UNIVERSITÄT DES SAARLANDES

FREE SPECTRAHEDRA AND THE COMPATIBILITY OF QUANTUM MEASUREMENTS

Master Thesis

submitted by

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Preface

In this thesis, we will discuss the connection between compatible POVMs (positive operator valued measures) and the inclusion of free spectrahedra which was discovered by Andreas Bluhm and Ion Nechita. We focus on their paper 'Compatibility of Quantum Measurements and Inclusion Constants for the Matrix Jewel' ([BN20]) from 2020 which is a continuation of 'Joint Measurability of Quantum Effects and the Matrix Diamond' ([BN18]) published in 2018. Both papers combine problems of Quantum Information Theory (QIT) and convex optimization. Bluhm and Nechita discovered that both problems are based on the same scaling problem. Not just the connection but also both fields on their own are of interest. Thus, we give the fields enough space to introduce before we present the core results which establish the aforementioned connection and illustrate them with the aid of a concrete example; the latter, to the best of my knowledge, has not yet been studied in the literature before.

In convex analysis, linear matrix inequalities (LMI) are often used to solve optimization problems (see [BEGFB94]). An often discussed problem are LMIs where the solution sets are tuples of matrices ([HKM12],[HKMS16]). A matrix-valued solution set of LMIs is called a free spectrahedron. That means, for a g-tuple of self-adjoint matrices $A = (A_1, ..., A_g)$, the free spectrahedron $\mathcal{D}_A$ is the set of all g-tuples of self-adjoint matrices $X = (X_1, ..., X_g)$ such that

$$I - \sum_{i=1}^{g} A_i \otimes X_i$$

is positive semi-definite. For $X \in \mathbb{R}^g$, then we denote the set by $\mathcal{D}_A(1)$. It is a fundamental and important problem to find an $s = (s_1, ..., s_g) \in [0,1]^g$ such that the implication

$$\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \qquad \Rightarrow \qquad s\mathcal{D}_A \subseteq \mathcal{D}_B$$

holds. This scaling problem is related to the compatibility of POVMs.

A POVM $\{E_1, ..., E_k\}$ is a collection of k positive, self-adjoint matrices which sum up to the identity matrix. A collection of g POVMs $\{E^{(1)}, ..., E^{(g)}\}$ is called compatible or jointly measurable if there is a single POVM which can express the collection. Compatibility of POVMs is indeed a current topic. For example in the papers [GHK$^+$23] and [Lou23] from 2023 you can find some connections to the CHSH-game or quantum steering. A known result is that you can make POVMs compatible by adding noise. For that we replace the collection $\{E^{(1)}, ..., E^{(g)}\}$ by $\{s_1 E^{(1)} + (1 - s_1)/k_1 I, ..., s_g E^{(g)} + (1 - s_g)/k_g I\}$ where $s_1, ..., s_g \in [0,1]$ is the noise level and $k_1, ..., k_g \in \mathbb{N}$ the number of effects of the POVMs $E^{(1)}, ..., E^{(g)}$ respectively. One can ask for which $s = (s_1, ..., s_g)$ a given collection of POVMs becomes compatible. This is also a scaling problem.

In this thesis we present the connection of these scaling problems, which was discovered in [BN20]. These results are for example useful to get more information about compatibility witnesses ([BN20, Section 8,9], [Lou23, Chapter 8]). The main object, which builds the bridge between these at first glance unrelated questions is a special spectrahedron, the matrix jewel denoted by $\mathcal{D}_{♦,\mathbf{k}}$ for a g-tuple $\mathbf{k} \in \mathbb{N}^g$. One of the connections is the following theorem.

Theorem. *Let $d \in \mathbb{N}$ and $g \in \mathbb{N}$. For $i \in \{1, ..., g\}$ take $E^{(i)} = (E_1^{(i)}, ..., E_{k_i-1}^{(i)}) \in (\mathcal{M}_d^{sa})^{k_i-1}$ and set $E_{k_i}^{(i)} := I_d - E_1^{(i)} - ... - E_{k_i-1}^{(i)}$. Let $\mathbf{k} = (k_1, ..., k_g)$ and define*

$$E := \left(2E^{(1)} - \frac{2}{k_1} I_d, ..., 2E^{(g)} - \frac{2}{k_g} I_d\right).$$

1. *It holds $\mathcal{D}_{♦,\mathbf{k}}(1) \subseteq \mathcal{D}_E(1)$ if and only if $\{E_1^{(i)}, ..., E_{k_i}^{(i)}\}$, $i \in \{1, ..., g\}$, are POVMs.*
2. *It holds $\mathcal{D}_{♦,\mathbf{k}} \subseteq \mathcal{D}_E$ if and only if $\{E_1^{(i)}, ..., E_{k_i}^{(i)}\}$, $i \in \{1, ..., g\}$, are jointly measurable POVMs.*

For given number and size of a collection of POVMs, Bluhm and Nechita showed that the amount of noise to make POVMs compatible is equivalent to the scalar s which solves the inclusion problem

$$\mathcal{D}_{♦,\mathbf{k}}(1) \subseteq \mathcal{D}_E(1) \qquad \Rightarrow \qquad s\mathcal{D}_{♦,\mathbf{k}} \subseteq \mathcal{D}_E.$$

With help of the proof of this statement we worked out a new theorem which refers to a given POVM.

Theorem. *Let $d \in \mathbb{N}$, $g \in \mathbb{N}$, $\mathbf{k} = (k_1, ..., k_g) \in \mathbb{N}^g$ and $s \in [0,1]^g$. For the given POVMs $\left\{E_1^{(i)}, ..., E_{k_i}^{(i)}\right\}$, $i \in \{1, ..., g\}$ we have, that*

$$\left\{s_i E^{(i)} + (1 - s_i) I_d / k_i\right\}_{i \in \{1,...,g\}}$$

is compatible if and only if

$$(s_1^{\times(k_1-1)}, ..., s_g^{\times(k_g-1)})\mathcal{D}_{♦,\mathbf{k}} \subseteq \mathcal{D}_{(2E^{(1)} - \frac{2}{k_1} I_d, ..., 2E^{(g)} - \frac{2}{k_g} I_d)}$$

holds true where $s_i^{\times(k_i-1)} = \underbrace{(s_i, ..., s_i)}_{k_i-1}$ for $i \in \{1, ..., g\}$.

To get an idea of the meaning of making POVMs compatible, we introduce an example POVM

$$E = \left\{ \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \right\}$$

where $\theta \in [0, \pi/2]$. Starting with this POVM, we are creating a collection of POVM where one POVM has the fixed angle $\theta = 0$ and the other one has an arbitrary angle $\theta \in [0, \pi]$. We observe the changing necessary noise level to make the two POVMs compatible when changing θ. We are presenting a numerical way and an analytical one to solve the problem. For the numerical solution we write a semi-definite program in Mathematica and visualize the problem. By interpreting the visualizations, we hypothesize that the set of all noise levels which make POVMs compatible is minimal in $\theta = \pi/4$. This hypothesize can be verified by the analytical solution which we also present in the thesis. Furthermore, the solution set we get in $\theta = \pi/4$ is known as the 'quarter-circle' which is a result from the theory of inclusion of free spectrahedra.

The thesis is organized as follows. In the first chapter we give an overview about matrix convex sets which are generalizations of convex sets. We introduce two important examples, namely free spectrahedra and matrix ranges. These are not only examples but each of these objects can be expressed by the respective other. These connections are explained in Chapter II. From this point on, we focus more on the free spectrahedra, especially their direct sum (in Chapter III) and the inclusion (in Chapter IV). We will see that inclusions of free spectrahedra can be characterized by a unital map. Positivity of the map gives information about on which level the inclusion holds and vice versa. We are primarily interested in one specific inclusion, that is if the matrix jewel is a subset of a given spectrahdron - and if not, how much do we have to shrink the matrix jewel such that the inclusion holds. The matrix jewel is a specific free spectrahedron, also discussed in Chapter IV. Before we can give the connection between the inclusion of these free spectrahedra and the POVMs, we give an introduction to the Quantum Information Theory (Chapter V). We not only give definitions of POVMs and their compatibility, we also show and visualize them in an example. We continue with the example in Chapter VI where we present the main results of this thesis. In Chapter VII we present further ideas about the example collection of POVMs. We are giving a numerical solution and an analytical one. Every self-written computer program can be found in the Appendix, Chapter A.

I would like to thank my advisor Dr. Tobias Mai for his support and his encouragement throughout the process of researching and writing this thesis. Thank you.

Contents

Chapter I.

Introduction to Matrix Convex Sets

1 Convex Analysis

A main object of this thesis are matrix convex sets which are a generalization of convex sets. In this section we recall a few definitions and properties of the convex sets polyhedra and polytopes. For brevity, let us fix $[n] := \{1, ..., n\}$ where $n \in \mathbb{N}$.

Definition I.1. Let $c_1, ..., c_m$ be vectors in $\mathbb{R}^d$ and let $\alpha_1, ..., \alpha_m \in \mathbb{R}$. The set

$$\mathcal{P} := \{x \in \mathbb{R}^d : \langle c_i, x\rangle \leq \alpha_i \text{ for all } i \in [m]\}$$

is called a **polyhedron**. The convex hull of a finite set of points in $\mathbb{R}^d$ is called a **polytope**.

Both polyhedra and polytopes are convex sets. One can show that every polytope is a polyhedron but the reverse is not always true since polyhedra can be unbounded. But when a polyhedron is bounded, then you can express a polyhedron by a polytope. Thus, we have two representations of a polyhedron.

Lemma I.2 (Weyl-Minkowski Theorem [Bar02, II.(4.3), IV.(1.3)])**.**

1. *A bounded polyhedron is a polytope.*
2. *A polytope is a polyhedron.*

The polar dual is an important object in convex analysis. Duality is often used to get another mathematical point of view.

Definition I.3 ([Bar02, IV.(1.1)])**.** Let $A \subseteq \mathbb{R}^d$ be a a non-empty set. Then we call the set

$$A^\bullet := \left\{c \in \mathbb{R}^d : \langle c, x\rangle \leq 1 \text{ for all } x \in A\right\}$$

the **polar (dual)** of A.

Remark I.4. Usually the polar set of A is denoted by A°. For the matrix convex sets we distinguish between the complex and the real polar dual, following [DDOSS17], the former will be denote by $A^\bullet$ and the latter by A° (see Definition I.36). To avoid confusion, we write $A^\bullet$ also the polar dual in the real scalar case which we are considering in this section.

One of the most important properties of the polar dual is the Bipolar Theorem.

Lemma I.5 (Bipolar Theorem [Bar02, IV.(1.2)]). *Let $A \subseteq \mathbb{R}^d$ a closed convex set containing the origin. Then $(A^\bullet)^\bullet = A$.*

Polar duals of polytopes can be spanned by the extreme points of the polytope.

Lemma I.6 ([BN20, Lemma 3.2]). *The polar dual of $\mathcal{P} = \operatorname{conv}(\{v_1, ..., v_m\}) \subseteq \mathbb{R}^d$ for an $m \in \mathbb{N}$ can be written as*

$$\mathcal{P}^\bullet := \left\{ x \in \mathbb{R}^d : \langle v_i, x \rangle \leq 1 \text{ for all } i \in [m] \right\}.$$

We can construct new convex sets for example with the Cartesian product or the the direct sum.

Definition I.7. Let $\mathcal{P}_1 \subseteq \mathbb{R}^{k_1}$ and $\mathcal{P}_2 \subseteq \mathbb{R}^{k_2}$ be two convex sets where $k_1, k_2 \in \mathbb{N}$. Then, their **Cartesian product** is defined by

$$\mathcal{P}_1 \times \mathcal{P}_2 := \left\{ (x, y) \in \mathbb{R}^{k_1+k_2} : x \in \mathcal{P}_1, y \in \mathcal{P}_2 \right\}.$$

We present a nice property of faces of a Cartesian product of two polytopes. You can find a definition of a face in [Bar02, II.(2.6)]. A face is not to be confused with a facet, see [Bar02, VI.(1.4)].

Lemma I.8 ([Bre98, Lemma 2.3]). *Let $k_i \in \mathbb{N}$ and let $\mathcal{P}_i \subseteq \mathbb{R}^{k_i}$ be two polytopes where $i \in [2]$. For $0 \leq l \leq k_1 + k_2$, the l-dimensional faces of $\mathcal{P}_1 \times \mathcal{P}_2$ are of the form $\mathcal{F}_1 \times \mathcal{F}_2$, where $\mathcal{F}_i$ is a j_i-dimensional face of $\mathcal{P}_i$ ($i \in [2]$) and $j_1 + j_2 = l$.*

Again we can construct new convex sets of higher dimension by summing up two convex sets. There are several definitions of the direct sum. The one we use is discussed for instance in [Bre98, Page 16] where it is called 'orthogonal sum'. For simplicity, we will use the name 'direct sum'.

Definition I.9. Let $\mathcal{P}_1 \subseteq \mathbb{R}^{k_1}$ and $\mathcal{P}_2 \subseteq \mathbb{R}^{k_2}$ be two convex sets where $k_1, k_2 \in \mathbb{N}$. Then, their **direct sum** is defined by

$$\mathcal{P}_1 \oplus \mathcal{P}_2 := \operatorname{conv}\left(\left\{ (x, 0) \in \mathbb{R}^{k_1+k_2} : x \in \mathcal{P}_1 \right\} \cup \left\{ (0, y) \in \mathbb{R}^{k_1+k_2} : y \in \mathcal{P}_2 \right\} \right).$$

It is obvious, that the direct sum of convex sets is again a convex set. Furthermore, the direct sum of two polytopes is again a polytope which can be expressed by the extreme points of the original polytopes.

Lemma I.10 ([Bre98, Corollary 2.2(a)]). *Let $k_i \in \mathbb{N}$ and let $\mathcal{P}_i \subseteq \mathbb{R}^{k_i}$ be two polytopes with $0 \in \operatorname{int}(P_i)$ where $i \in [2]$. Then x_i is an extreme point of $\mathcal{P}_i$ ($i \in [2]$) if and only if $x_1 \oplus x_2$ is an extreme point or $\mathcal{P}_1 \oplus \mathcal{P}_2$.*

We can find a further representation of the direct sum of polytopes by using the crossproduct and the polar dual.

Lemma I.11 ([Bre98, Lemma 2.4]). *Let $k_i \in \mathbb{N}$ and let $\mathcal{P}_i \subseteq \mathbb{R}^{k_i}$ be two polytopes with $0 \in \operatorname{int}(\mathcal{P}_i)$ where $i \in [2]$. Then*

$$\mathcal{P}_1 \oplus \mathcal{P}_2 = (\mathcal{P}_1^\bullet \times \mathcal{P}_2^\bullet)^\bullet.$$

We see that there are nice properties for convex sets especially for polytopes and polyhedra. We want to generalize some properties in the free analysis for example the connection between the polar dual, the direct sum and the Cartesian product.

