector spaces here, in fact only with maps between open sets of euclidean spaces even. In this situation, there are particulary favourable directional derivatives, namely those with respect to the elements of the canonical basis; and the linear map from Definition B.2.4 can be represented via a matrix, the so-called *Jacobian*.

Definition B.2.7 (Partial derivatives): Let $n \in \mathbb{N}$, let $U \subseteq \mathbb{R}^n$ be open, let $(V, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space, let $u \in U$ be a point and let $f\colon U \to V$ be a function. If they exist, we call the directional derivatives

$$\partial_i f(u) := \frac{\partial f}{\partial x^i}(u) := \frac{d}{dt} f(u+te_i)\bigg|_{t=0} = \lim_{t\to 0} \frac{f(u+te_i)-f(u)}{t}$$

the *partial derivatives of f in u.* The function f is said to be *partially differentiable in u*, if the partial derivatives $\partial_i f(u)$ exist for $1 \le i \le n$, f is said to be *partially differentiable*, if f is partially differentiable in every point in U and f is said to be *continuously partially differentiable*, if the maps $u \mapsto \partial_i f(u)$ are continuous for $1 \le i \le n$.

Note that if the partial derivatives in $u = (u^1, \dots, u^n)$ exist, they are, for $1 \le i \le n$, given by

$$\partial_i f(u) = \lim_{h\to 0} \frac{f(u^1,\dots,u^{i-1},u^i+h,u^{i+1},\dots,u^n) - f(u^1,\dots,u^n)}{h},$$

i.e., the function f is partially differentiable in u if and only if the maps $t \mapsto f(u^1,\dots,u^{i-1},t,u^{i+1},\dots,u^n)$ are differentiable in u^i as functions of a real variable for $1 \le i \le n$.

For a function $f\colon U \to V$ from an open subset of $\mathbb{R}^n$, where n is a natural number, to a normed $\mathbb{R}$-vector space $(V, \|\cdot\|)$ that is differentiable in $u \in U$, it is easy to see that the directional derivative of f in a point u in direction $h = (h^1, \ldots, h^n)$ due to Proposition B.2.6 is given by

$$\mathbf{D}f(u) \cdot h = \sum_{i=1}^{n} h^i \mathbf{D}f(u) \cdot e_i = \sum_{i=1}^{n} h^i \partial_i f(u).$$

Definition B.2.8 (Coordinate functions): Let n and m be natural numbers, let $U \subseteq \mathbb{R}^n$ be open and let $f\colon U \to \mathbb{R}^m$ be a map. For $1 \leq j \leq m$, the map

$$f^j := f \circ \mathrm{pr}_j \colon U \longrightarrow \mathbb{R}, \qquad (x^1, \ldots, x^n) \longmapsto \mathrm{pr}_j(f(x^1, \ldots, x^n))$$

is called the j-th coordinate function of f.

Using the coordinate functions, a function $f\colon U \to \mathbb{R}^m$, where $U \subseteq \mathbb{R}^n$ is open, can be identified with the m-tuple $(f^1, \ldots, f^m) \in \prod_{i=1}^m \mathbb{R}^U$.

Theorem B.2.9: *Let n and m be natural numbers, let $U \subseteq \mathbb{R}^n$ be open, let u be a point in U and $f = (f^1, \ldots, f^m)\colon U \to \mathbb{R}^m$ be a function that is partially differentiable in u. The matrix*

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1}(u) & \cdots & \frac{\partial f^1}{\partial x^n}(u) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(u) & \cdots & \frac{\partial f^m}{\partial x^n}(u) \end{pmatrix} \in M_{m\times n}(\mathbb{R})$$

is called the Jacobian of f in u. *If f is differentiable in u, then the Jacobian is the transformation matrix of $\mathbf{D}f(u)$ with respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$.*

Just as for functions of a real variable, there is a chain rule for functions between open sets of euclidean spaces:

Proposition B.2.10 (Chain rule): *Let m, n and ℓ be natural numbers, let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ and $Z \subseteq \mathbb{R}^\ell$ be open sets. Furthermore, let $f\colon X \to \mathbb{R}^m$, $g\colon Y \to \mathbb{R}^\ell$ be maps with $f(X) \subseteq Y$ and let f be differentiable in $x_0 \in X$, let g be differentiable in $f(x_0) \in Y$. Then the composition $g \circ f\colon X \to \mathbb{R}^\ell$ is differentiable in x_0 as well and it holds*

$$\mathbf{D}(g \circ f)(x_0) = \mathbf{D}g(f(x_0)) \circ \mathbf{D}f(x_0).$$

For the Jacobians, this means: The Jacobian of $g \circ f$ in x_0 is the product of the Jacobian of g in $f(x_0)$ and the Jacobian of f in x_0.

