Quantum Automorphism Groups of Graphs and Coherent Algebras

Bachelor's Thesis

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Statement in Lieu of an Oath

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

Ich versichere hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Saarbrücken, June 19, 2019

Luca Leon Junk

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Contents

1 Introduction

In this thesis we study quantum automorphism groups of finite graphs. Those are compact quantum groups that are meant to model the "quantum symmetries" of a graph, generalizing classical automorphism groups of graphs. Compact quantum groups in the sense of Woronowicz are C^*-algebraic generalizations of compact groups.

We will present recent results by Martino Lupini, Laura Mančinska and David Roberson about the asymptotic behaviour of quantum automorphism groups of graphs, when the size of the graph grows:
In the classical case it is known, that almost all graphs have trivial automorphism group in the following sense: When choosing a graph on n vertices uniformly at random, the probability, that its automorphism group is trivial, goes to 1 as n tends to infinity.
This result is due to Erdős and Rényi, who established it in [ER63]. By work of Lupini, Mančinska and Roberson (see [LMR17]), we now have a quantum analogue of this fact: Almost all graphs have trivial quantum automorphism group. The critical tool in the proof is the so-called "coherent algebra" of a graph, a notion originating from algebraic graph theory. We will present their proof of the above theorem and investigate, how coherent algebras can help to study the quantum automorphism group of a graph.

Erdős and Rényi proved in the aforementioned paper also a theorem about the asymptotic behaviour of automorphism groups of trees, i.e. connected graphs without cycles, namely: Almost all trees have non-trivial automorphism group. This shows, that trees behave vastly different in terms of symmetries as compared to general graphs.
In this thesis we will show, that this behaviour carries over to the quantum world: Almost all trees have quantum symmetries, i.e. their quantum automorphism group is strictly larger than their classical automorphism group. The proof of this is a combination of classical results of Erdős and Rényi about trees and a recent result by Schmidt that gives a sufficient criterion for a graph to have quantum symmetry.
Summarized we have the following theorem:

Theorem.

(i) [ER63] Almost all graphs have no symmetry.

(ii) [LMR17] Almost all graphs have no quantum symmetry.

(iii) [ER63] Almost all trees have symmetry.

(iv) Almost all trees have quantum symmetry.

In the next chapter we begin by giving the basic definitions of compact (matrix) quantum groups, quantum automorphism groups of graphs and provide some basics about graphs and their (quantum) symmetries.
In the third chapter we introduce coherent algebras and show, how they can be of use for studying the (quantum) automorphism group of graphs.
In the fourth chapter we present the proof of the main theorem from [LMR17] and discuss some characterizations of graphs with trivial quantum automorphism group. Furthermore we present some own experimental results about the quantum automorphism group of asymmetric graphs, which we obtained by computing the coherent algebra of small graphs.
In the fifth chapter we prove the new result that almost all trees have quantum symmetry.

2 Preliminaries

In this chapter we will provide some background about compact quantum groups, graphs, their (classical) automorphism groups, define their quantum automorphism groups and give meaning to the phrase "quantum symmetry".
Note: In the following, whenever we mention the tensor product of C^*-algebras, we mean the minimal one.

2.1 Compact (Matrix) Quantum Groups

Compact (matrix) quantum groups were first defined by Woronowicz in an attempt to generalize compact groups. As a general reference for compact quantum groups we refer the reader to [Tim08].

Definition 2.1.1. [Wor98] A **compact quantum group (CQG)** is a pair (A, Δ) where A is a unital C^*-algebra and $\Delta : A \to A \otimes A$ is a unital $*$-homomorphism such that:

(i) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$

(ii) $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are linearly dense in $A \otimes A$.

The map $\Delta : A \to A \otimes A$ is called **comultiplication**.

Example 2.1.2. Let G be a compact group and let $C(G)$ be the space of all continuous complex-valued functions on G. We define the map

$$\begin{aligned} \Delta : C(G) &\to C(G \times G) \cong C(G) \otimes C(G) \\ f &\mapsto f \circ \mu \end{aligned}$$

where $\mu : G \times G \to G$ is the group law. Then $(C(G), \Delta)$ is a compact quantum group and in fact all compact quantum groups (A, Δ_A) with commutative C^*-algebra A are of this form (see [Tim08, Proposition 5.1.3]). By identifying G with $(C(G), \Delta)$, one can think of compact quantum groups as a generalization of classical compact groups.

In view of this example, we will often denote the C^*-algebra (or "noncommutative function algebra") of a compact quantum group G by $C(G)$.

Definition 2.1.3. Let $G = (C(G), \Delta_G)$ and $H = (C(H), \Delta_H)$ be two compact quantum groups. We say that H is a **quantum subgroup** of G, if there is a surjective $*$-homomorphism $\varphi : C(G) \to C(H)$ such that $\Delta_H \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_G$. In this case we write "$H \subseteq G$".

Definition 2.1.4. [Wor87] A **compact matrix quantum group (CMQG)** is a pair (A, u) where A is a unital C^*-algebra and $u = (u_{ij})_{i,j=1}^n$ is a matrix with entries in A such that:

(i) A is generated (as a C^*-algebra) by the entries of u.

(ii) The $*$-homomorphism $\Delta : A \to A \otimes A$, $u_{ij} \mapsto \sum_{k=1}^n u_{ik} \otimes u_{kj}$ exists. (The map Δ is called **comultiplication**.)

(iii) The matrix u and its transpose u^t are invertible.

Example 2.1.5. Let $G \subseteq GL_n(\mathbb{C})$ be a compact matrix group and let $(u_{ij})_{i,j=1}^n$ be the coordinate functions on G, i.e.

$$\begin{aligned} u_{ij} &: G \to \mathbb{C} \\ A = (a_{kl})_{k,l=1}^n &\mapsto a_{ij} \end{aligned}$$

Then the pair $(C(G), u = (u_{ij})_{i,j=1}^n)$ is a compact matrix quantum group. Moreover, all compact matrix quantum groups (A, v) with commutative C^*-algebra A are of this form (see [Tim08, Proposition 6.1.11]). So under the identification of G with $(C(G), u)$, compact matrix quantum groups generalize classical compact matrix groups.

A justification for the name *compact matrix quantum group* is the following proposition, a proof of which can be found in [Tim08, Proposition 6.1.4].

Proposition 2.1.6. *Every compact matrix quantum group is a compact quantum group and the two notions of "comultiplication" coincide.*

Remark 2.1.7. If $G = (A, u)$ is a compact matrix quantum group and $H = (B, \Delta_H)$ is a quantum subgroup of G, then H naturally is a compact matrix quantum group. Namely let $\varphi : A \to B$ be a surjective $*$-homomorphism with $\Delta_H \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_G$ where Δ_G is the comultiplication of G and define the matrix $v = (v_{ij})_{i,j=1}^n$ with entries in B via $v_{ij} := \varphi(u_{ij})$. Then B is generated as a C^*-algebra by the entries of v, the matrices v and v^t are invertible and we have:

$$\begin{aligned} \Delta_H(v_{ij}) = \Delta_H(\varphi(u_{ij})) = (\varphi \otimes \varphi)(\Delta_G(u_{ij})) &= (\varphi \otimes \varphi)\left(\sum_{k=1}^n u_{ik} \otimes u_{kj}\right) \\ = \sum_{k=1}^n \varphi(u_{ik}) \otimes \varphi(u_{kj}) &= \sum_{k=1}^n v_{ik} \otimes v_{kj} \end{aligned}$$

Example 2.1.8. Another very important example of a compact matrix quantum group is the quantum symmetric group S_n^+ defined by Wang in [Wan98]:

$$C(S_n^+) := C^*(u_{ij},\ 1 \le i,j \le n \mid u_{ij}^* = u_{ij}^2 = u_{ij},\ \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1\ \forall\, i,j)$$

It is the quantum analogue of the classical symmetric group S_n which is a quantum subgroup of S_n^+ (in the sense of Definition 2.1.3).

We have that $C(S_n^+)$ is commutative for $n = 1, 2, 3$ and the map $\varphi : C(S_n^+) \to C(S_n)$ sending the generators u_{ij} to the corresponding coordinate functions on the permutation matrices is an isomorphism.
For $n \geq 4$, the algebra $C(S_n^+)$ is non-commutative and infinite-dimensional.

Definition 2.1.9. Let A be a unital C^*-algebra and $u = (u_{ij})_{i,j=1}^n$ be a matrix with entries in A. Then u is called a **magic unitary**, if

(i) u_{ij} is a projection (i.e. $u_{ij}^* = u_{ij}^2 = u_{ij}$) for all $1 \leq i, j \leq n$ and

(ii) $\sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{ki} = 1$ for all $1 \leq i \leq n$.

Remark 2.1.10. The second condition in Definition 2.1.9 implies that for each $1 \leq i \leq n$ the projections $\{u_{ij} \mid 1 \leq j \leq n\}$ (and respectively $\{u_{ji} \mid 1 \leq j \leq n\}$) are orthogonal.

