UNIVERSITÄT DES SAARLANDES

HYPERTREE C^*-ALGEBRAS

Bachelor Thesis

by

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Contents

Introduction

This thesis deals with the structural classification of C^*-algebras associated with a specific class of finite directed hypergraphs, which we introduce and analyze as *hyperbranches* and *hyperbranch trees*.

The study of C^*-algebras mathematically formalizes the non-commutativity inherent in quantum mechanics. In functional analysis, this is modeled via bounded linear operators on complex Hilbert spaces $\mathcal{H}$, with the space of such operators denoted as $\mathcal{B}(\mathcal{H})$. The non-commutative nature of operator composition in $\mathcal{B}(\mathcal{H})$ prompted pioneers like Francis Murray and John von Neumann in the 1930s to develop what is now known as non-commutative analysis.

In 1943, Israel Gelfand and Mark Naimark introduced the formal concept of what are now known as C^*-algebras. The terminology itself was established by Irving Segal in 1947, where the letter "C" originally referred to the fact that these algebras are closed in the norm topology as subalgebras of $\mathcal{B}(\mathcal{H})$. A monumental milestone in this field is the Gelfand-Naimark Theorem, which establishes that every commutative unital C^*-algebra is isometrically $*$-isomorphic to $C(X)$, the algebra of continuous functions on a compact Hausdorff space X. This result dictates that commutative C^*-algebras correspond exactly to classical topology. Consequently, the study of general (non-commutative) C^*-algebras is widely regarded as *non-commutative topology*. Furthermore, the Gelfand-Naimark-Segal (GNS) construction guarantees that any abstractly defined C^*-algebra can be concretely represented as a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

A graph is a structure consisting of a set of vertices and a set of edges. By equipping this structure with explicit source and range maps, one obtains a directed graph. These directed graphs serve as the fundamental combinatorial data for constructing graph C^*-algebras, a class of operator algebras that evolved from the Cuntz-Krieger algebras introduced by Joachim Cuntz and Wolfgang Krieger in 1980. Graph C^*-algebras have been extensively studied over the last four decades because the underlying directed graph provides an effective combinatorial tool for visually and structurally characterizing the properties of the associated C^*-algebra. In this framework, a directed graph can be represented by operators on a Hilbert space: vertices correspond to mutually orthogonal closed subspaces (modeled by projections), and edges correspond to partial isometries acting between these subspaces.

A fundamental result in this classical theory, which serves as the baseline for this thesis, is the classification of finite acyclic graphs (see Raeburn [5, Proposition 1.18]): Let $\Gamma = (V, E, r, s)$ be a finite, acyclic graph and $C^*(\Gamma)$ its graph C^*-algebra. Let $w_1, \ldots, w_k$ be the sinks of Γ. Then we have an isomorphism:

$$C^*(\Gamma) \cong \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}),$$

where $n_i := |\{\mu \in E^* \mid r(\mu) = w_i\}|$ denotes the number of paths ending at the sink w_i. Conversely, every finite-dimensional C^*-algebra can be realized as the C^*-algebra of such a finite directed acyclic graph up to isomorphism.

Recently, this concept has been generalized to directed hypergraphs by Weber, Trieb, and Zenner [9, 8], where edges can simultaneously connect multiple source vertices to multiple range vertices. This generalization introduces significant algebraic complexity that is no longer immediately apparent from the combinatorial data. While in the classical case the property of being "finite and acyclic" immediately guarantees finite dimensionality, this intuition fails for hypergraphs. Because hyperedges map entire sets of vertices to other sets, we will see that the projections corresponding to distinct vertices within the same source or range set have no inherent relations to each other. This stands in stark contrast to their behavior in classical graph C^*-algebras and therefore leads to a fundamentally different and more complex algebraic structure. As a result, even a strictly finite and acyclic hypergraph can generate an infinite-dimensional C^*-algebra. Furthermore, as demonstrated by Schäfer and Weber [6], such algebras may fail to be nuclear, a phenomenon that never occurs for classical graph C^*-algebras.

The central objective of this thesis is to investigate whether the classical structure theorem for finite acyclic graphs can be generalized to the realm of hypergraphs. However, translating the classical notion of "acyclicity" to hypergraphs is highly non-trivial, as properly defining paths and cycles in a multi-vertex setting introduces significant combinatorial complexity. Therefore, rather than establishing a universal concept of acyclic hypergraphs, we restrict our focus to a specific class of hypergraphs that possess a naturally strict, forward-directed structure. To the best of the author's knowledge, the structural classification of this specific class has not been previously established or proven in the literature.

By systematically decomposing these structures, we prove that the non-commutative interactions inside multi-vertex sets inevitably generate free products of C^*-algebras. We summarize our main results below.

As a fundamental building block, we introduce the concept of a *hyperbranch*, which is defined as a strictly sequential chain of disjoint vertex sets connected by directed hyperedges. This leads to our first main structure theorem:

Theorem A. *Let $H\Gamma$ be a hyperbranch of length k with vertex sets $V_1, \dots, V_k$ and edges $e_1, \dots, e_{k-1}$. Let $n_j = |V_j|$ denote the number of vertices in the set V_j. Then there is an isomorphism:*

$$C^*(H\Gamma) \cong M_k \left(\mathbb{C}^{n_1} *_{\mathbb{C}} \mathbb{C}^{n_2} *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathbb{C}^{n_k} \right),$$

where $*_{\mathbb{C}}$ denotes the unital free product of C^*-algebras.

This result demonstrates that the algebraic complexity is entirely captured by the free product of the multi-vertex sets, while the sequential path structure scales the matrix dimension. It explicitly shows how the classical path intuition generalizes to the hypergraph setting.

Building on this foundational result, we expand our scope to *hyperbranch trees*, effectively translating the classical graph-theoretic intuition of trees into the hypergraph setting. For hyperbranch out-trees satisfying a necessary singleton source assumption, we demonstrate that the global C^*-algebra naturally decouples. We formalize this in our second main result:

Theorem B. *Let $H\Gamma$ be a hyperbranch out-tree consisting of k branches diverging from a single source vertex. For each branch $i \in \{1, \dots, k\}$, let n_i denote its length and $m_{i,2}, \dots, m_{i,n_i}$ denote the sizes of its sequential vertex sets. Then there is an isomorphism:*

$$C^*(H\Gamma) \cong \bigoplus_{i=1}^{k} M_{n_i} \left(\mathbb{C} *_{\mathbb{C}} \mathbb{C}^{m_{i,2}} *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathbb{C}^{m_{i,n_i}} \right).$$

This theorem proves that as long as the tree diverges at a single vertex, the global algebra perfectly decouples into independent blocks. The overall structure is exactly the direct sum of the isolated branches, where each branch contributes its own matrix dimension and internal free product, mirroring the classical decomposition of finite acyclic graphs as presented by Raeburn [5].

The thesis is structured as follows. Chapter 1 recalls the foundational definitions of C^*-algebras, universal C^*-algebras, and classical graph C^*-algebras. For a comprehensive introduction to the general theory, we refer to standard literature on operator algebras [3, 1]. Furthermore, we review the structure theorem for finite acyclic graphs, a foundational result by Raeburn [5], and present a detailed proof of the isomorphism.

Chapter 2 introduces the mathematical generalization to hypergraph C^*-algebras. We establish their fundamental properties and formally define the concept of paths within the framework of directed hypergraphs.

In Chapter 3, we begin our structural analysis by introducing hyperbranches. By utilizing the concept of full corners, we establish our first main result, Theorem A. We then apply this theorem to several illustrative examples, analyzing one specific case in greater detail to highlight the underlying algebraic mechanics.

Chapter 4 extends these results to directed tree structures, specifically hyperbranch in-trees and out-trees. For in-trees, we show that converging branches naturally yield a single matrix block over a unified free product. For out-trees, we isolate the interactions between diverging branches to prove the strict direct sum decomposition established in Theorem B. Additionally, we explore hybrid hypergraphs formed by adjoining classical paths to study how the matrix dimension and the corner algebra behave when classical and hypergraph components interact.

Finally, Chapter 5 provides a brief outlook on general hypertrees, discussing further research directions and what remains to be investigated in the context of imperfect paths, modified relations, and more complex hypergraph topologies.

1 Preliminaries

We assume standard knowledge of functional analysis and bounded linear operators on Hilbert spaces. This chapter briefly recalls the foundational concepts of C^*-algebras, universal C^*-algebras, and graph C^*-algebras to establish the notation used throughout this thesis. For a comprehensive introduction to the general theory, we refer to standard literature on operator algebras (see, e.g., [3, 1]). For the specific theory of graph C^*-algebras, we refer to [5].

1.1 C^*-Algebras

Since the study of hypergraph C^*-algebras intrinsically relies on the abstract framework of operator algebras, we first recall the foundational definitions and structural properties of Banach and C^*-algebras.

Definition 1.1 (C^*-Algebras): Let A be an algebra over $\mathbb{C}$.

(i) An *involution* on A is an antilinear map $*\colon A \to A$ satisfying $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in A$. A complex algebra equipped with an involution is called a $*$-*algebra*.

(ii) A *Banach algebra* is a complex algebra equipped with a complete norm $\|\cdot\|$ that is submultiplicative, i.e., $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. A *Banach* $*$-*algebra* is a Banach algebra endowed with an involution.

(iii) A C^*-*algebra* is a Banach $*$-algebra A that satisfies the C^*-*identity*:

$$\|x^*x\| = \|x\|^2 \quad \text{for all } x \in A.$$

A subset $B \subseteq A$ is a C^*-*subalgebra* if it is a topologically closed $*$-subalgebra.

(iv) An element $x \in A$ is called *normal* if $x^*x = xx^*$, and *self-adjoint* if $x^* = x$.

(v) The algebra A is *unital* if it possesses a multiplicative identity element, and *commutative* if $xy = yx$ for all $x, y \in A$.

The defining characteristic that distinguishes C^*-algebras from general Banach $*$-algebras is the C^*-identity. This single axiom is extraordinarily restrictive and fundamentally shapes the structure of the algebra. As detailed in standard literature such as Blackadar [1], it automatically forces the involution to be an isometry ($\|x^*\| = \|x\|$).

A structural feature of C^*-algebras concerns their quotients, which heavily relies on the automatic compatibility of ideals with the involution.

Remark 1.2 (Ideals and Quotients): If I is a closed, two-sided ideal in a Banach algebra A, the quotient space A/I inherently becomes a Banach algebra when equipped with the standard quotient norm. In the setting of C^*-algebras, a much stronger statement holds: Any closed two-sided ideal I in a C^*-algebra A is automatically self-adjoint ($I = I^*$). Consequently, the quotient A/I naturally inherits the structure of a C^*-algebra under the quotient norm. This property is indispensable for studying extensions and ideal structures of operator algebras.

To classify commutative C^*-algebras, one relies on the spectrum of the algebra and the Gelfand transform.

Definition 1.3 (Gelfand Transform): Let A be a commutative, unital Banach algebra and let $\mathrm{Spec}(A)$ denote its spectrum, which consists of all non-zero multiplicative linear functionals $\varphi\colon A \to \mathbb{C}$. The *Gelfand transform* is the map $\chi\colon A \to C(\mathrm{Spec}(A))$ defined by $x \mapsto \widehat{x}$, where $\widehat{x}(\varphi) := \varphi(x)$.

The paramount importance of the Gelfand transform is established by the following classification theorem.

Theorem 1.4 (Commutative Gelfand-Naimark Theorem): *For any commutative, unital C^*-algebra A, the Gelfand transform $\chi\colon A \to C(\mathrm{Spec}(A))$ is an isometric $*$-isomorphism. Consequently, a unital C^*-algebra A is commutative if and only if there exists a compact Hausdorff space X such that $A \cong C(X)$.*

In this context, the topological space X is precisely $\mathrm{Spec}(A)$ equipped with the weak*-topology. For non-unital commutative C^*-algebras, the theorem holds with $A \cong C_0(X)$ for a locally compact Hausdorff space X.

For general, non-commutative C^*-algebras, concrete representations are obtained via states (positive linear functionals of norm 1).

Theorem 1.5 (GNS Construction): *Let A be a C^*-algebra and $\varphi\colon A \to \mathbb{C}$ be a state. Then there exists a unique (up to unitary equivalence) triple $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ consisting of a Hilbert space $\mathcal{H}_\varphi$, a $*$-representation $\pi_\varphi\colon A \to \mathcal{B}(\mathcal{H}_\varphi)$, and a cyclic unit vector $\xi_\varphi \in \mathcal{H}_\varphi$ such that*

$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle \quad \textit{for all } a \in A.$$

By taking the direct sum of the GNS representations corresponding to all possible states on A, one obtains a faithful (injective) representation. This bridges the abstract definition and the concrete operator-theoretic perspective, culminating in the Gelfand-Naimark-Segal Theorem.

Theorem 1.6 (Gelfand-Naimark-Segal Theorem): *Every C^*-algebra A admits a faithful $*$-representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. Consequently, any abstract C^*-algebra is isometrically $*$-isomorphic to a norm-closed C^*-subalgebra of bounded operators on a Hilbert space.*

1.2 Projections and Universal C^*-Algebras

Before defining graph C^*-algebras, we briefly recall the properties of projections and partial isometries, as these form the standard generating elements.

Definition 1.7: Let A be a C^*-algebra. An element $p \in A$ is called a *projection* if it is self-adjoint ($p^* = p$) and idempotent ($p^2 = p$).

Definition 1.8: Let A be a C^*-algebra. We call $s \in A$ a *partial isometry* if $s = ss^*s$. This condition is algebraically equivalent to s^*s being a projection, which in turn is equivalent to ss^* being a projection.

Let us quickly recall some properties of projections in C^*-algebras.

Remark 1.9: Let p and q be two projections in a C^*-algebra.

(i) The relation $p \leq q$ is defined by $pq = p = qp$.

(ii) A finite sum of projections is a projection if and only if the projections are mutually orthogonal (i.e., $p_i p_j = 0$ for $i \neq j$).

(iii) If $\{p_i\}_{i\in I}$ is a finite family of mutually orthogonal projections such that $p_i \leq q$ for all i and $q \leq \sum_{i\in I} p_i$, then $q = \sum_{i\in I} p_i$.

In the following, we define universal C^*-algebras. Again, for more detailed proofs and explanations, we refer to [3].

Definition 1.10 (Universal C^*-Algebra): Let $E = \{x_i \mid i \in I\}$ be a set of elements indexed by some index set I. From these we can construct noncommutative monomials and noncommutative polynomials by concatenation of letters from E. By adding another set $E^* = \{x_i^* \mid i \in I\}$ which is disjoint from E and by defining an involution on $E \cup E^*$, we obtain the free *-algebra $\mathcal{P}(E)$ on the generator set E.

We can view relations as a subset of polynomials $R \subset \mathcal{P}(E)$. By taking the two-sided *-ideal $J(R) \subset \mathcal{P}(E)$ generated by R, we define the universal *-algebra $\mathcal{A}(E|R) := \mathcal{P}(E)/J(R)$ as a quotient space.

Recall that a C^*-seminorm on a *-algebra A is given by a map $p\colon A \to [0, \infty)$, such that:

(a) $p(\lambda x) = |\lambda| p(x)$ and $p(x + y) \leq p(x) + p(y)$ for all $x, y \in A$ and $\lambda \in \mathbb{C}$,

(b) $p(xy) \leq p(x)p(y)$ for all $x, y \in A$,

(c) $p(x^* x) = p(x)^2$ for all $x \in A$.

We put

$$\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } \mathcal{A}(E|R)\}.$$

If $\|x\| < \infty$ for all $x \in \mathcal{A}(E|R)$, then $\| \cdot \|$ is a C^*-seminorm. In that case, we define the universal C^*-algebra $C^*(E|R)$ as the completion with respect to the norm $\| \cdot \|$:

$$C^*(E|R) := \overline{\mathcal{A}(E|R)/\{x \in \mathcal{A}(E|R) \mid \|x\| = 0\}}^{\|\cdot\|}.$$

This is a C^*-algebra. By abuse of notation, we simply write $x \in C^*(E|R)$ for the elements in $C^*(E|R)$.

Proposition 1.11 (Universal Property): *Let $E = \{x_i \mid i \in I\}$ be a generator set and $R \subset \mathcal{P}(E)$ relations, such that the universal C^*-algebra $C^*(E|R)$ exists. Let $E' = \{y_i \mid i \in I\}$ be a subset of some C^*-algebra B. If the elements in E' satisfy the relations R, then there exists a unique *-homomorphism $\varphi\colon C^*(E|R) \to B$, mapping x_i to y_i for all $i \in I$.*

Example 1.12 (Universal C^*-Algebras): The following examples illustrate how the universal construction allows us to define well-known operator algebras purely through generators and relations. These cases are fundamental for the theory of graph and hypergraph C^*-algebras.

(i) **Matrix Algebras:** For any $N \in \mathbb{N}$ with $N \geq 2$, the complex matrix algebra $M_N(\mathbb{C})$ is isomorphic to the universal C^*-algebra generated by N^2 elements e_{ij} for $i, j = 1, \ldots, N$. These generators are called *matrix units.* They abstractly represent matrices with a 1 in the i-th row and j-th column and 0 elsewhere, which is purely characterized by their involution and multiplication rules:

$$M_N(\mathbb{C}) \cong C^*(e_{ij}; i, j = 1, \ldots, N \mid e_{ij}^* = e_{ji},\ e_{ij} e_{kl} = \delta_{jk} e_{il}).$$

(ii) **The Cuntz Algebra $\mathcal{O}_N$:** For $N \geq 2$, the Cuntz algebra $\mathcal{O}_N$ is the universal C^*-algebra generated by N isometries $\{s_1, \ldots, s_N\}$ satisfying the relations:

$$\mathcal{O}_N \cong C^*(s_1, \ldots, s_N \mid s_i^* s_j = \delta_{ij} 1, \sum_{i=1}^{N} s_i s_i^* = 1).$$

In this thesis, we will frequently need a mathematical construction to combine independent C^*-algebras. The following definition of the free product serves exactly this purpose.

Definition 1.13 (Free Product): Let $A = C^*(E_1|R_1)$ and $B = C^*(E_2|R_2)$ be unital universal C^*-algebras. We call

$$A *_{\mathbb{C}} B := C^*(E_1, E_2 \mid R_1, R_2 \text{ and } 1_A = 1_B)$$

the *unital free product* of A and B.