2 Preparation for the Matrix Convex Sets

Before we can start to define matrix convex sets, we have to introduce some standard formalism with some properties. For brevity, we use $\mathbb{R}_+^g := \{x = (x_1, ..., x_g) \in \mathbb{R}^g : x_i \geq 0 \text{ for all } i \in [g]\}$ where $g \in \mathbb{N}$. For an $n \in \mathbb{N}$ we denote by $\mathcal{M}_n$ the set of complex $n \times n$ matrices. We write $\mathcal{M}_n^{sa}$ for the self-adjoint $n \times n$ matrices. We denote by I_n the $n \times n$ identity matrix. If the dimension is clear from the context, we sometimes drop the index. For $d, g \in \mathbb{N}$ we introduce the **operator system** generated by the g-tuple $A = (A_1, ..., A_g) \in (\mathcal{M}_d^{sa})^g$ which is defined by

$$\mathcal{OS}_A := \operatorname{span}\{I_d, A_i : i \in [g]\}.$$

In my thesis, all Hilbert spaces are finite dimensional. Let $\mathcal{H}$ be a Hilbert space, then $\mathcal{B}(\mathcal{H})$ are the bounded, linear operators on $\mathcal{H}$. Let $X, Y \in \mathcal{B}(\mathcal{H})$. We write $X \geq 0$ if X is positive semi-definite. Correspondingly, we write $X \geq Y$ if $X - Y$ is positive semi-definite. We are often working with positivity of operators on tensor products of Hilbert spaces. The next lemma is a generalization of [BN20, Lemma 3.20].

Lemma I.12. *Let $k \in \mathbb{N}$ and let $\mathcal{H}_1, ..., \mathcal{H}_k$ be Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_k)^{sa}$. If for all $n \in \mathbb{N}$ the inequality*

$$(P_1 \otimes ... \otimes P_k) A (P_1 \otimes ... \otimes P_k) \geq 0$$

holds true for all $P_1, ..., P_k$ orthogonal projections onto n-dimensional subspaces of $\mathcal{H}_1, ..., \mathcal{H}_k$, then

$$A \geq 0.$$

Proof. For $i \in [k]$, let $A = (A_1, ..., A_k)$ and $A_i \subseteq \mathcal{H}_i$ a not necessarily countable set and let $\{e_{\alpha_i}^{(i)}\}_{\alpha_i \in A_i}$ an orthonormal basis of $\mathcal{H}_i$. Then,

$$\left\{ e_{\alpha_1}^{(1)} \otimes ... \otimes e_{\alpha_k}^{(k)} : \alpha_i \in A_i, i \in [k] \right\}$$

is an orthonormal basis of $\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_k$ [RS03, Proposition 2 in Section II.4].

Let us assume, that A is not positive. Then, there exists a $\psi \in \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_k$ with $\|\psi\| = 1$ such that $\langle \psi, A\psi \rangle < 0$. Thus, we can write ψ in the basis introduced above as

$$\psi = \sum_{i_1, ..., i_k = 1}^{\infty} \psi_{i_1, ..., i_k} e_{i_1}^{(1)} \otimes ... \otimes e_{i_k}^{(k)}$$

where $\psi_{i_1, ..., i_k} \in \mathbb{C}$ for all $i_j \in \mathbb{N}$, $j \in [k]$. The sequence converges in norm, this means (by [RS03, Theorem II.6]) for every $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that $\|\psi - \psi_N\| \leq \varepsilon$ with

$$\psi_N = \sum_{i_1, ..., i_k = 1}^{N} \psi_{i_1, ..., i_k} e_{i_1}^{(1)} \otimes ... \otimes e_{i_k}^{(k)}$$

for $N \in \mathbb{N}$. Thus, by Bessel's inequality and the Cauchy-Schwarz-inequality we get

$$|\langle \psi_N, A\psi_N \rangle - \langle \psi, A\psi \rangle| \leq 2\varepsilon \|A\|_\infty.$$

We conclude that for N large enough $\langle \psi_N, A\psi_N \rangle < 0$. We choose $P_1, ..., P_k$ to be the orthogonal projections onto the subspace spanned by $\{e_i^{(1)}\}_{i\in[N]}$,..., $\{e_i^{(k)}\}_{i\in[N]}$, respectively. Then,

$$\langle \psi_N, (P_1 \otimes ... \otimes P_k)A(P_1 \otimes ... \otimes P_k)\psi_N \rangle = \langle \psi_N, A\psi_N \rangle < 0$$

which contradicts the assumption that $(P_1 \otimes ... \otimes P_k)A(P_1 \otimes ... \otimes P_k)$ is positive semi-definite. Thus, $A \geq 0$. □

In this thesis, we need not only the positivity of operator but also of maps of operators. We notate by id the identity function.

Definition I.13. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ a linear map.

1. Let $k \in \mathbb{N}$. The map T is ***k*-positive** if the map

$$T \otimes \mathrm{id} : \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}_k \to \mathcal{B}(\mathcal{H}_2) \otimes \mathcal{M}_k$$

   is positive.
2. A map T is called **completely positive** if it is k-positive for all $k \in \mathbb{N}$.
3. The map T is **unital** if $T(I_{\mathcal{H}_1}) = I_{\mathcal{H}_2}$.

For brevity, we use the notation $\mathrm{UCP}(\mathcal{B}(\mathcal{H}_1), \mathcal{B}(\mathcal{H}_2))$ for a set of unital, completely positive maps from $\mathcal{B}(\mathcal{H}_1)$ to $\mathcal{B}(\mathcal{H}_2)$.

Unital, completely positive maps are closed under compositions and direct sums. For convenience we show the closeness of direct sums.

Lemma I.14. *Let $m, n \in \mathbb{N}$ and $\mathcal{H}$ a Hilbert space. Let $\Phi_m \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_m)$ and $\Phi_n \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_n)$. Then the direct sum*

$$\Phi_{m+n} := \Phi_m \oplus \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{M}_{m+n}, \qquad A \mapsto \begin{pmatrix} \Phi_m(A) & 0 \\ 0 & \Phi_n(A) \end{pmatrix}$$

is again a unital completely positive map.

Proof. Since Φ_m and Φ_n are unital, it follows directly that Φ_{m+n} is unital. It remains to show that Φ_{m+n} is completely positive. By Definition I.13 we have to show that Φ_{m+n} is k-positive for all $k \in \mathbb{N}$. So, let us fix an $k \in \mathbb{N}$ and take an $(A_{i,j})_{i,j=1}^k \in \mathcal{B}(\mathcal{H})$. We calculate

$$(\Phi_{m+n} \otimes \mathrm{id})((A_{i,j})_{i,j=1}^k) = \begin{pmatrix} (\Phi_m \otimes \mathrm{id})((A_{i,j})_{i,j=1}^k) & 0 \\ 0 & (\Phi_n \otimes \mathrm{id})((A_{i,j})_{i,j=1}^k) \end{pmatrix}.$$

If we assume that $(A_{i,j})_{i,j=1}^k$ is positive, than it follows that the two blocks on the diagonal are positive, that means $(\Phi_{m+n} \otimes \mathrm{id})((A_{i,j})_{i,j=1}^k)$ is positive. Since $k \in \mathbb{N}$ was chosen arbitrary, the assertion holds. □

We often work with tuples of operators. For brevity, we use some conventions. Let $A \in \mathcal{B}(\mathcal{H})^g$ for $g \in \mathbb{N}$. Then we will write A as $A = (A_1, ..., A_g)$. If we write λA for a $\lambda \in \mathbb{C}$, than we mean $\lambda A := (\lambda A_1, ..., \lambda A_g)$. Additionally, consider $B \in \mathcal{B}(\mathcal{H})^g$, then we notate $A \pm B := (A_1 \pm B_1, ..., A_g \pm B_g)$.

3 Definition of Matrix Convex Sets

In convex analysis, you have convex sets defined in Euclidean spaces where the main idea is that the set contains the line segment connecting any two points within the set. Matrix convex sets are a generalization of convex sets. You can say that matrix convex sets are closed under convex combinations of matrices.

Definition I.15. For $g \in \mathbb{N}$, let $\mathcal{F}_n \subseteq (\mathcal{M}_n)^g$ for all $n \in \mathbb{N}$. Then we call $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ a **free set**. A free set $\mathcal{F}$ is a **matrix convex set** if it satisfies the following properties for all $m, n \in \mathbb{N}$.

1. If $X = (X_1, ..., X_g) \in \mathcal{F}_m$ and $Y = (Y_1, ..., Y_g) \in \mathcal{F}_n$, then
$$X \oplus Y := (X_1 \oplus Y_1, ..., X_g \oplus Y_g) \in \mathcal{F}_{m+n}.$$
2. If $X = (X_1, ..., X_g) \in \mathcal{F}_m$ and $\Psi : \mathcal{M}_m \to \mathcal{M}_n$ is a unital completely positive map, then $(\Psi(X_1), ..., \Psi(X_g)) \in \mathcal{F}_n$.

We see, that matrix convex sets are closed under direct sums by the first property. By the second property matrix convex sets are closed under transformations by completely positive maps.

Remark I.16. In this work, we mostly consider free sets or matrix convex sets as $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, where $\mathcal{F}_n \subseteq (\mathcal{M}_n^{sa})^g$ for all $n \in \mathbb{N}$.

We introduce some common properties of matrix convex sets which are defined intuitively.

Definition I.17. Let $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ a free set.

- The matrix convex set $\mathcal{F}$ is **open/ closed/ bounded** if all $\mathcal{F}_n$ $(n \in \mathbb{N})$ defining it have this property.
- The matrix convex set $\mathcal{F}$ is **uniformly bounded**, if there is some $0 < R \in \mathbb{R}$ such that $\|X_i\| < R$ for all $X = (X_1, ..., X_g) \in \mathcal{F}$ and $i \in [g]$.
- We say $\mathcal{F}$ **contains** 0 **in its interior** when there is a $\delta > 0$ such that for all $X = (X_1, ..., X_g) \in \mathcal{F}$ it holds: If $\|X_i\| < \delta$ for all $i \in [g]$ then $X \in \mathcal{F}$. If $\mathcal{F}$ contains 0 in its interior, we write $0 \in \operatorname{int}(\mathcal{F})$.

Remark I.18. We can easily see, if the matrix convex set $\mathcal{F}$ is uniformly bounded, then it is also bounded. The converse is not always true. But we will see (in Theorem II.6), if $0 \in \operatorname{int}(\mathcal{F})$, then $\mathcal{F}$ is uniformly bounded if and only if $\mathcal{F}$ is bounded (actually, it is enough to have boundedness in the first level).

Additionally, we want to define the largest matrix convex set such that the first level of the matrix convex set is fixed by a given convex set.

Definition I.19 ([DDOSS17, Definition 4.1])**.** Let $g \in \mathbb{N}$ and $\mathcal{F}$ a matrix convex set defined by $\mathcal{F}_n \subseteq (\mathcal{M}_n)^g$ for all $n \in \mathbb{N}$. Let $C \subseteq \mathbb{R}^g$ be a convex set such that $\mathcal{F}_1 = C$. We define the **largest matrix convex set** of $\mathcal{F}$ with $\mathcal{F}_1 = C$ as

$$\mathcal{W}_{\max}(C)(n) :=$$
$$\left\{ X \in (\mathcal{M}_n^{sa})^g : \sum_{i=1}^{g} c_i X_i \leq \alpha I, \quad \forall c = (c_1, ..., c_g) \in \mathbb{R}^g,\ \forall \alpha \in \mathbb{R} \text{ s.t. } C \subseteq H(c, \alpha) \right\}$$

where

$$H(c, \alpha) := \left\{ x \in \mathbb{R}^g : \sum_{i=1}^{g} c_i x_i \leq \alpha \right\}.$$

Note that $\mathcal{W}_{\max}(C)(1) = C$.

Remark I.20. You really can verify $\mathcal{W}_{\max}(C)(n)$ as 'maximal convex set' since: If $\mathcal{F} \subseteq \bigcup_{n\in\mathbb{N}} (\mathcal{M}_n^{sa})^g$ is a closed matrix convex set with $\mathcal{F}_1 = C$, then indeed

$$\mathcal{F}_n \subseteq \mathcal{W}_{\max}(C)(n)$$

for all $n \in \mathbb{N}$ as proven in [DDOSS17, Proposition 4.3].

There are two important examples of matrix convex sets that we will discuss in the next two sections. The first example is the free spectrahedron, which is essential for the connection to the POVMs. Thus, we look at the properties of free spectrahedra and express them using other objects. Therefor, we will also introduce the matrix range, which is also a very important example of a matrix convex set.

4 Free Spectrahedron

The free spectrahedra are geometric objects which are important for solving several problems in convex optimization.

Definition I.21. Let $g \in \mathbb{N}$ and $A = (A_1, ..., A_g) \in (\mathcal{B}(\mathcal{H})^{sa})^g$ be a g-tuple of self-adjoint bounded operators on a Hilbert space $\mathcal{H}$. Let $n \in \mathbb{N}$. The **free spectrahedron at level** n defined by A is the set

$$\mathcal{D}_A(n) := \left\{ X \in (\mathcal{M}_n^{sa})^g : \sum_{i=1}^{g} A_i \otimes X_i \leq I_{\mathcal{H}} \otimes I_n \right\}.$$

The **free spectrahedron** is then defined as the (disjoint) union of all these levels, i.e.

$$\mathcal{D}_A := \bigcup_{n\in\mathbb{N}} \mathcal{D}_A(n).$$

In some literature, it is not provided that the free spectrahedron is self-adjoint. In [DDOSS17] you can see that there are similar results for the non-self-adjoint spectrahedron.

Remark I.22. In some literature, for example [HKM12] or [DDOSS17], the free spectrahedron is defined with help of a linear matrix inequality. Let $g \in \mathbb{N}$ and $A = (A_1, ..., A_g) \in (\mathcal{B}(\mathcal{H})^{sa})^g$. For a fixed $n \in \mathbb{N}$, we can write the inequality in $\mathcal{D}_A(n)$ as

$$L(X) = I_{\mathcal{H}} \otimes I_n - \sum_{i=1}^{g} A_i \otimes X_i \geq 0.$$

This representation is called **linear matrix inequality** and is a generalization of a monic linear pencil or matrix pencil of degree g which has the form

$$L(x) = I_{\mathcal{H}} - \sum_{i=1}^{g} x_i A_i \geq 0.$$

This representation is often discussed in convex optimization. An overview over the history and applications of linear matrix inequalities can be found in [BEGFB94].