Proof. We proof the general result that projections $p_1, \dots, p_n$ with $\sum_{i=1}^n p_i = 1$ are orthogonal.
For every $j \in \{1, \dots, n\}$ we have

$$p_j = p_j^2 = p_j \left(\sum_{i=1}^n p_i \right) p_j = \sum_{i=1}^n p_j p_i p_j = p_j + \sum_{\substack{i=1 \\ i \neq j}}^n p_j p_i p_j$$

So $\sum_{\substack{i=1 \\ i \neq j}}^n p_j p_i p_j = 0$. For each $k \neq j$ the operator $p_j p_k p_j = (p_k p_j)^*(p_k p_j)$ is positive, hence also $-p_j p_k p_j = \sum_{\substack{i=1 \\ i \neq j,k}}^n p_j p_i p_j$ is positive. But this implies that $p_j p_k p_j = 0$ and since

$$\|p_k p_j\|^2 = \|(p_k p_j)^*(p_k p_j)\| = \|p_j p_k p_j\| = 0$$

we have that $p_k p_j = 0$. □

2.2 Graphs and their symmetries

Definition 2.2.1. A **graph** is a tuple $\Gamma = (V, E)$ where V is a non-empty set of vertices and $E \subseteq V \times V$ is a set of edges (in particular, we don't allow a graph to have multiple edges between the same pair of vertices). It is called **finite** if V is finite, and **undirected** if we have $(i, j) \in E \Rightarrow (j, i) \in E$ for all $i, j \in V$. An edge of the form $(i, i) \in E$ $(i \in V)$ is called a **loop**. If $v \in V$ is a vertex, we define its **degree** to be the number of neighbours of v, i.e. the number of vertices $u \in V$ such that $(v, u) \in E$.

In this thesis we will only be concerned with finite undirected graphs without loops. Furthermore, we will usually identify the vertex set V with the set $\{1, \dots, n\}$ where $n = \#V$.

Definition 2.2.2. A **path** of length k in a graph $\Gamma = (V, E)$ is a k-tuple $(e_1, \dots, e_k)$ of edges $e_j = (u_j, v_j) \in E$ such that $v_j = u_{j+1}$ $\forall j = 1, \dots k-1$. It is

called a **cycle** if $u_1 = v_k$.
A graph is called **connected** if for every pair of vertices $i, j \in V$ $(i \neq j)$ there is a path from i to j, i.e. $u_1 = i$ and $v_k = j$. A **tree** is a connected graph without cycles.

Definition 2.2.3. The **adjacency matrix** of a graph $\Gamma = (V, E)$ is the matrix $A = (a_{ij})_{i,j \in V}$ with entries

$$a_{ij} := \begin{cases} 1, & \text{if } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.2.4. An **automorphism** of a graph $\Gamma = (V, E)$ is a bijection $\sigma : V \to V$ that preserves adjacency and non-adjacency, i.e. $(i, j) \in E \Leftrightarrow (\sigma(i), \sigma(j)) \in E$ for all $i, j \in V$.
The set of all automorphisms of Γ forms a group $\mathrm{Aut}(\Gamma)$ under composition and is called the **automorphism group** of Γ. It can be identified with a subgroup of the symmetric group S_n (where $n = \#V$) which can in turn be embedded as permutation matrices in $M_n(\mathbb{C})$.
The automorphism group then has a nice description in terms of the adjacency matrix A of Γ:

$$\mathrm{Aut}(\Gamma) = \{\sigma \in S_n \mid \sigma A = A\sigma\}$$

Definition 2.2.5. We call a graph Γ **symmetric**, if there exists a non-trivial automorphism of Γ, and **asymmetric** otherwise.

2.3 Quantum symmetries

Definition 2.3.1. Let $G = (C(G), \Delta)$ be a compact quantum group and let A be a unital C^*-algebra. An **action** of G on A is a $*$-homomorphism $\alpha : A \to A \otimes C(G)$ such that:

(i) $(\Delta \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha) \circ \alpha$

(ii) The linear span of $\alpha(A)(1 \otimes C(G))$ is dense in $A \otimes C(G)$.

If X is a compact space and G is a compact quantum group acting on $C(X)$, we also say that G acts on X (following the paradigm of identifying spaces with their function algebras).

Example 2.3.2. Let $X_n := \{1, \ldots, n\}$. Then

$$C(X_n) = C^*(p_1, \ldots, p_n \mid p_i^2 = p_i^* = p_i \; \forall i = 1, \ldots, n, \; \sum_{j=1}^{n} p_j = 1)$$

and the quantum group S_n^+ acts on X_n via

$$\alpha : C(X_n) \to C(X_n) \otimes C(S_n^+)$$
$$p_j \mapsto \sum_{i=1}^{n} p_i \otimes u_{ij}$$

and this action is maximal in the sense that every other compact quantum group acting on X_n is a quantum subgroup of S_n^+ (for details see [Wan98]).

In the same way that classical group actions can be understood to model (classical) symmetries of spaces, we can interpret quantum group actions as *quantum symmetries*. With that viewpoint, S_n^+ can be seen as the quantum symmetry group of X_n. We call quantum subgroups of S_n^+ **quantum permutation groups**.
A natural question is now: What is the quantum symmetry group of a graph? As we have already seen, the classical symmetry group of a graph has a nice description in terms of permutation matrices and the adjacency matrix. Going over to function algebras we get:

$$\begin{aligned} C(\mathrm{Aut}(\Gamma)) = C^*(u_{ij}, 1 \leq i,j \leq n \mid \;\; & u_{ij}^2 = u_{ij}^* = u_{ij}, \sum_{s=1}^{n} u_{is} = \sum_{s=1}^{n} u_{sj} = 1, \\ & u_{ij}u_{kl} = u_{kl}u_{ij} \;\; \forall i,j,k,l, \quad uA = Au) \\ = {C(S_n)}\big/{< uA = Au >} & \end{aligned}$$

Dropping commutativity then leads to the following definition which is in analogy with the classical case.

Definition 2.3.3. [Ban05] Let $\Gamma = (V, E)$ be a graph on n vertices and let A be its adjacency matrix. Then the **quantum automorphism group** $\mathrm{QAut}(\Gamma)$ of Γ is defined as the compact matrix quantum group with

$$C(\mathrm{QAut}(\Gamma)) := {C(S_n^+)}\big/{< uA = Au >}$$

and generator matrix $\pi(u) = (\pi(u_{ij}))_{i,j=1}^n$ where $u = (u_{ij})_{i,j=1}^n$ is the generator matrix of $C(S_n^+)$ and

$$\pi : C(S_n^+) \to {C(S_n^+)}\big/{< uA = Au >}$$

is the canonical projection.
It is a quantum subgroup of S_n^+ and acts on V in the canonical way.
We say that Γ has **quantum symmetry**, if $C(\mathrm{QAut}(\Gamma))$ is non-commutative.

Proposition 2.3.4. *In the situation above, the relation $uA = Au$ is equivalent to the relations:*

(i) $u_{ij}u_{kl} = 0 = u_{kl}u_{ij}$ *whenever* $(i,k) \in E$ *and* $(j,l) \notin E$

(ii) $u_{ij}u_{kl} = 0 = u_{kl}u_{ij}$ *whenever* $(i,k) \notin E$ *and* $(j,l) \in E$

Proof. Assume that $uA = Au$ holds. That means that for all $1 \leq i, j \leq n$ we have that

$$\sum_{k:(k,j)\in E} u_{ik} = \sum_{l:(i,l)\in E} u_{lj}$$

Now let $(i,k) \notin E$ and $(j,l) \in E$. Then:

$$u_{ij}u_{kl} = u_{ij}\left(\sum_{r:(r,l)\in E} u_{ir}\right)u_{kl} = u_{ij}\left(\sum_{s:(s,i)\in E} u_{sl}\right)u_{kl} = 0$$

The first equality follows from the fact that u_{ij} appears in the first sum as (j,l) is an edge. The second equality follows from the assumption. The last expression is zero because u_{kl} does not appear in the second sum since (k,i) is not an edge. The relation $u_{kl}u_{ij} = 0$ then follows by applying the involution to $u_{ij}u_{kl} = 0$. The proof of (ii) is completely analogous.
Now for the reverse implication, assume that (i) and (ii) hold and let $i, j \in \{1, \ldots, n\}$ be arbitrary. Then:

$$\begin{aligned}
\sum_{k:(k,j)\in E} u_{ik} &= \left(\sum_{k:(k,j)\in E} u_{ik}\right)\left(\sum_{r=1}^{n} u_{rj}\right) \\
&= \left(\sum_{k:(k,j)\in E} u_{ik}\right)\left(\sum_{r:(i,r)\in E} u_{rj}\right) \\
&= \sum_{r:(i,r)\in E}\ \sum_{k:(k,j)\in E} u_{ik}u_{rj} \\
&= \sum_{r:(i,r)\in E}\left(\sum_{k=1}^{n} u_{ik}\right)u_{rj} \\
&= \sum_{r:(i,r)\in E} u_{rj}
\end{aligned}$$

Where we have used the relations (i) and (ii) in the second and fourth equality. This shows that $uA = Au$ as desired. □

Remark 2.3.5. Let Γ be a graph and let $u = (u_{ij})_{i,j=1}^{n}$ be the generator matrix of $C(\mathrm{QAut}(\Gamma))$. Then the canonical map $\varphi : C(\mathrm{QAut}(\Gamma)) \to C(\mathrm{Aut}(\Gamma))$ sending u_{ij} to the corresponding coordinate function on the permutation matrices makes $\mathrm{Aut}(\Gamma)$ a quantum subgroup of $\mathrm{QAut}(\Gamma)$ and factors through $C(\mathrm{QAut}(\Gamma))/\langle u_{ij}u_{kl} = u_{kl}u_{ij} \rangle$. The induced map

$$\widetilde{\varphi} : C(\mathrm{QAut}(\Gamma))/\langle u_{ij}u_{kl} = u_{kl}u_{ij} \rangle \to C(\mathrm{Aut}(\Gamma))$$

is an isomorphism.

Remark 2.3.6. Let $G = (A, u)$ be a quantum permutation group and let $A_0 \subseteq A$ be the dense $*$-algebra generated by the entries of u. Then the assignments

$$\varepsilon(u_{ij}) = \delta_{ij}$$
$$S(u_{ij}) = u_{ji}^*$$

define a **counit** and an **antipode** on A_0. Together with the comultiplication this makes A_0 into a Hopf-$*$-algebra.
The existence of a dense Hopf-$*$-algebra is also true in general compact quantum groups, for a proof of this see [Tim08, Theorem 5.4.1].