We want to finish with a crucial differentiability criterion:

Theorem B.2.11: *Let n be a natural number, let $U \subseteq \mathbb{R}^n$ be open, let $(V, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space and let $f: U \to V$ be a function. Then f is continuously differentiable if and only if f is continuously partially differentiable.*

This generalises to higher derivatives, but we will omit discussion of what those are exactly.

If $f: U \to V$ is q-times continously partially differentiable, we say "f is C^q" and write $f \in C^q(U, V)$.

3. Embedded submanifolds

This section aims at presenting a very brief overview over the main definitions of embedded submanifolds of $\mathbb{R}^n$ with the main goal being Theorem B.3.10 that characterises the tangent space of an embedded submanifold at a given point using tangent vectors of smooth curves through that point. It follows closely the presentation of this topic from [3].

Definition B.3.1 (Immersion): Let m and n be natural numbers and let $U \subseteq \mathbb{R}^n$ be open. If $f: U \to \mathbb{R}^m$ is differentiable such that, for every $u \in U$, $\mathbf{D}f(u) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ is injective, f is called an *immersion*.

Note that being an immersion is a weaker requirement than being injective.

For U, the map $\iota_U: U \to \mathbb{R}^n$, $x \mapsto x$ is the *canoncial embedding*; obviously ι_U is an immersion.

Definition B.3.2 (C^q-chart): Let m and n be natural numbers, let $U \subseteq \mathbb{R}^n$ be open, let $V \subseteq U$ be open in U and let u_0 be a point in V. Let $\varphi: V \to \mathbb{R}^m$ be a map. If $\varphi(V) \subseteq \mathbb{R}^m$ is open, $\varphi: V \to \varphi(V)$ is a homeomorphism, i.e., a bijective continuous map whose inverse map is continuous as well, and the composition $g := \iota_U \circ \varphi^{-1}$ is a C^q-immersion, we call φ an *m-dimensional (local) C^q-chart of U around u_0*.

We call V the *chart domain*, $\varphi(V)$ the *parameter domain* and g the *parametrisation of V with respect to φ*. Sometimes we write (φ, V) for the chart and $(g, \varphi(V))$ for the associated parametrisation.

Definition B.3.3 (Embedded submanifold): Let m, n and q be natural numbers and let $M \subseteq \mathbb{R}^n$ be a set. If for every point $x_0 \in M$ there is an open neighbourhood U (which is open in M) and an m-dimensional C^q-chart (φ, U)

of M around x_0, we call M an m-dimensional embedded C^q-submanifold of $\mathbb{R}^n$. A collection

$$\mathcal{A} := \{(\varphi_i, U_i)_{i\in I} \mid I \text{ is an index set, } \varphi_i \text{ is a } C^q\text{-chart of } M \text{ for all } i \in I\}$$

of charts with $M \subseteq \bigcup_{i\in I} U_i$ is called a C^q-atlas of M.

Definition B.3.4 (Change of charts): Let m and n be natural numbers, let M be an m-dimensional embedded C^q-submanifold of $\mathbb{R}^n$, where $q \in \mathbb{N}\cup\{\infty\}$. Furthermore let $\mathcal{A} = \{(\varphi_i, U_i)_{i\in I}\}$ be a C^q-atlas of M. Then, for $i, j \in I$, the maps

$$\varphi_i \circ \varphi_j^{-1} \colon \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

are called *changes of charts*.

Proposition B.3.5: *Let m and n be natural numbers, M be an m-dimensional embedded C^q-submanifold of $\mathbb{R}^n$, where $q \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{A} = \{(\varphi_i, U_i)_{i\in I}\}$ be a C^q-atlas of M. Then, for $i, j \in I$, the change of charts $\varphi_i \circ \varphi_j^{-1}$ is a C^q-diffeomorphism, i.e., q-times differentiable bijective map with q-times differentiable inverse, whose inverse is $\varphi_j \circ \varphi_i^{-1}$.*

Definition B.3.6 (Ad-hoc definitions for $\mathbb{R}^n$): Let n and ℓ be natural numbers, let $X \subseteq \mathbb{R}^n$ be open, let $Y \subseteq \mathbb{R}^\ell$ be open, let $f \in C^1(X, Y)$ be a map and let p be a point in X. Then we call the set $T_pX := \{p\} \times \mathbb{R}^n$, equipped with the euclidean structure

$$(p, v) + \lambda(p, w) := (p, v + \lambda w), \qquad ((p, v)|(p, w)) := (v|w)_{\mathbb{R}^n}$$

induced by $\mathbb{R}^n$, the *tangent space of X at p*. An element (p, v) of T_pX is called a *tangent vector*, sometimes denoted $(v)_p$, and v is called the *main part* of the tangent vector. The linear map

$$T_pf \colon T_pX \longrightarrow T_{f(p)}Y, \qquad (p, v) \longmapsto (p, \mathbf{D}f(p) \cdot v)$$

is called the *differential of f at p*.