The next lemma shows that the free spectrahedron is indeed a matrix convex set which is proven in [DDOSS17, Proposition 2.1] for the non-self-adjoint case. Furthermore we can show that the free spectrahedron is closed.

Lemma I.23. *The free spectrahedron is a closed matrix convex set.*

Proof. Let $g \in \mathbb{N}$ and $A = (A_1, ..., A_g) \in (\mathcal{B}(\mathcal{H})^{sa})^g$.

First, we want to verify for $\mathcal{D}_A$ the two conditions in Definition I.15.

1. For fixed $m, n \in \mathbb{N}$, choose $X = (X_1, ..., X_g) \in \mathcal{D}_A(m)$ and $Y = (Y_1, ..., Y_g) \in \mathcal{D}_A(n)$. By definition of $\mathcal{D}_A$ we get
$$\begin{aligned}\sum_{i=1}^{g} A_i \otimes (X_i \oplus Y_i) &= \left(\sum_{i=1}^{g} A_i \otimes X_i\right) \oplus \left(\sum_{i=1}^{g} A_i \otimes Y_i\right)\\ &\leq (I_{\mathcal{H}} \otimes I_m) \oplus (I_{\mathcal{H}} \otimes I_n) = I_{\mathcal{H}} \otimes (I_m \oplus I_n)\end{aligned}$$
and thus $X \oplus Y \in \mathcal{D}_A(m+n)$.
2. We take an $X = (X_1, ..., X_g) \in \mathcal{D}_A(m)$ and a $\Psi \in \mathrm{UCP}(\mathcal{M}_m, \mathcal{M}_n)$ and show $(\Psi(X_1), ..., \Psi(X_g)) \in \mathcal{D}_A(n)$. From $X \in \mathcal{D}_A(m)$ we get the positivity of
$$I_{\mathcal{H}} \otimes I_m - \sum_{i=1}^{g} A_i \otimes X_i.$$
Since $\Psi \in \mathrm{UCP}(\mathcal{M}_m, \mathcal{M}_n)$ we know that $\mathrm{id} \otimes \Psi : \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_m \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_n$ is a unital positive map. Thus,
$$I_{\mathcal{H}} \otimes I_n - \sum_{i=1}^{g} A_i \otimes \Psi(X_i) = (\mathrm{id} \otimes \Psi)\left(I_{\mathcal{H}} \otimes I_m - \sum_{i=1}^{g} A_i \otimes X_i\right)$$
is positive and hence $(\Psi(X_1), ..., \Psi(X_g)) \in \mathcal{D}_A(n)$.

Since the two conditions are fulfilled, the free spectrahdron is a matrix convex set.

In the second part of the proof we show that $\mathcal{D}_A$ is closed, which means that $\mathcal{D}_A(n)$ is closed for all $n \in \mathbb{N}$. Therefor, we fix $n \in \mathbb{N}$ and take the sequence $(X^k)_{k\in\mathbb{N}}$ with $X^k = (X_1^k, ..., X_g^k) \in \mathcal{D}_A(n)$ as a convergent sequence with limit $X = (X_1, ..., X_g) \in (\mathcal{M}_n)^g$. We claim that limit X is self-adjoint. To see this, we use that the involution $*$ on the C^*-algebra $\mathcal{B}(\mathcal{H})$ is isometric and that each X^k is a g-tuple of self-adjoint operators to get
$$X = \lim_{k\to\infty} X^k = \lim_{k\to\infty} (X^k)^* = X^*.$$
Thus, we obtain for the limit that $X \in (\mathcal{M}_n^{sa})^g$. But this means that for all $i \in [g]$ the sequence $(X_i^k)_{k\in\mathbb{N}}$ converges to $X_i \in (\mathcal{M}_n^{sa})^g$ as $k \to \infty$. We define the sequence $(P^k)_{k\in\mathbb{N}}$ as
$$P^k := I_{\mathcal{H}} \otimes I_n - \sum_{i=1}^{g} A_i \otimes X_i^k.$$
To prove that $\mathcal{D}_A$ is closed, we want to show that the limit of the sequence $(P^k)_{k\in\mathbb{N}}$ is positive.

First, we show that $(P^k)_{k\in\mathbb{N}}$ converges to
$$P := I_{\mathcal{H}} \otimes I_n - \sum_{i=1}^{g} A_i \otimes X_i.$$

Therefore, we estimate the matrix norm $\|P - P^k\|$. With the calculation

$$\begin{aligned} P - P^k &= \left(I_{\mathcal{H}} \otimes I_n - \sum_{i=1}^{g} A_i \otimes X_i \right) - \left(I_{\mathcal{H}} \otimes I_n - \sum_{i=1}^{g} A_i \otimes X_i^k \right) \\ &= \sum_{i=1}^{g} \left(A_i \otimes X_i - A_i \otimes X_i^k \right) = \sum_{i=1}^{g} A_i \otimes (X_i - X_i^k) \end{aligned}$$

we get

$$\|P - P^k\| = \left\| \sum_{i=1}^{g} A_i \otimes (X_i - X_i^k) \right\| \leq \sum_{i=1}^{g} \left\| A_i \otimes (X_i - X_i^k) \right\|.$$

Since the C^*-norm is a cross norm we can deduce that

$$\|A_i \otimes (X_i - X_i^k)\| = \|A_i\| \|X_i - X_i^k\|.$$

But X_i^k converges to X_i for every $i \in [g]$, such that P^k converges to P.

Now, we show that the limit P is positive. For all $\xi \in \mathcal{H} \otimes \mathbb{C}^n$ we know that $\langle P^k \xi, \xi \rangle \geq 0$, since P^k is positive. Since P^k converges to P it follows $\langle P\xi, \xi \rangle \geq 0$ for all ξ. Hence, P is positive. □

The next property is a characteristic one of free spectrahedra. Thus, we show this property for convenience.

Lemma I.24. *Let $g \in \mathbb{N}$ and $A = (A_1, ..., A_g) \in (\mathcal{B}(\mathcal{H})^{sa})^g$, then $0 \in \operatorname{int} \mathcal{D}_A$.*

Proof. Take a $\delta > 0$, such that

$$\sum_{i=1}^{g} \|A_i\| \leq \frac{1}{\delta} \quad \text{or equivalently} \quad \sum_{i=1}^{g} \delta \|A_i\| \leq 1. \tag{I.1}$$

We want to show $X \in \mathcal{D}_A$ whenever $\|X_i\| \leq \delta$ for all $i \in [g]$. So, take $X = (X_1, ..., X_g) \in (\mathcal{M}_n^{sa})^g$ for an arbitrary $n \in \mathbb{N}$ such that $\|X\| \leq \delta$. With this and the inequality (I.1), it follows that

$$\sum_{i=1}^{g} \|A_i\| \|X_i\| \leq 1$$

which is nothing else than

$$\sum_{i=1}^{g} \|A_i \otimes X_i\| \leq 1.$$

By the triangle inequality we also get

$$\left\| \sum_{i=1}^{g} A_i \otimes X_i \right\| \leq 1.$$

It follows for the spectrum σ that $\sigma(\sum_{i=1}^{g} A_i \otimes X_i) \subseteq [-1, 1]$, consequently

$$\sigma \left(I_{dn} - \sum_{i=1}^{g} A_i \otimes X_i \right) \subseteq [0, \infty)$$

such that

$$\sum_{i=1}^{g} A_i \otimes X_i \leq I_{dn}$$

and hence $X \in \mathcal{D}_A$. □

Remark I.25 ([BN20, Remark 3.13]). With the last property of free spectrahedra we can concretize the largest matrix convex set (Definition I.15). Let $g \in \mathbb{N}$ and $C \subseteq \mathbb{R}^g$ a convex set. Let $d \in \mathbb{N}$ and $A = (A_1, ..., A_g) \in (\mathcal{M}_d^{sa})^g$. The largest free spectrahedron $\mathcal{W}_{\max}(C)$ such that $\mathcal{D}_A(1) = C$ is again a free spectrahedron. This is true since $0 \in \operatorname{int}(\mathcal{D}_A(1)) = \operatorname{int}(C)$. Furthermore, the closed convex set C can be defined as the intersection of finitely many hyperplanes.

Let us recall that polyhedra can be unbounded (unlike polytopes). Similarly, spectrahedra can be unbounded.

Remark I.26. A free spectrahedron is not necessarily bounded. Take for example $A = \{1\}$, then

$$\mathcal{D}_A(1) = \{x \in \mathbb{R} : x \leq 1\}$$

is unbounded and so is $\mathcal{D}_A = \bigcup_{n \in \mathbb{N}} \mathcal{D}_A(n)$.

We finish this section by introducing two examples. The first example is one of the most important ones in this thesis.

Example I.27. Let $k \in \mathbb{N}$ and define the diagonal matrices

$$V_j^{(k)} := -\frac{2}{k} \operatorname{diag}(1, ..., 1, \overset{j\text{-th entry}}{\overset{\downarrow}{1-k}}, 1, ..., 1)$$

for $j \in [k-1]$. Notate $V^{(k)} = (V_1^{(k)}, ..., V_{k-1}^{(k)})$. The **matrix jewel base** $\mathcal{D}_{\diamondsuit,k} = \bigcup_{n=1}^{\infty} \mathcal{D}_{\diamondsuit,k}(n)$ defined by

$$\mathcal{D}_{\diamondsuit,k}(n) := \mathcal{D}_{V^{(k)}}(n) = \left\{ X \in (\mathcal{M}_n^{sa})^{k-1} : \sum_{j=1}^{k-1} V_j^{(k)} \otimes X_j \leq I_{kn} \right\}$$

is a free spectrahedron. This object is discussed in detail in Chapter IV Section 2 and used in the inclusion of the free spectrahedra which is connected with POVMs. For more information refer to Chapter VI.

The next presented object is the matrix cube which is the main object in the matrix cube problem formulated in [BTN02]. This problem is discussed in several papers for example [HKMS16]. Furthermore, the matrix cube is brought in connection with the maximal violent of the steering inequality [BN22].

Example I.28 ([DDOSS17, Example 2.2]). Let $g \in \mathbb{N}$ and E_{ii} the diagonal $g \times g$ matrix with 1 at the i-th place and 0 elsewhere. We define the $2g \times 2g$ matrices

$$A_i = \begin{pmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{pmatrix}$$

for $1 \leq i \leq g$ and set $A = (A_1, ..., A_g)$. Then $X = (X_1, ..., X_g) \in \mathcal{D}_A$ if and only if

$$\sum_{i=1}^{g} A_i \otimes X_i = \sum_{i=1}^{g} \begin{pmatrix} E_{ii} \otimes X_i & 0 \\ 0 & -E_{ii} \otimes X_i \end{pmatrix} \leq I_{2g} \otimes I_n.$$

This is equivalent to $-I_n \leq X_i \leq I$ for all $i \in [g]$. We call this free spectrahedron the **(g-dimensional) complex matrix cube**.

5 Matrix Range

A further example of a matrix convex set is the matrix range. We use the matrix range to get more information about the free spectrahedra.

Definition I.29. The **matrix range** $\mathcal{W}(A)$ of $A = (A_1, ..., A_g) \in (\mathcal{B}(\mathcal{H})^{sa})^g$ is defined as $\mathcal{W}(A) = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(A)$ where

$$\begin{aligned} \mathcal{W}_n(A) := \{&(X_1, ..., X_g) \in (\mathcal{M}_n^{sa})^g : \\ &\exists \Psi \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_n) \text{ such that } X_i = \Psi(A_i)\ \forall i \in [g]\} \end{aligned}$$

for all $n \in \mathbb{N}$.

Similar to the free spectrahedra, the matrix range is also a closed matrix convex set. Before, we can show this property, we need some preparations.

5.1 Short Introduction to the Compactness in the BW Topology

To prepare the proof of the closedness of the matrix range, we need that a unital, completely positive map which is compact in the bounded, weak topology. Thus, we introduce the bounded, weak topology along with some statements which are important for us. This introduction is based on [Pau02, Chapter 7].

Let X and Y be two Banach spaces, let Y^* the dual of Y. For fixed $x \in X$ and $y \in Y$ we define the linear functional $x \otimes y$ on $\mathcal{B}(X, Y^*)$ by

$$x \otimes y(L) = L(x)(y).$$

Let Z denote the closed linear span in $B(X, Y^*)^*$ of these tensors. Then, by [Pau02, Lemma 7.1], $B(X, Y^*)$ is isometrically isomorphic to Z^* with the duality

$$\langle L, x \otimes y\rangle = x \otimes y(L).$$

This allows us to endow $\mathcal{B}(X, Y^*)$ with the weak* topology defined as follow:

Definition I.30. We call the weak* topology that is induced on $\mathcal{B}(X, Y^*)$ the **BW topology** (bounded weak topology).

The name 'bounded weak topology' can be deduced from the statement of the next lemma.

Lemma I.31. *[Pau02, Theorem 7.2] A bounded net $\{L_\lambda\}$ in $\mathcal{B}(X, Y^*)$ converges to L in the BW topology if and only if $L_\lambda(x)$ converges weakly to $L(x)$ for all $x \in X$.*

There are various sets which are compact in the BW topology. We only need compactness for the set of unital, completely positive maps.

Theorem I.32. *[Pau02, Theorem 7.4] Let A be a C^*-algebra, let $\mathcal{OS}$ be a closed operator system contained in A. Then $\mathrm{UCP}(\mathcal{OS}_A, \mathbb{C}^n)$ is compact in the BW topology.*

5.2 Properties of the Matrix Range

Now, we have all preliminaries to show one of the main properties of the matrix range.

Lemma I.33. *The matrix range is a closed matrix convex set.*

Proof. For $g \in \mathbb{N}$, we fix $A = (A_1, ..., A_g) \in (\mathcal{B}(\mathcal{H})^{sa})^g$. For better clarity, we split the proof in two parts.

First, we show that $\mathcal{W}(A)$ is a matrix convex set. We check the two conditions of Definition I.15:

1. For $m, n \in \mathbb{N}$ let $X = (X_1, ..., X_g) \in \mathcal{W}_m(A)$ and $Y = (Y_1, ..., Y_g) \in \mathcal{W}_n(A)$. Then there exist unital completely positive maps $\Psi_m \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_m)$ and $\Psi_n \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_n)$ such that $X_i = \Psi_m(A_i)$ and $Y_j = \Psi_n(A_j)$ for all $i, j \in [g]$. Now we can construct $\Psi_{m+n} \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_{m+n})$ such that
$$(\Psi_{m+n}(A_1), ..., \Psi_{m+n}(A_g)) = (\Psi_m(A_1) \oplus \Psi_n(A_1), ..., \Psi_m(A_g) \oplus \Psi_n(A_g)).$$
By Lemma I.14 it holds $\Psi_{m+n} \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_{m+n})$ such that
$$X \oplus Y = (X_1 \oplus Y_1, ..., X_g \oplus Y_g) \in \mathcal{W}_{n+m}.$$
2. Every $X = (X_1, ..., X_g) \in \mathcal{W}_m(A)$ can be express by a unital, completely positive map $\Psi \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_m)$, that means
$$(X_1, ..., X_g) = (\Psi(A_1), ..., \Psi(A_g)).$$
Since UCP maps are closed under compositions we have $(\Psi(X_1), ..., \Psi(X_g)) \in \mathcal{W}_n(A)$ for an arbitrary function $\Psi \in \mathrm{UCP}(\mathcal{M}_m, \mathcal{M}_n)$.