3 Coherent configurations and coherent algebras

Coherent configurations and coherent algebras were first introduced by Higman in [Hig76]. His original motivation came from the representation theory of finite groups, but it turns out that coherent configurations/algebras arise naturally in the context of algebraic combinatorics and algebraic graph theory. As a general reference for this we recommend [Pec02] and [God10].
In this chapter we will see, how one can associate to a graph certain coherent configurations/algebras which "know" whether or not the graph has trivial (quantum) automorphism group. This idea is due to Lupini, Mančinska and Roberson and the results in this chapter are mostly taken from their paper [LMR17].

3.1 Basic definitions and first examples

Definition 3.1.1. [Hig76] Let X be a finite set. A partition $\mathcal{R} = \{R_i \mid i \in I\}$ of $X \times X$ is called a **coherent configuration** on X, if:

(i) There exists $J \subseteq I$ such that $\{R_j \mid j \in J\}$ is a partition of the diagonal $\mathcal{D} = \{(x,x) \mid x \in X\}$ in $X \times X$.

(ii) For each R_i, its transpose $\{(y,x) \mid (x,y) \in R_i\}$ is also in $\mathcal{R}$.

(iii) For all $i,j,k \in I$ and $(x,y) \in R_k$ the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant p_{ij}^k independent of the choice of x and y.

Furthermore, we define the characteristic matrices $A^{(i)}$ ($i \in I$) of $\mathcal{R}$ as

$$A^{(i)}_{xy} := \begin{cases} 1, & (x,y) \in R_i \\ 0, & \text{otherwise} \end{cases}$$

Example 3.1.2. Let G be a finite group acting on a finite set X. The orbits of the induced action on $X \times X$ are called the orbitals of G on X and they form a coherent configuration.

Proof. By identifying X with the diagonal in $X \times X$ we see that the orbits of G acting on X correspond to the orbitals contained in the diagonal. Hence (i) in Definition 3.1.1 is satisfied.

The fact that (ii) holds is immediate from the following:

$$\begin{aligned} &(x, y) \text{ and } (w, z) \text{ are in the same orbital} \\ \Leftrightarrow\ &\exists\, \sigma \in G : \sigma x = w \wedge \sigma y = z \\ \Leftrightarrow\ &(y, x) \text{ and } (z, w) \text{ are in the same orbital} \end{aligned}$$

For (iii), denote the orbitals by O_i ($i \in I$) and let $i, j, k \in I$ and $(x, y), (x', y') \in O_k$. There exists $\sigma \in G$ with $\sigma x = x'$ and $\sigma y = y'$. This induces then a bijection between

$$\{z \in X \mid (x, z) \in O_i \wedge (z, y) \in O_j\}$$

and

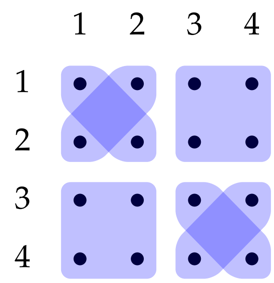
$$\{z \in X \mid (x', z) \in O_i \wedge (z, y') \in O_j\}$$

via $z \mapsto \sigma z$.
Hence

$$p_{ij}^k := \#\{z \in X \mid (x, z) \in O_i \wedge (z, y) \in O_j\}$$

does not depend on the choice of x and y. □

For example let $X = \{1, 2, 3, 4\}$ and $G = \{\mathrm{id}, (1, 2), (3, 4), (1, 2)(3, 4)\}$. Then the action of G on X yields the following coherent configuration:



Definition 3.1.3. A subset $\mathcal{A} \subseteq M_n(\mathbb{C})$ is called a **coherent algebra** if:

(i) $\mathcal{A}^* = \mathcal{A}$

(ii) $\mathcal{A}$ is a unital algebra with respect to ordinary matrix multiplication.

(iii) $\mathcal{A}$ is a unital algebra with respect to entrywise multiplication of matrices.

Proposition 3.1.4. *Let X be a finite set and $\mathcal{R} = \{R_i \mid i \in I\}$ be a coherent configuration on X. Then the linear span of the characteristic matrices $A^{(i)}$ ($i \in I$) of $\mathcal{R}$ is a coherent algebra.*

Proof. Let $\mathcal{A}$ denote the linear span of the $A^{(i)}$. Since the union of all classes of $\mathcal{R}$ is $X \times X$, the all-ones-matrix is the sum of all characteristic matrices, hence in $\mathcal{A}$. That the identity matrix is in $\mathcal{A}$ follows from (i) in Definition 3.1.1. The assertion that $\mathcal{A}$ is closed under the involution $*$ follows from (ii) in Definition 3.1.1. Closedness under entrywise products (denoted by "•") follows from the fact that $A^{(i)} \bullet A^{(j)} = 0$ if $i \neq j$ (since the classes of $\mathcal{R}$ are disjoint) and that $A^{(i)} \bullet A^{(i)} = A^{(i)}$ for all $i \in I$ (since $A^{(i)}$ is a 0-1-matrix). For closedness under ordinary matrix multiplication, let $i, j \in I$ and $x, y \in X$ and consider the (x, y)-entry $c_{xy} = \sum_{z \in X} a^{(i)}_{xz} a^{(j)}_{zy}$ of the product $A^{(i)} A^{(j)}$. Then c_{xy} counts the number of $z \in X$ such that $a^{(i)}_{xz} = 1$ and $a^{(j)}_{zy} = 1$, i.e. the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$. So we have $c_{xy} = p^k_{ij}$ where $k \in I$ such that $(x, y) \in R_k$. By (iii) in Definition 3.1.1, this does not depend on the choice of $(x, y) \in R_k$. So for all $(x, y) \in R_k$ we have that $c_{xy} = p^k_{ij}$ and hence $A^{(i)} A^{(j)} = \sum_{k \in I} p^k_{ij} A^{(k)} \in \mathcal{A}$. □

Proposition 3.1.5. *Let $\mathcal{A} \subseteq M_n(\mathbb{C})$ be a coherent algebra. Then $\mathcal{A}$ has a unique basis of* 0-1-*matrices which are the characteristic matrices of a coherent configuration.*

Proof. See for example Proposition 1.5 in [Pec02]. □

Remark 3.1.6. The two preceding propositions show, that coherent configurations and coherent algebras contain essentially the same information and merely reflect two different perspectives of the same thing. They can be used interchangeably and in the following we will frequently switch between those two perspectives.

3.2 The coherent algebra of a graph

Definition 3.2.1. Let Γ be a graph. The coherent algebra generated by the adjacency matrix of Γ is called the **coherent algebra of** Γ and is denoted by $\mathcal{CA}(\Gamma)$.
The **coherent configuration of** Γ is the coherent configuration corresponding to the coherent algebra of Γ via Proposition 3.1.5.

Example 3.2.2. A graph Γ is called *k-regular* ($k \in \mathbb{N}$) if every vertex of Γ has degree k, i.e. if every vertex has exactly k neighbours.
A k-regular graph is called *strongly regular*, if there exist $\lambda, \mu \in \mathbb{N}$ such that:

(i) Every pair of adjacent vertices has λ common neighbours.

(ii) Every pair of non-adjacent vertices has μ common neighbours.

Regularity can equivalently be encoded in terms of the adjacency matrix $A \in M_n(\mathbb{C})$ as the relation

$$AJ = JA = kJ$$

where $J \in M_n(\mathbb{C})$ denotes the all-ones-matrix.
Furthermore, strong regularity of a k-regular graph can be encoded in the relation

$$A^2 = kI + \lambda A + \mu(J - I - A)$$

where $I \in M_n(\mathbb{C})$ is the identity matrix.
These relations show that for a strongly regular graph every polynomial in

A and J is in the linear span of $\{I, J, A\}$. Moreover, this linear span is also closed under entrywise products because the diagonal of A is zero (since we only consider graphs without loops) so the entrywise product of I and A is the zero matrix. Hence the coherent algebra of a strongly regular graph is always given by $\mathrm{span}\{I, J, A\}$.

3.3 The orbital algebra of a graph

Definition 3.3.1. Let $\Gamma = (V, E)$ be a graph. Its automorphism group naturally acts on the vertices V and by Example 3.1.2 this yields a coherent configuration on V. The corresponding coherent algebra is called the **orbital algebra** of Γ and denoted by $\mathcal{O}(\Gamma)$.

The automorphism group of a graph is trivial if and only if the orbits of its action on the vertices (and also the orbitals) are singletons. Hence we have:

$$\Gamma \text{ has no symmetry} \iff \mathrm{Aut}(\Gamma) = \{\mathrm{id}\} \iff \mathcal{O}(\Gamma) = M_n(\mathbb{C})$$

Definition 3.3.2. For a subset $S \subseteq M_n(\mathbb{C})$ we define its **commutant** S' as

$$S' := \{M \in M_n(\mathbb{C}) \mid MN = NM \text{ for all } N \in S\}$$

The following characterization of the orbital algebra in terms of the automorphism group is mentioned without a proof in [LMR17]. For the reader's convenience we give here a proof of this result.