The image of the differential T_pf can be understood as a linear approximation of $f(X)$ at $f(p)$.

If in the situation of Definition B.3.6, s is another natural number, $Z \subseteq \mathbb{R}^s$ is another open set and $g \in C^1(Y, Z)$ is another map, we have the chain rule

$$T_p(g \circ f) = T_{f(p)}g \circ T_pf.$$

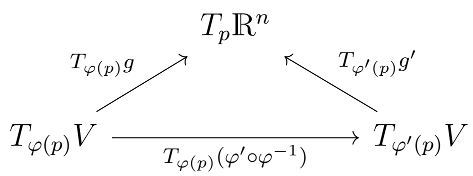
This follows immediately from Proposition B.2.10.

Definition B.3.7 (Tangent space and tangent bundle): Let n and m be natural numbers, let M be an embedded m-dimensional C^q-submanifold of $\mathbb{R}^n$ where $q \in \mathbb{N} \cup \{\infty\}$, let p be a point in M, let (φ, U) be a chart of M around p and (g, V) be the associated parametrisation. Then

$$T_pM := \operatorname{im}(T_{\varphi(p)}g) = T_{\varphi(p)}(g)(T_{\varphi(p)}V)$$

is called the *tangent space of M at p*. The elements of T_pM are called *tangent vectors at M in p* and the disjoint union $TM := \bigcup_{p \in M} T_pM \subseteq M \times \mathbb{R}^n$ is called the *tangent bundle of M*.

The chain rule grants that the definition of T_pM is independent of the choice of the chart (φ, U) of M around p: If we consider another chart (φ', U') of M around p, we have the diagram



which is commutative.

For open sets $M \subseteq \mathbb{R}^n$, the definitions of tangent space in Definition B.3.6 and Definition B.3.7 coincide.

Due to our requirements for charts and parametrisations, $\dim T_pM = m$ holds for all $p \in M$.

Definition B.3.8: Let m and n be natural numbers, let M be an m-dimensional embedded C^q-submanifold of $\mathbb{R}^n$, where $q \in \mathbb{N} \cup \{\infty\}$, let p be a point in M, let (φ, U) be a C^q-chart of M around p and let (g, V) be the associated parametrisation. For $1 \le j \le m$, let $\varepsilon > 0$ be such that $\varphi(p) + te_j \in V$ for $|t| < \varepsilon$. Then the path

$$\gamma_j \colon (-\varepsilon, \varepsilon) \longrightarrow M, \qquad t \longmapsto g(\varphi(p) + te_j)$$

is called the j-th *coordinate path*.

Proposition B.3.9: *Let m and n be natural numbers, let M be an m-dimensional embedded C^q-submanifold of $\mathbb{R}^n$, where $q \in \mathbb{N} \cup \{\infty\}$, let p be a point in M, let (φ, U) be a C^q-chart of M around p and let (g, V) be the associated parametrisation. Then $\{\dot\gamma_j(0) \mid 1 \le j \le m\} \subseteq T_pM$ is linearly independent with*

$$\operatorname{Lin}(\{\dot\gamma_j(0) \mid 1 \le j \le m\}) = T_pM,$$

i.e., the set $\{\dot\gamma_j(0) \mid 1 \le j \le m\}$ is a basis of T_pM.

This is true, since $\dim T_pM = m$ and the derivative of the j-th coordinate path in 0 is the j-th column of the Jacobian of $T_{\varphi(p)}g$ — due to our requirements to charts, the set of columns of this Jacobian is linearly independent.

We even can give the following (more general) characterisation of the tangent space in a point $p \in M$, which justifies the name "tangent space":

Theorem B.3.10: *Let m and n be natural numbers, let M be an m-dimensional embedded C^q-submanifold of $\mathbb{R}^n$, where $q \in \mathbb{N} \cup \{\infty\}$, let p be a point in M, let (φ, U) be a C^q-chart of M around p and let (g, V) be the associated parametrisation. Then we have the following characterisation of T_pM:*

$$T_pM = \{(v)_p \in T_p\mathbb{R}^n \mid \textit{There are } \varepsilon > 0 \textit{ and } \gamma \in C^1((-\varepsilon, \varepsilon), M) \\ \textit{such that } \gamma(-\varepsilon, \varepsilon) \subseteq M,\ \gamma(0) = p,\ \dot{\gamma}(0) = v\}.$$

A more general notion, in contrast with that of an embedded submanifold of $\mathbb{R}^n$, is the notion of a differentiable manifold. Basically all the ideas presented here transfer to these objects, one just has to pay more attention when defining things like the tangent space, because there is no surrounding vector space anymore and thus, one has to use intrinsic data of the manifold to define these notions.