We will also consider C^*-algebras which are generated by a family of mutually orthogonal projections. The following lemma gives us a useful characterization of such C^*-algebras.

Lemma 1.14: *Let $\{p_v \mid v \in W\}$ be a finite family of mutually orthogonal projections in a C^*-algebra, where W is a finite set. Then the C^*-subalgebra generated by these projections is isomorphic to $\mathbb{C}^{|W|}$.*

Proof: Since the projections $\{p_v \mid v \in W\}$ are mutually orthogonal, any element in the generated subalgebra can be written as a linear combination $\sum_{v \in W} \lambda_v p_v$ with $\lambda_v \in \mathbb{C}$. The map $\phi \colon \mathbb{C}^{|W|} \to C^*(\{p_v \mid v \in W\})$ defined by $\phi((\lambda_v)_{v \in W}) = \sum_{v \in W} \lambda_v p_v$ is clearly a *-homomorphism. It is injective since the projections p_v are linearly independent, and surjective by construction. $\square$

1.3 Graph C^*-Algebras

Graph C^*-algebras form a well-understood and extensively studied class of operator algebras. For a comprehensive treatment of how the combinatorial properties of a directed graph are reflected in the algebraic structure of its associated C^*-algebra, we refer to Raeburn [5].

1.3.1 Basics

Definition 1.15: A *directed finite graph* is a quadruple $\Gamma = (V, E, r, s)$, where V and E are finite sets. The elements $v \in V$ are called *vertices* and $e \in E$ are called *edges*. The map $r\colon E \to V$ is called the *range map* and $s\colon E \to V$ is called the *source map*. We call a vertex $v \in V$ a *source* if $r^{-1}(v) = \emptyset$ and we call v a *sink* if $s^{-1}(v) = \emptyset$.

Definition 1.16 (Graph C^*-Algebra): Let $\Gamma = (V, E, r, s)$ be a finite directed graph. The *graph C^*-algebra* $C^*(\Gamma)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v \mid v \in V\}$ and by partial isometries $\{s_e \mid e \in E\}$ satisfying the following relations, known as the *Cuntz-Krieger relations*:

- $s_e^* s_f = \delta_{ef} p_{r(e)}$ for all $e, f \in E$. (CK1)
- $p_v = \sum_{\substack{e \in E \\ s(e)=v}} s_e s_e^*$ for all $v \in V$ such that $s^{-1}(v) \neq \emptyset$. (CK2)

Remark 1.17 (Convention): Note that the standard literature on graph C^*-algebras, such as Raeburn [5], typically uses the *Australian convention*. In this thesis, we strictly follow the *European convention*, which is standard in recent works on hypergraph C^*-algebras (see, e.g., [8, 6]). The structural difference essentially lies in reversing the roles of the range map r and the source map s in the definitions and Cuntz-Krieger relations.

Remark 1.18: Every graph C^*-algebra $C^*(\Gamma)$ always exists. This follows directly from the existence criterion for universal C^*-algebras given in Definition 1.10. Since the generators of $C^*(\Gamma)$ are mutually orthogonal projections p_v and partial isometries s_e, their norms in any $*$-representation are uniformly bounded by 1 ($\|p_v\| \leq 1$ and $\|s_e\| \leq 1$). Consequently, the supremum over all C^*-seminorms is finite for every algebraic combination of the generators.

Proposition 1.19: *Let $\Gamma = (V, E, r, s)$ be a graph and $C^*(\Gamma)$ the associated graph C^*-algebra. Then, for all $e \in E$, the following holds:*

$$p_{s(e)} s_e = s_e = s_e p_{r(e)}.$$

Proof: Let $e \in E$. We first show the right-hand equality. Since s_e is a partial isometry, we have $s_e = s_e s_e^* s_e$. Using (CK1), we know $s_e^* s_e = p_{r(e)}$. Thus:

$$s_e = s_e(s_e^* s_e) = s_e p_{r(e)}.$$

For the left-hand equality, we use relation (CK2). Let $v = s(e)$. Since e starts at v, v is not a sink. Thus:

$$p_{s(e)}s_e = \left(\sum_{s(f)=s(e)} s_f s_f^*\right) s_e = \sum_{s(f)=s(e)} s_f(s_f^* s_e).$$

By relation (CK1), the term $s_f^* s_e$ is zero unless $f = e$. Therefore, the sum collapses to the single term where $f = e$:

$$p_{s(e)}s_e = s_e(s_e^* s_e) = s_e p_{r(e)} = s_e.$$

□

Proposition 1.20: *Let $\Gamma = (V, E, r, s)$ be a finite graph. Then the sum of all vertex projections is the identity of $C^*(\Gamma)$:*

$$\sum_{v \in V} p_v = 1.$$

Proof: Let $P = \sum_{v \in V} p_v$. Since the projections p_v are mutually orthogonal, P is a projection by Remark 1.9. To show $P = 1$, it suffices to show that P acts as the identity on the generators.

For any $w \in V$:

$$Pp_w = \left(\sum_{v \in V} p_v\right) p_w = p_w p_w = p_w.$$

For any edge $e \in E$, using Proposition 1.19:

$$Ps_e = \left(\sum_{v \in V} p_v\right) s_e = \sum_{v \in V} (p_v s_e).$$

Since $p_v s_e = p_v(p_{s(e)} s_e)$, this term is zero unless $v = s(e)$ due to orthogonality. Thus:

$$Ps_e = p_{s(e)} s_e = s_e.$$

Similarly, $s_e P = s_e$. It follows $P = 1$. □

1.3.2 Paths in Graphs

After revisiting the definition and the basic properties of graph C^*-algebras, we now want to look at paths and cycles in graphs. We will see that paths play an important role in understanding the structure of graph C^*-algebras, especially for finite directed graphs. For more details, we refer to [5].

Definition 1.21: Let Γ be a finite graph.

(i) A *path* μ is a tuple of edges $\mu = (\mu_1, \ldots, \mu_n)$ such that $r(\mu_j) = s(\mu_{j+1})$ for all $j \in \{1, \ldots, n-1\}$.
We denote the set of all paths by E^*. Vertices $v \in V$ are regarded as paths of length 0, so $V \subset E^*$.

(ii) For a path $\mu = (\mu_1, \ldots, \mu_n)$, we define:

$$s(\mu) := s(\mu_1), \quad r(\mu) := r(\mu_n).$$

For a vertex v (path of length 0), we set $s(v) = r(v) = v$.

(iii) If a path $\mu = (\mu_1, \ldots, \mu_n)$ with $|\mu| \geq 1$ satisfies $r(\mu_n) = s(\mu_1)$, we call μ a *cycle*.

(iv) If a graph contains no cycles, we call it an *acyclic graph*.

We will primarily be interested in finite acyclic graphs, since their associated C^*-algebras are finite-dimensional.

Proposition 1.22: *Let Γ be a finite graph and $C^*(\Gamma)$ its graph C^*-algebra. For every path $\mu = (\mu_1, \ldots, \mu_n) \in E^*$, we define the element*

$$s_\mu := s_{\mu_1} s_{\mu_2} \cdots s_{\mu_n}.$$

Then s_μ is a partial isometry.

Proof: To show that s_μ is a partial isometry, we must verify $s_\mu s_\mu^* s_\mu = s_\mu$. We compute the product by iteratively applying the Cuntz-Krieger relation (CK1) and the property $p_{s(e)} s_e = s_e$:

$$\begin{aligned} s_\mu^* s_\mu &= (s_{\mu_n}^* \cdots s_{\mu_2}^* s_{\mu_1}^*)(s_{\mu_1} s_{\mu_2} \cdots s_{\mu_n}) \\ &= s_{\mu_n}^* \cdots s_{\mu_2}^* \underbrace{(s_{\mu_1}^* s_{\mu_1})}_{= p_{r(\mu_1)}} s_{\mu_2} \cdots s_{\mu_n} \\ &= s_{\mu_n}^* \cdots s_{\mu_3}^* s_{\mu_2}^* p_{r(\mu_1)} s_{\mu_2} \cdots s_{\mu_n}. \end{aligned}$$

Since μ is a path, we have $r(\mu_1) = s(\mu_2)$. By Proposition 1.19, we know $p_{s(\mu_2)} s_{\mu_2} = s_{\mu_2}$. Thus:

$$s_{\mu_2}^* (p_{r(\mu_1)} s_{\mu_2}) = s_{\mu_2}^* s_{\mu_2} = p_{r(\mu_2)}.$$

Repeating this process inductively for all edges in the path, we obtain:

$$s_\mu^* s_\mu = p_{r(\mu_n)}.$$

Finally, since $r(\mu_n) = r(\mu)$, we have:

$$s_\mu(s_\mu^* s_\mu) = s_\mu p_{r(\mu)} = s_\mu.$$

Thus, s_μ is a partial isometry. □

The following proposition shows that the multiplication of path operators behaves as expected.

Proposition 1.23: *Let $\Gamma = (V, E, r, s)$ be a finite directed graph and $C^*(\Gamma)$ its associated C^*-algebra. Let $\nu, \alpha \in E^*$ be two paths. Then:*

$$s_\nu^* s_\alpha = \begin{cases} s_{\nu'}^* & \text{if } \nu = \alpha\nu' \ (\nu \text{ extends } \alpha), \\ s_{\alpha'} & \text{if } \alpha = \nu\alpha' \ (\alpha \text{ extends } \nu), \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We distinguish the three cases:

(i) *Case $\nu = \alpha\nu'$:* Substituting ν, we get:

$$s_{\alpha\nu'}^* s_\alpha = (s_\alpha s_{\nu'})^* s_\alpha = s_{\nu'}^* s_\alpha^* s_\alpha.$$

Using the fact that s_α is a partial isometry ($s_\alpha^* s_\alpha = p_{r(\alpha)}$):

$$s_{\nu'}^* p_{r(\alpha)} = s_{\nu'}^* p_{s(\nu')} = s_{\nu'}^*.$$

(Note: Since $\nu = \alpha\nu'$, the start of ν' is the end of α, so $r(\alpha) = s(\nu')$).

(ii) *Case $\alpha = \nu\alpha'$:* Substituting α, we get:

$$s_\nu^* s_{\nu\alpha'} = s_\nu^* s_\nu s_{\alpha'} = p_{r(\nu)} s_{\alpha'}.$$

Since $\alpha = \nu\alpha'$, the path α' starts at $r(\nu)$, so $p_{r(\nu)} s_{\alpha'} = p_{s(\alpha')} s_{\alpha'} = s_{\alpha'}$.

(iii) *Case "Otherwise":* Let $\nu = (\nu_1, \ldots, \nu_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$. Without loss of generality, assume $|\alpha| \geq |\nu|$. Since neither path extends the other, they must differ at some point. Let i be the smallest index such that $\nu_i \neq \alpha_i$. We can expand the product:

$$s_\nu^* s_\alpha = s_{\nu_n}^* \cdots s_{\nu_i}^* \cdots s_{\nu_1}^* s_{\alpha_1} \cdots s_{\alpha_i} \cdots s_{\alpha_k}.$$

The terms for $j < i$ collapse. At index i, we have $s_{\nu_i}^* s_{\alpha_i}$. Since $\nu_i \neq \alpha_i$, by relation (CK1) we have $s_{\nu_i}^* s_{\alpha_i} = 0$. Thus, the entire product is 0. □

We now have all the tools to show how the C^*-algebra of a finite graph can be spanned by path operators.

Corollary 1.24: *Let Γ be a finite graph and $C^*(\Gamma)$ its graph C^*-algebra. Then:*

$$C^*(\Gamma) = \overline{\operatorname{span}}\{s_\mu s_\nu^* \mid \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}.$$

Proof: Let $\mathcal{S} := \operatorname{span}\{s_\mu s_\nu^* \mid \mu, \nu \in E^*, r(\mu) = r(\nu)\}$ denote the algebraic span of the path operators. To prove the claim, it suffices to show that $\mathcal{S}$ forms a *-subalgebra containing the generators of $C^*(\Gamma)$. First, we observe that $\mathcal{S}$ is closed under multiplication and involution. For any two elements $x = s_\mu s_\nu^*$ and $y = s_\alpha s_\beta^*$ in $\mathcal{S}$, their product xy remains a linear combination of elements in $\mathcal{S}$ due to the Cuntz-Krieger relations (specifically, the interaction of s_ν^* and s_α either vanishes or collapses to a path operator). Moreover, the involution maps $\mathcal{S}$ into itself, as $(s_\mu s_\nu^*)^* = s_\nu s_\mu^*$, which is clearly of the required form.

Furthermore, $\mathcal{S}$ contains the generators of $C^*(\Gamma)$. For any vertex $v \in V$, the projection corresponds to a path of length zero, i.e., $p_v = s_v s_v^* \in \mathcal{S}$. Similarly, for any edge $e \in E$, we can write $s_e = s_e p_{r(e)} = s_e s_{r(e)}^*$, which lies in $\mathcal{S}$.

Since $C^*(\Gamma)$ is defined as the universal C^*-algebra generated by these elements, and $\overline{\mathcal{S}}$ is a closed *-subalgebra containing them, we necessarily have $C^*(\Gamma) \subseteq \overline{\mathcal{S}}$. The reverse inclusion is trivial. Thus, equality holds. □

1.3.3 Finite Acyclic Graph C^*-Algebras

Now we outline the structure theorem for classical graph C^*-algebras, which serves as the primary motivation for this thesis. The following theorem and its proof are classical results and closely follow the presentation by Raeburn [5, Proposition 1.18].

Theorem 1.25: *Let $\Gamma = (V, E, r, s)$ be a finite, acyclic graph and $C^*(\Gamma)$ its graph C^*-algebra. Let $w_1, \dots, w_k$ be the sinks of Γ. Then we have an isomorphism:*

$$C^*(\Gamma) \cong \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}),$$

where $n_i := |\{\mu \in E^* \mid r(\mu) = w_i\}|$ denotes the number of paths ending at the sink w_i.

Proof: First, we confirm that the graph has sinks. Start at any vertex v_0. If it is not a sink, move to v_1 via an edge. Repeat. Since Γ is finite and acyclic, this path must terminate at a sink. By Corollary 1.24, $C^*(\Gamma)$ is spanned by $s_\mu s_\nu^*$

with $r(\mu) = r(\nu) = v$. If v is not a sink, we use (CK2) to write $p_v = \sum_{s(e)=v} s_e s_e^*$. This extends the paths μ and ν to μe and νe. Repeating this process (which terminates due to acyclicity), every element can be written as a sum of terms where the paths end at a sink. Fix a sink w_i. Let $\mathcal{P}_i$ be the set of paths ending at w_i. Consider $s_\mu s_\nu^*$ and $s_\alpha s_\beta^*$ with paths in $\mathcal{P}_i$. The product depends on $s_\nu^* s_\alpha$. Since both ν and α end at the sink w_i, neither can extend the other unless they are identical. Thus:

$$s_\nu^* s_\alpha = \delta_{\nu,\alpha} p_{w_i}.$$

Consequently, $(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \delta_{\nu,\alpha} s_\mu s_\beta^*$. This corresponds exactly to the multiplication of matrix units $E_{\mu\nu} E_{\alpha\beta} = \delta_{\nu\alpha} E_{\mu\beta}$. Thus, the span associated with w_i is isomorphic to $M_{n_i}(\mathbb{C})$. Finally, if we have $(s_\mu s_\nu^*)$ with $r(\mu) = r(\nu) = w_i$ and $(s_\alpha s_\beta^*)$ with $r(\alpha) = r(\beta) = w_j$ for $i \neq j$, then the paths cannot extend each other, yielding by Proposition 1.23:

$$(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = 0.$$

This shows that the algebra decomposes as the direct sum of the matrix blocks.□

Example 1.26: Consider a simple directed graph Γ consisting of n vertices arranged in a line (see Figure 1.1):

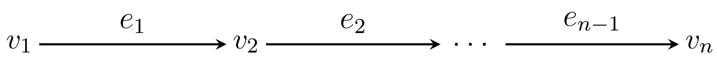


Figure 1.1: Finite directed graph with n vertices.

This graph has a single sink, v_n. The number of paths ending at v_n is exactly n, corresponding to the paths:

$$(v_n), (e_{n-1}), (e_{n-2}, e_{n-1}), \ldots, (e_1, e_2, \ldots, e_{n-1}).$$

By Theorem 1.25, we conclude:

$$C^*(\Gamma) \cong M_n(\mathbb{C}).$$

One can also construct this isomorphism purely by utilizing the universal property. This alternative approach is demonstrated by Zenner [9, Proposition 2.11].

Example 1.27: Consider a graph that branches into two directions (see Figure 1.2). Let $V = \{v_1, v_2, v_3, v_4\}$ with edges as shown below:

This graph has two sinks: v_2 and v_4. We analyze the paths ending at each sink separately:

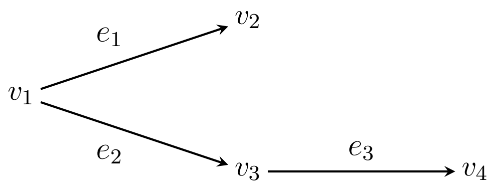


Figure 1.2: The fork graph with two sinks v_2 and v_4.

- *Sink v_2:* The paths ending at v_2 are (v_2) and (e_1). Since there are 2 such paths, this component generates $M_2(\mathbb{C})$.
- *Sink v_4:* The paths ending at v_4 are (v_4), (e_3), and (e_2, e_3). Since there are 3 such paths, this component generates $M_3(\mathbb{C})$.

By Theorem 1.25, the graph C^*-algebra is the direct sum of these matrix blocks:

$$C^*(\Gamma) \cong M_2(\mathbb{C}) \oplus M_3(\mathbb{C}).$$

For a detailed treatment of infinite or cyclic graph C^*-algebras, we refer the reader to Raeburn [5]. The overarching goal of the subsequent chapters will be to generalize the structural findings of Theorem 1.25.

Remark 1.28: An acyclic finite graph which is connected (i.e., there is a way to go from each vertex to any other vertex by following edges, ignoring their direction) is often called a *directed tree*. For our investigation, connectedness is not strictly required, since we still have a good theory for "disconnected" graphs.

2 Hypergraph C^*-Algebras

Hypergraphs generalize classical directed graphs by allowing a single edge to connect multiple source vertices to multiple range vertices. In the following, we will define directed hypergraphs and their associated C^*-algebras. The underlying hypergraph (Cuntz-Krieger) relations naturally generalize those of classical graph C^*-algebras; however, the resulting class of hypergraph C^*-algebras is strictly larger, as shown in [8].