Proposition 3.3.3. *The orbital algebra of a graph Γ is the commutant of its automorphism group as a subgroup of the permutation matrices:*

$$\mathcal{O}(\Gamma) = \mathrm{Aut}(\Gamma)'$$

Proof. We first show the inclusion "$\subseteq$":
We denote the orbitals of $\mathrm{Aut}(\Gamma)$ by $O_1, \ldots, O_r \subseteq V \times V$ and their characteristic matrices by $A^{(1)}, \ldots, A^{(r)} \in M_n(\mathbb{C})$. They have the entries

$$A^{(k)}_{ij} = \begin{cases} 1, & \text{if } (i,j) \in O_k \\ 0, & \text{otherwise} \end{cases}$$

Since these matrices span the orbital algebra $\mathcal{O}(\Gamma)$, it suffices to show that each $A^{(k)}$ commutes with each element of $\mathrm{Aut}(\Gamma)$. So let $\sigma \in \mathrm{Aut}(\Gamma)$ be an automorphism of Γ (seen as a permutation matrix). Then we have for all $1 \le k \le t$ and $1 \le i, j \le n$ (where n is the number of vertices in Γ):

$$(A^{(k)}\sigma)_{ij} = \sum_{l=1}^{n} A^{(k)}_{il}\sigma_{lj} = A^{(k)}_{i\sigma^{-1}(j)} = \begin{cases} 1, & \text{if } (i, \sigma^{-1}(j)) \in O_k \\ 0, & \text{otherwise} \end{cases}$$

$$(\sigma A^{(k)})_{ij} = \sum_{l=1}^{n} \sigma_{il}A^{(k)}_{lj} = A^{(k)}_{\sigma(i)j} = \begin{cases} 1, & \text{if } (\sigma(i), j) \in O_k \\ 0, & \text{otherwise} \end{cases}$$

But certainly $(i, \sigma^{-1}(j))$ and $(\sigma(i), j)$ are in the same orbital of $\mathrm{Aut}(\Gamma)$, so the expressions above are equal, i.e. $A^{(k)}$ and σ commute, which implies that $\mathcal{O}(\Gamma) \subseteq \mathrm{Aut}(\Gamma)'$.
Now for the reverse inclusion let $M \in \mathrm{Aut}(\Gamma)'$ and let $(i, j), (k, l)$ be in the same orbital of $\mathrm{Aut}(\Gamma)$, i.e. there is $\sigma \in \mathrm{Aut}(\Gamma)$ such that $\sigma(i) = k$ and $\sigma(j) = l$. Then, since M and σ commute:

$$M_{ij} = (\sigma M \sigma^{-1})_{ij} = \sum_{r,s=1}^{n} \sigma_{is} M_{sr} \sigma_{jr} = M_{\sigma(i)\sigma(j)} = M_{kl}$$

This shows that M is constant on the orbitals of $\mathrm{Aut}(\Gamma)$, hence a linear combination of the $A^{(k)}$, hence in $\mathcal{O}(\Gamma)$. □

Corollary 3.3.4. *For a graph* Γ *we have:*

$$\mathcal{CA}(\Gamma) \subseteq \mathcal{O}(\Gamma)$$

Proof. Let A be the adjacency matrix of Γ. By the characterization of the automorphism group of Γ as the set of permutation matrices that commute with A, we know that $A \in \mathrm{Aut}(\Gamma)'$. Hence, by the preceding proposition, $A \in \mathcal{O}(\Gamma)$. And because A generates $\mathcal{CA}(\Gamma)$ as a coherent algebra, the claim follows. □

Remark 3.3.5. Together with the characterization

$$\mathrm{Aut}(\Gamma) = \{\mathrm{id}\} \iff \mathcal{O}(\Gamma) = M_n(\mathbb{C})$$

this yields a sufficient condition for a graph to be asymmetric:

$$\mathcal{CA}(\Gamma) = M_n(\mathbb{C}) \Longrightarrow \mathrm{Aut}(\Gamma) = \{\mathrm{id}\}$$

Note however that the reverse implication does not hold in general, as the inclusion in Corollary 3.3.4 can be strict:
As we have shown in Example 3.2.2, the coherent algebra of a strongly regular graph is (at most) 3-dimensional but there exists an asymmetric strongly regular graph Γ with 25 vertices, i.e. $\dim(\mathcal{CA}(\Gamma)) = 3$ while $\dim(\mathcal{O}(\Gamma)) = 25^2$.

3.4 The quantum orbital algebra of a graph

In this section we aim to construct a quantum analogue of the orbital algebra of a graph in order to decide whether or not its quantum automorphism group is trivial. The construction will be very similar to that of the orbital algebra, using coherent configurations and coherent algebras. In order to be able to define a quantum version of the orbital algebra, we first need to find a quantum analogue of orbits and orbitals:
For this, consider a classical permutation group $G \subseteq S_n$ acting on the set

$X_n = \{1, \ldots, n\}$. Then the equivalence relation of the orbits is:

$$i \sim j \Leftrightarrow \exists\, \sigma \in G : \ \sigma(i) = j \qquad (i, j \in X_n)$$

In terms of coordinate functions $(u_{ij})_{i,j=1}^n$ of permutation matrices in G this can be reformulated as:

$$i \sim j \Leftrightarrow u_{ij} \neq 0 \qquad (i, j \in X_n)$$

Which motivates the following definition due to Banica and Freslon:

Definition 3.4.1. [BF18] Let G be a quantum permutation group acting on the set $X_n = \{1, \ldots, n\}$ and let $u = (u_{ij})_{i,j=1}^n$ be the magic unitary defining $C(G)$. On X_n we define the relation $\sim_1$ via:

$$i \sim_1 j :\Leftrightarrow u_{ij} \neq 0 \qquad (i, j \in X_n)$$

Lemma 3.4.2. *The relation $\sim_1$ as defined in Definition 3.4.1 is an equivalence relation.*

Proof. To see that $\sim_1$ is reflexive, we apply the counit to u_{ii}:

$$\varepsilon(u_{ii}) = \delta_{ii} = 1$$

Hence $u_{ii} \neq 0$, i.e. $i \sim_1 i$ for all $i \in X_n$.
For the symmetry of $\sim_1$, assume that $i \nsim_1 j$, i.e. $u_{ij} = 0$. Then

$$0 = S(u_{ij}) = u_{ji}$$

i.e. $j \nsim_1 i$.
The transitivity of $\sim_1$ can be seen as follows: Let $i \sim_1 j$ and $j \sim_1 k$, i.e. $u_{ij} \neq 0 \neq u_{jk}$. Then also $u_{ij} \otimes u_{jk} \neq 0$. And therefore

$$\begin{aligned}(u_{ij} \otimes u_{jk})\Delta(u_{ik}) = (u_{ij} \otimes u_{jk}) \sum_{l=1}^{n} u_{il} \otimes u_{lk} &= \sum_{l=1}^{n} \left(u_{ij}u_{il}\right) \otimes \left(u_{jk}u_{lk}\right) \\ &= u_{ij} \otimes u_{jk} \neq 0\end{aligned}$$

which implies that $\Delta(u_{ik}) \neq 0$, so $u_{ik} \neq 0$, i.e. $i \sim_1 k$. □

Definition 3.4.3. [BF18] Let G be a quantum permutation group acting on $X_n = \{1, \ldots, n\}$. We define the **(quantum) orbits** of G on X to be the equivalence classes of the equivalence relation $\sim_1$.

In the case when G is a classical permutation group (i.e. $C(G)$ is commutative), this definition agrees with the usual definition of orbits.

Now that we have a quantum version of orbits, we aim to find a quantum version of orbitals.
Consider again the situation from the beginning of the section. The equivalence relation of the orbitals is:

$$(i, j) \sim (k, l) \Leftrightarrow \exists\, \sigma \in G : \ \sigma(i) = k \wedge \sigma(j) = l \qquad (i, j, k, l \in X_n)$$

which can be reformulated in terms of coordinate functions $(u_{ij})_{i,j=1}^n$ of permutation matrices in G as:

$$(i,j) \sim (k,l) \Leftrightarrow u_{ik}u_{jl} \neq 0 \qquad (i,j,k,l \in X_n)$$

This motivates the following definition due to Lupini, Mančinska and Roberson.

Definition 3.4.4. [LMR17] Let G be a quantum permutation group acting on $X_n = \{1,\dots,n\}$ and let $u = (u_{ij})_{i,j=1}^n$ be the magic unitary defining $C(G)$. On $X_n \times X_n$ we define the relation $\sim_2$ via:

$$(i,j) \sim_2 (k,l) :\Leftrightarrow u_{ik}u_{jl} \neq 0 \qquad (i,j,k,l \in X_n)$$

Lemma 3.4.5. *The relation $\sim_2$ as defined in Definition 3.4.4 is an equivalence relation.*

Proof. For the reflexivity, note that

$$\varepsilon(u_{ii}u_{jj}) = \varepsilon(u_{ii})\varepsilon(u_{jj}) = \delta_{ii}\delta_{jj} = 1$$

since the counit is an algebra homomorphism. This implies that $u_{ii}u_{jj} \neq 0$, i.e. $(i,j) \sim_2 (i,j)$ for all $(i,j) \in X_n \times X_n$.
For the symmetry of $\sim_2$ let $(i,j),(k,l) \in X_n \times X_n$ with $(i,j) \sim_2 (k,l)$, i.e. $u_{ik}u_{jl} \neq 0$. Then by applying the antipode to $u_{ki}u_{lj}$ we see that

$$S(u_{ki}u_{lj}) = S(u_{ki})S(u_{lj}) = u_{ik}u_{jl} \neq 0$$

so $u_{ki}u_{lj} \neq 0$, i.e. $(k,l) \sim_2 (i,j)$.
For the transitivity, let $(i,j),(k,l),(r,s) \in X_n \times X_n$ with $(i,j) \sim_2 (k,l)$ (i.e. $u_{ik}u_{jl} \neq 0$) and $(k,l) \sim_2 (r,s)$ (i.e. $u_{kr}u_{ls} \neq 0$). We have

$$\begin{aligned} \Delta(u_{ir}u_{js}) &= \Delta(u_{ir})\Delta(u_{js}) \\ &= \left(\sum_{p=1}^n u_{ip} \otimes u_{pr}\right)\left(\sum_{q=1}^n u_{jq} \otimes u_{qs}\right) \\ &= \sum_{p,q=1}^n u_{ip}u_{jq} \otimes u_{pr}u_{qs} \end{aligned}$$

and therefore

$$\begin{aligned} u_{ik} \otimes u_{kr}\Delta(u_{ir}u_{js})u_{jl} \otimes u_{ls} &= \sum_{p,q=1}^n u_{ik}u_{ip}u_{jq}u_{jl} \otimes u_{kr}u_{pr}u_{qs}u_{ls} \\ &= u_{ik}u_{jl} \otimes u_{kr}u_{ls} \\ &\neq 0 \end{aligned}$$

So in particular $\Delta(u_{ir}u_{js}) \neq 0$, which implies that $u_{ir}u_{js} \neq 0$, i.e. $(i,j) \sim_2 (r,s)$. ∎

Remark 3.4.6. In the case of a classical group G acting on a set X, one can also define higher order orbits (or *k-orbits*) as the orbits of the induced action of G on $X^k = X \times \ldots \times X$. We have found quantum analogues of this for $k = 1$ and $k = 2$. But already for $k = 3$ it is absolutely not clear how to even define something similar to the relations $\sim_1$ or $\sim_2$ because of the non-commutativity in the quantum case. One could try to define $\sim_3$ in a similar way as $\sim_1$ and $\sim_2$ via a product of three elements u_{ij}, u_{kl}, u_{rs} but there is no canonical order in which one should multiply them.