Definition B.3.11 (Smooth manifold): An *n-dimensional topological manifold* is a Hausdorff topological space M which is *locally euclidean*, i.e., each point $x \in M$ has an open neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

A *(local) chart* (U, φ) of M consists of an open subset $U \subseteq M$ and a homeomorphism $\varphi\colon U \to \varphi(U) \subseteq \mathbb{R}^n$.

A family $\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$ of charts satisfying $M = \bigcup_{i \in I} U_i$ is called an *atlas of M*. The homeomorphisms $\psi_{i,j}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ given by $\psi_{i,j} := \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_i \cap U_j)}$ are called *changes of chart.*

An atlas $\mathcal{A}$ of M is called *smooth* if all its transition maps are smooth, i.e., arbitrarily often differentiable. A chart (U, φ) is said to be *smooth with respect to a smooth atlas* $\mathcal{A}$, if $\mathcal{A} \cup \{(U, \varphi)\}$ is again a smooth atlas. A smooth atlas $\mathcal{A}$ is called *maximal*, if every chart (U, φ) that is smooth with respect to $\mathcal{A}$ already belongs to $\mathcal{A}$. Every smooth atlas $\mathcal{A}$ induces a maximal one by

$$\mathcal{A}_{\max} := \{(U, \varphi) \mid (U, \varphi) \text{ is a chart smooth with respect to } \mathcal{A}\}.$$

An *n-dimensional smooth manifold* is an n-dimensional topological manifold $\mathcal{M}$ with a maximal smooth atlas $\mathcal{A}$.

4. Hilbert spaces

In analysis, often times one works with normed vector spaces that are infinite-dimensional, e.g., function spaces. Most often, Banach and Hilbert spaces are encountered. To develop a similar theory as in the finite-dimensional setting, one has to consider topological aspects. As it turns out, many results transfer because the maps behave nicely with the topology – for example, the canonical embedding of a normed vector space into its dual space (equipped with the dual norm) is isometric and thus continuous. Many central theorems rely on the important Hahn-Banach type theorems that grant extensions of linear maps defined on subspaces of normed vector spaces under certain circumstances and separations of certain subsets of locally convex vector spaces in terms of continuous functionals under certain circumstances.

Infinite-dimensional Hilbert spaces behave similarly to finite-dimensional Hilbert spaces, i.e., euclidean and unitary spaces. Just like for euclidean and unitary spaces, Hilbert spaces are canonically isomorphic to their (in this case topological) dual spaces and many constructions like the adjoint operator (see Definition A.4.19) carry over to the infinite-dimensional setting.

Due to Baire's theorem, there is no infinite-dimensional Banach space (and thus in particular, no infinite-dimensional Hilbert space) with a countable vector space basis, meaning vector space bases in this case are quite useless. To end up with a more useful notion of "basis", one has to soften the requirements a bit to end up with the more useful notion of "orthonormal basis".

Hilbert spaces are of vital importance for the understanding of C^*-algebras, which is what we need them for.

Remark B.4.1: Let $M \subseteq H$ be a subset of the Hilbert space H over $\mathbb{K}$. We already established that $M^\perp$ is a linear subspaces of H. In addition, $M^\perp$ also is closed: For a sequence $(x_n)_{n\in\mathbb{N}}$ in $M^\perp$ with $x_n \to x$ we have for all $m \in M$ that

$$(x|m) = (\lim_{n\to\infty} x_n|m) = \lim_{n\to\infty}(x_n|m) = 0,$$

i.e., $x \in M^\perp$. Hence it holds $\mathrm{cl}(M) \subseteq M^{\perp\perp}$.

In fact, one can show that if M is a linear subspace, then also $M^{\perp\perp} \subseteq \mathrm{cl}(M)$ holds, i.e., $M^{\perp\perp} = \mathrm{cl}(M)$.

In Hilbert spaces, the distance to a closed convex subset always is realised, i.e., if $A \subseteq H$ is a convex subset and $x \in H - A$ is some vector, there is $x_0 \in A$ such that $\mathrm{dist}(x, A) = \inf\{\|x - a\| \mid a \in A\} = \|x - x_0\|$. Furthermore, if A is a linear subspace as well, it holds $x - x_0 = \mathrm{dist}(x, A)$ if and only if

$x - x_0 \in A^\perp$. Those are the ingredients we need for the proof of the projection theorem:

Theorem B.4.2 (Projection Theorem): *Let H be a Hilbert space over $\mathbb{K}$ and let $K \subseteq H$ be a closed linear subspace. Then we have $H = K \oplus K^\perp$, i.e., every element $x \in H$ has a unique decomposition $x = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in K^\perp$. The parts x_i are called* best approximations of x in K_i *for $i \in \{1, 2\}$.*

We remind of Definition B.1.31, where we defined the normed space of bounded linear operators between two normed spaces.