2.1 Basics

To formally capture the hypergraph structure, we define a directed hypergraph by replacing the codomain of the classical source and range maps with the power set $\mathcal{P}(V)$. This modification allows each edge to originate from and terminate at multiple vertices simultaneously.

Definition 2.1: A *finite directed hypergraph* is a quadruple $H\Gamma = (V, E, r, s)$, where V is a finite set of vertices, E is a finite set of (hyper-)edges, and $r \colon E \to \mathcal{P}(V) \setminus \{\emptyset\}$ and $s \colon E \to \mathcal{P}(V) \setminus \{\emptyset\}$ are the range and source maps, respectively. Here, $\mathcal{P}(V)$ denotes the power set of V. We call a vertex $v \in V$ a *source* if there is no edge $e \in E$ such that $v \in r(e)$, and we call v a *sink* if there is no edge $e \in E$ such that $v \in s(e)$.

In this thesis, all hypergraphs are finite, i.e. the set of vertices and edges are both finite. For better understanding, we look at an example of a hypergraph.

Example 2.2: Consider the hypergraph $H\Gamma = (V, E, r, s)$ with vertices $V = \{v_1, v_2, v_3, v_4\}$ and a single edge $e \in E$ defined by:

- $s(e) = \{v_1, v_2\}$.
- $r(e) = \{v_3, v_4\}$.

This hypergraph, which is listed as a 'Mysterious Example' in [9], will be analyzed in detail in Chapter 3, where we establish its associated C^*-isomorphism

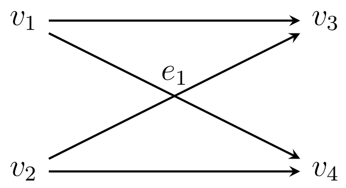


Figure 2.1: A simple hypergraph.

Definition 2.3 (Hypergraph C^*-Algebra): Let $H\Gamma = (V, E, r, s)$ be a finite directed hypergraph. The *hypergraph C^*-algebra* $C^*(H\Gamma)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v \mid v \in V\}$ and partial isometries $\{s_e \mid e \in E\}$ satisfying the following relations:

- $s_e^* s_f = \delta_{ef} \sum_{v \in r(e)} p_v$ for all $e, f \in E$. (HR1)
- $s_e s_e^* \leq \sum_{v \in s(e)} p_v$ for all $e \in E$. (HR2a)
- $p_v \leq \sum_{\substack{e \in E \\ v \in s(e)}} s_e s_e^*$ for all $v \in V$, where v is not a sink. (HR2b)

Remark 2.4: Every hypergraph C^*-algebra $C^*(H\Gamma)$ always exists since a universal C^*-algebra generated by partial isometries and projections always exists, similar to graph C^*-algebras.

Note that every graph $\Gamma = (V, E, r, s)$ is also a hypergraph $H\Gamma = (V, E, r', s')$ by defining $r' \colon E \to \mathcal{P}(V), e \mapsto \{r(e)\}$ and $s' \colon E \to \mathcal{P}(V), e \mapsto \{s(e)\}$. In other words, graphs are hypergraphs with the restriction $|s(e)| = |r(e)| = 1$ for all edges $e \in E$.

The following three propositions and their proofs, due to Trieb, Weber and Zenner [8], show that hypergraph C^*-algebras truly generalize classical graph C^*-algebras and establish their basic algebraic properties. We reproduce the proofs in this section for completeness, as the calculation techniques are helpful for our later analysis.

Proposition 2.5: *Consider a graph $\Gamma = (V, E, r, s)$ and interpret it as a hypergraph $H\Gamma = (V, E, r', s')$ in the sense described above. Then we have $C^*(\Gamma) \cong C^*(H\Gamma)$.*

Proof: We denote the generators of $C^*(\Gamma)$ by $\tilde{s}_e$ and $\tilde{p}_v$, and the generators of $C^*(H\Gamma)$ by s_e and p_v.

First, we check that the generators of $C^*(\Gamma)$ fulfill the relations of $C^*(H\Gamma)$. Since the only element in the set $r'(e)$ is the vertex $r(e)$, we have

$$\tilde{s}_e^* \tilde{s}_f = \delta_{ef} \tilde{p}_{r(e)} = \delta_{ef} \sum_{\substack{v \in V \\ v \in r'(e)}} \tilde{p}_v.$$

We see that Relation (HR1) is fulfilled. For the same reasons it follows for $v \in V$ with $s^{-1}(v) \neq \emptyset$ that

$$\tilde{p}_v = \sum_{\substack{e \in E \\ v = s(e)}} \tilde{s}_e \tilde{s}_e^* = \sum_{\substack{e \in E \\ v \in s'(e)}} \tilde{s}_e \tilde{s}_e^*.$$

Hence, Relation (HR2b) is satisfied, while (HR2a) follows from Proposition 1.19.

Conversely, the generators of $C^*(H\Gamma)$ satisfy the Relations (CK1) and (CK2) of $C^*(\Gamma)$: Using the same argument as in the above direction, we have

$$s_e^* s_f = \delta_{ef} \sum_{\substack{v \in V \\ v \in r'(e)}} p_v = \delta_{ef} p_{r(e)}.$$

We see that Relation (CK1) is satisfied. To show that (CK2) is fulfilled we need Relations (HR2a) and (HR2b). Let $v \in V$ with $s^{-1}(v) \neq \emptyset$ and hence there exists at least one $f \in E$ with $s(f) = v$. With (HR2a) it follows

$$s_e s_e^* \leq \sum_{\substack{v \in V \\ v \in s'(e)}} p_v = p_{s(e)}$$

and using Relation (HR2b) we have

$$p_v \leq \sum_{\substack{e \in E \\ v \in s'(e)}} s_e s_e^* = \sum_{\substack{e \in E \\ v = s(e)}} s_e s_e^*.$$

Combining both inequalities yields:

$$p_v = \sum_{\substack{e \in E \\ v = s(e)}} s_e s_e^*.$$

By the universal properties of $C^*(\Gamma)$ and $C^*(H\Gamma)$, we find an isomorphism mapping $\tilde{s}_e$ to s_e and $\tilde{p}_v$ to p_v. □

Analogous to finite graph C^*-algebras, we show that for finite hypergraph C^*-algebras the sum of all vertex projections is the identity.

Proposition 2.6: *For every hypergraph $H\Gamma = (V, E, r, s)$ and hypergraph C^*-algebra $C^*(H\Gamma)$ we have that $\sum_{v \in V} p_v$ is the unit element in $C^*(H\Gamma)$ and therefore, $\sum_{v \in V} p_v = 1$.*

Proof: Using Relation (HR1), we have

$$\begin{aligned} s_e \sum_{v\in V} p_v &= s_e s_e^* s_e \sum_{v\in V} p_v \\ &= s_e \sum_{v\in r(e)} p_v \sum_{v\in V} p_v \\ &= s_e \sum_{v\in r(e)} p_v \\ &= s_e. \end{aligned}$$

Using the same trick as in the argument before but with Relation (HR2a), we also have $\Big(\sum_{v\in V} p_v\Big) s_e = s_e$. Notice that we have $\sum_{v\in V} p_v p_w = p_w = p_w \sum_{v\in V} p_v$ for all $w \in V$ and $\Big(\sum_{v\in V} p_v\Big)^2 = \sum_{v\in V} p_v = \Big(\sum_{v\in V} p_v\Big)^*$. We conclude that $\sum_{v\in V} p_v$ is the unit element in $C^*(H\Gamma)$. □

Next, we establish an analogue of the classical range projection property for hypergraph C^*-algebras.

Proposition 2.7: *Let $H\Gamma = (V, E, r, s)$ be a hypergraph and $C^*(H\Gamma)$ be the associated hypergraph C^*-algebra. Then for every edge $e \in E$ we have*

$$\left(\sum_{v\in s(e)} p_v\right) s_e = s_e = s_e \left(\sum_{v\in r(e)} p_v\right).$$

In particular, $s_e s_f^* = 0$, if $r(e) \cap r(f) = \emptyset$.

Proof: By (HR2a), we have

$$\left(\sum_{v\in s(e)} p_v\right) s_e = \left(\sum_{v\in s(e)} p_v\right) s_e s_e^* s_e = s_e s_e^* s_e = s_e$$

and by (HR1), we have

$$s_e \left(\sum_{v\in r(e)} p_v\right) = s_e s_e^* s_e = s_e.$$

To see the final statement, assume $r(e) \cap r(f) = \emptyset$. Then the projections $P_{r(e)} = \sum_{v\in r(e)} p_v$ and $P_{r(f)} = \sum_{w\in r(f)} p_w$ are orthogonal, meaning $P_{r(e)} P_{r(f)} = 0$. Using the right-hand identity established above, we get:

$$s_e s_f^* = (s_e P_{r(e)})(P_{r(f)} s_f^*) = s_e P_{r(e)} P_{r(f)} s_f^* = 0.$$ □

2.2 Paths in Hypergraphs

In classical directed graphs, the notion of a path is highly intuitive: it is simply a sequence of edges where the target of one edge is exactly the source of the next. In the realm of hypergraphs, however, this concept becomes more complex. Because hyperedges connect entire sets of vertices rather than single nodes, there is no single obvious way to concatenate them. To address this, Trieb, Weber and Zenner [8] introduced three distinct definitions of paths in hypergraphs, depending on how strictly the range and source sets of consecutive edges overlap.

Definition 2.8: Let $\mu = (\mu_1, \ldots, \mu_n)$ be a tuple of edges in $H\Gamma$. Then we call μ:

(i) a *perfect path*, if $s(\mu_{j+1}) = r(\mu_j)$ for all $j \in \{1, \ldots, n-1\}$;

(ii) a *quasi perfect path*, if $s(\mu_{j+1}) \subseteq r(\mu_j)$ for all $j \in \{1, \ldots, n-1\}$;

(iii) a *partial path*, if $s(\mu_{j+1}) \cap r(\mu_j) \neq \emptyset$ for all $j \in \{1, \ldots, n-1\}$.

We define $s_\mu := s_{\mu_1} \cdots s_{\mu_n}$ and $s_v = p_v$ for all $v \in V$. We define $s(\mu) := s(\mu_1)$ and $r(\mu) := r(\mu_n)$ for all paths μ longer than one edge, and for vertices $v \in V$ we look at the trivial path $\mu = (v)$ and define $s(\mu) = r(\mu) = \{v\}$. In the following, we write $P_{s(\mu)} := \sum_{v \in s(\mu)} p_v$ and $P_{r(\mu)} := \sum_{v \in r(\mu)} p_v$ for the projections associated to the source and range of μ.

To visualize the strict hierarchy of these path definitions, consider the following illustrative examples of two sequential edges e_1 and e_2.

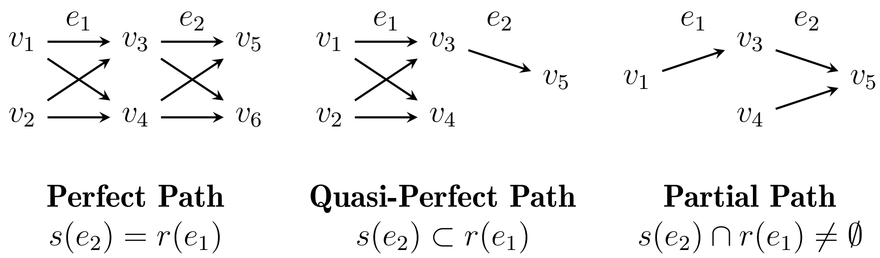


Figure 2.2: Illustration of the three different path types in hypergraphs.

Remark 2.9: The hierarchy of paths is strict: every perfect path is quasi perfect, and every quasi perfect path is partial. However, the converses do not hold in general. In this thesis, we will primarily work with hypergraphs that consist only of perfect paths, as they have a more tractable structure and allow for clearer insights into the associated C^*-algebras. We denote by E^* the set of all perfect paths in $H\Gamma$.

The complexity of defining paths fundamentally complicates the translation of the classical concept of “acyclic graphs” to the hypergraph setting. In a standard finite graph, an acyclic structure guarantees that the generated C^*-algebra is finite-dimensional, as paths cannot infinitely extend. However, in hypergraphs, the existence of quasi-perfect and partial paths creates intersecting topologies where “cycles” can form purely through overlapping sub-components rather than a strict return to an initial vertex. Consequently, formulating a universal, structurally meaningful definition of an “acyclic hypergraph” remains highly non-trivial.

3 Hyperbranch C^*-Algebras

In the previous chapters, we reviewed the established theory of graph and hypergraph C^*-algebras. In this chapter, we begin our original contribution by providing a structural classification for a specific class of hypergraphs: the *hyperbranches*. While the general theory of hypergraph C^*-algebras is well-defined, a concrete description of their isomorphism classes, analogous to the classical Theorem 1.25, has not been established in the literature. Our objective is to characterize hyperbranches as the hypergraph counterparts to the line graphs shown in Figure 1.1.

To achieve this, we will proceed step by step. First, we define hyperbranches and their algebraic relations. We will then need an important result regarding full corners in C^*-algebras. Building on this, we present our main result, Theorem A, which parallels the classical graph C^*-algebra decomposition. Finally, we examine a special case of a hyperbranch where the associated C^*-algebra admits a more explicit description. For this last part, we will briefly recall key facts about the C^*-algebra generated by two projections.

3.1 Definitions and Notation

To approach the classification of hypergraph C^*-algebras, we begin by analyzing their simplest building blocks. Analogous to the directed line graph in classical graph theory, which generates a simple matrix algebra, we introduce the concept of a *hyperbranch*. This structure consists of a linear sequence of vertex sets connected by hyperedges, devoid of any branching or cycles. By restricting our attention to this linear case, we can isolate the effects of the hyperedges on the algebra structure before tackling more complex geometries.

Definition 3.1 (Hyperbranch): Let $H\Gamma = (V, E, r, s)$ be a hypergraph. We call $H\Gamma$ a *hyperbranch of length k* if the vertex set V can be partitioned into k disjoint sets $V_1, \dots, V_k$ such that:

(i) The vertex set is the disjoint union $V = \dot\bigcup_{i=1}^{k} V_i$.

(ii) The edge set consists of $k-1$ edges, denoted by $E = \{e_1, \ldots, e_{k-1}\}$.

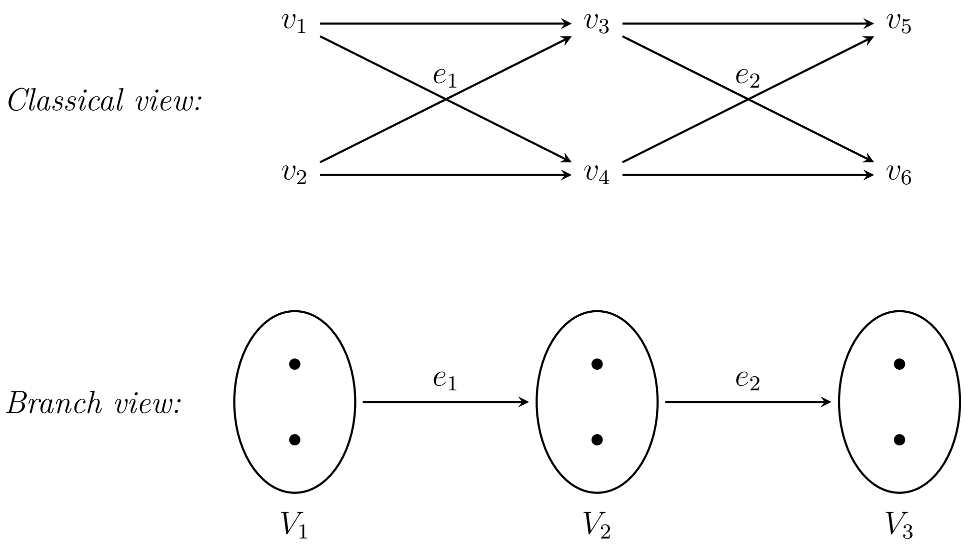
(iii) For each edge e_i, the source is the set V_i and the range is the set V_{i+1}. That is:

$$s(e_i) = V_i \quad \text{and} \quad r(e_i) = V_{i+1}$$

for all $1 \leq i \leq k-1$.

Hyperbranches can be visualized as a sequence of vertex sets connected by hyperedges, where each hyperedge connects all vertices in one set to all vertices in the next set. This structure allows us to explore the properties of hypergraph C^*-algebras in a controlled setting, providing insights into the more general case.

Standard representations of hypergraphs typically use the same visual language as for graphs. While intuitive for small examples, this notation becomes cumbersome for algebraic analysis. For hyperbranches and for the rest of the thesis, we use a representation introduced in [6] and call it the *branch view representation*. Since the interaction happens strictly between the sets V_i and V_{i+1}, we can simplify the visualization by grouping vertices into explicit ellipses and representing the hyperedge e_i as a directed link between these ellipses, as illustrated in Figure 3.1.



$$V_1 = \{v_1, v_2\}, \quad V_2 = \{v_3, v_4\}, \quad V_3 = \{v_5, v_6\}$$

Figure 3.1: Comparison of visualizations for a hyperbranch of length 3.

Remark 3.2: Note that a direct consequence of the linear structure of a hyperbranch is the existence of a unique path. For any vertex set V_i, there is exactly one path μ connecting it to the sink V_k.

One might argue that even without a formal definition of generalized cycles, this finite hypergraph appears simple, as it clearly possesses no classical cycles and has a strictly forward-directed finite vertex set. However, as we will demonstrate, the associated C^*-algebra exhibits a remarkably rich structure. This motivates our focus on hyperbranches.

Before investigating the structure theorem, we must connect the combinatorial object to its operator algebra and fix our notation.

For the remainder of this chapter, let $H\Gamma = (V, E, r, s)$ be a hyperbranch of length $k \geq 1$ with vertex sets $V_1, \ldots, V_k$ and edges $e_1, \ldots, e_{k-1}$. We study its associated hypergraph C^*-algebra $C^*(H\Gamma)$ as defined in Definition 2.3. We will uniformly denote the generators of $C^*(H\Gamma)$ by the partial isometries $\{s_{e_i} \mid 1 \leq i \leq k-1\}$ and the mutually orthogonal vertex projections $\{p_v \mid v \in V\}$.