It remains to show that the matrix range $\mathcal{W}(A)$ is closed. For a fixed $n \in \mathbb{N}$, we take the sequence $(X^k)_{k\in\mathbb{N}}$ with $X^k := (X_1^k, ..., X_g^k) \in \mathcal{W}_n(A)$ which converges to an $X := (X_1, ..., X_g) \in (\mathcal{M}_n)^g$. Similar to the proof of Lemma I.23 one can show that the limit X is self-adjoint. Now, we show that the limit $X \in (\mathcal{M}_n^{sa})^g$ is also an element in $\mathcal{W}_n(A)$.

Since $X^k \in \mathcal{W}_n(A)$, we know that for every X^k we can find a unital map $\Psi^k \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_n)$ such that $X_i^k = \Psi^k(A_i)$ for all $i \in [g]$. Now we want to show, that we can find a $\Psi \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_n)$ such that $X_i = \Psi(A_i)$ for the limit $X \in (\mathcal{M}_n^{sa})^g$.

We can calculate with the linearity and unitality of Ψ
$$\Psi^k\left(\alpha_0 I_{\mathcal{H}} + \sum_{i=1}^{g} \alpha_i A_i\right) = \alpha_0 I_n + \sum_{i=1}^{g} \alpha_i X_i^k$$
for arbitrary $\alpha_i \in \mathbb{C}$. We know that this term converges to
$$\alpha_0 I_n + \sum_{i=1}^{g} \alpha_i X_i.$$
Consequently, for
$$\mathcal{OS}_A := \left\{ Z \in \mathcal{B}(\mathcal{H}) \mid Z = \alpha_0 I_{\mathcal{H}} + \sum_{i=1}^{g} \alpha_i A_i \text{ and } \alpha_i \in \mathbb{C} \text{ for } i = 1, ..., g \right\}$$

the sequence $(\Psi^k|_{\mathcal{OS}_A})_{k\in\mathbb{N}}$ converges pointwise.

In the next step, we want to show that there is a sub-net of the sequence $(\Psi^k)_{k\in\mathbb{N}}$ such that

$$\lim_{\lambda\in\Lambda} \Psi^{k_\lambda}|_{\mathcal{OS}_A} \in \mathrm{UCP}(\mathcal{OS}_A, \mathcal{M}_n).$$

From Theorem I.32 we know, that a map in $\mathrm{UCP}(\mathcal{OS}_A, \mathcal{M}N$ is compact in the BW topology. Therefor, for the sequence $\Psi^k|_{\mathcal{OS}_A} \in \mathrm{UCP}(\mathcal{OS}_A, \mathcal{M}_n)$ exists a sub-net $(\Psi^{k_\lambda}|_{\mathcal{OS}_A})_{\lambda\in\Lambda}$ of $(\Psi^k|_{\mathcal{OS}_A})_{k\in\mathbb{N}}$ which converges with respect to the BW topology to a limit $\Psi|_{\mathcal{OS}_A} \in \mathrm{UCP}(\mathcal{OS}_A, \mathcal{M}_n)$, that means

$$\Psi|_{\mathcal{OS}_A} = \lim_{\lambda\in\Lambda} \Psi^{k_\lambda}|_{\mathcal{OS}_A} \in \mathrm{UCP}(\mathcal{OS}_A, \mathcal{M}_n).$$

By Lemma I.31, this means for all A_i, $i \in [g]$ we have the pointwise convergence $\Psi|_{\mathcal{OS}_A}(A_i) = \lim_{\lambda\in\Lambda} \Psi^{k_\lambda}|_{\mathcal{OS}_A}(A_i)$. But $\Psi^{k_\lambda}|_{\mathcal{OS}_A}(A_i)$ is by construction nothing else than $X_i^{k_\lambda}$. And by construction of X^k it holds $\lim_{\lambda\in\Lambda} X_i^{k_\lambda} = X_i$. Thus, $X = \Psi|_{\mathcal{OS}_A}(A_i)$. By Arveson we can $\Psi|_{\mathcal{OS}_A} \in \mathrm{UCP}(\mathcal{OS}_A, \mathcal{M}_n)$ extend to a $\Psi \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_n)$. Then $X \in \mathcal{W}_n(A)$ and the matrix range is closed. □

We recall that the free spectrahedra is also closed but not bounded. However, in contrast, we can show that the matrix range is not only bounded but uniformly bounded.

Lemma I.34. *The matrix range is uniformly bounded.*

Proof. For a fixed $n \in \mathbb{N}$, the matrix range is a set of elements which can be expressed by a unital, completely positive map Φ. By [Pau02, Corollary 2.9 (Russo-Dye)], we know that Φ is bounded and $\|\Phi\| \leq \|\Phi(I_n)\|$. Since Φ is unital, the map Φ is bounded by 1. So, for every $n \in \mathbb{N}$ the elements of the matrix range are bounded by 1 and thus the matrix range is uniformly bounded. □

Unlike free spectrahedra, which has the origin in their interior (see Lemma I.24), the matrix range does not necessarily contain the origin.

Remark I.35. The matrix range does not necessarily contains 0. Take for example $A = \{1\}$, then for $n \in \mathbb{N}$ we have $\mathcal{W}_n(A) = \{I_n\}$ since the map $\Psi \in \mathrm{UCP}(\mathbb{R}, \mathcal{M}_n)$ is unital. Thus, $0 \notin \mathcal{W}(A) = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(A)$.

There are a lot of connections between the matrix range and the free spectrahedron for example we can represent the matrix range by the polar dual of the free spectrahedron and reversed. To see this, we have to introduce the polar dual of free sets.

6 Polar Dual

Similar to the Definition I.3 we define the polar dual of free sets. This section is a collection of statements of [DDOSS17, Chapter 3].

Definition I.36. Let $g \in \mathbb{N}$.

1. Let $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n \subseteq \bigcup_{n\in\mathbb{N}} (\mathcal{M}_n)^g$ be a free set. Its **polar dual** is defined as $\mathcal{F}^\circ = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n^\circ \subseteq \bigcup_{n\in\mathbb{N}} (\mathcal{M}_n)^g$, where

$$\mathcal{F}_n^\circ = \left\{ X \in (\mathcal{M}_n)^g : \mathrm{Re}\left(\sum_{i=1}^g A_i \otimes X_i\right) \leq I \text{ for all } A \in \mathcal{F} \right\}.$$

2. Likewise, let $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n \subseteq (\mathcal{M}_n^{sa})^g$ be a free set. Its **polar dual** is defined as $\mathcal{F}^\bullet = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n^\bullet \subseteq (\mathcal{M}_n^{sa})^g$, where
$$\mathcal{F}_n^\bullet = \left\{ X \in (\mathcal{M}_n^{sa})^g : \sum_{i=1}^g A_i \otimes X_i \leq I \text{ for all } A \in \mathcal{F} \right\}.$$

Similar to the real case at the beginning, we want to characterize the bipolar of a free set. We again require the condition that the origin is contained in the free set.

Lemma I.37 ([DDOSS17, Lemma 3.2]). *If $\mathcal{F} \subseteq \bigcup_n (\mathcal{M}_n)^g$ is a closed matrix convex set containing 0, then*
$$(\mathcal{F}^\circ)^\circ = \mathcal{F}.$$
Likewise, if $\mathcal{F}$ is a closed matrix convex set in $\bigcup_n (\mathcal{M}_n^{sa})^g$ containing 0, then
$$(\mathcal{F}^\bullet)^\bullet = \mathcal{F}.$$

Proof. For the first part we refere you to the bipolar theorem of Effros and Winkler [EW97, Corollary 5.5].

To show $(\mathcal{F}^\bullet)^\bullet = \mathcal{F}$, we first show the inclusion $\mathcal{F} \subseteq (\mathcal{F}^\bullet)^\bullet$. Let $n \in \mathbb{N}$. We know that
$$\mathcal{F}_n^\bullet = \left\{ X \in (\mathcal{M}_n^{sa})^g : \sum_{i=1}^g A_i \otimes X_i \leq I \text{ for all } A \in \mathcal{F} \right\}$$
and
$$((\mathcal{F}^\bullet)^\bullet)_n = \left\{ Y \in (\mathcal{M}_n^{sa})^g : \sum_{i=1}^g X_i \otimes Y_i \leq I \text{ for all } X \in \mathcal{F}_n^\bullet \right\}.$$
Recognize, that the eigenvalues not change by switching the matrices in a tensor product, that means that the eigenvalues of $M \otimes N$ are the same as for $N \otimes M$ for arbitrary $M, N \in \mathcal{M}_n$. For an arbitrary $A \in \mathcal{F}_n$, we see, that the inequality in $((\mathcal{F}^\bullet)^\bullet)_n$ is fulfilled.

It remains to show $(\mathcal{F}^\bullet)^\bullet \subseteq \mathcal{F}$. We use the fact, that $\mathcal{F} = (\mathcal{F}^\circ)^\circ$. By that, it is sufficient to show that $(\mathcal{F}^\bullet)^\bullet \subseteq (\mathcal{F}^\circ)^\circ$. Take an $Y \in ((\mathcal{F}^\bullet)^\bullet)_n$, then $Y \in (\mathcal{F}^\bullet)^\circ \cap (\mathcal{M}_n^{sa})^g$. This means
$$\sum_{j=1}^g X_j \otimes Y_j = \operatorname{Re}\left(\sum_{j=1}^g X_j \otimes Y_j \right) \leq I$$
for all $X \in \mathcal{F}_n^\bullet \subseteq ((\mathcal{M}_n)^{sa})^g$. Now, let $\widetilde{X} \in \mathcal{F}_n^\circ$, this means
$$\operatorname{Re}\left(\sum_{j=1}^g \widetilde{X}_j \otimes A_j \right) \leq I$$
for all $A \in \mathcal{F}_n$. By assumption of $\mathcal{F}_n$, every A is self-adjoint, this means that
$$\sum_{j=1}^g (\operatorname{Re} \widetilde{X}_j) \otimes A_j = \operatorname{Re}\left(\sum_{j=1}^g \widetilde{X}_j \otimes A_j \right) \leq I$$

such that $\mathrm{Re}(\widetilde{X}) \in \mathcal{F}_n^{\bullet}$. By using that $Y \in (\mathcal{F}^{\bullet})^{\circ} \cap (\mathcal{M}_n^{sa})^g$ is self-adjoint, we get

$$\mathrm{Re}\left(\sum_{j=1}^{g} \widetilde{X}_j \otimes Y_j\right) = \sum_{j=1}^{g} (\mathrm{Re}\,\widetilde{X}_j) \otimes Y_j \leq I$$

and thus $Y \in (\mathcal{F}^{\circ})^{\circ} = \mathcal{F}$. $\square$

Chapter II.

Connection between Matrix Range and Free Spectrahedra

1 Connection between Matrix Range and Free Spectrahedra by using the Polar Dual

The matrix range and the free spectrahedra are not just examples of matrix convex sets. With a few conditions an arbitrary matrix convex set can be expressed by the matrix range or free spectrahedron. Furthermore, they are connected with each other. We can express the matrix range by the polar dual of the free spectrahedra and reversed.

Proposition II.1 ([DDOSS17, Proposition 3.1]). *Let $A \in (B(\mathcal{H})^{sa})^g$, then*

$$(\mathcal{W}(A) \cup \{0\})^\bullet = \mathcal{W}(A)^\bullet = \mathcal{D}_A.$$

Proof. The equation $(\mathcal{W}(A) \cup \{0\})^\bullet = \mathcal{W}(A)^\bullet$ follows directly from the definition.

To verify $\mathcal{W}(A)^\bullet = \mathcal{D}_A$, we first show that $X \in \mathcal{W}(A)^\bullet$ whenever $X \in \mathcal{D}_A$. Let $m, n \in \mathbb{N}$ and $\Psi \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_n)$ arbitrary, then also

$$\Psi \otimes \mathrm{id} \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_m, \mathcal{M}_n \otimes \mathcal{M}_m).$$

Furthermore,

$$I_{\mathcal{H}} \otimes I_m - \sum_{i=1}^{g} A_j \otimes X_j$$

is positive by definition of $\mathcal{D}_A$. Since $\Psi \otimes I_m$ is unital completely positive, we know

$$(\Psi \otimes \mathrm{id}) \left(I_{\mathcal{H}} \otimes I_m - \sum_{i=1}^{g} A_j \otimes X_j \right) = I_n \otimes I_m - \sum_{i=1}^{g} \Psi(A_j) \otimes X_j$$

is also positive. This means, for all elements in $\mathcal{W}(A)$ we know, that the inequality is fulfilled.

It remains to show the reversed direction: For all $X \in \mathcal{W}(A)^\bullet$ it follows $X \in \mathcal{D}_A$. So, let $X \in \mathcal{W}(A)^\bullet$, then

$$I_n \otimes I_m - \sum_{i=1}^{g} \Psi(A_i) \otimes X_i \geq 0$$

for all $m, n \in \mathbb{N}$ and $\Psi \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}), \mathcal{M}_n)$. For $\Psi \otimes \mathrm{id} \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_m, \mathcal{M}_n \otimes \mathcal{M}_m)$ we get

$$(\Psi \otimes \mathrm{id})\left(I_{\mathcal{H}} \otimes I_m - \sum_{i=1}^{g} A_i \otimes X_i\right) \geq 0.$$

Now, we can use the argument of Lemma I.12 to get

$$I_{\mathcal{H}} \otimes I_m - \sum_{i=1}^{g} A_i \otimes X_i \geq 0$$

and thus $X \in \mathcal{D}_A$. □

We can ask if the polar of the free spectrahedra is again the matrix range. For this statement we need a further condition, namely that the origin is in the matrix range.