To transfer the Riesz Representation Theorem, we need to think about the topological part of the statements Definition A.4.17 and Theorem A.4.18: The associated functional f_v to $v \in H$ is linear, as we already know, but due to the Cauchy-Schwarz inequality it is also bounded with $\|f_v\| = \|v\|$, i.e., an element of the topological dual space of H'. This makes j an isometry (it holds $\|j(v)\| = \|f_v\| = \|v\|$ for all $v \in H$).

Theorem B.4.3 (Riesz representation theorem): *Let H be a Hilbert space over $\mathbb{K}$. Then the map*

$$j \colon H \longrightarrow H', \qquad v \longmapsto f_v := (w \mapsto (w|v))$$

is an isometric anti-linear isomorphism, i.e., $H \cong H'$ canonically.

For the proof, we only have to take care of surjectivity, which we got for free in the finite-dimensional setting. Therefore we take $f \in H'$, decompose $H = \ker f \oplus \ker f^\perp$ and construct a vector $v \in H$ such that $f_v(w) = f(w)$ for all $w \in H$.

We note that the proof for the existence of the adjoint map, in this setting called *adjoint operator*, goes through word-for-word for Hilbert spaces, too, since we only needed the Riesz representation theorem that also holds for Hilbert spaces. Some easy properties of the adjoint are collected the following lemma, which can be found in [23], Satz V.5.2:

Lemma B.4.4: *Let H_1, H_2, H_3 be Hilbert spaces over $\mathbb{K}$, let $\lambda \in \mathbb{K}$, let $S, T \in B(H_1, H_2)$ and let $R \in B(H_2, H_3)$. Then it holds*

(i) $(S + T)^* = S^* + T^*$,

(ii) $(\lambda S)^* = \lambda^* S^*$,

(iii) $(RS)^* = S^* R^*$,

(iv) S^* *is continuous, i.e.,* $S \in B(H_2, H_1)$*, with* $\|S^*\| = \|S\|$,

(v) $S^{**} = S$,

(vi) $\|SS^*\| = \|S^*S\| = \|S\|^2$,

(vii) $\ker S = \operatorname{im} S^{*\perp}$, $\operatorname{im} S = \ker S^{*\perp}$*; in particular* S *is injective if and only if* $\operatorname{im} S^{*\perp}$ *is dense in* H_1.

In the special case $H_1 = H_2$, the properties of the adjoint mean that the map

$$^*\colon B(H, H) \longrightarrow B(H, H), \qquad S \longmapsto S^*$$

is an anti-linear isometry, i.e., continuous in particular.

Definition B.4.5: Let H be a Hilbert space over $\mathbb{K}$ and let $A \in B(H)$.

(i) A is called *selfadjoint* (sometimes *hermitian*), if $A = A^*$,

(ii) A is called *normal*, if $AA^* = A^*A$,

(iii) A is called *isometry*, if $A^*A = \mathrm{id}_H$,

(iv) A is called *unitary*, if $A^*A = AA^* = \mathrm{id}_H$, i.e., $A^{-1} = A$,

(v) A is called *orthogonal projection*, if $A = A^* = A^2$,

(vi) A is called *partial isometry*, if $A = AA^*A$.

Compared to the finite-dimensional setting, we now have to distinguish between unitaries and isometries (see Definition A.4.11).

A partial isometry P can be characterised in the following way: There is a subspace $K \subseteq H$ such that $P|_K\colon K \to P(K)$ is an isometry and $P|_{K^\perp} \equiv 0$. The compositions P^*P and PP^* both are orthogonal projections, P^*P is called the initial projection, PP^* is called the final projection.

The relation between orthogonal projections and closed linear subspaces K of a Hilbert space H is the following: If K is a closed linear subspace, then there is an orthogonal projection $P \in B(H)$ such that $\operatorname{im} P = K$, $\ker P = K^\perp$. If $P \in B(H)$ is an orthogonal projection, it is easy to see that $\operatorname{im} P$ is closed and we know from Lemma B.4.4 that $\ker P = \operatorname{im} P^\perp$, i.e., $H = \ker P \oplus \operatorname{im} P$.

On the bounded linear operators, there are different notions for convergence.

Definition B.4.6: Let H be a Hilbert space over $\mathbb{K}$, let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $B(H)$ and let T be a bounded linear operator on H. By $\|\cdot\|$ denote the operator norm on $B(H)$.

(i) The sequence converges *uniformly* to T, if $\|T_n - T\| \to 0$.

(ii) The sequence converges *strongly* to T, if $T_n\xi \to T\xi$ for all $\xi \in H$.

(iii) The sequence converges *weakly* to T, if $(T_n\xi|\eta) \to (T\xi|\eta)$ for all $\xi, \eta \in H$.