The following fundamental properties follow directly from the defining relations of hypergraph C^*-algebras (HR1) and (HR2a) applied to the specific topology of a hyperbranch.

Proposition 3.3: *For every $i \in \{1, \ldots, k-1\}$, the following relations hold in $C^*(H\Gamma)$:*

(i) $s_{e_i}^* s_{e_i} = \sum_{v \in V_{i+1}} p_v$,

(ii) $s_{e_i} s_{e_i}^* = \sum_{v \in V_i} p_v$.

Proof: The first identity (i) follows immediately from the relation (HR1), which states that for any edge e, we have

$$s_e^* s_e = \sum_{v \in r(e)} p_v.$$

Since for a hyperbranch $r(e_i) = V_{i+1}$, equation (i) holds.

For the second identity (ii), we utilize the inequalities given by (HR2). First, (HR2a) gives us an upper bound:

$$s_{e_i} s_{e_i}^* \leq \sum_{v \in s(e_i)} p_v = \sum_{v \in V_i} p_v.$$

Second, we consider (HR2b). Since $H\Gamma$ is a hyperbranch, for any vertex $v \in V_i$ (where $i < k$), there is exactly one outgoing edge, namely e_i. The vertex v is not a sink. Thus, the relation (HR2b) simplifies to:

$$p_v \leq \sum_{\substack{e \in E \\ v \in s(e)}} s_e s_e^* = s_{e_i} s_{e_i}^*.$$

With Remark 1.9 we get:

$$\sum_{v \in V_i} p_v = s_{e_i} s_{e_i}^*.$$

□

Remark 3.4: In the following we will write $P_i := \sum_{v \in V_i} p_v$ for the projection associated to the vertex set V_i. With this notation, we can rewrite the relations from Proposition 3.3 as $s_{e_i}^* s_{e_i} = P_{i+1}$ and $s_{e_i} s_{e_i}^* = P_i$ for all i.

3.2 Structure of Hyperbranch C^*-Algebras

To determine the structure of hyperbranch C^*-algebras, our overarching strategy is to identify a full corner within the algebra that is isomorphic to a simpler structure, and then use matrix units to reconstruct the entire algebra.

As a foundational tool for this approach, we recall a standard result concerning full corners. We refer to Blackadar [1, Example II.9.4.3] for the following fact. Recall from Chapter 1 that a system of matrix units $\{e_{ij} \mid 1 \leq i, j \leq N\}$ satisfies $e_{ij}^* = e_{ji}$, $e_{ij} e_{kl} = \delta_{jk} e_{il}$, and $\sum e_{ii} = 1_A$. Furthermore, for a projection $p \in A$, the corner algebra is defined as $pAp := \{pap \mid a \in A\}$.

Lemma 3.5 (Full Corner Isomorphism): *Let A be a unital C^*-algebra and suppose there exist a system of matrix units $\{e_{ij} \mid 1 \leq i, j \leq N\}$ in A and $p = e_{kk}$ for any $k \in \{1, \ldots, N\}$. Then*

$$M_N(pAp) \cong A$$

via the isomorphism

$$\Phi \colon M_N(pAp) \to A, \quad (b_{ij})_{i,j=1,\ldots,N} \mapsto \sum_{i,j=1}^{N} e_{ik} b_{ij} e_{kj}.$$

Proof: Φ is well-defined since $e_{ik}b_{ij}e_{kj} \in A$ for all i, j, k. Linearity is clear. For the multiplication, we compute for $(b_{ij})_{i,j}, (c_{ij})_{i,j} \in M_N(pAp)$:

$$\begin{aligned}
\Phi((b_{ij})_{i,j})\Phi((c_{ij})_{i,j}) &= \left(\sum_{i,j=1}^{N} e_{ik}b_{ij}e_{kj}\right)\left(\sum_{\ell,m=1}^{N} e_{\ell k}c_{\ell m}e_{km}\right) \\
&= \sum_{i,j,\ell,m=1}^{N} e_{ik}b_{ij}(e_{kj}e_{\ell k})c_{\ell m}e_{km} \\
&= \sum_{i,j,\ell,m=1}^{N} e_{ik}b_{ij}\delta_{j\ell}e_{kk}c_{\ell m}e_{km} \\
&= \sum_{i,j,m=1}^{N} e_{ik}b_{ij}e_{kk}c_{jm}e_{km} \\
&= \sum_{i,j,m=1}^{N} e_{ik}b_{ij}c_{jm}e_{km} \\
&= \Phi((b_{ij}c_{jm})_{i,m}).
\end{aligned}$$

For the involution, we notice that

$$\begin{aligned}
\Phi((b_{ij})_{i,j})^* &= \left(\sum_{i,j=1}^{N} e_{ik}b_{ij}e_{kj}\right)^* \\
&= \sum_{i,j=1}^{N} e_{kj}^*b_{ij}^*e_{ik}^* \\
&= \sum_{i,j=1}^{N} e_{jk}b_{ij}^*e_{ki} \\
&= \Phi((b_{ij}^*)_{i,j}).
\end{aligned}$$

So we get a $*$-homomorphism. Now we construct the inverse map:

$$\Psi\colon A \to M_N(pAp), \quad a \mapsto (e_{ki}ae_{jk})_{i,j=1,\ldots,N}.$$

We check that Ψ is indeed the inverse of Φ:

$$\begin{aligned}\Phi(\Psi(a)) &= \Phi((e_{ki}ae_{jk})_{i,j})\\ &= \sum_{i,j=1}^{N} e_{ik}(e_{ki}ae_{jk})e_{kj}\\ &= \sum_{i,j=1}^{N} (e_{ik}e_{ki})a(e_{jk}e_{kj})\\ &= \sum_{i,j=1}^{N} e_{ii}ae_{jj}\\ &= \left(\sum_{i=1}^{N} e_{ii}\right) a \left(\sum_{j=1}^{N} e_{jj}\right)\\ &= 1_A a 1_A = a,\end{aligned}$$

and

$$\begin{aligned}\Psi(\Phi((b_{ij})_{i,j})) &= \Psi\left(\sum_{i,j=1}^{N} e_{ik}b_{ij}e_{kj}\right)\\ &= \left(e_{k\ell}\left(\sum_{i,j=1}^{N} e_{ik}b_{ij}e_{kj}\right)e_{mk}\right)_{\ell,m=1,\dots,N}\\ &= \left(\sum_{i,j=1}^{N} (e_{k\ell}e_{ik})b_{ij}(e_{kj}e_{mk})\right)_{\ell,m=1,\dots,N}\\ &= \left(\sum_{i,j=1}^{N} \delta_{\ell i}b_{ij}\delta_{jm}\right)_{\ell,m=1,\dots,N}\\ &= (b_{\ell m})_{\ell,m=1,\dots,N}.\end{aligned}$$

Thus, Φ is an isomorphism with inverse Ψ. □

Our strategy to prove Theorem A relies heavily on Lemma 3.5. However, to apply this lemma, we must first deeply understand the internal structure of the corner algebra associated with the sink of the hyperbranch. The following lemmas will establish this structure step by step before we formulate our main result.

Lemma 3.6: *For any $1 \leq i, j \leq k$, let μ_i and μ_j denote the unique paths from V_i and V_j to the sink V_k, respectively. If $i \leq j$ and γ is the unique path from V_i to V_j, then:*

$$s_\gamma = s_{\mu_i}s_{\mu_j}^*.$$

Consequently, if $i > j$ and γ is the unique path from V_j to V_i, we have $s_\gamma^* = s_{\mu_i} s_{\mu_j}^*$. In particular, for $i = j$, the projection $P_i = \sum_{v \in V_i} p_v$ satisfies $P_i = s_{\mu_i} s_{\mu_i}^*$.

Proof: For $i = j$, the path γ has length zero, and $s_\gamma = P_i$. The identity $P_i = s_{\mu_i} s_{\mu_i}^*$ follows by repeated application of Proposition 3.3.

For $i < j$, the path from V_i to the sink can be factored as the concatenation $\mu_i = \gamma \mu_j$. In the C^*-algebra, this corresponds to $s_{\mu_i} = s_\gamma s_{\mu_j}$. Multiplying both sides by $s_{\mu_j}^*$ from the right yields:

$$s_{\mu_i} s_{\mu_j}^* = s_\gamma s_{\mu_j} s_{\mu_j}^* = s_\gamma P_j = s_\gamma.$$

For $i > j$, the path γ goes from V_j to V_i. By the previous case, $s_\gamma = s_{\mu_j} s_{\mu_i}^*$. Taking the adjoint on both sides immediately yields:

$$s_\gamma^* = (s_{\mu_j} s_{\mu_i}^*)^* = s_{\mu_i} s_{\mu_j}^*. \qquad \square$$

The following lemma identifies the specific generators of the corner algebra $P_k C^*(H\Gamma) P_k$ viewed as a C^*-subalgebra of $C^*(H\Gamma)$.

Lemma 3.7: *The corner algebra associated with the sink V_k coincides with the C^*-subalgebra generated by the set of all path-derived projections terminating in V_k. Explicitly, as subalgebras of $C^*(H\Gamma)$:*

$$P_k C^*(H\Gamma) P_k = C^*\Big(\Big\{ s_\mu^* p_v s_\mu \mid \mu \in E^*,\ r(\mu) \subseteq V_k,\ v \in s(\mu) \Big\}\Big).$$

Proof: Recall that $r(\mu) \subseteq V_k$ is equivalent to $r(\mu) = V_k$ or $r(\mu) = \{v\}$ for some $v \in V_k$. The inclusion "$\supseteq$" is immediate: for any path μ with $r(\mu) \subseteq V_k$, we get $P_k s_\mu^* = s_\mu^*$ and $s_\mu P_k = s_\mu$. Therefore,

$$P_k (s_\mu^* p_v s_\mu) P_k = s_\mu^* p_v s_\mu.$$

For "$\subseteq$": The universal C^*-algebra $C^*(H\Gamma)$ is the closed linear span of words in the fundamental generators $\{p_v \mid v \in V\} \cup \{s_e, s_e^* \mid e \in E\}$. It suffices to show that for any fully reduced word w in these generators, the element $P_k w P_k$ is generated by elements of the form $s_\mu^* p_v s_\mu$ with $r(\mu) \subseteq V_k$.

Assume $P_k w P_k \neq 0$. Since w is evaluated inside the corner P_k and the absolute sink V_k has no outgoing edges, w cannot begin with a forward edge s_e or end with an adjoint edge s_e^*. A fully reduced word must therefore alternate between traveling upstream via adjoint paths and downstream via forward paths, interspersed with vertex projections. Suppose w contains m vertex projections $p_{v_1}, \dots, p_{v_m}$ located in layers $V_{j_1}, \dots, V_{j_m}$. The general form of such a non-zero word is:

$$P_k w P_k = s^*_{\alpha_1} p_{v_1} s_{\alpha_1} s^*_{\alpha_2} p_{v_2} s_{\alpha_2} \dots s^*_{\alpha_m} p_{v_m} s_{\alpha_m},$$

where each α_i is the unique path from the vertex set containing v_i to the sink V_k. The fact that we can restrict our attention to paths terminating exclusively at the sink is a direct consequence of Lemma 3.6. Since all factors are now explicitly of the form $s^*_\mu p_v s_\mu$ with $r(\mu) \subseteq V_k$, it follows that $P_k w P_k$ is contained in the right-hand side algebra, which completes the proof. □

With the generators of the corner algebra identified, we can now determine its isomorphism class. The next lemma shows that the algebraic independence of the vertex sets V_i leads to a free product structure within the sink corner.

Lemma 3.8: *The corner algebra $P_k C^*(H\Gamma) P_k$ decomposes into a free product of finite-dimensional algebras. Specifically:*

$$C^*\Big(\Big\{ s^*_\mu p_v s_\mu \mid \mu \in E^*,\ r(\mu) \subseteq V_k,\ v \in s(\mu) \Big\}\Big) \cong \mathbb{C}^{n_1} *_{\mathbb{C}} \mathbb{C}^{n_2} *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathbb{C}^{n_k},$$

where $n_j = |V_j|$ is the number of vertices in the set V_j.

Proof: Each element $s^*_\mu p_v s_\mu$ is a projection, so the algebra is generated by projections. For fixed i, the projections $\{s^*_{\mu_i} p_v s_{\mu_i} \mid v \in s(\mu_i) = V_i\}$ are mutually orthogonal, i.e., for $v, w \in V_i$ with $v \neq w$ we have:

$$(s^*_{\mu_i} p_v s_{\mu_i})(s^*_{\mu_i} p_w s_{\mu_i}) = s^*_{\mu_i} p_v P_i p_w s_{\mu_i} = s^*_{\mu_i} p_v p_w s_{\mu_i} = 0.$$

Additionally, the sum of these projections is the identity on the corner algebra:

$$\sum_{v \in V_i} s^*_{\mu_i} p_v s_{\mu_i} = s^*_{\mu_i} P_i s_{\mu_i} = s^*_{\mu_i} s_{\mu_i} = P_k.$$

Since the vertex sets V_i are pairwise disjoint, projections arising from different paths μ_i and μ_j (with $i \neq j$) share no non-trivial relations beyond having common unit P_k. By the universal property of the free product, the algebra generated by all these projections is isomorphic to

$$\mathbb{C}^{n_1} *_{\mathbb{C}} \mathbb{C}^{n_2} *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathbb{C}^{n_k}.$$

□

We have now characterized the corner algebra $P_k C^*(H\Gamma) P_k$ as a free product. To extend this structure to the entire algebra $C^*(H\Gamma)$, we construct a system of matrix units that connects the different layers of the hyperbranch. This allows us to apply the Full Corner Isomorphism.

Theorem A (Structure of Hyperbranch C^*-Algebras): *Let $H\Gamma$ denote a hyperbranch of length k with vertex sets $V_1, \dots, V_k$ and edges $e_1, \dots, e_{k-1}$. Let $n_j = |V_j|$ denote the number of vertices in the set V_j. Then there is an isomorphism:*

$$C^*(H\Gamma) \cong M_k\left(\mathbb{C}^{n_1} *_{\mathbb{C}} \mathbb{C}^{n_2} *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathbb{C}^{n_k}\right),$$

where $*_{\mathbb{C}}$ denotes the unital free product of C^*-algebras.

Proof: For each $1 \leq i \leq k$, define the projection

$$e_{ii} := P_i = \sum_{v \in V_i} p_v.$$

By Remark 1.9, each e_{ii} is a projection. These projections are mutually orthogonal (since the V_i are disjoint) and sum to the identity by Proposition 2.6. For $1 \leq i < j \leq k$, define the partial isometry $e_{ij} := s_{\mu_{ij}} := s_{e_i} s_{e_{i+1}} \cdots s_{e_{j-1}}$, where μ_{ij} is the unique path from V_i to V_j. For $i > j$, define $e_{ij} := e_{ji}^*$.

We verify that $\{e_{ij}\}$ forms a system of matrix units. Self-adjointness of the diagonal elements and the sum to identity are clear. For the multiplication, consider $e_{ij} e_{\ell m}$ for arbitrary i, j, ℓ, m. Using $e_{ij} = e_{ij} P_j$ and $e_{\ell m} = P_\ell e_{\ell m}$, we have

$$e_{ij} e_{\ell m} = e_{ij} P_j P_\ell e_{\ell m} = \delta_{j\ell} e_{ij} e_{\ell m}.$$

Thus, if $j \neq \ell$, then $e_{ij} e_{\ell m} = 0$. On the other hand, if $j = \ell$ and $i < j < m$, then

$$e_{ij} e_{jm} = (s_{e_i} \cdots s_{e_{j-1}})(s_{e_j} \cdots s_{e_{m-1}}) = s_{e_i} \cdots s_{e_{m-1}} = e_{im}.$$

For $i, m > j$, we have

$$e_{ij} e_{jm} = (s_{e_j} \cdots s_{e_{i-1}})^* (s_{e_j} \cdots s_{e_{m-1}}) = (s_{e_{i-1}}^* \cdots s_{e_j}^*)(s_{e_j} \cdots s_{e_{m-1}}).$$

By repeatedly applying Proposition 3.3, which states $s_{e_n}^* s_{e_n} = P_{n+1}$, the inner terms cancel out up to $\min(i, m)$, yielding e_{im}. For the remaining cases $m < j < i$ and $i, m < j$, the result follows directly by taking adjoints. Thus, $\{e_{ij}\}$ is a system of matrix units.

Now we can apply Lemma 3.5 with $p = e_{kk}$ and use Lemmas 3.7 and 3.8 to conclude that

$$C^*(H\Gamma) \cong M_k\left(\mathbb{C}^{n_1} *_{\mathbb{C}} \mathbb{C}^{n_2} *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathbb{C}^{n_k}\right). \qquad \square$$

Remark 3.9: The key insight of this theorem is that the hyperbranch C^*-algebra decomposes into a $k \times k$ matrix algebra over the free product of the algebras $\mathbb{C}^{n_j}$ generated by the mutually orthogonal projections $\{p_v \mid v \in V_j\}$

in each vertex set. The matrix structure arises from the linear ordering of the vertex sets, while the free product captures the fact that projections from different vertex sets have no additional relations beyond those imposed by the hypergraph structure.

Importantly, this result is perfectly consistent with the classical structure theorem for acyclic graph C^*-algebras (Theorem 1.25). If the hyperbranch is a classical directed line graph, then each vertex set consists of exactly one vertex ($|V_j| = 1$). Consequently, $n_j = 1$ for all j, yielding factors of $\mathbb{C}^1 \cong \mathbb{C}$. Since $\mathbb{C}$ is the neutral element for the unital free product of C^*-algebras, the entire free product collapses to $\mathbb{C}$. The structure thus simplifies to $M_k(\mathbb{C})$, which exactly matches the classical result.