Definition 3.4.7. Let G be a quantum permutation group acting on $X_n = \{1, \ldots, n\}$. We define the **(quantum) orbitals** of G on X_n to be the equivalence classes of the equivalence relation $\sim_2$.

In the construction of the orbital algebra of a graph we used the fact, that the orbitals of the automorphism group form a coherent configuration. We will now show the quantum analogue of this which is due to Lupini, Mančinska and Roberson ([LMR17]).

Proposition 3.4.8. *Let G be a quantum permutation group acting on $X_n = \{1, \ldots, n\}$. Then the quantum orbitals of G form a coherent configuration.*

Proof. Denote by $u = (u_{ij})_{i,j=1}^n$ the magic unitary defining $C(G)$ and let $\mathcal{R} = \{R_i \mid i \in I\}$ be the quantum orbitals of G on X_n. Since the R_i are the classes of the equivalence relation $\sim_2$ it is clear that they form a partition of $X_n \times X_n$. Now let $(i,j), (k,k) \in X_n \times X_n$ with $(i,j) \sim_2 (k,k)$, i.e. $u_{ik}u_{jk} \neq 0$. Then the orthogonality of the rows of u implies that $i = j$. Thus every orbital intersecting the diagonal of $X_n \times X_n$ is contained in it. This shows that condition (i) in Definition 3.1.1 is satisfied.
Now assume that $(i,j) \sim_2 (k,l)$ for some $(i,j), (k,l) \in X_n \times X_n$. Then $u_{ik}u_{jl} \neq 0$, so also $u_{jl}u_{ik} = (u_{ik}u_{jl})^* \neq 0$, which implies that $(j,i) \sim_2 (l,k)$. Hence, condition (ii) in Definition 3.1.1 is satisfied.
For verifying condition (iii) in Definition 3.1.1 let $r, s, t \in I$, $(i,j), (i',j') \in R_t$ and put

$$S := \{k \in X \mid (i,k) \in R_r \text{ and } (k,j) \in R_s\}$$
$$S' := \{k' \in X \mid (i',k') \in R_r \text{ and } (k',j') \in R_s\}$$

Since $u_{ii'}u_{jj'} \neq 0$ we can conclude from

$$\begin{aligned} \#S \cdot u_{ii'}u_{jj'} &= u_{ii'} \left(\sum_{k \in S} 1 \right) u_{jj'} = u_{ii'} \left(\sum_{k \in S} \sum_{k' \in X} u_{kk'} \right) u_{jj'} \\ &= \sum_{k \in S} \sum_{k' \in X} u_{ii'}u_{kk'}u_{jj'} = \sum_{k \in S} \sum_{k' \in S'} u_{ii'}u_{kk'}u_{jj'} \\ &= \sum_{k' \in S'} \sum_{k \in S} u_{ii'}u_{kk'}u_{jj'} = \sum_{k' \in S'} \sum_{k \in X} u_{ii'}u_{kk'}u_{jj'} \\ &= u_{ii'} \left(\sum_{k' \in S'} \sum_{k \in X} u_{kk'} \right) u_{jj'} = u_{ii'} \left(\sum_{k' \in S'} 1 \right) u_{jj'} \\ &= \#S' \cdot u_{ii'}u_{jj'} \end{aligned}$$

that $\#S = \#S'$. So the numbers $p_{rs}^t := \#S$ do not depend on the choice of $(i,j) \in R_t$. This concludes the proof. $\square$

With this fact in hand, we can now make the following definition:

Definition 3.4.9. Let Γ be a graph. The coherent algebra corresponding to the coherent configuration formed by the quantum orbitals of $\mathrm{QAut}(\Gamma)$ is called the **quantum orbital algebra** of Γ and denoted by $\mathcal{QO}(\Gamma)$.

The next result is a refinement of Corollary 3.3.4.

Proposition 3.4.10. *For a graph Γ we have the following chain of inclusions:*

$$\mathcal{CA}(\Gamma) \subseteq \mathcal{QO}(\Gamma) \subseteq \mathcal{O}(\Gamma)$$

Proof. Denote by $u = (u_{ij})_{i,j=1}^n$ the magic unitary defining $C(\mathrm{QAut}(\Gamma))$ and by A the adjacency matrix of Γ.
Recall that by Proposition 2.3.4 we have $u_{ij}u_{kl} = 0$ whenever (i,k) is an edge and (j,l) is not. Thus, we can write the adjacency matrix of Γ as a sum of characteristic matrices of the coherent configuration formed by the quantum orbitals. Therefore $A \in \mathcal{QO}(\Gamma)$ and hence $\mathcal{CA}(\Gamma) \subseteq \mathcal{QO}(\Gamma)$.
The second inclusion follows from the observation, that the orbitals of $\mathrm{Aut}(\Gamma)$ are a refinement of the quantum orbitals of $\mathrm{QAut}(\Gamma)$. This is true because $\mathrm{Aut}(\Gamma)$ is a quantum subgroup of $\mathrm{QAut}(\Gamma)$ so we have a surjective homomorphism $\varphi : C(\mathrm{QAut}(\Gamma)) \to C(\mathrm{Aut}(\Gamma))$ and if $\varphi(u_{ij}u_{kl})$ is non-zero, then already $u_{ij}u_{kl}$ must have been non-zero, i.e. if (i,k) and (j,l) are in the same orbital, they are also in the same quantum orbital. $\square$

Corollary 3.4.11. *Let Γ be a graph and let $u = (u_{ij})_{i,j=1}^n$ be the magic unitary defining $C(\mathrm{QAut}(\Gamma))$. We have $u_{ij}u_{kl} = 0$ unless (i,k) and (j,l) are in the same class of the coherent configuration of Γ.*

Proof. This follows from the first inclusion in Proposition 3.4.10. $\square$

Recall from Proposition 3.3.3 that we have the following characterization of the orbital algebra of a graph Γ:

$$\mathcal{O}(\Gamma) = \mathrm{Aut}(\Gamma)'$$

If we write $\mathrm{Aut}(\Gamma)$ as a compact matrix quantum group $(C(\mathrm{Aut}(\Gamma)), u)$, where $u = (u_{ij})_{i,j=1}^n$ are as usual the coordinate functions on the permutation matrices, we can reformulate the above equation as

$$\forall M \in M_n(\mathbb{C}) : \quad M \in \mathcal{O}(\Gamma) \iff Mu = uM$$

We have the exact quantum analogue of this, proved by Lupini, Mančinska and Roberson:

Theorem 3.4.12. *[LMR17] Let Γ be a graph and let $u = (u_{ij})_{i,j=1}^n$ be the magic unitary defining $C(\mathrm{QAut}(\Gamma))$. Then we have:*

$$\forall M \in M_n(\mathbb{C}) : \quad M \in \mathcal{QO}(\Gamma) \iff Mu = uM$$

Proof. Let $M = (M_{ij})_{i,j=1}^n \in \mathcal{QO}(\Gamma)$. Then M is constant on the quantum orbitals of $\mathrm{QAut}(\Gamma)$, i.e. for all $(i,k),(l,j)$ with $(i,k) \sim_2 (l,j)$ we have that $M_{ik} = M_{lj}$. Using this we compute:

$$\begin{aligned}
(Mu)_{ij} &= \sum_{k=1}^n M_{ik}u_{kj} = \left(\sum_{l=1}^n u_{il}\right)\left(\sum_{k=1}^n M_{ik}u_{kj}\right) \\
&= \sum_{k,l=1}^n M_{ik}u_{il}u_{kj} = \sum_{\substack{k,l=1 \\ (i,k)\sim_2(l,j)}}^n M_{ik}u_{il}u_{kj} \\
&= \sum_{\substack{k,l=1 \\ (i,k)\sim_2(l,j)}}^n M_{lj}u_{il}u_{kj} = \sum_{k,l=1}^n M_{lj}u_{il}u_{kj} \\
&= \left(\sum_{l=1}^n u_{il}M_{lj}\right)\left(\sum_{k=1}^n u_{kj}\right) = \sum_{l=1}^n u_{il}M_{lj} \\
&= (uM)_{ij}
\end{aligned}$$

This shows that indeed $Mu = uM$.
Conversely, let $M \in M_n(\mathbb{C})$ be arbitrary and assume that M and u commute. Then by the same calculation as above we have for all $1 \leq i,j \leq n$:

$$\sum_{\substack{k,l=1 \\ (i,k)\sim_2(l,j)}}^n M_{ik}u_{il}u_{kj} = (Mu)_{ij} = (uM)_{ij} = \sum_{\substack{k,l=1 \\ (i,k)\sim_2(l,j)}}^n M_{lj}u_{il}u_{kj}$$

Let $(i,p),(q,j)$ be in the same quantum orbital. Then by multiplying both sides of the above equation with u_{iq} from the left and u_{pj} from the right we obtain:

$$M_{ip}u_{iq}u_{pj} = M_{qj}u_{iq}u_{pj}$$

And since $u_{iq}u_{pj} \neq 0$ we can conclude that $M_{ip} = M_{qj}$, i.e. M is constant on the quantum orbitals of $\mathrm{QAut}(\Gamma)$ and hence in $\mathcal{QO}(\Gamma)$. □

4 (Quantum) Asymmetric Graphs

In this chapter we present the proof of the main result from [LMR17] and discuss some characterizations of graphs with trivial quantum automorphism group. Such graphs will be called **quantum asymmetric**. Furthermore we present some own experimental results about the quantum automorphism group of asymmetric graphs, which we obtained by computing the coherent algebra of small graphs.