Lemma II.2 ([DDOSS17, Proposition 3.3])**.** *Let $A \in (\mathcal{B}(\mathcal{H})^{sa})^g$ and $0 \in \mathcal{W}(A)$, then*

$$(\mathcal{D}_A)^\bullet = \mathcal{W}(A).$$

Proof. Since $0 \in \mathcal{W}(A)$, we can use the bipolar theorem of matrix convex sets, Lemma I.37, to get $\mathcal{W}(A) = (\mathcal{W}(A)^\bullet)^\bullet$. With Proposition II.1 we get the assertion which is $((\mathcal{W}(A)^\bullet)^\bullet)^\bullet = \mathcal{D}_A^\bullet$. □

2 Equivalent Statements of Free Spectrahedra and Matrix Range

In Chapter I we have seen that the origin is contained in the interior of the free spectrahedron but not necessarily in the matrix range. On the other hand the matrix range is uniformly bound but the free spectrahedron is not. But one can show that the interior of the matrix range contains the origin if and only if the free spectrahedron is uniformly bounded. Before we introduce this theorem we need some preparations.

2.1 Preparations

We need some preparations in spectral theory. We are interested in the spectrum of a normal tuple. You can find more information about the spectral theory in [Mül03].

Definition II.3 ([Mül03, I.2 Definition 14])**.** Let $\mathcal{A}$ be a commutative Banach algebra and denote by $\mathcal{M}(\mathcal{A})$ the set of all multiplicative functionals on $\mathcal{A}$. For $x_1, ..., x_n \in \mathcal{A}$ the **joint spectrum** $\sigma(x_1, ..., x_n)$ is the set

$$\sigma(x_1, ..., x_n) = \{(\varphi(x_1), ..., \varphi(x_n)) : \varphi \in \mathcal{M}(\mathcal{A})\}.$$

Definition II.4 ([DDOSS17, Chapter 4])**.** We say a tuple $N = (N_1, ..., N_d)$ is a **normal tuple** if $N_1, ..., N_g$ are normal, commuting operators. We denote by $\sigma(N)$ the joint spectrum of a normal tuple N in the sense of Definition II.3.

We need a connection between the matrix range and the joint spectrum of a normal tuple. We present the connection beginning with the first level over the n-the level up to the union over all levels.

Lemma II.5 ([DDOSS17, Theorem 2.7, Corollary 2.8]). *Let N be a normal g-tuple with $\sigma(N) \subseteq \mathbb{C}^g$. Then*

$$\mathcal{W}_1(N) = \operatorname{conv}(\sigma(N))$$

and

$$\mathcal{W}_n(N) = \left\{ \sum_{i=1}^{m} \lambda^{(i)} K_i : i \in [m], \lambda^{(i)} \in \sigma(N), m \in \mathbb{N}, K_i \in \mathcal{M}_n, K_i \geq 0, \sum_{i=1}^{m} K_i = I_n \right\}$$

for $n \geq 2$. Furthermore, $\mathcal{W}(N)$ is the smallest matrix convex set containing $\sigma(N)$.

2.2 The Theorem about the Equivalence of Free Spectrahedron and Matrix Range

Theorem II.6 ([DDOSS17, Lemma 3.4]). *For $A \in \mathcal{B}((\mathcal{H})^{sa})^g$ the following terms are equivalent:*

1. $0 \in \operatorname{int}(\mathcal{W}(A))$.
2. $0 \in \operatorname{int}(\mathcal{W}_1(A))$.
3. $\mathcal{D}_A(1)$ *is bounded.*
4. $\mathcal{D}_A$ *is uniformly bounded.*

Proof. We show this by proving the following chain of implications: (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (4).

(4) $\Rightarrow$ (3):
This statement is clear by using the definition of $\mathcal{D}_A$.

(3) $\Rightarrow$ (2):
We prove this part by contradiction. Suppose that $0 \notin \operatorname{int}(\mathcal{W}_1(A))$. Since $\mathcal{W}_1(A)$ is convex we can use the Hahn-Banach separation theorem. Thus, there exist $a_1, ..., a_g \in \mathbb{R}$ such that

$$\sum_{i=1}^{g} a_i x_i \geq 0$$

for all $x = (x_1, ..., x_g) \in \mathcal{W}_1(A)$. Furthermore, the inequality

$$\sum_{i=1}^{g} t a_i x_i \leq 0 < 1$$

holds true for all $t < 0$. Thus, we have $(ta_1, ..., ta_g) \in \mathcal{D}_A(1)$ for all $t < 0$. But this contradicts that $\mathcal{D}_A(1)$ is bounded.

(2) $\Rightarrow$ (1):
Suppose $0 \in \operatorname{int}(\mathcal{W}_1(A))$. Then there is an $\varepsilon > 0$ such $\varepsilon[-1,1]^g \subseteq \mathcal{W}_1(A)$. We recognize, that the cube with radius ε is inside $\mathcal{W}_1(A)$, that is $\varepsilon[-1,1]^d \subseteq \mathcal{W}_1(A)$. Then there is a normal d-tuple N with

$$\sigma(N) = \varepsilon[-1,1]^d.$$

By Lemma II.5 we know that $\mathcal{W}(N)$ is the smallest convex set containing $\sigma(N)$ such that $\mathcal{W}(N) \subseteq \mathcal{W}(A)$. Hence, to show that $0 \in \mathcal{W}(A)$ it is enough to show, that $0 \in \operatorname{int}(\mathcal{W}(N))$ for a normal d-tuple with $\sigma(N) = [-1,1]^d$. But this follows by the representation in Lemma II.5.

(1) $\Rightarrow$ (4):
Let $\delta > 0$ and assume, that $0 \in \operatorname{int}(\mathcal{W}(A))$. Fix some $i \in [g]$ and let $X = (X_1, ..., X_g)$ be defined as

$$X_j = \begin{cases} \frac{1}{2}\delta I_n, & \text{if } j = i, \\ 0, & \text{if } j \neq i \end{cases}$$

for $j \in [g]$. By assumption and the definition of $\operatorname{int}(\mathcal{W}(A))$, we know that $\pm X \in \mathcal{W}(A)$. Since $\mathcal{D}_A = \mathcal{W}(A)^\bullet$ and by definition of the polar dual, Definition I.36, we know that for every $Y \in \mathcal{D}_A$

$$\pm X_i \otimes Y_i = \pm \sum_{j=1}^{g} X_j \otimes Y_j \leq I_{n^2}$$

holds. Thus, we get

$$1 \geq \|X_j \otimes Y_j\| = \frac{1}{2}\delta \|Y_j\|$$

respectively $\|Y_j\| \leq 2/\delta$ for all $j \in [g]$. Thus, $\mathcal{D}_A$ is bounded. □

2.3 A further Connection between Matrix Range and Free Spectrahedra by using the Polar Dual

We can use the shown equivalence of Proposition II.6 to get another connection between matrix range and free spectrahedra at the first leve by using the polar dual which is not necessarily self-adjoint.

Lemma II.7 ([BN20, Lemma 3.22])**.** *Let $\mathcal{H}$ be a Hilbert space and let $A \in (\mathcal{B}(\mathcal{H})^{sa})^g$ for $g \in \mathbb{N}$ such that $\mathcal{D}_A(1)$ is bounded. Then*

$$(\mathcal{D}_A(1))^\circ = \mathcal{W}_1(A).$$

Proof. By Definition I.21, $x \in \mathcal{D}_A(1)$ if and only if

$$I_{\mathcal{H}} - \sum_{i=1}^{g} x_i A_i \geq 0.$$

By applying an arbitrary unital, completely positive map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ this is equivalent to

$$1 - \sum_{i=1}^{g} x_i \Psi(A_i) \geq 0.$$

Thus, we have the equivalence

$$\mathcal{D}_A(1) = \mathcal{W}_1(A)^\circ.$$

Since $\mathcal{D}_A(1)$ is bounded we know with Theorem II.6 that $0 \in \mathcal{W}_1(A)$. Thus, we can use the Bipolar Theorem, Theorem I.5, such that we get

$$\mathcal{D}_A(1)^\circ = (\mathcal{W}_1(A)^\circ)^\circ = \mathcal{W}_1(A).$$

□

3 Representing a Matrix Convex Set by Free Spectrahedra or Matrix Range

We recall that the matrix range is uniformly bounded. This is a characteristic property of a matrix range. One can say that if an arbitrary closed matrix convex set is bounded then you can write it as a matrix range.

Proposition II.8 ([DDOSS17, Prop. 3.5]). *Let $\mathcal{F} \subseteq \bigcup_n (\mathcal{M}_n^{sa})^d$ be a closed matrix convex set. One can show that $\mathcal{F}$ has the form $\mathcal{F} = \mathcal{W}(A)$ for some $A \in (\mathcal{B}(\mathcal{H})^{sa})^d$ if and only if $\mathcal{F}$ is uniformly bounded.*

In this whole chapter we see the connection between the matrix range and the free spectrahedron. So, for the free spectrahedron we can make a similar statement. Therefore, we use the property that the origin is in the interior of a free spectrahedron and the connection between the matrix range and the free spectrahedron.

Proposition II.9 ([DDOSS17, Prop. 3.5]). *A closed matrix convex set $\mathcal{F} \subseteq \bigcup_n (\mathcal{M}_n^{sa})^d$ has the form $\mathcal{F} = \mathcal{D}_A$ for some $A \in (\mathcal{B}(\mathcal{H})^{sa})^d$ if and only if $0 \in \operatorname{int}(\mathcal{F})$.*

Proof. First we show $0 \in \operatorname{int}(\mathcal{F})$ whenever $\mathcal{F} = \mathcal{D}_A$ for some $A \in (\mathcal{B}(\mathcal{H})^{sa})^d$. Since $\mathcal{D}_A$ is a matrix convex set, the assertion follows from $0 \in \operatorname{int} \mathcal{D}_A$, refer by Lemma I.24

For the other direction we assume $0 \in \operatorname{int}(\mathcal{F})$ to conclude that $\mathcal{F}$ has the form $\mathcal{F} = \mathcal{D}_A$ for some $A \in (\mathcal{B}(\mathcal{H})^{sa})^d$. By Definition I.17 $0 \in \operatorname{int}(\mathcal{F})$ means that there is a $\delta > 0$ such that $\|X_i\| \leq \delta$ for all $i \in [d]$ implies $X \in \mathcal{F}$. Let this $\delta > 0$ be fixed and define $X = (X_1, ..., X_d)$ by

$$X_j = \begin{cases} 0, & j \neq k, \\ \frac{1}{2}\delta I, & j = k, \end{cases}$$

then $X \in \mathcal{F}$. Take $Y \in \mathcal{F}^\bullet$. With Definition I.36 it follows

$$\frac{\delta}{2}\|Y_k\| = \left\| \sum_{j=1}^{d} Y_j \otimes X_j \right\| \leq 1$$

for all $k \in [d]$. Thus, $\|Y_k\| \leq 2/\delta$ and $\mathcal{F}^\bullet$ is bounded. With Proposition II.8, there is an $A \in (\mathcal{B}(\mathcal{H})^{sa})^d$ such that $\mathcal{F}^\bullet = \mathcal{W}(A)$. Finally with Lemma I.6 and Theorem II.6 we conclude

$$\mathcal{F} = (\mathcal{F}^\bullet)^\bullet = (\mathcal{W}(A))^\bullet = \mathcal{D}_A.$$

□

Chapter III.

Direct Sum of Matrix Convex Sets

In Chapter I Section 1 we introduced the direct sum of polytopes. We want to generalize this definition for matrix convex sets and show representations for direct sums between free spectrahedra.

1 Cartesian Product

The Cartesian product is necessary for defining the direct sum of matrix convex sets.

Definition III.1. Let $\mathcal{F}, \mathcal{G}$ be two free sets. Their **Cartesian product** is defined as $\mathcal{F} \hat{\times} \mathcal{G} = \bigcup_{n \in \mathbb{N}} (\mathcal{F} \hat{\times} G)_n$ where

$$(\mathcal{F} \hat{\times} G)_n := \{(X, Y) : X \in \mathcal{F}_n, Y \in \mathcal{G}_n\}$$

for all $n \in \mathbb{N}$.

Remark III.2 ([BN20, Section 3.3])*.* Let $\mathcal{F}, \mathcal{G}$ be two matrix convex sets. Then their Cartesian product $(\mathcal{F} \hat{\times} G)_n$ is again matrix convex.

The Cartesian product of matrix convex sets at level $n = 1$ is the ordinary Cartesian product of convex sets, i.e.

$$(\mathcal{F} \hat{\times} G)_1 = \mathcal{F}_1 \times \mathcal{G}_1.$$

Let $\mathcal{F}, \mathcal{G}$ be two free spectrahedra. Then their Cartesian product $\mathcal{F} \hat{\times} G$ is also a free spectrahedron. We see this in the next proposition:

Proposition III.3 ([BN20, Proposition 3.18])**.** *Let $A \in (\mathcal{B}(\mathcal{H}_1)^{sa})^{k_1}$, $B \in (\mathcal{B}(\mathcal{H}_2)^{sa})^{k_2}$ ($k_1, k_2 \in \mathbb{N}$) be two tuples of self-adjoint bounded operators. Then $\mathcal{D}_A \hat{\times} \mathcal{D}_B$ is the free spectrahedron defined as $\mathcal{D}_A \hat{\times} \mathcal{D}_B = \bigcup_{n \in \mathbb{N}} (\mathcal{D}_A \hat{\times} \mathcal{D}_B)(n)$ where*

$$(\mathcal{D}_A \hat{\times} \mathcal{D}_B)(n) = \left\{ X \in (\mathcal{M}_n^{sa})^{k_1 + k_2} : \sum_{i=1}^{k_1} (A_i \oplus 0_{\mathcal{H}_2}) \otimes X_i + \sum_{j=1}^{k_2} (0_{\mathcal{H}_1} \oplus B_j) \otimes X_{k_1 + j} \leq I_{\mathcal{H}_1 \oplus \mathcal{H}_2} \otimes I_n \right\}$$

for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. Since the isomorphism

$$(\mathcal{H}_1 \oplus \mathcal{H}_2) \otimes \mathbb{C}^n \simeq (\mathcal{H}_1 \otimes \mathbb{C}^n) \oplus (\mathcal{H}_2 \otimes \mathbb{C}^n),$$

holds true, the inequality in the set $(\mathcal{D}_A \hat{\times} \mathcal{D}_B)(n)$

$$\sum_{i=1}^{k_1} (A_i \oplus 0_{\mathcal{H}_2}) \otimes X_i + \sum_{j=1}^{k_2} (0_{\mathcal{H}_2} \oplus B_j) \otimes X_{k_1+j} \leq I_{\mathcal{H}_1 \oplus \mathcal{H}_2} \otimes I_n$$

holds true if and only if

$$\left(\sum_{i=1}^{k_1} A_i \otimes X_i \right) \oplus \left(\sum_{j=1}^{k_2} B_j \otimes X_{k_1+j} \right) \leq (I_{\mathcal{H}_1} \otimes I_n) \oplus (I_{\mathcal{H}_2} \otimes I_n).$$

But the last inequality is true if and only if $(X_1, ..., X_{k_1}) \in \mathcal{D}_A(n)$ and $(X_{k_1+1}, ..., X_{k_1+k_2}) \in \mathcal{D}_A(n)$. □

2 Direct Sum

Now, we have all preparations to define the direct sum.