3.3 Examples and the Special Case of Two Projections

Example 3.10: Consider the hyperbranch of length 3 with vertex sets $V_1 = \{v_1\}$, $V_2 = \{v_2, v_3\}$, and $V_3 = \{v_4\}$, and edges e_1 from V_1 to V_2 and e_2 from V_2 to V_3 (see Figure 3.2). Then by Theorem A, we have

$$C^*(H\Gamma) \cong M_3 \left(\mathbb{C} *_{\mathbb{C}} \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}\right).$$

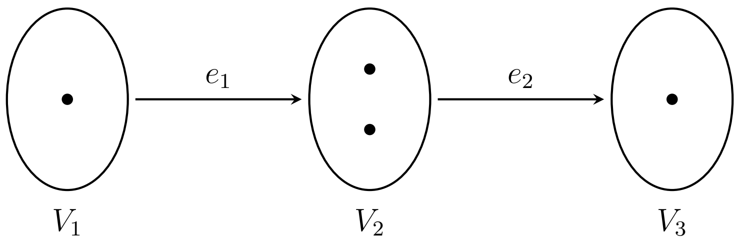


Figure 3.2: Example 3.10

Example 3.11: Consider the hyperbranch of length 2 with vertex sets $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$, and a single edge e_1 from V_1 to V_2 (see Figure 3.3). By Theorem A, we have

$$C^*(H\Gamma) \cong M_2 \left(\mathbb{C}^3 *_{\mathbb{C}} \mathbb{C}^3\right).$$

Another interesting example is the C^*-algebra generated by two projections. This specific case has been studied in depth, originally formulated by Halmos

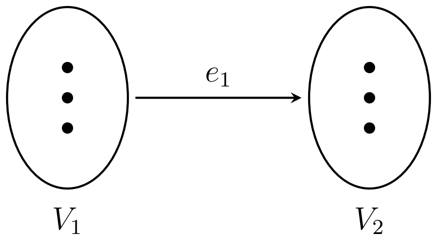


Figure 3.3: Example 3.11

[2] in the context of Hilbert spaces, and explicitly established for abstract C^*-algebras by Pedersen [4].

In their work, the structure of the C^*-algebra generated by two projections p and q is classified. A key result is that if p and q are projections with no additional relations, the generated algebra can be completely described by a continuous parameter $\theta \in [0, \pi/2]$, which corresponds to the angle between the ranges of the projections.

Theorem 3.12 (Halmos, Pedersen): *Let $C^*(p, q)$ be the unital C^*-algebra generated by two projections p and q. Then there is an isomorphism of $C^*(p, q)$ onto*

$$A = \{f \in C([0, \pi/2], M_2(\mathbb{C})) \mid f(0), f(\pi/2) \text{ are diagonal}\},$$

which carries the generating projections into the matrix-valued functions:

$$p(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q(\theta) = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

For the proof, we refer to the original works [2, 4]. We can now apply these results to our running example, the simple hyperbranch from Figure 3.4.

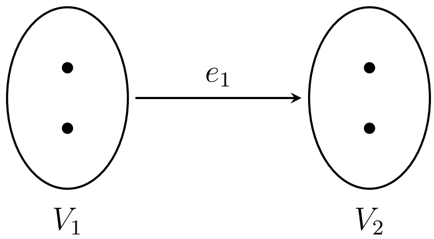


Figure 3.4: The "simple hyperbranch" ($|V_1| = 2, |V_2| = 2$).

Proposition 3.13: *Let $H\Gamma$ be the hyperbranch of length 2 with $|V_1| = |V_2| = 2$. Then*

$$C^*(H\Gamma) \cong M_2\left(\{f \in C([0, \pi/2], M_2(\mathbb{C})) \mid f(0) \text{ and } f(\pi/2) \text{ are diagonal}\}\right).$$

Proof: By Theorem A we have

$$C^*(H\Gamma) \cong M_2\left(\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2\right).$$

Note that the algebra $\mathbb{C}^2$ is generated by the identity and a single projection. Thus, the free product $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2$ is universally generated by two projections p and q without any further relations. The claim then follows directly from applying Theorem 3.12. □

Remark 3.14: This example serves as a crucial warning: even the simplest non-trivial hyperbranch generates an infinite-dimensional algebra with continuous trace. This stands in contrast to graph C^*-algebras, where finite acyclic graphs always yield finite-dimensional algebras. The case of three or more projections (e.g., $|V_1| = 3$) is significantly more complicated, as the algebra becomes "wild" and contains the group C^*-algebra of the free group $\mathbb{F}_2$, as described in [7].

4 Hyperbranch Tree C^*-Algebras

After characterizing the structure of isolated hyperbranch C^*-algebras, we now turn to more complex directed networks. A natural approach to building these larger structures is conceptually simple: we use hyperbranches as fundamental building blocks and "glue" them together to form trees.

In classical graph theory, this naturally leads to the study of directed rooted trees. Depending on the orientation of the edges relative to the root vertex, these are classified as either *in-trees* (where multiple paths converge into exactly one sink) or *out-trees* (where multiple paths diverge from exactly one source). In this chapter, we construct and analyze the hypergraph analogues of these structures.

It is important to note that the hyperbranch trees we investigate represent a specific, strict subclass of general rooted (hyper-)trees. To maintain algebraic tractability, we require that the constituent hyperbranches are entirely disjoint everywhere except at the unique global source or sink. While this restriction means our framework does not cover all possible rooted hypertrees, it allows us to isolate the exact algebraic consequences of branching and converging paths. Furthermore, all classes studied in this chapter share a crucial combinatorial property: they contain no generalized cycles and admit only perfect paths.

We proceed step by step. First, we investigate hyperbranch in-trees, establishing the structural consequences of multiple branches converging into a single sink. Following this, we study hyperbranch out-trees, where a single source diverges into multiple independent branches, culminating in our second main result, Theorem B. Finally, we explore the algebraic obstructions that arise in more general trees and examine hybrid structures.

4.1 Hyperbranch In-Tree C^*-Algebras

We begin by analyzing the hypergraph counterpart to an in-tree. By joining the terminal ends of multiple distinct hyperbranches, we construct a structure that flows into a single, shared sink (the root). We investigate the resulting hypergraph and its associated C^*-algebra. Let us formally define this class of hypergraphs.

Definition 4.1 (Hyperbranch In-Tree): Let $H\Gamma = (V, E, r, s)$ be a finite directed hypergraph. We call $H\Gamma$ a *hyperbranch in-tree with k branches* if there exist k hyperbranches $HB_1, \dots, HB_k$, where each $HB_i = (V^{(i)}, E^{(i)}, r_i, s_i)$ has length n_i with $V^{(i)} = \dot\bigcup_{j=1}^{n_i} V_{i,j}$ and $E^{(i)} = \{e_{i,1}, \dots, e_{i,n_i-1}\}$, satisfying the following conditions:

(i) All hyperbranches share the same final vertex set, i.e., $V_{\text{sink}} := V_{i,n_i}$ and $r_i(e_{i,n_i-1}) = V_{\text{sink}}$ for all $i \in \{1, \dots, k\}$.

(ii) The hyperbranches are disjoint except for the sink, i.e., $E^{(i)} \cap E^{(j)} = \emptyset$ and $V^{(i)} \cap V^{(j)} = V_{\text{sink}}$ for $i \neq j$ with $i, j \in \{1, \dots, k\}$.

(iii) The total vertex set is $V = \bigcup_{i=1}^{k} V^{(i)}$ and the total edge set is $E = \dot\bigcup_{i=1}^{k} E^{(i)}$.

To better understand this definition, it is helpful to visualize the structure as a set of distinct "branches" flowing into a common sink vertex set. The condition that the vertex sets intersect only at the sink ensures that there is no interaction between the branches HB_i and HB_j prior to the final step.

The notation $V_{i,j}$ effectively acts as a coordinate system: the index i identifies the specific branch (hyperbranch), while j denotes the layer or distance from the start of that branch. For instance, $V_{i,1}$ represents the starting vertex set of the i-th hyperbranch.

We illustrate this construction with a concrete example in Figure 4.1, where three hyperbranches of different lengths merge into a single sink.

Remark 4.2: A direct consequence of the hyperbranch in-tree structure is the existence of unique paths. By Definition 4.1, the component branches are purely linear and disjoint except at the sink. Therefore, for any vertex set $V_{i,j}$ located in any branch, there is exactly one unique path μ connecting it to the common sink V_{sink}.

Before investigating the structure theorem, we must connect the combinatorial object to its operator algebra and fix our notation.

For the remainder of this section, let $H\Gamma = (V, E, r, s)$ be a hyperbranch in-tree with $k \geq 1$ branches $HB_1, \dots, HB_k$ of lengths $n_1, \dots, n_k$. We denote its vertex sets by $V_{i,j}$ and V_{sink}, and its edges by $e_{i,j}$ as introduced in Definition 4.1. We study its associated hypergraph C^*-algebra $C^*(H\Gamma)$. We will uniformly denote the fundamental generators of $C^*(H\Gamma)$ by the partial isometries $\{s_{e_{i,j}}\}$ and the mutually orthogonal vertex projections $\{p_v \mid v \in V\}$.

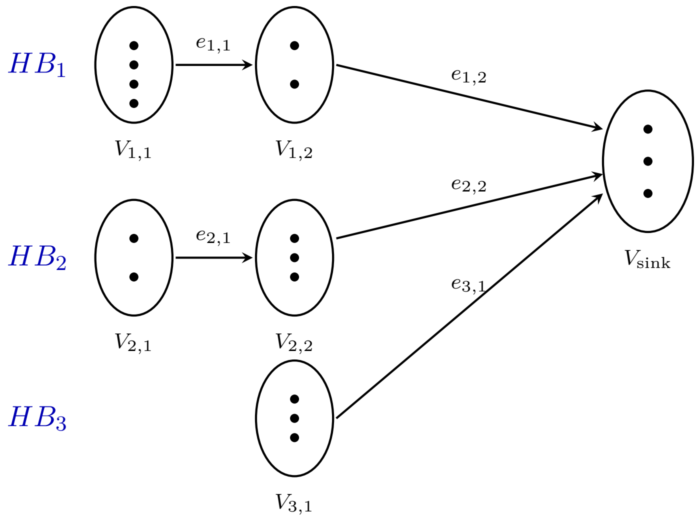


Figure 4.1: A hyperbranch in-tree with $k = 3$ hyperbranches of lengths $n_1 = 3$, $n_2 = 3$, and $n_3 = 2$.

Similar to the isolated hyperbranch case, the following fundamental properties follow directly from the defining relations of hypergraph C^*-algebras.

Proposition 4.3: *For every i with $1 \leq i \leq k$, the following holds in $C^*(H\Gamma)$:*

(i) $s_{e_{i,j}}^* s_{e_{i,j}} = P_{i,j+1} := \sum_{v \in V_{i,j+1}} p_v$ *for all* $1 \leq j < n_i - 1$,

(ii) $s_{e_{i,j}} s_{e_{i,j}}^* = P_{i,j} := \sum_{v \in V_{i,j}} p_v$ *for all* $1 \leq j \leq n_i - 1$,

(iii) $s_{e_{i,n_i-1}}^* s_{e_{i,n_i-1}} = P_{sink} := \sum_{v \in V_{sink}} p_v$.

Proof: The calculations for (i) and (ii) are identical to the proof of Proposition 3.3, as each individual branch is a hyperbranch. For (iii), we use the fact that $r(e_{i,n_i-1}) = V_{\text{sink}}$ for all i by the definition of a hyperbranch in-tree. $\square$

We now turn to the structure theorem for hyperbranch in-trees. Since the entire structure converges into a single global sink, we can apply the exact same strategy as in Chapter 3. The following lemmas adapt the analysis of the sink corner algebra to account for the multiple incoming branches, ultimately yielding a very similar structural result.

Lemma 4.4: *The corner algebra associated with the sink V_{sink} coincides with the C^*-subalgebra generated by the set of all path-derived projections terminating in V_{sink}. Explicitly, as subalgebras of $C^*(H\Gamma)$:*

$$P_{sink} C^*(H\Gamma) P_{sink} = C^*\Big(\Big\{ s_\mu^* p_v s_\mu \mid \mu \in E^*,\ r(\mu) \subseteq V_{sink},\ v \in s(\mu) \Big\}\Big).$$

Proof: Recall that $r(\mu) \subseteq V_{\text{sink}}$ is equivalent to $r(\mu) = V_{\text{sink}}$ or $r(\mu) = \{v\}$ for some $v \in V_{\text{sink}}$. The inclusion "$\supseteq$" is immediate: for any path μ with $r(\mu) \subseteq V_{\text{sink}}$, the relation (HR1) implies $s_\mu^* s_\mu \leq P_{\text{sink}}$. Therefore, $P_{\text{sink}} s_\mu^* = s_\mu^*$ and $s_\mu P_{\text{sink}} = s_\mu$, which yields

$$P_{\text{sink}}(s_\mu^* p_v s_\mu) P_{\text{sink}} = s_\mu^* p_v s_\mu.$$

For "$\subseteq$": The universal C^*-algebra $C^*(H\Gamma)$ is the closed linear span of fully reduced words in the fundamental generators. It suffices to show that for any fully reduced word w, the element $P_{\text{sink}} w P_{\text{sink}}$ lies in the right-hand side algebra B.

Assume $P_{\text{sink}} w P_{\text{sink}} \neq 0$. Because w is evaluated inside the corner P_{sink} and V_{sink} has no outgoing edges, w must begin with an adjoint path and end with a forward path. Suppose w contains m vertex projections $p_{v_1}, \ldots, p_{v_m}$. Between two projections p_{v_l} and $p_{v_{l+1}}$ within the same branch (where $v_l, v_{l+1} \in V^{(i)}$), the transition is given by $s_{\alpha_l} s_{\alpha_{l+1}}^*$ by Lemma 3.6. If the vertices lie in different branches, meaning $v_l \in V^{(i)}$ and $v_{l+1} \in V^{(j)}$ with $i \neq j$, we use Remark 4.2 stating that there exists a unique path to the sink. To have a transition from v_l to v_{l+1}, we need to transit through V_{sink}. Therefore, we use the unique paths α_l and α_{l+1} respective to their branches and obtain the transition $s_{\alpha_l} s_{\alpha_{l+1}}^*$.

In both scenarios, we get for any word w:

$$P_{\text{sink}} w P_{\text{sink}} = (s_{\alpha_1}^* p_{v_1} s_{\alpha_1})(s_{\alpha_2}^* p_{v_2} s_{\alpha_2}) \ldots (s_{\alpha_m}^* p_{v_m} s_{\alpha_m}).$$

Each bracketed factor is an explicit generator of the right-hand side algebra B, which completes the proof. □

The following lemma identifies the isomorphism class of this corner algebra.

Lemma 4.5: *Let $m_{i,j} := |V_{i,j}|$ and $m_{sink} := |V_{sink}|$. Then:*

$$C^*\left(\left\{s_\mu^* p_v s_\mu \mid \mu \in E^*,\ r(\mu) \subseteq V_{sink},\ v \in s(\mu)\right\}\right) \cong \mathbb{C}^{m_{sink}} *_{\mathbb{C}} \left(\mathop{*_{\mathbb{C}}}_{i=1}^{k} \mathop{*_{\mathbb{C}}}_{j=1}^{n_i-1} \mathbb{C}^{m_{i,j}} \right).$$

Proof: The proof is analogous to the proof of Lemma 3.8 for hyperbranches. For fixed i, j, the elements $s_\mu^* p_v s_\mu$ with $r(\mu) \subseteq V_{\text{sink}}$ and $v \in V_{i,j}$ generate a commutative C^*-algebra isomorphic to $\mathbb{C}^{m_{i,j}}$. The elements with $v \in V_{\text{sink}}$ generate a commutative C^*-algebra isomorphic to $\mathbb{C}^{m_{\text{sink}}}$. Since the branches are disjoint except for the sink, there are no non-trivial relations between these algebras beyond sharing the common unit P_{sink}. By the universal property of the free product, we conclude that the entire algebra is isomorphic to the stated free product. □

Lemma 4.6: *Let*

$$\mathcal{V} := \{V_{i,j} \mid 1 \le i \le k,\ 1 \le j \le n_i - 1\} \cup \{V_{sink}\}$$

be the collection of all vertex sets in the in-tree, and let P_{sink} be the projection onto the unique sink. For each vertex set $W \in \mathcal{V}$, let μ_W be the unique path from W to the sink, with the convention that $s_{\mu_{V_{sink}}} := P_{sink}$. Define the elements

$$e_{UW} := s_{\mu_U} s_{\mu_W}^* \quad \text{for all } U, W \in \mathcal{V}.$$

Then the family $\{e_{UW}\}_{U,W\in\mathcal{V}}$ forms a system of matrix units in $C^*(H\Gamma)$.

Proof: We verify the three defining properties of matrix units. First, the involution property holds immediately, as $e_{UW}^* = (s_{\mu_U} s_{\mu_W}^*)^* = s_{\mu_W} s_{\mu_U}^* = e_{WU}$.

Second, since the collection $\mathcal{V}$ forms a partition of the total vertex set V, Proposition 2.6 implies that the sum of all vertex projections is the identity. Thus, we have $\sum_{W\in\mathcal{V}} e_{WW} = \sum_{W\in\mathcal{V}} s_{\mu_W} s_{\mu_W}^* = \sum_{W\in\mathcal{V}} P_W = \sum_{v\in V} p_v = 1_{C^*(H\Gamma)}$.

Finally, to verify the multiplication rule, we consider the product $e_{UW}e_{ZY} = s_{\mu_U}(s_{\mu_W}^* s_{\mu_Z})s_{\mu_Y}^*$ and analyze the middle term $s_{\mu_W}^* s_{\mu_Z}$. Both paths μ_W and μ_Z end at the same sink V_{sink}. If $W = Z$, Proposition 4.3 (i) applied to the branch containing W yields $s_{\mu_W}^* s_{\mu_W} = P_{\text{sink}}$. If $W \neq Z$, the uniqueness of paths in the in-tree structure and the domain relations $s_{\mu_W} = P_W s_{\mu_W}$ allow us to rewrite the term as $s_{\mu_W}^* s_{\mu_Z} = (P_W s_{\mu_W})^*(P_Z s_{\mu_Z}) = s_{\mu_W}^* P_W P_Z s_{\mu_Z}$. Since distinct vertex sets in $\mathcal{V}$ are disjoint, $P_W P_Z = 0$, which implies $s_{\mu_W}^* s_{\mu_Z} = 0$. Consequently, we obtain $s_{\mu_W}^* s_{\mu_Z} = \delta_{WZ} P_{\text{sink}}$, leading to $e_{UW}e_{ZY} = s_{\mu_U}(\delta_{WZ}P_{\text{sink}})s_{\mu_Y}^* = \delta_{WZ} s_{\mu_U} s_{\mu_Y}^* = \delta_{WZ} e_{UY}$. Thus, $\{e_{UW}\}_{U,W\in\mathcal{V}}$ is a system of matrix units. □

Now with these lemmas in place, we can prove the structure theorem for hyperbranch in-trees in an analogous fashion.