Note that (iii) from Definition B.4.6 can be rephrased to "$(T_n\xi|\xi) \to (T\xi|\xi)$ for all $\xi \in H$" due to the polarisation identity.

Remark B.4.7 (Operator topologies in finite dimension): If a sequence of operators $(T_n)_{n\in\mathbb{N}}$ converges uniformly to an operator T, then $(T_n)_{n\in\mathbb{N}}$ converges strongly to T and if $(T_n)_{n\in\mathbb{N}}$ converges strongly to T, then $(T_n)_{n\in\mathbb{N}}$ converges weakly to T.

If the Hilbert space H is finite-dimensional, all notions for convergence from Definition B.4.6 coincide, since weak convergence then implies uniform convergence. This can be seen like this: If n is a natural number and H is an n-dimensional Hilbert space over $\mathbb{K}$ equipped with an orthonormal basis $(b_1, \ldots, b_n)$, any operator T on H can be identified with its transformation matrix $D(T) := D_{B,B}(T)$ in $M_n(\mathbb{K})$.

If now $(T_n)_{n\in\mathbb{N}}$ converges weakly to T, for $i, j \in \mathbb{N}_n$ we in particular have

$$(T_n b_i|b_j) = (D(T_n)e_i|e_j) = D(T_n)^i_j \to D(T)^i_j = (D(T)e_i|e_j) = (Tb_i|b_j),$$

i.e., we have componentwise convergence for the transformation matrices and thus $\|D(T_n) - D(T)\|_{\mathrm{op}} \to 0$, which means $T_n \to T$.

The strong and weak convergence allow for the definition of locally convex topologies (see Definition B.1.27) on $B(H)$: For $\xi \in H$, the maps

$$p_\xi \colon B(H) \longrightarrow [0, \infty), \qquad T \longmapsto \|T\xi\|,$$
$$q_\xi \colon B(H) \longrightarrow [0, \infty), \qquad T \longmapsto |(T\xi|\xi)|$$

are seminorms on $B(H)$ (see Definition A.4.4). The topologies declared by the families $(p_\xi)_{\xi\in H}$, $(q_\xi)_{\xi\in H}$ are called *strong operator topology* respectively *weak operator topology*.

In the finite-dimensional setting, since all notions for convergence coincide, they induce the same topologies. In the infinite-dimensional setting, the weak operator topology is finer than the strong operator topology which is finer than the topology induced by the operator norm.

Definition B.4.8 (Orthonormal basis): Let $(H, (\cdot|\cdot))$ be a Hilbert space, let I be an index set and let $\{e_i \mid i \in I\}$ be an orthonormal system. If it holds $\mathrm{cl}(\mathrm{Lin}(\{e_i \mid i \in I\})) = H$, we call the orthonormal system $\{e_i \mid i \in I\}$ an *orthonormal basis of H*. If the index set I is countable, H is called *separable Hilbert space.*

With this new notion of orthonormal bases, we have a handy generalisation of Remark A.4.10:

Proposition B.4.9: *Let $(H, (\cdot|\cdot))$ be a Hilbert space, let $\{e_i \mid i \in I\}$ be an orthonormal basis of H and let $v \in H$ be a vector. Then it holds*

$$v = \sum_{i\in I}(v|e_i)e_i, \qquad \|v\|^2 = \sum_{i\in I}|(v|e_i)|^2.$$

Since we made no assumptions on the index set in Definition B.4.8, we could end up having to consider not only series, but summable families even. Luckily, in a summable summable family, only countably many terms are non-zero, so series are sufficient. More precisely: For each element v of H, there is a countable subset J of our index set I such that $\sum_{i\in I}(v|e_i)e_i = \sum_{j\in J}(v|e_j)e_j$.

If a Hilbert space has an orthonormal basis, every other orthonormal basis of said Hilbert space has the same cardinality. This cardinality is then called the *Hilbert space dimension*. Two Hilbert spaces are isomorphic if and only if they have the same Hilbert space dimension.

On the other hand, each Hilbert space allows for an orthonormal basis; this statement is proven using Zorns lemma. For each separable Hilbert space, one can construct an orthonormal basis using the Gram-Schmidt algorithm.

5. Hilbert space tensor products

Remark B.5.1: (i) Let $(H_1, (\cdot|\cdot)_1)$ and $(H_2, (\cdot|\cdot)_2)$ be (complex) Hilbert spaces. On their algebraic tensor product (over $\mathbb{C}$)

$$H_1 \otimes H_2 = \left\{ \sum_{i=1}^{n} \xi_i \otimes \eta_i : n \in \mathbb{N}, \xi_i \in H, \eta_i \in H \right\}$$

we may introduce an inner product $(\cdot|\cdot)$, which is uniquely determined by

$$(\xi_1 \otimes \eta_1|\xi_2 \otimes \eta_2) := (\xi_1|\xi_2)_1(\eta_1|\eta_2)_2$$

for all $\xi_1, \xi_2 \in H_1; \eta_1, \eta_2 \in H_2$. This yields a pre-Hilbert space, whose completion will be denoted by $H_1 \hat{\otimes} H_2$ and is called the *Hilbert space tensor product of H_1 and H_2*.