Definition III.4. Let $\mathcal{F}, \mathcal{G}$ be two matrix convex sets defined by $\mathcal{F}_n \in (\mathcal{M}_n^{sa})^{g_1}$ and $\mathcal{G}_n \in (\mathcal{M}_n^{sa})^{g_2}$ for all $n \in \mathbb{N}$. Their **direct sum** is defined as

$$(\mathcal{F} \hat{\oplus} \mathcal{G}) := ((\mathcal{F}^\bullet \otimes I) \hat{\times} (I \otimes \mathcal{G}^\bullet))^\bullet$$

where

$$(\mathcal{F}^\bullet \otimes I)_m := \begin{cases} \{(X_1 \otimes I_1, \ldots, X_{g_1} \otimes I_n) : X \in \mathcal{F}_n^\bullet\} & \exists n \in \mathbb{N} \text{ s.t. } m = n^2, \\ \emptyset & \text{all other } m \in \mathbb{N}. \end{cases}$$

We discuss direct sums of various sets. We start with the direct sum of free spectrahedra at an arbitrary level $n \in \mathbb{N}$.

Proposition III.5 ([BN20, Proposition 3.21])**.** *Let $A \in (\mathcal{B}(\mathcal{H}_1)^{sa})^{k_1}$, $B \in (\mathcal{B}(\mathcal{H}_2)^{sa})^{k_2}$ ($k_1, k_2 \in \mathbb{N}$) be two tuples of self-adjoint operators. Moreover, let $\mathcal{D}_A(1)$ and $\mathcal{D}_B(1)$ be bounded. Then the direct sum $\mathcal{D}_A \hat{\oplus} \mathcal{D}_B$ is the free spectrahedron defined by*

$$\begin{aligned}&(\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(n) \\ &= \left\{ X \in (\mathcal{M}_n^{sa})^{k_1+k_2} : \sum_{i=1}^{k_1} (A_i \otimes I_{\mathcal{H}_2}) \otimes X_i + \sum_{j=1}^{k_2} (I_{\mathcal{H}_1} \otimes B_j) \otimes X_{k_1+j} \leq I_{\mathcal{H}_1 \oplus \mathcal{H}_2} \otimes I_n \right\}.\end{aligned}$$

Proof. We show each direction separately.

First, let $X = (X_1, ..., X_{k_1+k_2}) \in (\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)_n$, $n \in \mathbb{N}$. Then

$$\sum_{i=1}^{k_1} ((\Psi_1 \otimes \Psi_2)(A_i \otimes I_{\mathcal{H}_2}) \otimes X_i) + \sum_{j=1}^{k_2} ((\Psi_1 \otimes \Psi_2)(I_{\mathcal{H}_1} \otimes B_j)) \otimes X_{k_1+j} \leq I_{m^2 n}$$

for all UCP maps $\Psi_i : \mathcal{B}(\mathcal{H}) \to \mathcal{M}_m$, $m \in \mathbb{N}$ and $i \in [2]$. We rewrite the last equation to

$$(\Psi_1 \otimes \Psi_2 \otimes I_n)(I_{\mathcal{H}_1 \oplus \mathcal{H}_2} \otimes I_n) - \sum_{i=1}^{k_1} (\Psi_1 \otimes \Psi_2 \otimes I_n)(A_i \otimes I_{\mathcal{H}_2} \otimes X_i) \\ - \sum_{j=1}^{k_2} (\Psi_1 \otimes \Psi_2 \otimes I_n)(I_{\mathcal{H}_1} \otimes B_j \otimes X_{k_1+j}) \geq 0.$$

Consider orthogonal projections P_1, P_2 onto m-dimensional subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We remark, that the maps $X \mapsto PXP$, $X \mapsto QXQ$ and $X \mapsto I_n X I_n$ are UCP maps. Thus, we can imply that

$$\begin{aligned} &(P_1 \otimes P_2 \otimes I_n)(I_{\mathcal{H}_1 \otimes \mathcal{H}_2} \otimes I_n)(P_1 \otimes P_2 \otimes I_n) \\ &- \sum_{i=1}^{k_1} (P_1 \otimes P_2 \otimes I_n)(A_i \otimes I_{\mathcal{H}_2} \otimes X_i)(P_1 \otimes P_2 \otimes I_n) \\ &- \sum_{j=1}^{k_2} (P_1 \otimes P_2 \otimes I_n)(I_{\mathcal{H}_1} \otimes B_j \otimes X_{k_1+j})(P_1 \otimes P_2 \otimes I_n) \geq 0. \end{aligned}$$

We rewrite the equation to

$$(P_1 \otimes P_2 \otimes I_n) \left(I_{\mathcal{H}_1 \oplus \mathcal{H}_2} \otimes I_n - \sum_{i=1}^{k_1} A_i \otimes I_{\mathcal{H}_2} \otimes X_i - \sum_{j=1}^{k_2} I_{\mathcal{H}_1} \otimes B_j \otimes X_{k_1+j} \right) \\ (P_1 \otimes P_2 \otimes I_n) \geq 0$$

to use Lemma I.12 to get

$$I_{\mathcal{H}_1 \oplus \mathcal{H}_2} \otimes I_n - \sum_{i=1}^{k_1} (A_i \otimes I_{\mathcal{H}_2}) \otimes X_i - \sum_{j=1}^{k_2} (I_{\mathcal{H}_1} \otimes B_j) \otimes X_{k_1+j} \geq 0$$

respectively

$$\sum_{i=1}^{k_1} (A_i \otimes I_{\mathcal{H}_2}) \otimes X_i + \sum_{j=1}^{k_2} (I_{\mathcal{H}_1} \otimes B_j) \otimes X_{k_1+j} \leq I_{\mathcal{H}_1 \oplus \mathcal{H}_2} \otimes I_n.$$

Thus, we conclude $\mathcal{D}_A \hat{\oplus} \mathcal{D}_B \subseteq \mathcal{D}_{(A \otimes I_{\mathcal{H}_2}, I_{\mathcal{H}_1} \otimes B)}$.

For the other direction we use, that by Theorem II.6 the boundness of $\mathcal{D}_A(1)$ and $\mathcal{D}_B(1)$ implies that $0 \in \mathcal{W}(A)$ and $0 \in \mathcal{W}(B)$. Thus, we can use Lemma II.2 to get $\mathcal{D}_A^\bullet = \mathcal{W}(A)$ and $\mathcal{D}_B^\bullet = \mathcal{W}(B)$. When we now calculate $((\mathcal{D}_A^\bullet \otimes I) \hat{\times} (I \otimes \mathcal{D}_B^\bullet))_m$ for an arbitrary $m \in \mathbb{N}$, then we get by Definition III.4

$$\begin{aligned} &((\mathcal{D}_A^\bullet \otimes I) \hat{\times} (I \otimes \mathcal{D}_B^\bullet))_{n^2} \\ &= \big\{ (X \otimes I_n, I_n \otimes Y) : \forall i \in [2] \, \exists \Psi_i \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}_i), \mathcal{M}_n) \text{ such that} \\ &\ (X \otimes I_n, I_n \otimes Y) = (\Psi(A_1 \otimes I_{\mathcal{H}_2}), \Psi(A_2 \otimes I_{\mathcal{H}_2}), ..., \Psi(I_{\mathcal{H}_1} \otimes B_{k_2})), \Psi := \Psi_1 \otimes \Psi_2 \big\} \end{aligned}$$

when there is an $n \in \mathbb{N}$ such that $n^2 = m$ and

$$((\mathcal{D}_A^\bullet \otimes I) \hat{\times} (I \otimes \mathcal{D}_B^\bullet))_m = \emptyset$$

for all other $m \in \mathbb{N}$. Thus, we have the inclusion

$$(\mathcal{D}_A^\bullet \otimes I) \hat{\times} (I \otimes \mathcal{D}_B^\bullet) \subseteq \mathcal{W}(A \otimes I_{\mathcal{H}_2}, I_{\mathcal{H}_1} \otimes B)$$

which implies

$$\mathcal{D}_{(A \otimes I_{\mathcal{H}_2}, I_{\mathcal{H}_1} \otimes B)} \subseteq \mathcal{D}_A \hat{\oplus} \mathcal{D}_B$$

by Definition III.4 and Lemma II.1.

□

2.1 Direct sum at level 1

With help of the direct sum of the general free spectrahedra we can find an expression for spectrahedra at the first level. We see that the direct sums at level one are objects from convex analysis.

Proposition III.6 ([BN20, Prop 3.24])**.** *Let $A \in (\mathcal{B}(\mathcal{H}_1)^{sa})^{k_1}$, $B \in (\mathcal{B}(\mathcal{H}_2)^{sa})^{k_2}$ ($k_1, k_2 \in \mathbb{N}$) be two tuples of self-adjoint operators. Moreover, let $\mathcal{D}_A(1)$ and $\mathcal{D}_B(1)$ be polytopes. Then*

$$(\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(1) = \mathcal{D}_A(1) \oplus \mathcal{D}_B(1).$$

Proof. By Proposition III.5, we know for all $x \in \mathcal{D}_A(1)$ and $y \in \mathcal{D}_B(1)$ that $(x, 0)$ and $(0, y)$ are elements in $(\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(1)$. Thus, the inclusion

$$\mathcal{D}_A(1) \oplus \mathcal{D}_B(1) \subseteq (\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(1)$$

holds true.

It remains to show the reversed direction. Let $(x, y) \in (\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(1)$. Then, by Proposition III.5

$$I_{\mathcal{H}_1 \oplus \mathcal{H}_2} - \sum_{i=1}^{k_1} (A_i \otimes I_{\mathcal{H}_1}) x_i - \sum_{i=1}^{k_2} (I_{\mathcal{H}_2} \otimes B_j) y_j \geq 0.$$

Let $\Psi_i \in \mathrm{UCP}(\mathcal{B}(\mathcal{H}_i), \mathbb{C})$, $i \in [2]$ arbitrary. By an application of $\Psi_1 \otimes \Psi_2 \otimes \mathrm{id}$ we get

$$1 - \sum_{i=1}^{k_1} x_i \Psi_1(A_i) - \sum_{i=1}^{k_1} y_j \Psi_2(B_j) \geq 0.$$

Thus, $(\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(1)^\bullet \subseteq \mathcal{W}_1(A) \times \mathcal{W}_1(B)$. Since $(\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(1)$ is closed and convex, we can applying the Bipolar Theorem I.5 to get

$$((\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(1)^\bullet)^\bullet = (\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(1).$$

Now, we can apply Lemma I.11 and Lemma II.7 to see that the inclusion

$$(\mathcal{D}_A \hat{\oplus} \mathcal{D}_B)(1) \subseteq (\mathcal{W}_1(A) \times \mathcal{W}_1(B))^\bullet = \mathcal{D}_A(1) \oplus \mathcal{D}_B(1)$$

holds true. □

In Proposition II.9 we see that we can represent a matrix convex set by free spectrahedra. We can use this representation to expand the last statement. Thereby, we see that the direct sum at the first level is the direct sum as usual.

Lemma III.7 ([BN20, Cor 3.25])**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be closed matrix convex sets with 0 in their interior and let $\mathcal{F}_1$ and $\mathcal{G}_1$ polytopes. Then*

$$(\mathcal{F}\hat{\oplus}\mathcal{G})_1 = \mathcal{F}_1 \oplus \mathcal{G}_1.$$

Proof. By Proposition II.9 we know that for $\mathcal{F}$ and $\mathcal{G}$ there are $g_1, g_2 \in \mathbb{N}$, Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and $A \in (\mathcal{B}(\mathcal{H}_1)^{sa})^{g_1}$, $B \in (\mathcal{B}(\mathcal{H}_2)^{sa})^{g_2}$ such that

$$\mathcal{F} = \mathcal{D}_A \quad \text{and} \quad \mathcal{G} = \mathcal{D}_B.$$

By using the last Proposition III.6 we get the assertion. □

3 Further Properties of the Direct Sum

In Chapter I Section I we introduced the largest matrix convex set. There are many free spectrahedra which are polytopes at the first level. Thus, it makes sense to have a nice property about the direct sum of the matrix convex sets of polytopes.

Lemma III.8 ([BN20, Lemma 3.26])**.** *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two polytopes with zero in their interiors. Then*

$$\mathcal{W}_{\max}(\mathcal{P}_1 \oplus \mathcal{P}_2) = \mathcal{W}_{\max}(\mathcal{P}_1)\hat{\oplus}\mathcal{W}_{\max}(\mathcal{P}_2).$$

Proof. Let $i \in [2]$, $k_i \in \mathbb{N}$ and $\mathcal{P}_i \subseteq \mathbb{R}^{k_i}$. By the secound part of the Weyl-Minkowski theorem, Theorem I.2, every polytope is a polyhedron. Thus, there exist $c_{s_i}^{(i)} \in \mathbb{R}^{k_i}$ and $\alpha_{s_i}^{(i)} \in \mathbb{R}$ such that

$$\mathcal{P}_i = \left\{ x \in \mathbb{R}^{k_i} : \langle c_{s_i}^{(i)}, x\rangle \leq \alpha_{s_i}^{(i)} \qquad \forall s_i \in [m_i] \right\}$$

where $m_i \in \mathbb{N}$. Since $0 \in \operatorname{int}(\mathcal{P}_i)$ by assumption, the inequality can only be fulfilled, if $\alpha_{s_i}^{(i)} > 0$. Furthermore, the facets of $\mathcal{P}_i$ are defined by

$$\mathcal{F}_{s_i}^{(i)} = \left\{ p_i \in \mathcal{P}_i : \langle c_{s_i}^{(i)}, p_i\rangle = \alpha_{s_i}^{(i)} \right\}$$

for $s_i \in [m_i]$. Therefore, we can write

$$\mathcal{P}_i = \left\{ x \in \mathbb{R}^{k_i} : \left\langle h_{s_i}^{(i)}, x \right\rangle \leq 1 \quad \forall s_i \in [m_i] \right\} = \left\{ x \in \mathbb{R}^{k_i} : \sum_{j=1}^{k_i} x_j P_j^{(i)} \leq I_{m_i} \right\}$$

where $h_{s_i}^{(i)} = c_{s_i}^{(i)}/\alpha_{s_i}^{(i)}$ and $P_j^{(i)} = \operatorname{diag}(h_1^{(i)}(j), ..., h_{m_i}^{(i)}(j)) \in \mathbb{R}^{m_i \times m_i}$ such that $P_j^{(i)}(s) := h_{s_i}^{(i)}(j)$ for $j \in [k_i]$. By Lemma I.8, we know, that $\mathcal{F}_{s_1}^{(1)} \times \mathcal{F}_{s_2}^{(2)}$ is the facet of $\mathcal{P}_1 \times \mathcal{P}_2$, such that we get $(h_{s_1}^{(1)}, h_{s_2}^{(2)})$ as the extreme points of $\mathcal{P}_1^\bullet \times \mathcal{P}_2^\bullet$. By Lemma I.11 we know that $(\mathcal{P}_1^\bullet \times \mathcal{P}_2^\bullet)^\bullet = \mathcal{P}_1 \oplus \mathcal{P}_2$ such that we can represent $\mathcal{P}_1 \oplus \mathcal{P}_2$ with the extreme points of $\mathcal{P}_1^\bullet \times \mathcal{P}_2^\bullet$:

$$\mathcal{P}_1 \oplus \mathcal{P}_2 = \left\{ (x_1, x_2) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} : \langle (h_{s_1}^{(1)}, h_{s_2}^{(2)}), (x_1, x_2)\rangle \leq 1 \quad \forall s_i \in [m_i] \right\}.$$