Theorem 4.7 (Structure of Hyperbranch In-Trees): *Let $\ell := \sum_{i=1}^{k}(n_i - 1) + 1$ be the total number of vertex sets in the in-tree. Then*

$$C^*(H\Gamma) \cong M_\ell\left(\mathbb{C}^{m_{sink}} *_{\mathbb{C}} \left(\mathop{*_{\mathbb{C}}}_{i=1}^{k} \mathop{*_{\mathbb{C}}}_{j=1}^{n_i-1} \mathbb{C}^{m_{i,j}} \right)\right).$$

Proof: By Lemma 4.6, the algebra $C^*(H\Gamma)$ contains a system of matrix units $\{e_{UW}\}$ of size $\ell \times \ell$. The projection corresponding to the sink is $p = e_{V_{\text{sink}},V_{\text{sink}}} = P_{\text{sink}}$. We can therefore apply the Full Corner Isomorphism (Lemma 3.5), which yields:

$$C^*(H\Gamma) \cong M_\ell(P_{\text{sink}} C^*(H\Gamma) P_{\text{sink}}).$$

By Lemma 4.4 and Lemma 4.5, the corner algebra $P_{\text{sink}} C^*(H\Gamma) P_{\text{sink}}$ is isomorphic to the free product of the algebras $\mathbb{C}^{|W|}$ for all vertex sets W. Inserting this into the matrix algebra gives the desired isomorphism. □

Remark 4.8: Our formal definition of a hyperbranch in-tree (Definition 4.1) restricts the structure to branches that intersect exclusively at the final sink. However, the algebraic classification extends naturally to arbitrary hyperbranch in-trees where branches may merge at intermediate stages before reaching the global sink. The crucial property driving the matrix unit construction in Lemma 4.6 is the existence of a unique path from every vertex set to the global sink. As long as this holds, the algebra will consistently decompose into a single matrix block over the free product of all vertex set projections.

Importantly, this behavior is perfectly consistent with Raeburn's classical theory of graph C^*-algebras, where a directed tree flowing into a single sink yields a single matrix algebra rather than a direct sum. We restricted our explicit formulation to the final-sink merging case solely to maintain algebraic tractability and readable indexing.

Example 4.9: Consider the hyperbranch in-tree in Figure 4.1 with $k = 3$ branches:

- HB_1 has length $n_1 = 3$ with $m_{1,1} = 4$ and $m_{1,2} = 2$,
- HB_2 has length $n_2 = 3$ with $m_{2,1} = 2$ and $m_{2,2} = 3$,
- HB_3 has length $n_3 = 2$ with $m_{3,1} = 3$,
- The shared sink has $m_{\text{sink}} = 3$.

Then $\ell = (3-1) + (3-1) + (2-1) + 1 = 6$ and:

$$C^*(H\Gamma) \cong M_6\left(\mathbb{C}^3 *_{\mathbb{C}} (\mathbb{C}^4 *_{\mathbb{C}} \mathbb{C}^2) *_{\mathbb{C}} (\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3) *_{\mathbb{C}} \mathbb{C}^3\right).$$

4.2 Hyperbranch Out-Tree C^*-Algebras

We now consider the dual construction to the hyperbranch in-tree case: the hypergraph counterpart to an out-tree. Here, multiple hyperbranches share a common source (the root) but terminate at distinct sinks.

Definition 4.10 (Hyperbranch Out-Tree): A finite directed hypergraph $H\Gamma = (V, E, r, s)$ is called a *hyperbranch out-tree with k branches* if there exist k hyperbranches $HB_1, \ldots, HB_k$, where each $HB_i = (V^{(i)}, E^{(i)}, r_i, s_i)$ has length n_i, such that:

(i) There exists a vertex set V_{source} that is the source of the first edge in each hyperbranch. That is, for all $i \in \{1, \ldots, k\}$, we have $s(e_{i,1}) = V_{\text{source}}$.

(ii) The edge sets are pairwise disjoint: $E^{(i)} \cap E^{(j)} = \emptyset$ for $i \neq j$, and $E = \dot{\bigcup}_{i=1}^{k} E^{(i)}$.

(iii) Any two distinct hyperbranches intersect only at the source vertex set:

$$V^{(i)} \cap V^{(j)} = V_{\text{source}} \quad \text{for } i \neq j,$$

and $V = \bigcup_{i=1}^{k} V^{(i)}$.

We denote the vertex sets of the i-th branch by $V_{\text{source}}, V_{i,2}, \ldots, V_{i,n_i}$, where $V_{i,n_i} := V_{i,\text{sink}}$ is the terminal vertex set of branch i. We define the collection of vertex sets for the i-th branch as

$$\mathcal{V}_i := \{V_{\text{source}}, V_{i,2}, \ldots, V_{i,n_i}\},$$

and the collection of all vertex sets in the entire tree as

$$\mathcal{V} := \bigcup_{i=1}^{k} \mathcal{V}_i.$$

In the theory of standard directed graph C^*-algebras, a tree with multiple distinct sinks naturally decomposes into a strict direct sum of matrix algebras. This decoupling is enforced by the Cuntz-Krieger relation (CK1), which dictates that for distinct edges e and f emitting from the same vertex, the corresponding isometries are orthogonal: $s_e^* s_f = 0$. Because any algebraic path connecting two distinct branches must eventually traverse such a word $s_e^* s_f$ of distinct edges, the cross-terms vanish identically, algebraically isolating the branches.

For hypergraphs, the analogous relation (HR1) still enforces global orthogonality of distinct edges sharing a source set: $s_e^* s_f = 0$. However, this orthogonality only applies relative to the total source projection $P_{s(e)} = \sum_{v \in s(e)} p_v$. If the shared source set contains multiple vertices ($|s(e)| > 1$), the global orthogonality $s_e^* P_{s(e)} s_f = 0$ does not universally imply local orthogonality at individual vertices. Specifically, there can exist a single vertex $v \in s(e)$ such that the localized transition $s_e^* p_v s_f$ is strictly non-zero.

Algebraically, this implies that elements can "leak" from one branch to another through isolated vertices within the shared source layer, preventing the C^*-algebra from trivially decomposing into a direct sum. The following example explicitly constructs a finite-dimensional representation to demonstrate this non-decoupling phenomenon.

Example 4.11 (Non-Orthogonality of Branches via Source Vertices): Let $H\Gamma$ be a hyperbranch out-tree consisting of a shared source layer $V_{\text{source}} = \{v_1, v_2\}$

and two outgoing branches HB_1 and HB_2 of length 1. The branches consist of the edges e_1 and e_2, terminating in the distinct sinks $V_{1,\text{sink}} = \{w_{1,1}, w_{1,2}\}$ and $V_{2,\text{sink}} = \{w_{2,1}, w_{2,2}\}$.

By the hypergraph relations, we require:

(i) $P_{\text{source}} = p_{v_1} + p_{v_2}$

(ii) $s_{e_1}^* s_{e_2} = 0$ (since $s(e_1) = s(e_2)$ and $e_1 \neq e_2$)

(iii) $s_{e_1} s_{e_1}^* + s_{e_2} s_{e_2}^* \leq P_{\text{source}}$

We construct a valid representation of this C^*-algebra on the finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_{\text{source}} \oplus \mathcal{H}_{1,\text{sink}} \oplus \mathcal{H}_{2,\text{sink}}$, where $\mathcal{H}_{\text{source}} \cong \mathbb{C}^4$ and $\mathcal{H}_{i,\text{sink}} \cong \mathbb{C}^2$.

Define the source projections as block matrices in $M_4(\mathbb{C})$:

$$p_{v_1} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad p_{v_2} = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}.$$

This satisfies $p_{v_1} p_{v_2} = 0$ and $P_{\text{source}} = p_{v_1} + p_{v_2} = I_4$.

Let the isometries $s_{e_1} : \mathcal{H}_{1,\text{sink}} \to \mathcal{H}_{\text{source}}$ and $s_{e_2} : \mathcal{H}_{2,\text{sink}} \to \mathcal{H}_{\text{source}}$ be represented by the following 4×2 block matrices:

$$s_{e_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 \\ I_2 \end{pmatrix}, \quad s_{e_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 \\ -I_2 \end{pmatrix}.$$

First, we verify that these operators are valid generators. They are isometries:

$$s_{e_1}^* s_{e_1} = \frac{1}{2} \begin{pmatrix} I_2 & I_2 \end{pmatrix} \begin{pmatrix} I_2 \\ I_2 \end{pmatrix} = \frac{1}{2}(2I_2) = I_2 = P_{1,\text{sink}},$$

$$s_{e_2}^* s_{e_2} = \frac{1}{2} \begin{pmatrix} I_2 & -I_2 \end{pmatrix} \begin{pmatrix} I_2 \\ -I_2 \end{pmatrix} = \frac{1}{2}(2I_2) = I_2 = P_{2,\text{sink}}.$$

Next, we check the branch orthogonality relation (HR1):

$$s_{e_1}^* s_{e_2} = \frac{1}{2} \begin{pmatrix} I_2 & I_2 \end{pmatrix} \begin{pmatrix} I_2 \\ -I_2 \end{pmatrix} = \frac{1}{2}(I_2 - I_2) = 0.$$

We also verify that the sum of their range projections respects the source bound (HR2):

$$s_{e_1} s_{e_1}^* = \frac{1}{2} \begin{pmatrix} I_2 \\ I_2 \end{pmatrix} \begin{pmatrix} I_2 & I_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I_2 & I_2 \\ I_2 & I_2 \end{pmatrix},$$

$$s_{e_2}s_{e_2}^* = \frac{1}{2}\begin{pmatrix} I_2 \\ -I_2 \end{pmatrix}\begin{pmatrix} I_2 & -I_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix}.$$

Summing these yields:

$$s_{e_1}s_{e_1}^* + s_{e_2}s_{e_2}^* = \frac{1}{2}\begin{pmatrix} 2I_2 & 0 \\ 0 & 2I_2 \end{pmatrix} = I_4 = P_{\text{source}}.$$

Thus, all relations hold perfectly.

Now, we evaluate the cross-branch transition through the localized vertex v_1:

$$\begin{aligned} s_{e_1}^* p_{v_1} s_{e_2} &= \left(\frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & I_2 \end{pmatrix}\right)\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} I_2 \\ -I_2 \end{pmatrix}\right) \\ &= \frac{1}{2}\begin{pmatrix} I_2 & 0 \end{pmatrix}\begin{pmatrix} I_2 \\ -I_2 \end{pmatrix} \\ &= \frac{1}{2}I_2 \neq 0. \end{aligned}$$

This demonstrates conclusively that $s_{e_1}^* p_{v_1} s_{e_2} \neq 0$, proving that elements originating in $V_{2,\text{sink}}$ do not strictly decouple from $V_{1,\text{sink}}$ when localized at individual vertices within the shared source layer.

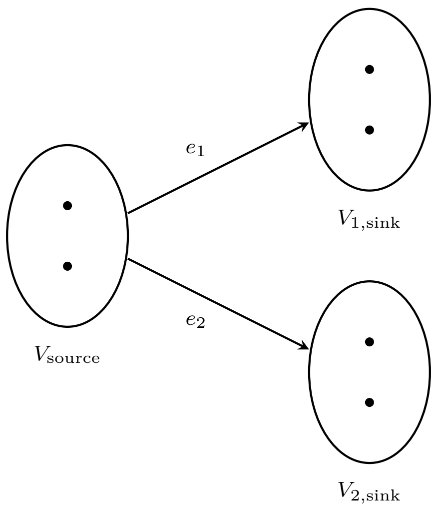


Figure 4.2: A minimal hyperbranch out-tree demonstrating non-orthogonality.

To circumvent the non-orthogonality demonstrated in Example 4.11, we must restrict the cardinality of the shared source. Before proceeding to the structural analysis, we fix our notation for the remainder of this section.

Let $H\Gamma = (V, E, r, s)$ be a hyperbranch out-tree with $k \geq 1$ branches $HB_1, \ldots, HB_k$ of lengths $n_1, \ldots, n_k$. We impose the *singleton source assumption*: we assume that the shared global source consists of exactly one vertex, meaning $V_{\text{source}} = \{v_0\}$. We denote the vertex sets of the individual branches by $\mathcal{V}_i$ and their edges by $e_{i,j}$ as introduced in Definition 4.10. We study the associated hypergraph C^*-algebra $C^*(H\Gamma)$, uniformly denoting its fundamental generators by the partial isometries $\{s_{e_{i,j}}\}$ and the mutually orthogonal vertex projections $\{p_v \mid v \in V\}$.

Because of the singleton source assumption ($p_{v_0} = P_{\text{source}}$), the hypergraph relations at the root behave exactly like those in classical graph C^*-algebras. Our overarching strategy is to algebraically "cut" the out-tree at this source vertex to decompose the global C^*-algebra into mutually orthogonal branch subalgebras. We achieve this by explicitly defining a set of cut-projections that will serve as the local identities for these isolated branches.

Definition 4.12 (Branch Cut-Projections): For each branch HB_i with vertex sets $\mathcal{V}_i = \{V_{\text{source}}, V_{i,2}, \ldots, V_{i,n_i}\}$, we define the *branch cut-projection* as:

$$P_{\text{Branch},i} := s_{e_{i,1}} s_{e_{i,1}}^* + \sum_{V \in \mathcal{V}_i \setminus \{V_{\text{source}}\}} P_V,$$

where $e_{i,1}$ is the initial edge of the branch HB_i originating at V_{source}.

Lemma 4.13: *The branch cut-projections $\{P_{Branch,i}\}_{i=1}^k$ form a family of mutually orthogonal projections that partition the identity of $C^*(H\Gamma)$. That is:*

(i) $P_{Branch,i} P_{Branch,j} = 0$ *for all* $i \neq j$.

(ii) $\sum_{i=1}^k P_{Branch,i} = 1_{C^*(H\Gamma)}$.

Proof: To prove (1), let $i \neq j$. Distinct branches HB_i and HB_j are disjoint except at the shared source V_{source}. Therefore, all projections corresponding to strictly internal vertex sets are mutually orthogonal. At the shared source, the branch cut-projections utilize the initial edge range projections $s_{e_{i,1}} s_{e_{i,1}}^*$ and $s_{e_{j,1}} s_{e_{j,1}}^*$. Since $e_{i,1} \neq e_{j,1}$ share the same source, relation (HR1) enforces their orthogonality: $(s_{e_{i,1}} s_{e_{i,1}}^*)(s_{e_{j,1}} s_{e_{j,1}}^*) = 0$. Consequently, $P_{\text{Branch},i} P_{\text{Branch},j} = 0$.

To prove (2), we sum over all k branches:

$$\sum_{i=1}^k P_{\text{Branch},i} = \sum_{i=1}^k \left(s_{e_{i,1}} s_{e_{i,1}}^* \right) + \sum_{i=1}^k \sum_{V \in \mathcal{V}_i \setminus \{V_{\text{source}}\}} P_V.$$

By relations (HR2a) and (HR2b) at the singleton source $V_{\text{source}} = \{v_0\}$, the sum of the initial edge range projections equals the total source projection:

$\sum_{i=1}^{k} s_{e_{i,1}} s_{e_{i,1}}^* = P_{\text{source}}$. Thus, the total sum evaluates to the partition of unity:

$$P_{\text{source}} + \sum_{i=1}^{k} \sum_{V \in \mathcal{V}_i \setminus \{V_{\text{source}}\}} P_V = \sum_{V \in \mathcal{V}} P_V = 1_{C^*(H\Gamma)}. \qquad \square$$

Using these cut-projections, we can formally define the isolated algebraic structure of each branch.

Definition 4.14 (Branch Subalgebras): For each branch HB_i, we define the corresponding *branch subalgebra* $\mathcal{B}_i$ as the localized corner algebra determined by its branch cut-projection:

$$\mathcal{B}_i := P_{\text{Branch},i} C^*(H\Gamma) P_{\text{Branch},i}.$$

Equipped with these isolated branch subalgebras, we can now establish the crucial decoupling property. The following lemma proves that the cross-terms between different branches vanish completely, paving the way for the structure theorem of this section.

Lemma 4.15: *For any distinct branches $i \neq j$, the cross-terms vanish:*

$$P_{\text{Branch},i} C^*(H\Gamma) P_{\text{Branch},j} = 0.$$

Consequently, the global C^*-algebra decomposes into the direct sum:

$$C^*(H\Gamma) = \bigoplus_{i=1}^{k} \mathcal{B}_i.$$

Proof: Since $C^*(H\Gamma)$ is the closed linear span of fully reduced words in its generators, it suffices to evaluate an arbitrary word x. For the cross-term $P_{\text{Branch},i} x P_{\text{Branch},j}$ to be non-zero for $i \neq j$, the word x must represent an algebraic path traversing from branch HB_j to branch HB_i.

Because distinct branches are disjoint everywhere except at the shared singleton source $V_{\text{source}} = \{v_0\}$, any connecting path must strictly transition through this vertex. Algebraically, this requires the word x to contain the subword $s_{e_{i,1}}^* p_{v_0} s_{e_{j,1}} = s_{e_{i,1}}^* s_{e_{j,1}} = 0$. Hence, x vanishes and $P_{\text{Branch},i} C^*(H\Gamma) P_{\text{Branch},j} = 0$.

By Lemma 4.13, the projections $P_{\text{Branch},i}$ are mutually orthogonal and partition the identity ($\sum_{i=1}^{k} P_{\text{Branch},i} = 1_{C^*(H\Gamma)}$). Expanding the algebra yields:

$$C^*(H\Gamma) = 1_{C^*(H\Gamma)} C^*(H\Gamma) 1_{C^*(H\Gamma)} = \sum_{i=1}^{k} \sum_{j=1}^{k} P_{\text{Branch},i} C^*(H\Gamma) P_{\text{Branch},j}.$$

Since all off-diagonal terms ($i \neq j$) are strictly zero, the double sum collapses to its diagonal, forming the direct sum of the branch subalgebras:

$$C^*(H\Gamma) = \bigoplus_{i=1}^{k} P_{\text{Branch},i} C^*(H\Gamma) P_{\text{Branch},i} = \bigoplus_{i=1}^{k} \mathcal{B}_i.$$

□

Theorem B (Structure of Hyperbranch Out-Tree C^*-Algebras): *Let $H\Gamma$ be a hyperbranch out-tree with k branches $HB_1, \ldots, HB_k$, satisfying the singleton source assumption $V_{source} = \{v_0\}$. Then there is an isomorphism:*

$$C^*(H\Gamma) \cong \bigoplus_{i=1}^{k} M_{n_i}(\mathcal{A}_{i,corn}),$$

where

$$\mathcal{A}_{i,corn} := P_{i,sink} C^*(HB_i) P_{i,sink} \cong \mathbb{C} *_{\mathbb{C}} \mathbb{C}^{m_{i,2}} *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathbb{C}^{m_{i,n_i}}$$

is the terminal corner algebra of branch HB_i, with $m_{i,j} = |V_{i,j}|$ denoting the number of vertices in the respective layers, and $*_{\mathbb{C}}$ denoting the unital free product of C^*-algebras.