4.1 Characterizations and a quantitative result

We begin by presenting the proof of Lupini, Mančinska and Roberson, that almost all graphs are quantum asymmetric. This fact is in analogy with the following classical result of Erdős and Rényi:

Theorem 4.1.1. *Almost all graphs are asymmetric, in the following sense: Among all graphs on n vertices, the proportion of those with trivial automorphism group tends to 1 as $n \to \infty$.*

Theorem 4.1.2. *[LMR17] Let Γ be a graph on n vertices. We have:*

$$\mathrm{QAut}(\Gamma) = \{\mathrm{id}\} \iff \mathcal{QO}(\Gamma) = M_n(\mathbb{C})$$

Proof. Assume that $\mathrm{QAut}(\Gamma) = \{\mathrm{id}\}$. Then the magic unitary defining $C(\mathrm{QAut}(\Gamma))$ is just the identity matrix and commutes with every element of $M_n(\mathbb{C})$. Hence by Theorem 3.4.12 we have that $\mathcal{QO}(\Gamma) = M_n(\mathbb{C})$.
Conversely assume that $\mathcal{QO}(\Gamma) = M_n(\mathbb{C})$, i.e. that the quantum orbitals are singletons. Since the quantum orbits are exactly the quantum orbitals contained in the diagonal, these are singletons too. So - by the very definition of the quantum orbits - this means that $u_{ij} \neq 0$ if and only if $i = j$. Thus u is the identity matrix, i.e. $\mathrm{QAut}(\Gamma) = \{\mathrm{id}\}$. □

The following corollary is the crucial observation of Lupini, Mančinska and Roberson. It provides a sufficient (and computable) criterion for a graph to be quantum asymmetric.

Corollary 4.1.3. *[LMR17] Let Γ be a graph on n vertices. We have:*

$$\mathcal{CA}(\Gamma) = M_n(\mathbb{C}) \Longrightarrow \mathrm{QAut}(\Gamma) = \{\mathrm{id}\}$$

Proof. From Proposition 3.4.10 we know that the coherent algebra is contained in the quantum orbital algebra. So if $\mathcal{CA}(\Gamma) = M_n(\mathbb{C})$, then also $\mathcal{QO}(\Gamma) = M_n(\mathbb{C})$. The result then follows from Theorem 4.1.2. □

So in fact we have the following chain of implications:

$$\mathcal{CA}(\Gamma) = M_n(\mathbb{C}) \Rightarrow \mathrm{QAut}(\Gamma) = \{\mathrm{id}\} \Rightarrow \mathrm{Aut}(\Gamma) = \{\mathrm{id}\}$$

One can ask whether one of these implications is actually an equivalence. By Remark 3.3.5 we know, that this can be true for at most one of them and not for both. This question is discussed in more detail for the second implication in the next section.

The following statement follows from Theorem 4.1 in [BK79].

Theorem 4.1.4. *Almost all graphs $\Gamma = (V, E)$ have coherent algebra $\mathcal{CA}(\Gamma)$ equal to $M_n(\mathbb{C})$ (where $n = \#V$).*

Combining this with the preceding corollary, we obtain:

Corollary 4.1.5. *[LMR17] Almost all graphs are quantum asymmetric.*

Remark 4.1.6. From both Theorem 4.1.4 and Corollary 4.1.5 one can deduce Theorem 4.1.1.

The following proposition and its proof are essentially also contained in [LMR17], but for the sake of exposition we collect these facts here together in one statement.

Proposition 4.1.7. *Let Γ be a graph on n vertices and let $u = (u_{ij})_{i,j=1}^n$ be the magic unitary generating $C(\mathrm{QAut}(\Gamma))$. Then the following are equivalent:*

(i) $\mathrm{QAut}(\Gamma) = \{\mathrm{id}\}$

(ii) $\mathcal{QO}(\Gamma) = M_n(\mathbb{C})$

(iii) Every subspace $K \subseteq \mathbb{C}^n$ is invariant for u, i.e. $u(K) \subseteq K \otimes C(\mathrm{QAut}(\Gamma))$.

(iv) $\mathrm{Hom}(1, u^{\otimes 2}) := \{f \in \mathbb{C}^n \otimes \mathbb{C}^n \mid u^{\otimes 2}(f) = f \otimes 1\} = \mathbb{C}^n \otimes \mathbb{C}^n$

Where we regard u as a linear map[1] $u : \mathbb{C}^n \to \mathbb{C}^n \otimes C(\mathrm{QAut}(\Gamma))$ defined on the standard basis vectors as

$$u(e_i) = \sum_{j=1}^{n} e_j \otimes u_{ji}$$

and $u^{\otimes 2} : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n \otimes C(\mathrm{QAut}(\Gamma))$ is the linear map defined as

$$u^{\otimes 2}(e_i \otimes e_j) = \sum_{k,l=1}^{n} e_k \otimes e_l \otimes u_{ki} u_{lj}$$

[1] It is in fact a (finite-dimensional) *representation* of the compact quantum group $\mathrm{QAut}(\Gamma)$, i.e. an element in $M_n(C(\mathrm{QAut}(\Gamma)))$ satisfying $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ for all $i, j = 1, \ldots, n$. One can define tensor products and morphisms of such representations, which explains the notation used in the proposition. (For details see for example [NT13])

Proof. The equivalence of (i) and (ii) was already shown in Theorem 4.1.2. For the equivalence of (ii) and (iii) note that by Theorem 2.3 in [Ban05] a subspace K of $\mathbb{C}^n$ is invariant for u if and only if u commutes with the orthogonal projection onto K. If (ii) holds, u commutes with every such projection by Theorem 3.4.12 and thus (iii) follows. If (iii) holds, u commutes with every projection in $M_n(\mathbb{C})$ and since every matrix is a linear combination of projections, u commutes with every matrix in $M_n(\mathbb{C})$ and (ii) follows by Theorem 3.4.12.
The equivalence of (ii) and (iv) amounts to the claim that for $f \in \mathbb{C}^n \otimes \mathbb{C}^n$ we have that $u^{\otimes 2}(f) = f \otimes 1$ if and only if f is constant on the quantum orbitals of QAut(Γ).
For $f = \sum_{i,j=1}^n f_{ij}(e_i \otimes e_j)$ we have

$$u^{\otimes 2}(f) = \sum_{i,j=1}^n f_{ij} u^{\otimes 2}(e_i \otimes e_j) = \sum_{i,j,k,l=1}^n f_{ij}(e_k \otimes e_l \otimes u_{ki}u_{lj}) = \sum_{k,l=1}^n e_k \otimes e_l \otimes \left(\sum_{i,j=1}^n f_{ij} u_{ki} u_{lj} \right)$$

and

$$f \otimes 1 = \left(\sum_{i,j=1}^n f_{ij}(e_i \otimes e_j) \right) \otimes 1 = \sum_{k,l=1} e_k \otimes e_l \otimes (f_{kl} 1)$$

So $u^{\otimes 2}(f) = f \otimes 1$ if and only if

$$\sum_{i,j=1}^n f_{ij} u_{ki} u_{lj} = f_{kl} 1 \quad \forall k, l = 1, \ldots, n$$

Now assume that this holds and let $k, l \in \{1, \ldots, n\}$ be arbitrary and let (r, s) be in the same quantum orbital as (k, l), i.e. $u_{kr}u_{ls} \neq 0$. Then

$$f_{rs} u_{kr} u_{ls} = u_{kr} \left(\sum_{i,j=1}^n f_{ij} u_{ki} u_{lj} \right) u_{ls} = u_{kr} (f_{kl} 1) u_{ls} = f_{kl} u_{kr} u_{ls}$$

implies that $f_{rs} = f_{kl}$. Hence f is constant on the quantum orbitals of QAut(Γ). Conversely, assume that f is constant on the quantum orbitals, then we have for all $k, l = 1, \ldots, n$

$$\sum_{i,j=1}^n f_{ij} u_{ki} u_{lj} = \sum_{i,j=1}^n f_{kl} u_{ki} u_{lj} = f_{kl} \sum_{i,j=1}^n u_{ki} u_{lj} = f_{kl} 1$$

and hence $u^{\otimes 2}(f) = f \otimes 1$. □

4.2 Experimental results

All the characterizations of quantum asymmetric graphs in the last section have in common, that none of them can be formulated purely in terms of classical graph theory. But it would be nice to have a characterization of quantum asymmetry in terms of classical graph invariants. One could argue that

$$\mathrm{QAut}(\Gamma) = \{\mathrm{id}\} \iff \mathrm{Aut}(\Gamma) = \{\mathrm{id}\}$$

would be a plausible characterization of quantum asymmetry, since the implication "$\Rightarrow$" is known to be true (since $\mathrm{Aut}(\Gamma) \subseteq \mathrm{QAut}(\Gamma)$) and a counterexample to the reverse implication would yield a compact quantum group, wich is a quantum analogue of the trivial group in the sense of Remark 2.3.5. The existence of such a "quantum trivial group" would be rather counterintuitive and surprising and is not to be expected. Unfortunately, the proof of the above equivalence still eludes us. Nevertheless we managed to verify it with a computer for all graphs with at most 10 vertices using the sufficient condition from Corollary 4.1.3 and obtained the following theorem.