Thus, $(h_{s_1}^{(1)}, h_{s_2}^{(2)})$ are the hyperplanes defining $\mathcal{P}_1 \oplus \mathcal{P}_2$ such that we represent

$$\mathcal{P}_1 \oplus \mathcal{P}_2 = \left\{ x \in \mathbb{R}^{k_1+k_2} : \sum_{j=1}^{k_1+k_2} x_j Q_j \leq I_{m_1 m_2} \right\}$$

where $Q_j \in \mathbb{R}^{m_1 m_2}$ with $Q_j(s_1, s_2) := (h^{(1)}_{s_1}, h^{(2)}_{s_2})_j$ and $j \in [k_1 + k_2]$. With the definition of the maximal spectrahedron, it follows

$$W_{\max}(\mathcal{P}_1 \oplus \mathcal{P}_2)(n) = \left\{ X \in (\mathcal{M}_n^{sa})^{k_1+k_2} : \sum_{j=1}^{k_1+k_2} Q_j \otimes X_j \leq I_{nm_1m_2} \right\}.$$

But when we now evaluate the expression for Q_j, we get

$$\begin{aligned} Q_j(s_1, s_2) &= \begin{cases} h^{(1)}_{s_1}(j) = P^{(1)}_j(s_1) & 1 \leq j \leq k_1 \\ h^{(2)}_{s_2}(j - k_1) = P^{(2)}_{j-k_1}(s_2) & k_1 + 1 \leq j \leq k_1 + k_2 \end{cases} \\ &= \begin{cases} (P^{(1)}_j \otimes I_{k_2})(s_1, s_2) & 1 \leq j \leq k_1 \\ (I_{k_1} \otimes P^{(2)}_{j-k_1})(s_1, s_2) & k_1 + 1 \leq j \leq k_1 + k_2 \end{cases} \end{aligned}$$

such that the assertion follows. □

The next lemma is again about the direct sum of polytopes. We combine the direct sum with an inclusion of free spectrahedra at level one. The next property is useful to check the inclusion of a direct sum of polytopes for each polytope separately.

Lemma III.9 ([BN20, Lemma 3.28])**.** *Let $k_i \in \mathbb{N}$, $d \in \mathbb{N}$ and $A^{(i)} \in (\mathcal{M}_d^{sa})^{k_i}$ where $i \in [2]$. Let $\mathcal{P}_i \subseteq \mathbb{R}^{k_i}$ be two polytopes for $i \in [2]$. Then*

$$\mathcal{P}_1 \oplus \mathcal{P}_2 \subseteq \mathcal{D}_{(A^{(1)}, A^{(2)})}(1) \qquad \Leftrightarrow \qquad \mathcal{P}_i \subseteq \mathcal{D}_{A^{(i)}}(1).$$

for all $i \in [2]$.

Proof. Since $\mathcal{P}_i$ are two polytopes, we can characterize them and their direct sum by their extreme points. By Lemma I.10 $\{w^{(i)}_j\}_{j=1}^{m_i}$ the set of the $m_i \in \mathbb{N}$ extreme points of $\mathcal{P}_i$ if and only if

$$\left\{ (w^{(1)}_{j_1}, 0), (0, w^{(2)}_{j_2}) : j_i \in [m_i], m_i \in \mathbb{N}, i \in [2] \right\}$$

are the extreme points of $\mathcal{P}_1 \oplus \mathcal{P}_2$. Thus, the assertion follows directly. □

Chapter IV.

Inclusion of Free Spectrahedra and Introduction of the Matrix Jewel

1 Inclusion of Free Spectrahedra

Our aim in this thesis is to give a connection between POVMs and the inclusion of free spectrahedra at level one and the connection between compatibility of POVMs and the inclusion of free spectrahedra in general. One can ask if the inclusion of free spectrahedra holds at the first level, can we follow that the inclusion holds for an arbitrary level. In general, this is not the case which we can see in the presented example.

Example IV.1 ([HKM12, Example 3.1]). We set

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By calculation of the determinant we get

$$\begin{aligned} \mathcal{D}_A(1) &= \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot x_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot x_2 \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \\ &= \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{pmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix} \geq 0 \right\} \\ &= \{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1 \} \end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_B(1) &= \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot x_1 + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot x_2 \leq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ &= \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix} \geq 0 \right\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}.\end{aligned}$$

With this calculation we see that $\mathcal{D}_A(1) = \mathcal{D}_B(1)$. Now, we want to see that $\mathcal{D}_A(2) \neq \mathcal{D}_B(2)$. The element

$$(X_1, X_2) = \left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{3}{4} & 0 \end{pmatrix} \right)$$

is obviously an element of $(\mathcal{M}_2^{sa})^2$. For this element, we calculate the matrix

$$\begin{aligned}A_1 \otimes X_1 + A_2 \otimes X_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{3}{4} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & -\frac{1}{2} & 0 \\ \frac{3}{4} & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

and its eigenvalues

$$\frac{-\sqrt{10} - 1}{4}, \ \frac{-\sqrt{10} + 1}{4}, \ \frac{\sqrt{10} - 1}{4} \text{ and } \frac{\sqrt{10} + 1}{4}.$$

We recognize that $\sqrt{10} + 1/4 > 1$ such that $(X_1, X_2) \notin \mathcal{D}_A(2) = \sum_{i=1}^{2} A_i \otimes X_i \leq I_2$. We also calculate the matrix

$$\begin{aligned}B_1 \otimes X_1 + B_2 \otimes X_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{3}{4} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

and its eigenvalues

$$-\sqrt{13}/4, \ -3/4, \ 0, \ 3/4 \text{ and } \sqrt{13}/4$$

which are all less than 1. Thus, (X_1, X_2) is an element of $\mathcal{D}_B(2)$ but not of $\mathcal{D}_A(2)$.

The preceding example shows, there are sets A and B such that $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$ but $\mathcal{D}_A \nsubseteq \mathcal{D}_B$. The idea is to shrink $\mathcal{D}_A$ such that the inclusion $\mathcal{D}_A \subseteq \mathcal{D}_B$ holds true. Therefor, we introduce the scaled free spectrahedron.

Definition IV.2. Let $A \in (\mathcal{B}(\mathcal{H})^{sa})^g$ and $\mathcal{D}_A$ a free spectrahedron. For an $s = (s_1, ..., s_g) \in \mathbb{R}^g$ we define the **(asymmetrically) scaled free spectrahedron** as

$$s \cdot \mathcal{D}_A := \{(s_1 X_1, ..., s_g X_g) : X = (X_1, ..., X_g) \in \mathcal{D}_A\}.$$

As you can see in the definition, we allow that $s \in \mathbb{R}^g$ (where $g \in \mathbb{N}$). To shrink a free spectrahedron it makes sense to choose an $s \in [0,1]^g$. Furthermore, taking the scalar $s \in \mathbb{R}^g$ as a tuple of zeros would be a trivial choice, but often one can choose tuple entries larger than zero. To determine how large s can be, we define the inclusion set. For $k \in \mathbb{N}$ we notate $(s)^{\times k} := \underbrace{(s, ..., s)}_{k \text{ times}}$.

Definition IV.3. Let $n \in \mathbb{N}$, $g \in \mathbb{N}$ and $A = (A^{(1)}, ..., A^{(g)})$ where $A^{(i)} \in (\mathcal{M}_n^{sa})^{k_i - 1}$, $k_i \in \mathbb{N}$ and $i \in [g]$. Let $\mathcal{D}_A$ be the associated free spectrahedron. The **inclusion set** is defined as

$$\begin{aligned}&\Delta_{\mathcal{D}_A}(g, d, \mathbf{k}) := \Big\{ s \in \mathbb{R}_+^g : \\ &\forall B \in (\mathcal{M}_d^{sa})^{\sum_{i=1}^g (k_i - 1)}, \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \Rightarrow (s_1^{\times (k_1 - 1)}, ..., s_g^{\times (k_g - 1)}) \cdot \mathcal{D}_A \subseteq \mathcal{D}_B \Big\}\end{aligned}$$

where $\mathbf{k} = (k_1, ..., k_g)$ and $d \in \mathbb{N}$.

We recognize that in the last definition A has $\sum_{i=1}^g (k_i - 1)$ elements, but we only take g scalars $s_1, ..., s_g$. For us, it is important that every tuple entry of A_{k_i}, $i \in [g]$, is multiplied by the same scalar.

Remark IV.4 ([BN18, Proposition 4.3]). Let us consider the conditions from Definition IV.3. Then $\Delta_{\mathcal{D}_A}(g, d, \mathbf{k})$ is a convex set.

You can describe inclusions of free spectrahedra by a special unital map.

Definition IV.5. Let $g \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $A = (A_1, ..., A_g) \in (\mathcal{M}_n^{sa})^g$ such that I_n and $A_1, ..., A_g$ are linearly independent define the free spectrahedron $\mathcal{D}_A$. Let $d \in \mathbb{N}$ and $B \in (\mathcal{M}_d^{sa})^g$ define the free spectrahedron $\mathcal{D}_B$. We define the unital map $\Phi : \mathcal{OS}_A \to \mathcal{M}_d$ as

$$\Phi : A_i \mapsto B_i$$

for all $i \in [g]$.

Recognize, that the linearly independence is important to make the map well-defined. Now, we present the connection between the positivity of Φ and the inclusion of free spectrahedra.

Lemma IV.6 ([BN18, Lemma 4.4])**.** *Let $A = (A_1, ..., A_g) \in (\mathcal{M}_{d_A}^{sa})^g$ such that I_{d_A} and $A_1, ..., A_g$ are linearly independent define the free spectrahedra $\mathcal{D}_A$. Let $B \in (\mathcal{M}_d^{sa})^g$ define the free spectrahedra $\mathcal{D}_B$. Furthermore, let $\mathcal{D}_A(1)$ be bounded and $n \in \mathbb{N}$. Then $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ if and only if $\Phi : \mathcal{OS}_A \to \mathcal{M}_{d_B}$ is n-positive. Furthermore, $\mathcal{D}_A \subseteq \mathcal{D}_B$ if and only if $\Phi : \mathcal{OS}_A \to \mathcal{M}_{d_B}$ is completely positive.*

Proof. By definition, it is enough to show the equivalence $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ if and only if $\Phi : \mathcal{OS}_A \to \mathcal{M}_{d_B}$ is n-positive for all $n \in \mathbb{N}$. We show each direction seperately.

For a fixed $n \in \mathbb{N}$, assume that $\Phi : \mathcal{OS}_A \to \mathcal{M}_{d_B}$ is n-positive. The inclusion $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ can be shown by definition of a free spectrahedron. Let $X = (X_1, ..., X_g) \in \mathcal{D}_A(n)$, then

$$I_{d_A} \otimes I_n - \sum_{i=1}^{g} A_i \otimes X_i \geq 0.$$

Since Φ is n-positive, we get

$$I_{d_B} \otimes I_n - \sum_{i=1}^{g} \Phi(A_i) \otimes X_i = (\Phi \otimes \mathrm{id}) \left(I_{d_A} \otimes I_n - \sum_{i=1}^{g} A_i \otimes X_i \right) \geq 0$$

such that $X \in \mathcal{D}_B(n)$.

Let us show the other direction. Therefor, let $Y \in \mathcal{M}_n^{sa}(\mathcal{OS}_A)$. We can write Y as

$$Y = I_{d_A} \otimes X_0 - \sum_{i=1}^{g} A_i \otimes X_i$$

for certain $\widetilde{X} = (X_0, ..., X_g) \in \mathcal{M}_n^g$. For positive Y we want to show $(\Phi \otimes \mathrm{id})Y \geq 0$ to verify that Φ is n-positive.

We first show that $X_0, ..., X_g$ are self-adjoint. For that we look at the equation

$$(I_{d_A} \otimes e_i^*)(Y - Y^*)(I_{d_A} \otimes e_j) = 0 \tag{IV.1}$$

for an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{C}^n$. We transform the left-hand side as follows

$$\begin{aligned}
&(I_{d_A} \otimes e_i^*)(Y - Y^*)(I_{d_A} \otimes e_j) \\
&= (I_{d_A} \otimes e_i^*)(I_{d_A} \otimes (X_0 - X_0^*) - \sum_{i=1}^{k} A_i \otimes (X_i - X_i^*))(I_{d_A} \otimes e_j) \\
&= e_i^*(X - X_0^*)e_j I_{d_A} - \sum_{i=1}^{k} e_i^*(X_i - X_i^*)e_j A_i.
\end{aligned}$$

Since I_{d_A} and $A_1, ..., A_g$ are linearly independent, it follows with equation IV.1

$$e_i^*(X_k^* - X_k)e_j = 0$$

for all $i, j \in [n]$, $k = 0, ..., g$. Thus, $X_k^* = X_k$ for $k = 0, ..., g$ such that each entry in $\widetilde{X}$ is self-adjoint.

Let us now assume that $Y \geq 0$, then we can show that $X_0 \geq 0$. Assume, $X_0 < 0$, then there exists an $x \in \mathbb{C}^n$ such that $\langle x, X_0 x \rangle < 0$. Since $Y = I_D \otimes X_0 - \sum_{i=1}^{g} A_i \otimes X_i$ is positive it follows

$$-\sum_{i=1}^{g} \langle x, X_i x \rangle A_i > 0$$

such that $\lambda(\langle x, X_1 x \rangle, ..., \langle x, X_g x \rangle) \in \mathcal{D}_A(1)$ for all $\lambda \geq 0$. But this contradicts the assumption that $\mathcal{D}_A(1)$ is bounded.