Proof: By Lemma 4.15, the global algebra decomposes into the direct sum of the isolated branch subalgebras:

$$C^*(H\Gamma) = \bigoplus_{i=1}^{k} \mathcal{B}_i.$$

Since each $\mathcal{B}_i$ is structurally the C^*-algebra of a single hyperbranch HB_i of length n_i, we can directly apply Theorem A. This yields $\mathcal{B}_i \cong M_{n_i}(\mathcal{A}_{i,\text{corn}})$ for each branch. Substituting these isomorphisms into the direct sum completes the proof. □

Example 4.16: Consider the hyperbranch out-tree in Figure 4.3 with $k = 2$ branches satisfying the singleton source assumption $|V_{\text{source}}| = 1$. Branch HB_1 consists of $n_1 = 3$ vertex layers (including the source) with intermediate dimension $m_{1,2} = 2$ and sink dimension $m_{1,3} = 2$. Branch HB_2 consists of $n_2 = 2$ vertex layers with sink dimension $m_{2,2} = 3$. Following Theorem B the matrix block dimensions correspond directly to the number of vertex layers in each branch ($n_1 = 3$ and $n_2 = 2$). This yields:

$$C^*(H\Gamma) \cong M_3(\mathcal{A}_{1,\text{corn}}) \oplus M_2(\mathcal{A}_{2,\text{corn}}),$$

where the terminal corner algebras are structurally determined by the vertex set dimensions of each branch:

$$\mathcal{A}_{1,\text{corn}} \cong \mathbb{C} *_{\mathbb{C}} \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2 \quad \text{and} \quad \mathcal{A}_{2,\text{corn}} \cong \mathbb{C} *_{\mathbb{C}} \mathbb{C}^3.$$

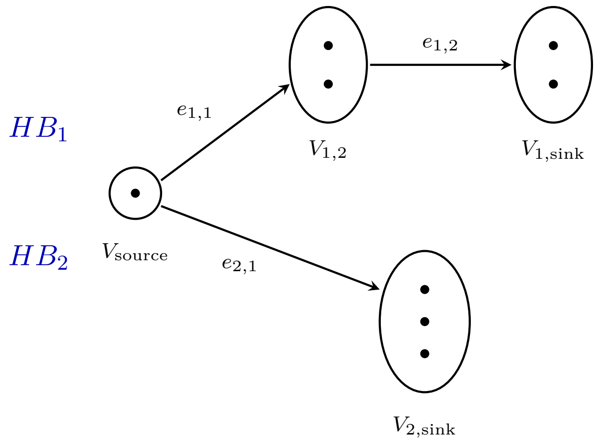


Figure 4.3: A hyperbranch out-tree illustrating Example 4.16.

4.3 Problem with General Hyperbranch Trees

In Section 4.2, we established the direct sum decomposition for a specific class of hyperbranch trees, the hyperbranch out-trees. Based on the behavior of standard directed graph C^*-algebras, one might conjecture that an arbitrary hyperbranch tree naturally decomposes into a direct sum of its branches.

However, the following example demonstrates that the strict orthogonality between distinct branches depends fundamentally on the cardinality of the upstream vertex sets. Specifically, suppose branches diverge at a singleton vertex v. An upstream vertex set is any set V connected to v via a directed path μ, such that $s(\mu) = V$ and $r(\mu) = \{v\}$. Because the hypergraph relations enforce orthogonality only relative to total vertex set projections, they are significantly weaker at the localized vertex level than their standard graph counterparts. This allows for an "upstream leaking" phenomenon: a multi-vertex source higher up in the tree algebraically entangles distinct branches lower down, even if their immediate branching point is a singleton.

Example 4.17: Consider the hypergraph tree illustrated in Figure 4.4. The tree consists of a trunk edge e_0 originating from a non-singleton source $V_0 = \{w_1, w_2\}$ and terminating at a singleton branching vertex $V_1 = \{v_1\}$. From V_1, the tree splits into two distinct branches via the edges $e_{1,1}$ and $e_{2,1}$, terminating in the respective sinks $V_{1,\text{sink}}$ and $V_{2,\text{sink}}$.

The algebraic paths representing the two branches from the global source are $\mu_1 = (e_0 e_{1,1})$ and $\mu_2 = (e_0 e_{2,1})$. To determine if the global C^*-algebra decomposes into a direct sum, we evaluate the cross-branch orthogonality.

Locally at the singleton branching point V_1, the relation holds. Because V_1 is a singleton, the local vertex projection p_{v_1} coincides with the total source projection P_{V_1}. By relation (HR1), the outgoing edges are strictly orthogonal:

$$s_{e_{1,1}}^* p_{v_1} s_{e_{2,1}} = s_{e_{1,1}}^* s_{e_{2,1}} = 0.$$

However, evaluating the cross-branch transition localized at a single vertex $w_1 \in V_0$ high up in the trunk yields a different result. The inner product of the branch isometries evaluated at p_{w_1} is:

$$s_{\mu_1}^* p_{w_1} s_{\mu_2} = (s_{e_0} s_{e_{1,1}})^* p_{w_1} (s_{e_0} s_{e_{2,1}}) = s_{e_{1,1}}^* \left(s_{e_0}^* p_{w_1} s_{e_0} \right) s_{e_{2,1}}.$$

Because $|V_0| > 1$, the projection p_{w_1} is strictly smaller than the total source projection P_{V_0}. The hypergraph relations do not force the cross-term $s_{e_{1,1}}^* Y s_{e_{2,1}}$ associated with the inner element $Y := s_{e_0}^* p_{w_1} s_{e_0}$ to be zero. Similar to the non-orthogonality demonstrated at the source layer in Example 4.11, this upstream element allows paths to "leak" between branches.

This implies that $P_{i,\text{sink}} C^*(H\Gamma) P_{j,\text{sink}}$ for $i \neq j$ is generally non-zero, which destroys the direct sum structure.

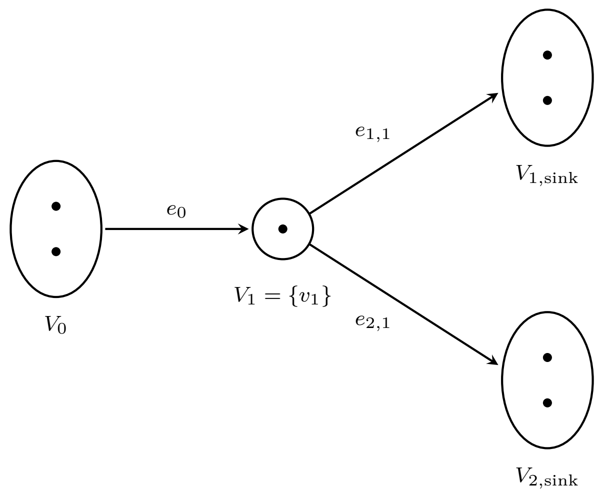


Figure 4.4: Hyperbranch tree with a non-singleton source V_0 branching at a singleton vertex V_1.

This observation establishes a strict boundary for the structural analysis of hypergraph C^*-algebras: the direct sum decomposition is determined not merely by the local properties of the branching point, but by the entire upstream history of the tree.

Suppose a hyperbranch tree contains multiple distinct sinks, and let V_{branch} denote the vertex set where their respective branches diverge. For the global C^*-algebra to decouple into a direct sum, it is insufficient for V_{branch} to be a singleton. Instead, V_{branch} and every vertex set upstream must strictly be singletons ($|V| = 1$).

Consequently, the direct sum decomposition theorem exclusively applies to the specific class of strict singleton-trunk hyperbranch out-trees, along with certain hybrid extensions that we will investigate in the following section.

4.4 Hybrid Hyperbranch Trees

We now investigate hybrid hyperbranch trees constructed by adjoining classical finite acyclic directed graphs to hypergraph trees. A standard directed graph $\Gamma = (V_\Gamma, E_\Gamma, r_\Gamma, s_\Gamma)$ can be embedded into our framework as a hypergraph where every edge $e \in E_\Gamma$ satisfies $|s_\Gamma(e)| = |r_\Gamma(e)| = 1$.

We analyze the structural effect of appending classical directed path graphs to the extremal vertices of both hyperbranch in-trees and out-trees. Let $\Gamma_m = (V_\Gamma, E_\Gamma, s_\Gamma, r_\Gamma)$ denote the classical linear graph (Example 1.26) on m vertices, consisting of $V_\Gamma = \{w_1, \ldots, w_m\}$ and edges $E_\Gamma = \{f_1, \ldots, f_{m-1}\}$, with $s_\Gamma(f_j) = \{w_j\}$ and $r_\Gamma(f_j) = \{w_{j+1}\}$ for all $1 \leq j < m$.

4.4.1 Path Attachments to Hyperbranch In-Trees

We first apply the attachment to the hyperbranch in-trees classified in Theorem 4.7. To rigorously define the hybrid structure, we construct the disjoint union of the hypergraph and the path, linked by a new connecting edge.

Theorem 4.18: *Let $H\Gamma = (V, E, r, s)$ be a hyperbranch in-tree with k branches of lengths $n_1, \ldots, n_k$. Let V_{sink} be its unique terminal sink with dimension $m_{sink} = |V_{sink}|$, and let $\ell = \sum_{i=1}^{k}(n_i - 1) + 1$ be the total number of vertex sets in $H\Gamma$.*

Suppose we adjoin the path Γ_m to $H\Gamma$ by introducing a connecting edge e_{conn}. Formally, we define the hybrid hypergraph $H\Gamma' = (V', E', r', s')$ where:

- $V' = V \cup V_\Gamma$,
- $E' = E \cup E_\Gamma \cup \{e_{conn}\}$,

and the extended maps are given by $s'|_E = s$, $r'|_E = r$, $s'|_{E_\Gamma} = s_\Gamma$, and $r'|_{E_\Gamma} = r_\Gamma$, with the connection defined as:

$$s'(e_{conn}) = V_{sink} \quad \text{and} \quad r'(e_{conn}) = \{w_1\}.$$

Then the C^*-algebra of the hybrid tree is isomorphic to:

$$C^*(H\Gamma') \cong M_{\ell+m}(\mathcal{A}_{corn}),$$

where $\mathcal{A}_{corn} = \mathbb{C}^{m_{sink}} *_{\mathbb{C}} (*_{\mathbb{C}}{}_{i=1}^{k} (*_{\mathbb{C}}{}_{j=1}^{n_i-1} \mathbb{C}^{m_{i,j}}))$ *is the corner algebra of the original tree* $H\Gamma$.

Proof: To explicitly determine the structure of $C^*(H\Gamma')$, we construct the system of matrix units for the extended graph. In the original tree $H\Gamma$, each of the ℓ vertex sets X has exactly one path ν_X to the original sink V_{sink}. In the hybrid graph $H\Gamma'$, the new terminal sink is w_m. Let $\mu_{\text{ext}} = e_{\text{conn}} f_1 \dots f_{m-1}$ be the unique path from V_{sink} to w_m. We extend the original paths by defining $\nu'_X = \nu_X \mu_{\text{ext}}$. Additionally, for the m appended singleton vertices $w_j \in V_\Gamma$, there exists exactly one path $\lambda_j = f_j \dots f_{m-1}$ to w_m, where λ_m is the empty path at w_m.

This yields exactly $\ell + m$ unique paths in $H\Gamma'$ terminating at w_m. For any two such paths μ and ν, we consider the elements $s_\mu s_\nu^* = s_\mu p_{w_m} s_\nu^*$. They satisfy self-adjointness, and their diagonal elements sum up to the identity. Since all paths terminate at the same sink w_m and the hypergraph relations imply $s_\nu^* s_\gamma = \delta_{\nu,\gamma} p_{w_m}$ for any two paths ν, γ from this collection, the multiplication of two such elements simplifies directly to:

$$(s_\mu s_\nu^*)(s_\gamma s_\delta^*) = \delta_{\nu,\gamma} s_\mu s_\delta^*.$$

Thus, the elements $s_\mu s_\nu^*$ form a complete system of $(\ell+m) \times (\ell+m)$ matrix units. Applying the Full Corner Isomorphism (Lemma 3.5) to the sink projection p_{w_m} immediately yields the spatial isomorphism:

$$C^*(H\Gamma') \cong M_{\ell+m}(p_{w_m} C^*(H\Gamma') p_{w_m}).$$

To evaluate the corner algebra $p_{w_m} C^*(H\Gamma') p_{w_m}$, we use the partial isometry $s_{\mu_{\text{ext}}}$. Because the appended path consists strictly of singletons with a single outgoing edge, the relations imply $s_{\mu_{\text{ext}}}^* s_{\mu_{\text{ext}}} = p_{w_m}$ and $s_{\mu_{\text{ext}}} s_{\mu_{\text{ext}}}^* = P_{\text{sink}}$.

Consider the map

$$\Phi\colon P_{\text{sink}} C^*(H\Gamma) P_{\text{sink}} \to p_{w_m} C^*(H\Gamma') p_{w_m}$$

defined by $\Phi(a) = s_{\mu_{\text{ext}}}^* a s_{\mu_{\text{ext}}}$. If $\Phi(a) = 0$, multiplying by $s_{\mu_{\text{ext}}}$ from the left and $s_{\mu_{\text{ext}}}^*$ from the right yields $P_{\text{sink}} a P_{\text{sink}} = 0$, hence $a = 0$. Thus, Φ is injective.

For surjectivity, any element $x = p_{w_m} x p_{w_m}$ in the new corner algebra is the image of $a := s_{\mu_{\text{ext}}} x s_{\mu_{\text{ext}}}^* \in P_{\text{sink}} C^*(H\Gamma) P_{\text{sink}}$, since $\Phi(a) = s_{\mu_{\text{ext}}}^* s_{\mu_{\text{ext}}} x s_{\mu_{\text{ext}}}^* s_{\mu_{\text{ext}}} = p_{w_m} x p_{w_m} = x$.

Therefore, Φ is a C^*-algebra isomorphism. By Theorem 4.7, the original corner algebra is $\mathcal{A}_{\text{corn}} = \mathbb{C}^{m_{\text{sink}}} *_{\mathbb{C}} *_{\mathbb{C}}{}_{i=1}^{k} *_{\mathbb{C}}{}_{j=1}^{n_i-1} \mathbb{C}^{m_{i,j}}$, which is strictly preserved under Φ. □

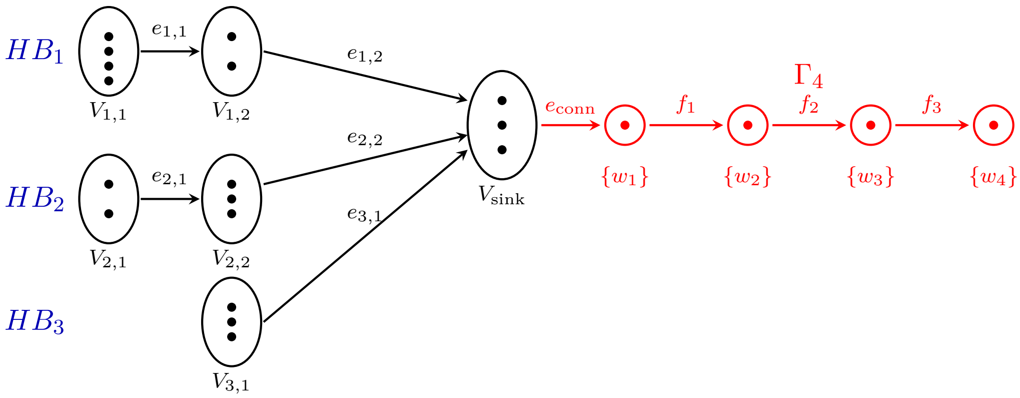


Figure 4.5: Hybrid hyperbranch in-tree $H\Gamma'$ with a classical path Γ_4 appended to the original terminal sink V_{sink}.

Remark 4.19: The construction and proof presented in Theorem 4.18 generalize directly to appending an arbitrary hyperbranch instead of a classical path graph Γ_m. By constructing new matrix units for the new terminal sink of $H\Gamma'$, we again obtain an $M_{\ell+m}(\mathcal{A}'_{\mathrm{corn}})$ structure, where the new corner algebra $\mathcal{A}'_{\mathrm{corn}}$ is expanded by taking the free product with the corner algebra of the appended hyperbranch.

Attaching a classical path (or even a hyperbranch) to a source of the hyperbranch in-tree is structurally straightforward. Such an extension only modifies the specific branch, allowing us to directly apply the structure theorem (Theorem 4.7).

Example 4.20: Consider the hyperbranch in-tree $H\Gamma$ from Figure 4.5 with $k = 3$ branches terminating at the common sink V_{sink}. Let ℓ be the total number of vertex sets in $H\Gamma$.

We append a classical path graph Γ_4 of length $m = 4$ to the sink. The path consists of the singleton vertices $\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}$ and the classical edges f_1, f_2, f_3. The connection is established by the connecting edge e_{conn} with $s(e_{\mathrm{conn}}) = V_{\mathrm{sink}}$ and $r(e_{\mathrm{conn}}) = \{w_1\}$.

In the hybrid hypergraph $H\Gamma'$, the topological structure changes as follows:

- The original sink V_{sink} becomes an intermediate multi-vertex set.
- The new unique global terminal sink is the singleton $\{w_4\}$.