Theorem 4.2.1. *For all graphs* Γ *with at most* 10 *vertices we have:*

$$\mathrm{QAut}(\Gamma) = \{\mathrm{id}\} \iff \mathrm{Aut}(\Gamma) = \{\mathrm{id}\}$$

The statement we have actually shown is a classical one, namely:
For all graphs Γ with at most $n = 10$ vertices we have:

$$\mathcal{CA}(\Gamma) = M_n(\mathbb{C}) \iff \mathrm{Aut}(\Gamma) = \{\mathrm{id}\}$$

But by Corollary 4.1.3 this yields the quantum statement in Theorem 4.2.1.

In contrast to the quantum orbital algebra, the coherent algebra of a graph can be computed in polynomial time using the two-dimensional *Weisfeiler-Lehman algorithm*. This algorithm (and its higher-dimensional extensions) determine a strong graph invariant and are used by many graph software packages to test whether or not two given graphs are isomorphic[2].
We will briefly describe the two-dimensional Weisfeiler-Lehman algorithm. For details we refer to [Für17].

[2]Two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are called *isomorphic*, if there is a bijection $\sigma : V_1 \to V_2$ which preserves adjacency and non-adjacency, i.e. for all $u, v \in V_1$ we have that $(u, v) \in E_1 \iff (\sigma(u), \sigma(v)) \in E_2$.

The two-dimensional Weisfeiler-Lehman algorithm:

Given: $\Gamma = (V, E)$

We denote by $\Delta := \{(u,u) \mid u \in V\} \subseteq V \times V$ the diagonal in $V \times V$.

1. Partition $V \times V$ into the self-loops, edges and non-edges of Γ, i.e. put $\mathcal{R} = \{\Delta, E, E^c \setminus \Delta\}$.

2. For every $(u,v) \in V \times V$ and $R, S \in \mathcal{R}$ put

$$p(u,v;R,S) := \#\{w \in V \mid (u,w) \in R \text{ and } (w,v) \in S\}.$$

   Define a new partition $\mathcal{R}'$ of $V \times V$ by declaring $(u,v),(u',v') \in V \times V$ to be in the same class if

$$p(u,v;R,S) = p(u',v';R,S) \quad \forall R, S \in \mathcal{R}.$$

3. If $\mathcal{R}' = \mathcal{R}$, then output $\mathcal{R}$, else repeat step 2 with $\mathcal{R}'$ in place of $\mathcal{R}$.

Result: The coherent configuration $\mathcal{R}$ of Γ.

The idea behind Step 2 is to calculate the intersection numbers $p(u,v;R,S)$ and refine the partition until they only depend on the class in which (u,v) is and not on the specific choice of u and v. The resulting numbers will then be the p_{ij}^k from Definition 3.1.1.

The canonical implementation of this algorithm is unfortunately quite slow. In our calculations we have used an optimized implementation from [Rei]. As a first step we generated a list of all graphs of size at most 10 using the graph software package nauty[3]. Then we singled out those with trivial automorphism group, also using nauty. For all resulting graphs we calculated their coherent configuration using the two-dimensional Weisfeiler-Lehman algorithm. The resulting configuration was always the partition of $\{1,\dots,n\} \times \{1,\dots,n\}$ into singletons, i.e. the coherent algebra was always the full matrix algebra. Hence by Corollary 4.1.3 these graphs are quantum asymmetric.

[3] `http://pallini.di.uniroma1.it/`

5 Quantum Symmetries of Trees

In this chapter we prove the new result that almost all trees have quantum symmetry. This is a quantum analogue of a classical theorem by Erdős and Rényi:

Theorem 5.1. *[ER63] Almost all trees are symmetric in the following sense: Among all trees on n vertices, the proportion of those with non-trivial automorphism group tends to* 1 *as* $n \to \infty$.

Definition 5.2. Let $\Gamma = (V, E)$ be a graph. A triple (u_1, u_2, v) of vertices $u_1, u_2, v \in V$ is called a **cherry**, if

(i) u_1, u_2 and v are pairwise distinct,

(ii) u_1 and u_2 are adjacent to v,

(iii) u_1 and u_2 have degree 1 and

(iv) v has degree 3.

Note, that in their paper [ER63], Erdős and Rényi used a slightly different notion of cherries. In their definition, the requirement (iv) from above is missing. This changes the formulas in the proof of Lemma 5.5 compared their proof of Theorem 5.1 by a small degree.
The key fact about cherries is, that if a graph has a cherry (u_1, u_2, v), then it has a non-trivial automorphism that swaps u_1 and u_2 and fixes every other vertex. Erdős and Rényi proved Theorem 5.1 by showing that almost all trees contain at least one cherry.
To show that almost all trees have quantum symmetry, it is interestingly enough to show that almost all trees have *two* cherries! The key fact in the quantum case is the following result by Schmidt ([Sch18, Theorem 2.2]). To state it, we need to define the support of a graph automorphism.

Definition 5.3. Let $\Gamma = (V, E)$ be a graph and let $\sigma : V \to V$ be an automorphism of Γ. The set

$$\{v \in V \mid \sigma(v) \neq v\}$$

is called the **support** of σ.

Proposition 5.4. *[Sch18, Theorem 2.2] Let* Γ *be a graph. If there exist two non-trivial automorphisms* σ, τ *of* Γ *that have disjoint support, then* Γ *has quantum symmetry.*

So if a graph has two cherries, it has two disjoint automorphisms and thus has quantum symmetry. We will now prove, that almost all trees contain at least two cherries. Luckily, Erdős and Rényi's proof can easily be adjusted to yield the existence of two cherries instead of just one. Our proof closely follows the argument of Erdős and Rényi, but we get slightly different formulas in (5.1) and (5.2) because of our modified definition of cherries. The main difference between our proof and their proof is in the last step, which provides the existence of two cherries and not just one.

Lemma 5.5. *Almost all trees contain at least two cherries in the following sense: The probability, that a tree - drawn uniformly at random from the set of all trees on n vertices - has at least two cherries, goes to 1 as n tends to infinity.*

Proof. For every $n \in \mathbb{N}$ we denote by $\mathbb{P}$ the uniform probability measure on the set of all trees on n vertices.
Let T_n be a random tree on n vertices and denote these vertices by $v_1, \dots, v_n$. For every choice of indices $i_1, i_2, j \in \{1, \dots, n\}$ we define

$$\varepsilon_{i_1,i_2,j}(T_n) := \begin{cases} 1 & \text{if } (v_{i_1}, v_{i_2}, v_j) \text{ is a cherry in } T_n \\ 0 & \text{otherwise} \end{cases}$$

It is a well-known fact that the number of trees on n labelled vertices is equal to n^{n-2}. Now let $i_1, i_2, j \in \{1, \dots, n\}$ be pairwise distinct labels and let $\Gamma = (V, E)$ be a tree on $n-3$ vertices labelled with $\{1, \dots, n\} \setminus \{i_1, i_2, j\}$. By attaching a cherry (v_{i_1}, v_{i_2}, v_j) at any vertex $u \in V$ we can construct a tree on n vertices with a cherry at (i_1, i_2, j). On the other hand, any tree on n vertices with a cherry at (i_1, i_2, j) can be constructed in this way. Since Γ has $n-3$ vertices, we have $n-3$ possibilities for choosing u, thus there are $(n-3)(n-3)^{n-5} = (n-3)^{n-4}$ trees on n vertices with a cherry at (i_1, i_2, j). Hence:

$$\mathbb{E}[\varepsilon_{i_1,i_2,j}] = \frac{(n-3)^{n-4}}{n^{n-2}} \tag{5.1}$$

for all pairwise distinct $i_1, i_2, j \in \{1, \dots, n\}$.
Similarly, let $j_1, i_1, i_2, j_2, i_3, i_4 \in \{1, \dots, n\}$ be labels and let $\Gamma = (V, E)$ be a tree with vertices labelled with $\{1, \dots, n\} \setminus \{j_1, i_1, i_2, j_2, i_3, i_4\}$. In the case that all labels $j_1, i_1, i_2, j_2, i_3, i_4$ are different from each other, we can attach cherries $(v_{i_1}, v_{i_2}, v_{j_1})$ and $(v_{i_3}, v_{i_4}, v_{j_2})$ at any two vertices u_1 and u_2 of Γ and thereby construct a tree on n vertices with two cherries at (i_1, i_2, j_1) and (i_3, i_4, j_2). On the other hand, every tree on n vertices with two cherries at (i_1, i_2, j_1) and (i_3, i_4, j_2) can be constructed in this way. Since Γ has $n-6$ vertices, we have $n-6$ possibilities for choosing u_1 and u_2 respectively. Thus there are $(n-6)(n-6)(n-6)^{n-8} = (n-6)^{n-6}$ trees on n vertices with two cherries at (i_1, i_2, j_1) and (i_3, i_4, j_2). In the case that $j_1 = j_2$ and either $i_1 = i_3$ and $i_2 = i_4$ or $i_1 = i_4$ and $i_2 = i_3$ and j_1, i_1, i_2 are distinct, we can conclude as in the case of 3 labels that the number of trees on n vertices with a cherry at (i_1, i_2, j_1) is $(n-3)^{n-4}$. In all other cases, there is no tree on n vertices with cherries at (i_1, i_2, j_1) or (i_3, i_4, j_2). Hence:

$$\mathbb{E}[\varepsilon_{i_1,i_2,j_1}\varepsilon_{i_3,i_4,j_2}] = \begin{cases} \frac{(n-6)^{n-6}}{n^{n-2}} & \text{if } i_1, i_2, i_3, i_4, j_1, j_2 \text{ are all different} \\ \frac{(n-3)^{n-4}}{n^{n-2}} & \text{if } j_1 = j_2 \text{ and } i_1 = i_3,\ i_2 = i_4 \\ & \text{or } j_1 = j_2 \text{ and } i_1 = i_4,\ i_2 = i_3 \\ & \text{and } j_1, i_1, i_2 \text{ are all different} \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