(ii) If $(\xi_i)_{i\in I}$ and $(\eta_j)_{j\in J}$ are orthonormal bases of H_1 and H_2 respectively, then $(\xi_i \otimes \eta_j)_{(i,j)\in I\times J}$ is an orthonormal basis of $H_1 \hat{\otimes} H_2$.

(iii) If $x \in B(H_1)$ and $y \in B(H_2)$ are given, then there is a linear operator $x \otimes y \colon H_1 \otimes_{\mathbb{K}} H_2 \to H_1 \otimes_{\mathbb{K}} H_2$ such that for all $\xi \in H_1, \eta \in H_2$ it holds

$$(x \otimes y)(\xi \otimes \eta) := (x\xi) \otimes (y\eta).$$

This operator extends uniquely to an operator $x \mathbin{\hat{\otimes}} y \in B(H_1 \mathbin{\hat{\otimes}} H_2)$ with $\|x \mathbin{\hat{\otimes}} y\| = \|x\|\|y\|$.

6. Operator algebras

In this section, we will introduce the concept of C^*-algebras, discuss how C^*-algebras can be understood, define what universal C^*-algebras are and talk about tensor products of C^*-algebras.

There are two (as it turns out, but this requires work) equivalent definitions for C^*-algebras.

Definition B.6.1 (C^*-algebra): Let $(A, \|\cdot\|)$ be a Banach algebra over $\mathbb{K}$, i.e., an algebra over $\mathbb{K}$ that as a $\mathbb{K}$-vector space is a complete normed space such that for all $a, b \in A$ it holds $\|ab\| \leq \|a\|\|b\|$. If there is an anti-linear map ${}^* \colon A \to A$ that, for all $a, b \in A$, satisfies

$$a^{**} = a, \qquad (ab)^* = b^*a^*, \qquad \|a^*a\| = \|aa^*\| = \|a\|^2,$$

A is called a *C^*-algebra*. If A is unital as algebra, A is called a *unital C^*-algebra*.

If A and B are C^*-algebras and $\varphi \colon A \to B$ is a linear map that respects the algebra structure and the involution, i.e., for all $a, b \in A$ it holds

$$\varphi(ab) = \varphi(a)\varphi(b), \qquad \varphi(a^*) = \varphi(a)^*,$$

φ is called a **-homomorphism*.

Definition B.6.2 (C^*-algebra): Let H be a Hilbert space over $\mathbb{K}$ and let $A \subseteq H$ be a subalgebra. If A is closed with respect to the topology on $B(H)$ induced by the operator norm and if ${}^*(A) \subseteq A$, then A is called a *C^*-algebra*.

Remark B.6.3: One standard example of a C^*-algebra is the algebra of bounded linear operators on a Hilbert space (see Lemma B.4.4). The Gelfand-Naimark-Segal construction grants that any C^*-algebra can be embedded into the bounded linear operators on some Hilbert space H. This theorem is the reason, why Definition B.6.1 and Definition B.6.2 coincide. It is useful however, to have the abstract definition at hand. Firstly because a C^*-algebra doesn't travel with a Hilbert space in whose bounded linear operators it can be embedded into and secondly, because it allows for interesting constructions.

Important for what we want to do with C^*-algebras is the theorem of Gelfand-Naimark that characterises commutative C^*-algebras: If A is a commutative unital C^*-algebra, then there is some compact Hausdorff space K such that $A \cong C(K)$, i.e., commutative unital C^*-algebras arise as algebras of continuous functions on some compact Hausdorff space.

Definition B.6.4 (Ideal): Let A be a C^*-algebra and let $I \subseteq A$ be a linear subspace. If I is closed under multiplication with elements from A from the left respectively from the right, i.e., $AI = \{ai \mid a \in A, i \in I\} \subseteq I$, $IA = \{ia \mid i \in I, a \in A\} \subseteq I$ respectively, then I is called a *left ideal in A* respectively a *right ideal in A*. If I is both a left- and a right ideal, then I is called a *two-sided ideal in A*. If I is a closed two-sided ideal in A, we write $I \lhd A$.

If A is a C^*-algebra and $M \subseteq A$ is any subset, then

$$\langle M \rangle := \bigcap \{I \subseteq A \mid I \text{ is a two-sided ideal with } M \subseteq I\}$$

is the smallest two-sided ideal in A that contains M, the so-called *ideal generated by M*.