By Theorem 4.18, the matrix dimension increases by the length of the appended path. Since $m = 4$ new paths are introduced from V_{sink} to the new sink w_4, the total number of paths from all vertex sets in $H\Gamma'$ to w_4 is $\ell + 4$. This expands the global matrix dimension to $M_{\ell+4}$ and we get

$$C^*(H\Gamma') \cong M_{10}\left(\mathbb{C}^3 *_{\mathbb{C}} (\mathbb{C}^4 *_{\mathbb{C}} \mathbb{C}^2) *_{\mathbb{C}} (\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3) *_{\mathbb{C}} \mathbb{C}^3\right).$$

4.4.2 Path Attachments to Hyperbranch Out-Trees

Definition 4.21: Let $H\Gamma = (V, E, r, s)$ be a hyperbranch tree with a distinguished initial source $V_{\text{source}} \in V$. Let Γ_m be a directed path graph of length m with vertices $V_\Gamma = \{w_1, w_2, \ldots, w_m\}$ and edges $E_\Gamma = \{f_1, \ldots, f_{m-1}\}$, equipped with source and range maps s_Γ and r_Γ such that $s_\Gamma(f_j) = \{w_j\}$ and $r_\Gamma(f_j) = \{w_{j+1}\}$.

The *path attachment* of Γ_m to $H\Gamma$ at V_{source} is defined as the hybrid hypergraph $H\Gamma' = (V', E', r', s')$, where the vertex and edge sets are given by the disjoint unions:

$$V' = V \cup\!\!\!\!\cdot\;\; V_\Gamma$$

$$E' = E \cup\!\!\!\!\cdot\;\; E_\Gamma \cup\!\!\!\!\cdot\;\; \{e_{\text{conn}}\}$$

The extended maps s' and r' restrict to s, r on E and s_Γ, r_Γ on E_Γ. The connecting edge e_{conn} is defined by:

$$s'(e_{\text{conn}}) = \{w_m\} \quad \text{and} \quad r'(e_{\text{conn}}) = V_{\text{source}}.$$

We now apply a similar proof strategy as in Theorem B.

Definition 4.22 (Extended Branch Cut-Projections): Let $H\Gamma'$ be the hybrid hyperbranch out-tree obtained by adjoining the upstream path Γ_m to the singleton source v_0 of $H\Gamma$. For each $w_j \in V_\Gamma$, let μ_j be the unique path from w_j to v_0. For each branch HB_i, we define the *extended branch cut-projection* as:

$$P^{\text{new}}_{\text{Branch},i} := P_{\text{Branch},i} + \sum_{j=1}^{m} s_{\mu_j}(s_{e_{i,1}} s^*_{e_{i,1}}) s^*_{\mu_j},$$

where $P_{\text{Branch},i}$ is the original cut-projection of HB_i from Definition 4.12. The element $P^{\text{new}}_{\text{Branch},i}$ is a projection because it is the sum of $P_{\text{Branch},i}$ and the terms $s_{\mu_j}(s_{e_{i,1}} s^*_{e_{i,1}}) s^*_{\mu_j}$, which form a family of mutually orthogonal projections. This mutual orthogonality follows directly from the hypergraph relations, as the initial spaces of these paths are localized at distinct vertices w_j strictly upstream from the original tree.

Lemma 4.23: *The extended branch cut-projections $\{P^{new}_{Branch,i}\}_{i=1}^{k}$ form a family of mutually orthogonal projections that partition the identity of $C^*(H\Gamma')$.*

Proof: For orthogonality (1), let $i \neq r$. We evaluate the product:

$$P^{\text{new}}_{\text{Branch},i} P^{\text{new}}_{\text{Branch},r} = \left(P_{\text{Branch},i} + \sum_{j=1}^{m} s_{\mu_j} s_{e_{i,1}} s^*_{e_{i,1}} s^*_{\mu_j} \right) \cdot \left(P_{\text{Branch},r} + \sum_{q=1}^{m} s_{\mu_q} s_{e_{r,1}} s^*_{e_{r,1}} s^*_{\mu_q} \right).$$

Expanding this expression yields four terms, all of which vanish:

- $P_{\text{Branch},i} P_{\text{Branch},r} = 0$ by Lemma 4.13.
- The cross-terms $P_{\text{Branch},i} \sum_{q=1}^{m}(\dots)$ and $\sum_{j=1}^{m}(\dots) P_{\text{Branch},r}$ are zero because the original branch projections are localized at or downstream from the source v_0, whereas the newly added terms are localized strictly upstream at the vertices $w_j \in V_\Gamma$. Thus, their underlying vertex projections are mutually orthogonal.
- For the product of the two sums, we evaluate the inner terms. Since μ_j and μ_q are paths on the appended classical line graph terminating at v_0, the relations dictate $s^*_{\mu_j} s_{\mu_q} = \delta_{j,q} p_{v_0}$. The product therefore collapses to:

  $$\sum_{j=1}^{m} s_{\mu_j} s_{e_{i,1}} (s^*_{e_{i,1}} s_{e_{r,1}}) s^*_{e_{r,1}} s^*_{\mu_j}.$$

  By the hypergraph relation (HR1) at the branching source v_0, the initial edges of distinct branches are orthogonal, meaning $s^*_{e_{i,1}} s_{e_{r,1}} = 0$. Hence, this sum evaluates to zero.

Thus, $P^{\text{new}}_{\text{Branch},i} P^{\text{new}}_{\text{Branch},r} = 0$.

For completeness (2), we sum over all k branches:

$$\sum_{i=1}^{k} P^{\text{new}}_{\text{Branch},i} = \sum_{i=1}^{k} P_{\text{Branch},i} + \sum_{j=1}^{m} s_{\mu_j} \left(\sum_{i=1}^{k} s_{e_{i,1}} s^*_{e_{i,1}} \right) s^*_{\mu_j}.$$

By Lemma 4.13, the first term sums to the identity of the original graph $1_{C^*(H\Gamma)}$. In the second term, by (HR2a), the sum over the initial edge range projections equals the source projection: $\sum_{i=1}^{k} s_{e_{i,1}} s^*_{e_{i,1}} = p_{v_0}$. Applying the relations along the non-branching classical path yields $s_{\mu_j} p_{v_0} s^*_{\mu_j} = p_{w_j}$. Thus, the total sum evaluates to the partition of unity:

$$1_{C^*(H\Gamma)} + \sum_{j=1}^{m} p_{w_j} = 1_{C^*(H\Gamma')}.$$

□

Definition 4.24 (Extended Branch Subalgebras): For each branch HB_i, the *extended branch subalgebra* $\mathcal{B}'_i$ is defined as the localized corner algebra determined by the extended branch cut-projection:

$$\mathcal{B}'_i := P^{\text{new}}_{\text{Branch},i} C^*(H\Gamma') P^{\text{new}}_{\text{Branch},i}.$$

Lemma 4.25: *For any distinct branches $i \neq r$, the cross-terms vanish:*

$$P^{new}_{Branch,i} C^*(H\Gamma') P^{new}_{Branch,r} = 0.$$

Consequently, the global C^*-algebra decomposes into the direct sum:

$$C^*(H\Gamma') = \bigoplus_{i=1}^{k} \mathcal{B}'_i.$$

Proof: Since $C^*(H\Gamma')$ is generated by words, consider an arbitrary word $x \in C^*(H\Gamma')$. We expand the cross-term $P^{\text{new}}_{\text{Branch},i} x P^{\text{new}}_{\text{Branch},r}$ for $i \neq r$ into four distinct terms:

$$\left(P_{\text{Branch},i} + \sum_{j=1}^{m} s_{\mu_j} s_{e_{i,1}} s^*_{e_{i,1}} s^*_{\mu_j} \right) x \left(P_{\text{Branch},r} + \sum_{q=1}^{m} s_{\mu_q} s_{e_{r,1}} s^*_{e_{r,1}} s^*_{\mu_q} \right).$$

We evaluate each term:

(i) $P_{\text{Branch},i} x P_{\text{Branch},r} = 0$: By Lemma 4.15, any word transitioning between distinct original branches must cross the shared source v_0, necessitating the evaluation $s^*_{e_{i,1}} s_{e_{r,1}} = 0$.

(ii) $\left(\sum_{j=1}^{m} s_{\mu_j} s_{e_{i,1}} s^*_{e_{i,1}} s^*_{\mu_j}\right) x P_{\text{Branch},r} = 0$: The word x must connect the original branch HB_r to the upstream vertices w_j. Any algebraic path traversing from HB_r to the extended upstream section must pass through the original source v_0. This requires x to contain the transition from $e_{r,1}$ to $e_{i,1}$, yielding a subword evaluation that includes $s^*_{e_{i,1}} s^*_{\mu_j} s_{\mu_j} s_{e_{r,1}} = s^*_{e_{i,1}} p_{v_0} s_{e_{r,1}} = 0$ by (HR1).

(iii) $P_{\text{Branch},i} x \left(\sum_{q=1}^{m} s_{\mu_q} s_{e_{r,1}} s^*_{e_{r,1}} s^*_{\mu_q}\right) = 0$: Symmetrical to case 2, transitioning from the upstream extension of branch r back downstream to HB_i enforces the crossing $s^*_{e_{i,1}} s_{e_{r,1}} = 0$.

(iv) $\left(\sum_{j=1}^{m} s_{\mu_j} s_{e_{i,1}} s^*_{e_{i,1}} s^*_{\mu_j}\right) x \left(\sum_{q=1}^{m} s_{\mu_q} s_{e_{r,1}} s^*_{e_{r,1}} s^*_{\mu_q}\right) = 0$: For the product to be non-zero, x must evaluate non-trivially between the localized projections at w_j and w_q, which forces $j = q$. Any such connecting word x within the extended path simplifies to p_{w_j}. The remaining outer terms thus inevitably enforce the inner product $s^*_{e_{i,1}} s_{e_{r,1}}$, which is strictly 0 by (HR1) since $i \neq r$.

Since all four terms vanish, $P^{\text{new}}_{\text{Branch},i} C^*(H\Gamma') P^{\text{new}}_{\text{Branch},r} = 0$. The direct sum decomposition then follows immediately from the partition of unity established in Lemma 4.23:

$$C^*(H\Gamma') = \bigoplus_{i=1}^{k} P^{\text{new}}_{\text{Branch},i} C^*(H\Gamma') P^{\text{new}}_{\text{Branch},i} = \bigoplus_{i=1}^{k} \mathcal{B}'_i.$$

□

Theorem 4.26 (Source Attachment for Hyperbranch Out-Trees): *Let $H\Gamma'$ be the hybrid hyperbranch out-tree constructed by appending Γ_m upstream to the singleton source v_0 of $H\Gamma$. Then the C^*-algebra is isomorphic to:*

$$C^*(H\Gamma') \cong \bigoplus_{i=1}^{k} M_{n_i+m}(\mathcal{A}_{i,corn}),$$

where

$$\mathcal{A}_{i,corn} := P_{i,sink} C^*(HB_i) P_{i,sink} \cong \mathbb{C} *_{\mathbb{C}} \mathbb{C}^{m_{i,2}} *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathbb{C}^{m_{i,n_i}}$$

is the terminal corner algebra of branch HB_i, with $m_{i,j} = |V_{i,j}|$ denoting the number of vertices in the respective layers, and $*_{\mathbb{C}}$ denoting the unital free product of C^*-algebras.

Proof: By Lemma 4.25, $C^*(H\Gamma') = \bigoplus_{i=1}^{k} \mathcal{B}'_i$. Each subalgebra $\mathcal{B}'_i$ corresponds exactly to the C^*-algebra of the original branch HB_i elongated upstream by the classical path Γ_m. The length of this extended branch is $n_i + m$. Since each $\mathcal{B}'_i$ is structurally a single hyperbranch C^*-algebra, applying Theorem A yields $\mathcal{B}'_i \cong M_{n_i+m}(\mathcal{A}_{i,\text{corn}})$. Taking the direct sum completes the proof. □

Example 4.27: Consider the hyperbranch out-tree $H\Gamma$ from Figure 4.3 with $k = 2$ branches originating at the shared singleton source $V_{\text{source}} = \{v_0\}$. The first branch HB_1 has length $n_1 = 3$ and the second branch HB_2 has length $n_2 = 2$.

We append a classical path graph Γ_3 of length $m = 3$ upstream to the source v_0. The appended path consists of the singleton vertices $\{w_1\}, \{w_2\}, \{w_3\}$ and the classical edges f_1, f_2. The connection is established by the connecting edge e_{conn} with $s(e_{\text{conn}}) = \{w_3\}$ and $r(e_{\text{conn}}) = \{v_0\}$.

In the extended hybrid hypergraph $H\Gamma'$ (see Figure 4.6), the topological structure changes as follows:

- The original global source V_{source} becomes an intermediate branching vertex.

- The new unique global source is the singleton $\{w_1\}$.
- The path length of every original branch extending from the new source $\{w_1\}$ is uniformly increased by $m = 3$.

By Theorem 4.26, the matrix dimension of each branch's corresponding factor in the direct sum increases by the length of the appended path. The new branch subalgebras evaluated at the respective sinks yield the same terminal corner algebras $\mathcal{A}_{1,\text{corn}}$ and $\mathcal{A}_{2,\text{corn}}$. Thus, the C^*-algebra of the extended graph is strictly isomorphic to:

$$C^*(H\Gamma') \cong M_{3+3}(\mathcal{A}_{1,\text{corn}}) \oplus M_{2+3}(\mathcal{A}_{2,\text{corn}}) = M_6(\mathcal{A}_{1,\text{corn}}) \oplus M_5(\mathcal{A}_{2,\text{corn}}).$$

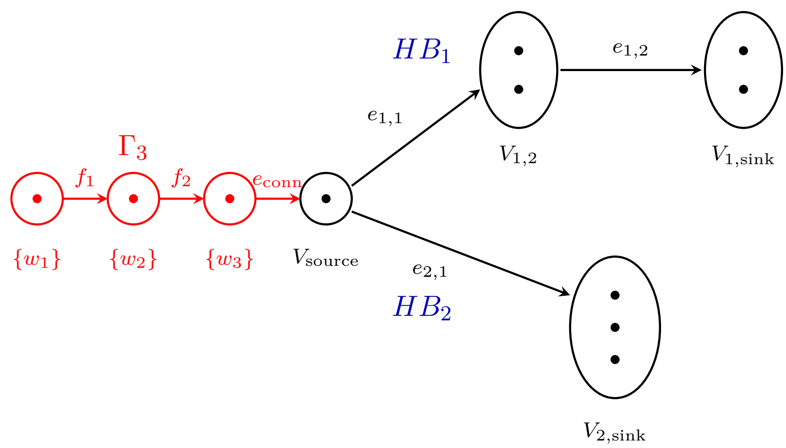


Figure 4.6: Hybrid hyperbranch out-tree $H\Gamma'$ with a classical path Γ_3 appended upstream to the shared singleton source V_{source}.

Remark 4.28: Adjoining structures upstream to a shared source v_0 is restricted to classical directed graphs (i.e., structures consisting exclusively of singleton vertex sets, such as classical paths or directed trees). If a multi-vertex set were introduced upstream, the associated partial isometries would generate non-trivial cross-terms between the extended branches, leading to upstream leaking and thereby breaking the direct sum decomposition. Conversely, adjoining arbitrary hyperbranches (including those with multi-vertex sets) to downstream terminal sinks is always permissible, as this operation elongates the localized branches while strictly preserving the mutual orthogonality of the branch cut-projections.

5 Outlook

The structural theorems established in this thesis for hyperbranches and hyperbranch trees rely strictly on finite vertex sets, the absence of cycles, and the exclusivity of perfect paths. Relaxing any of these constraints opens up significant avenues for future research in the theory of hypergraph C^*-algebras.

Problem 5.1 (Modified Cuntz-Krieger Relations and Orthogonality): A key insight from the out-tree architectures is that branches do not naturally decouple if their sources are not isolated. A potential research direction is the modification of the hypergraph C^*-algebra relations to force orthogonality. By inserting specific projection operators between non-disjoint structural components, one could enforce a zero-product relation. Investigating how this modified ideal changes the universal C^*-algebra could provide a systematic way to "tame" hypergraph C^*-algebras, potentially recovering nuclearity in structures that are otherwise wildly infinite.

Problem 5.2 (Imperfect Paths and Complex Branching): The current classification is restricted to perfect paths, where the range of one edge matches the source of the next exactly. If we consider a graph where a vertex v_1 connects to v_2, but a subsequent hyperedge requires both v_2 and v_3 to transition to v_4, the perfect path property is broken.

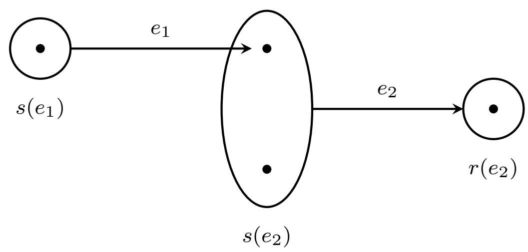


Figure 5.1: A partial path structure.

The resulting paths are only "quasi-perfect" or "partial". In such cases, the partial isometries do not cleanly concatenate, and the matrix unit construction applied in Chapter 4 fails. Analyzing the algebraic structure of such partial path transitions requires extending the combinatorial machinery beyond standard full corners.

Problem 5.3 (General Acyclic Graph Attachments): In Chapter 4, we demonstrated that attaching a simple classical path upstream to the singleton source of a hyperbranch out-tree preserves the direct sum decomposition, merely increasing the matrix dimensions of the summands. A natural combinatorial extension is to attach an arbitrary finite directed acyclic graph to this source. Since classical acyclic graph C^*-algebras decompose into direct sums of matrix algebras governed by path counting, such an attachment would intricately combine classical path combinatorics with the free product structure of the hyperbranches. Investigating this would yield a generalized structure theorem where the matrix dimensions over the hypergraph corner algebras are strictly determined by the total number of classical paths feeding into the shared branching source.

Problem 5.4 (Infinite Acyclic Hypergraphs): In classical graph C^*-algebra theory, infinite acyclic graphs lead to AF-algebras via direct limits of finite-dimensional algebras. Extending the hyperbranch tree construction to countably infinite vertex sets or infinitely long branches would require realizing the hypergraph C^*-algebra as a direct limit of the free products identified in this thesis.

Problem 5.5 (Introduction of Loops): Attaching a single loop to the terminal sink of a classical finite graph yields the continuous functions on the circle, $C(\mathbb{T})$, tensored with a matrix algebra. If a loop is adjoined to the multi-vertex sink of a hyperbranch, the algebraic consequences are vastly different. Due to the lack of commutativity among the projections in the multi-vertex sink, such a loop is expected to generate free group C^*-algebras or Cuntz-like structures. Understanding how classical loops behave when applied to hypergraph components remains an open and highly non-trivial question.

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