Let $C_n(T_n)$ denote the number of cherries in T_n, i.e.

$$\begin{aligned} C_n(T_n) &= \sum_{j=1}^{n} \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1} \varepsilon_{i_1,i_2,j}(T_n) \\ &= \sum_{j=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq j}}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq j}}^{i_1-1} \varepsilon_{i_1,i_2,j}(T_n) \end{aligned}$$

The number of 3-tuples $(j, i_1, i_2) \in \{1, \dots, n\}^3$ such that all entries are distinct is $n(n-1)(n-2)$. The further condition that $i_2 < i_1$ halves this number, so the above sum has $\frac{n(n-1)(n-2)}{2}$ summands. Hence:

$$\begin{aligned} \mathbb{E}[C_n] &= \sum_{j=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq j}}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq j}}^{i_1-1} \mathbb{E}[\varepsilon_{i_1,i_2,j}] \\ &= \sum_{j=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq j}}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq j}}^{i_1-1} \frac{(n-3)^{n-4}}{n^{n-2}} \\ &= \frac{n(n-1)(n-2)}{2} \frac{(n-3)^{n-4}}{n^{n-2}} \\ &= \frac{n}{2} + O(1) \end{aligned}$$

We now want to calculate the variance of C_n. For this we need the second moment. We first compute:

$$\begin{aligned} C_n(T_n)^2 &= \sum_{j_1=1}^{n} \sum_{i_1=1}^{n} \sum_{i_2=1}^{i_1-1} \sum_{j_2=1}^{n} \sum_{i_3=1}^{n} \sum_{i_4=1}^{i_3-1} \varepsilon_{i_1,i_2,j_1}(T_n)\varepsilon_{i_3,i_4,j_2}(T_n) \\ &= \sum_{j_1=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq j_1}}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq j_1}}^{i_1-1} \sum_{j_2=1}^{n} \sum_{\substack{i_3=1 \\ i_3 \neq j_2}}^{n} \sum_{\substack{i_4=1 \\ i_4 \neq j_2}}^{i_3-1} \varepsilon_{i_1,i_2,j_1}(T_n)\varepsilon_{i_3,i_4,j_2}(T_n) \end{aligned}$$

Taking expectation we get:

$$\mathbb{E}[C_n^2] = \sum_{j_1=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq j_1}}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq j_1}}^{i_1-1} \sum_{j_2=1}^{n} \sum_{\substack{i_3=1 \\ i_3 \neq j_2}}^{n} \sum_{\substack{i_4=1 \\ i_4 \neq j_2}}^{i_3-1} \mathbb{E}[\varepsilon_{i_1,i_2,j_1}\varepsilon_{i_3,i_4,j_2}]$$

To apply the formulas from (5.2) we split this sum into the two cases where $j_1, i_1, i_2, j_2, i_3, i_4$ are all different and where either $j_1 = j_2$ and $i_1 = i_3$, $i_2 = i_4$ or $j_1 = j_2$ and $i_1 = i_4$, $i_2 = i_3$ and j_1, i_1, i_2 are different.

$$\begin{aligned}
\mathbb{E}[C_n^2] &= \sum_{j_1=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq j_1}}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq j_1}}^{i_1-1} \sum_{\substack{j_2=1 \\ j_2 \neq j_1 \\ j_2 \neq i_1 \\ j_2 \neq i_2}}^{n} \sum_{\substack{i_3=1 \\ i_3 \neq j_1 \\ i_3 \neq i_1 \\ i_3 \neq i_2 \\ i_3 \neq j_2}}^{n} \sum_{\substack{i_4=1 \\ i_4 \neq j_1 \\ i_4 \neq i_1 \\ i_4 \neq i_2 \\ i_4 \neq j_2}}^{i_3-1} \mathbb{E}[\varepsilon_{i_1,i_2,j_1}\varepsilon_{i_3,i_4,j_2}] \\
&\quad + 2 \sum_{j=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq j}}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq j}}^{i_1-1} \mathbb{E}[\varepsilon_{i_1,i_2,j_1}\varepsilon_{i_3,i_4,j_2}] \\
&= \sum_{j_1=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq j_1}}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq j_1}}^{i_1-1} \sum_{\substack{j_2=1 \\ j_2 \neq j_1 \\ j_2 \neq i_1 \\ j_2 \neq i_2}}^{n} \sum_{\substack{i_3=1 \\ i_3 \neq j_1 \\ i_3 \neq i_1 \\ i_3 \neq i_2 \\ i_3 \neq j_2}}^{n} \sum_{\substack{i_4=1 \\ i_4 \neq j_1 \\ i_4 \neq i_1 \\ i_4 \neq i_2 \\ i_4 \neq j_2}}^{i_3-1} \frac{(n-6)^{n-6}}{n^{n-2}} \qquad (5.3) \\
&\quad + 2 \sum_{j=1}^{n} \sum_{\substack{i_1=1 \\ i_1 \neq j}}^{n} \sum_{\substack{i_2=1 \\ i_2 \neq j}}^{i_1-1} \frac{(n-3)^{n-4}}{n^{n-2}} \qquad (5.4)
\end{aligned}$$

The number of 6-tuples $(j_1, i_1, i_2, j_2, i_3, i_4) \in \{1, \dots, n\}^6$ such that all entries are different is $n(n-1)(n-2)(n-3)(n-4)(n-5) = \frac{n!}{(n-6)!}$. Each of the further conditions that $i_2 < i_1$ and $i_4 < i_3$ halves this number. So the expression in (5.3) has $\frac{1}{4}\frac{n!}{(n-6)!}$ summands. The expression in (5.4) is of order $O(n)$ since it can be bounded from above by $2n^3 \frac{(n-3)^{n-4}}{n^{n-2}} = O(n)$, so we get:

$$\begin{aligned}
\mathbb{E}[C_n^2] &= \frac{1}{4}\frac{n!}{(n-6)!}\frac{(n-6)^{n-6}}{n^{n-2}} + O(n) \\
&= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{4}\frac{(n-6)^{n-6}}{n^{n-2}} + O(n) \\
&= \frac{n^2}{4} + O(n)
\end{aligned}$$

Combining this with the previous calculation yields

$$\mathbb{V}\mathrm{ar}[C_n] = \mathbb{E}[C_n^2] - \mathbb{E}[C_n]^2 = O(n)$$

and hence

$$\frac{\mathbb{V}\mathrm{ar}[C_n]}{\mathbb{E}[C_n]^2} = O\left(\frac{1}{n}\right).$$

Moreover

$$\mathbb{E}[C_n - 1] = \mathbb{E}[C_n] - \mathbb{E}[1] = \frac{n}{2} + O(1) - 1 = \frac{n}{2} + O(1)$$

since the expectation is linear and

$$\mathbb{V}\mathrm{ar}[C_n - 1] = \mathbb{V}\mathrm{ar}[C_n] = O(n)$$

since the variance is invariant under addition of constants. Therefore:

$$\frac{\mathbb{V}\mathrm{ar}[C_n - 1]}{\mathbb{E}[C_n - 1]^2} = O\left(\frac{1}{n}\right)$$

We now want to see that $\mathbb{P}[C_n \geq 2]$ goes to 1 or equivalently that

$$\mathbb{P}[C_n = 0 \text{ or } C_n = 1] = \mathbb{P}[C_n = 0] + \mathbb{P}[C_n = 1]$$

goes to 0 as $n \to \infty$.
Let $N \in \mathbb{N}$ such that $\mathbb{E}[C_n] > 1$ for all $n \geq N$. Using Chebyshev's inequality, it follows that for all $n \geq N$:

$$\mathbb{P}[C_n = 0] \leq \mathbb{P}\left[|C_n - \mathbb{E}[C_n]| \geq \mathbb{E}[C_n]\right] \leq \frac{\mathbb{V}\mathrm{ar}[C_n]}{\mathbb{E}[C_n]^2} = O\left(\frac{1}{n}\right)$$

and

$$\begin{aligned}\mathbb{P}[C_n = 1] = \mathbb{P}[C_n - 1 = 0] &\leq \mathbb{P}\left[|C_n - 1 - \mathbb{E}[C_n - 1]| \geq \mathbb{E}[C_n - 1]\right] \\ &\leq \frac{\mathbb{V}\mathrm{ar}[C_n - 1]}{\mathbb{E}[C_n - 1]^2} = O\left(\frac{1}{n}\right)\end{aligned}$$

So $\mathbb{P}[C_n \geq 2] = 1 - \mathbb{P}[C_n = 0] - \mathbb{P}[C_n = 1] \xrightarrow{n\to\infty} 1$ which completes the proof. □

Combining this lemma with the result by Schmidt (Proposition 5.4) yields the following theorem.

Theorem 5.6. *Almost all trees have quantum symmetry.*

Proof. By the preceding lemma we know that almost all trees have at least two cherries. So in particular, almost all trees have two non-trivial automorphisms with disjoint support. So by Proposition 5.4 the result follows. □

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