Construction B.6.5 (Universal C^*-algebra): Let I be an arbitrary index set, let $E := \{x_i \mid i \in I\}$ be a set of generators, let $E^* := \{x_i^* \mid i \in I\}$ be the set of adjoints and

$$P(E) := \langle \{x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \mid i_1, \dots, i_k \in I, \alpha_i \in \{1, *\}, x_i \in E \text{ for } i \in I\} \rangle$$

be the involutive $\mathbb{C}$-algebra of non-commutative polynomials in $E \cup E^*$, where $(\lambda x_{i_1} \cdots x_{i_k})^* := \lambda^* x_{i_1}^* \cdots x_{i_k}^*$.

For a subset $R \subseteq P(E)$, denote by $J(R)$ the two-sided ideal in $P(E)$ generated by R and put $A(E, R) := P(E)/J(R)$, the so-called *universal involutive algebra with generators E and relations R*. For $x \in A(E, R)$, put

$$\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E, R)\},$$

where $p\colon A(E, R) \longrightarrow [0, \infty)$ is a C^*-seminorm, if for all $x, y \in A(E, R)$ and $\lambda \in \mathbb{C}$ it holds $p(\lambda x) = |\lambda| p(x)$, $p(x + y) \leq p(x) + p(y)$, $p(xy) \leq p(x)p(y)$ and $p(x^*x) = p(x)^2$. If for all $x \in A(E, R)$ it holds $\|x\| \leq \infty$, the completion

$$C^*(E|R) := \mathrm{cl}_{\|\cdot\|}(\{A(E, R)/\{x \in A(E, R) \mid \|x\| = 0\}\})$$

is called the *universal C^*-algebra with generators E relations R*.

As opposed to the tensor product of vector spaces, the universal C^*-algebra with generators E and relations R does not always exist. But there is a useful criterion to determine, if it does:

Lemma B.6.6: *Let $E = \{x_i \mid i \in I\}$ be a set of generators and let $R \subseteq P(E)$ be the involutive $\mathbb{C}$-algebra of non-commutative polynomials in $E \cup E^*$. If there is a constant $C > 0$ such that $p(x_i) < C$ for all $i \in E$ and all C^*-seminorms on $A(E, R)$, then $C^*(E|R)$ exists, i.e., for all $x \in A(E, R)$ it holds $\|x\| < \infty$.*

The universal C^*-algebra with generators E and relations R has the following universal property:

Theorem B.6.7: *Let $E = \{x_i \mid i \in I\}$ be a set of generators, let $R \subseteq P(E)$ be the involutive $\mathbb{C}$-algebra of non-commutative polynomials in $E \cup E^*$ and assume that $C^*(E|R)$ exists. If B is another C^*-algebra and $E' := \{y_i \mid i \in I\}$ is a subset of B such that E' satisfies the relations R, there is a unique *-homomorphism $\varphi\colon C^*(E|R) \to B$ with $\varphi(x_i) = y_i$.*

Finally, one can "build" new C^*-algebras from old ones via tensor products. As for Hilbert space tensor products, C^*-algebra tensor products are certain completions of the algebraic tensor product of C^*-algebras. However, for C^*-algebras, there are far more norms available, with respect to which one can complete the algebraic tensor product.

Definition B.6.8: Let A and B be C^*-algebras, let H and K be Hilbert spaces and let $\pi_1\colon A \to B(K)$, $\pi_2\colon H \to B(K)$ be two faithful representations, which in this context means injective *-homomorphisms.

The algebraic tensor product $A_1 \otimes_{\mathbb{K}} A_2$ has a unique involution such that $a^* \otimes b^* = (a \otimes b)^*$ for all elementary tensors, making $A \otimes_{\mathbb{K}} B$ an involutive algebra.

The following declares norms on $A \otimes_{\mathbb{K}} B$:

(i) $\|\sum_{i=1}^n a_i \otimes b_i\|_{\min} := \|\sum_{i=1}^n \pi_1(a) \otimes \pi_2(b)\|$,

(ii) $\|x\|_{\max} := \sup\{\|\pi(x)\| \mid \pi\colon A \otimes_{\mathbb{K}} B \longrightarrow B(H) \text{ is a } ^*\text{-homomorphism}\}$.

The completions $A \otimes_{\min} B := \mathrm{cl}_{\|\cdot\|_{\min}}(A \otimes_{\mathbb{K}} B)$, $A \otimes_{\max} B := \mathrm{cl}_{\|\cdot\|_{\max}}(A \otimes_{\mathbb{K}} B)$ are called the *minimal tensor product* (sometimes *spatial tensor product*) respectively *maximal tensor product of A and B*.

It is noteworthy that the definition of the norm in (i) does not depend on the representations π_1, π_2, see [8], Remark 3.3.6.

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