Let us assume that $Y \geq 0$, then we know that $X_0 \geq 0$. It remains to show that also $(\Phi \otimes I_n)Y \geq 0$. Let us first consider that $Y \geq 0$ and $X_0 > 0$ such that X_0 is invertible. Since $Y \geq 0$ it follows

$$I_{d_A} \otimes X_0^{-1/2} X_0 X_0^{-1/2} - \sum_{i=1}^{g} A_i \otimes X_0^{-1/2} X_i X_0^{-1/2} \geq 0$$

and thus

$$X_0^{-1/2} X X_0^{-1/2} \in \mathcal{D}_A(n)$$

for $X = (X_1, ..., X_g) \in (\mathcal{M}_n^{sa})^g$. By assumption, we have $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ such that $(\Phi \otimes \mathrm{id})Y \geq 0$. For $Y \geq 0$ and $X_0 \geq 0$ we exchange X_0 by $X_0 + \varepsilon I_n$ while $\varepsilon > 0$. By using the last argument and letting ε go to zero we get again $(\Phi \otimes \mathrm{id})Y \geq 0$. This completes the proof. $\square$

The next lemma tells us: If you are interested in having the inclusion $\mathcal{D}_A \subseteq \mathcal{D}_B$, there is a better way than of checking the inclusion for all levels $n \in \mathbb{N}$.

Theorem IV.7 ([BN18, Corollary 4.6])**.** *Let $g, n, d \in \mathbb{N}$. Let $A \in (\mathcal{M}_n^{sa})^g$ and $B \in (\mathcal{M}_d^{sa})^g$ define the free spectrahedra $\mathcal{D}_A$ and $\mathcal{D}_B$. Let $\mathcal{D}_A(1)$ be bounded. Then $\mathcal{D}_A(d) \subseteq \mathcal{D}_B(d)$ if and only if $\mathcal{D}_A \subseteq \mathcal{D}_B$.*

Proof. From Lemma IV.6 we know that $\mathcal{D}_A(d) \subseteq \mathcal{D}_B(d)$ is equivalent to $\Phi : \mathcal{OS}_A \to \mathcal{M}_d$ being d-positive. Since Φ maps to $\mathcal{M}_d$ we can use [Pau02, Theorem 6.1] whereby d-positive is equivalent to completely positive. Another application of Lemma IV.6 yields the claim. $\square$

2 The Matrix Jewel

In this section, we recall and discuss Example I.27 of the free spectrahedron. The matrix jewel is necessary to have a connection between the inclusion of free spectrahedra and the POVMs what will be discussed in Chapter VI.

2.1 The general Definition of the Matrix Jewel

We recall the matrix jewel base from Example I.27 and extend the object for $\mathbf{k} = (k_1, ..., k_g) \in \mathbb{N}^g$.

Definition IV.8. Let $k \in \mathbb{N}$. We define the vectors $v_j^{(k)} \in \mathbb{R}^k$ as

$$v_j^{(k)}(\varepsilon) := -\frac{2}{k} + 2\delta_{\varepsilon,j}$$

for all $j \in [k-1]$ and and for all $\varepsilon \in [k]$. Let $k \in \mathbb{N}$ and define the diagonal matrices

$$V_j^{(k)} := \mathrm{diag}\left(v_j^{(k)}\right) = -\frac{2}{k}\,\mathrm{diag}(1, ..., 1,\ \overset{\substack{j\text{-th entry}\\ \downarrow}}{1-k}\ , 1, ..., 1)$$

for $j \in [k-1]$. Notate $V^{(k)} = (V_1^{(k)}, ..., V_{k-1}^{(k)})$. The **matrix jewel base** $\mathcal{D}_{\diamond,k} = \bigcup_{n=1}^{\infty} \mathcal{D}_{\diamond,k}(n)$ defined by

$$\mathcal{D}_{\diamond,k}(n) := \mathcal{D}_{V^{(k)}}(n) = \left\{ X \in (\mathcal{M}_n^{sa})^{k-1} : \sum_{j=1}^{k-1} V_j^{(k)} \otimes X_j \leq I_{kn} \right\}$$

for all $n \in \mathbb{N}$.

For $\mathbf{k} = (k_1, ..., k_g) \in \mathbb{N}^g$, we define the **matrix jewel** $\mathcal{D}_{\text{💎},\mathbf{k}} = \bigcup_{n=1}^{\infty} \mathcal{D}_{\text{💎},\mathbf{k}}(n)$ as the direct sum of the free spectrahedra

$$\mathcal{D}_{\text{💎},\mathbf{k}}(n) := \mathcal{D}_{\text{💎},k_1}(n) \hat{\oplus} ... \hat{\oplus} \mathcal{D}_{\text{💎},k_g}(n)$$

for $n \in \mathbb{N}$.

Remark IV.9. There is a equivalent definition of the matrix jewel base which is sometimes useful too, for example in the proof of VI.5. For an $g \in \mathbb{N}$ we identify the subalgebra of $g \times g$ diagonal matrices with $\mathbb{C}^g$. After defining the vectors $v_j^{(k)} \in \mathbb{C}^k$ ($k \in \mathbb{N}$) as

$$v_j^{(k)}(\varepsilon) := -\frac{2}{k} + 2\delta_{\varepsilon,j}$$

for all $j \in [k-1]$ and for all $\varepsilon \in [k]$, we can define the matrix jewel base $\mathcal{D}_{\text{💎},k} = \bigcup_{n=1}^{\infty} \mathcal{D}_{\text{💎},k}(n)$ defined by

$$\mathcal{D}_{\text{💎},k}(n) := \left\{ X \in (\mathcal{M}_n^{sa})^{k-1} : \sum_{j=1}^{k-1} v_j^{(k)} \otimes X_j \leq I_{kn} \right\}$$

for all $n \in \mathbb{N}$.

Remark IV.10 ([BN20, Remark 4.2]). Let $g \in \mathbb{N}$ and $\mathbf{k} = (k_1, ..., k_g)$. With Lemma III.5 we can rewrite the jewel base $\mathcal{D}_{\text{💎},\mathbf{k}}(n)$, as

$$\mathcal{D}_{\text{💎},\mathbf{k}}(n) = \Bigg\{ X \in (\mathcal{M}_n^{sa})^{\sum_{i=1}^{g}(k_i-1)} : \\ \sum_{i=1}^{g} \sum_{j=1}^{k_i-1} (I_{k_i}^{\otimes(i-1)} \otimes V_j^{(k_i)} \otimes I_{k_i}^{\otimes(g-i)}) \otimes X_{i,j} \leq I_{(\prod_{s=1}^{g} k_i)n} \Bigg\}$$

for $n \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^g$.

Remark IV.11. Let $g \in \mathbb{N}$. The matrix jewel is a generalization of the **matrix diamond** $\mathcal{D}_{\diamond,g}$ defined by

$$\mathcal{D}_{\diamond,g}(n) = \left\{ X \in (\mathcal{M}_n^{sa})^g : \sum_{i=1}^{g} \varepsilon_i X_i \leq I_n \quad \forall \varepsilon \in \{-1,+1\}^g \right\}.$$

Since

$$V_1^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we can choose $\mathbf{k} = (k_1, ..., k_g) = (2, ..., 2)$ to see the matrix jewel is indeed a generalization of the matrix diamond. In some cases the matrix diamond is enough to get connections to the Quantum Information Theory ([BN22]). The next example can seen as an example for both the matrix jewel $\mathcal{D}_{\text{💎},(2,2)}$ and the matrix diamond $\mathcal{D}_{\diamond,2}$.

Example IV.12. Let $\mathbf{k} = (2, 2)$. For fixed $n \in \mathbb{N}$ we want to calculate

$$\mathcal{D}_{\text{💎},(2,2)}(n) = \left\{ (X_1, X_2) \in (\mathcal{M}_n^{sa})^2 : \left(V_1^{(2)} \otimes I_2\right) \otimes X_1 + \left(I_2 \otimes V_1^{(2)}\right) \otimes X_2 \leq I_{4n} \right\}.$$

For this, we first calculate

$$V_1^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to get for the left-hand side of the inequality

$$\begin{aligned} &\operatorname{diag}(1,1,-1,-1) \otimes X_1 + \operatorname{diag}(1,-1,1,-1) \otimes X_2 \\ &= \operatorname{diag}(X_1 + X_2, X_1 - X_2, -X_1 + X_2, -X_1 - X_2). \end{aligned}$$

Thus,

$$\mathcal{D}_{\diamondsuit,(2,2)}(n) = \left\{ (X_1, X_2) \in (\mathcal{M}_n^{sa})^2 : \pm X_1 \pm X_2 \le I_n, \pm X_1 \mp X_2 \le I_n \right\}.$$

2.2 The Matrix Jewel at the First Level

We want to discuss and visualize some matrix jewel at the first level. We know that they are objects from convex analysis and we can specify that they are polytopes. To see this, we start with the simplest case, namely that k is just a fixed, natural number (and not a tuple of numbers).

Remark IV.13. For a fixed $k \in \mathbb{N}$ we can see that $\mathcal{D}_{\diamondsuit,k}(1) \subseteq \mathbb{R}^{k-1}$ and recall

$$\begin{aligned} \mathcal{D}_{\diamondsuit,k}(1) &= \left\{ x \in \mathbb{R}^{k-1} : \sum_{j=1}^{k-1} V_j^{(k)} \cdot x_j \le I_k \right\} \\ &= \left\{ x \in \mathbb{R}^{k-1} : -\frac{2}{k} \sum_{j=1}^{k-1} \operatorname{diag} \underbrace{(1, ..., 1, \overset{j\text{-th entry}}{\overset{\downarrow}{1-k}}, 1, ..., 1)}_{k \text{ entries}} x_j \le I_k \right\}. \end{aligned}$$

The next lemma shows that $\mathcal{D}_{\diamondsuit,k}(1)$ is bounded. One can be argued with the linear independence of the defining matrices of the free spectrahedra but we show it more detailed.

Lemma IV.14. *The jewel base $\mathcal{D}_{\diamondsuit,k}(1) \subseteq \mathbb{R}^{k-1}$, $k \in \mathbb{N}$, is a polytope.*

Proof. With the Definition I.1 we directly see, that the jewel base is a polyhedron. Since by Lemma I.2 a bounded polyhedron is a polytope we show that $\mathcal{D}_{\diamondsuit,k}(1)$ is bounded via induction. Since $\mathcal{D}_{\diamondsuit,1}(1)$ is an empty set, we start with $\mathcal{D}_{\diamondsuit,2}$. By

$$\mathcal{D}_{\diamondsuit,2}(1) = \{x \in \mathbb{R} : \operatorname{diag}(-x, x) \le I_2\}$$

it follows that $-1 \le x \le 1$ and thus $\mathcal{D}_{\diamondsuit,2}(1)$ is bounded. Assume now that $\mathcal{D}_{\diamondsuit,k}$ is bounded for fixed $k \ge 2$. We calculate

$$\mathcal{D}_{\diamondsuit,k+1}(1) = \left\{ x \in \mathbb{R}^k : -\frac{2}{k+1} \sum_{j=1}^{k} \operatorname{diag} \underbrace{(1, ..., 1, 1 - \overset{j\text{-th entry}}{\overset{\downarrow}{(k+1)}}, 1, ..., 1)}_{k+1 \text{ entries}} x_j \le I_{k+1} \right\}.$$

In the last column we have the inequality $-2/(k+1) \sum_{j=1}^{k} x_j \le 1$. Thus, we get by the induction hypothesis that x_{k+1} is bounded from above. In the penultimate column we have $-2/(k+1)(\sum_{j=1}^{k-1} x_j - kx_{j+1}) \le 1$ whereby x_{k+1} is bounded from below by the same argument. Thus, $\mathcal{D}_{\diamondsuit,k+1}$ is bounded, which proves the claim. $\square$

An important property of polytopes is that they can be represented by finitely many extreme points. We can calculate them for the matrix jewel at level one.

Lemma IV.15 ([BN20, Lemma 4.3])**.** *Let $k \in \mathbb{N}$. The extreme points of the jewel base $\mathcal{D}_{\diamondsuit,k}(1) \in \mathbb{R}^{k-1}$ are*

$$\begin{aligned} x_i^{(k)} &:= -\frac{k}{2} e_i && \text{for } i \in [k-1] \text{ and} \\ x_k^{(k)} &:= \frac{k}{2} \underbrace{(1, ..., 1)}_{k-1 \text{ times}} \end{aligned}$$

where e_i are the elements of the standard orthonormal basis in $\mathbb{R}^{k-1}$.

Proof. By Lemma IV.14 we know that $\mathcal{D}_{\diamondsuit,k}(1)$ is a polyhedron. The hyperplanes $(v_1(\varepsilon), ..., v_{k-1}(\varepsilon))_{\varepsilon=1}^{k}$ are such that each $k-1$ of them linearly span $\mathbb{R}^{k-1}$. Thus, we can use [Bar02, II.(4.2)] and show that each point fulfills $k-1$ of the constraints with equality: We calculate for a fixed $\varepsilon \in [k]$

$$\sum_{j=1}^{k-1} V_j(\varepsilon) \left(x_i^{(k)}\right)_j = \sum_{j=1}^{k-1} V_j(\varepsilon) \left(-\frac{k}{2} e_i\right)_j = -\frac{k}{2} V_i(\varepsilon) = 1 - k\delta_{\varepsilon,i}$$

for $i \in [k-1]$ and

$$\sum_{j=1}^{k-1} V_j(\varepsilon) \left(x_k^{(k)}\right)_j = \sum_{j=1}^{k-1} V_j(\varepsilon) \frac{k}{2} (1, ..., 1)_j = 1 - k \left(1 - \sum_{j=1}^{k-1} \delta_{\varepsilon,j}\right) = 1 - k\delta_{\varepsilon,k}$$

which proves the claim. □

Example IV.16. With the explicit representation of the extreme points by Lemma IV.15, we want to visualize $\mathcal{D}_{\diamondsuit,k}(1)$ for $k \in [4]$. We recognize that there are exactly k extreme points for $\mathcal{D}_{\diamondsuit,k}(1) \subseteq \mathbb{R}^{k-1}$. Thus, the objects are not only polytopes but also simplexes. Furthermore, it is now easy to visualize $\mathcal{D}_{\diamondsuit,k}(1)$ for small $k \in \mathbb{N}$. Since the extreme points of $\mathcal{D}_{\diamondsuit,2}(1)$ are $x_1 = -1$ and $x_2 = 1$, the set is an interval between -1 and 1. For $k = 3$, we can calculate the extreme points

$$x_1 = -\frac{3}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_2 = -\frac{3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad x_3 = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $k = 4$ we get the extreme points

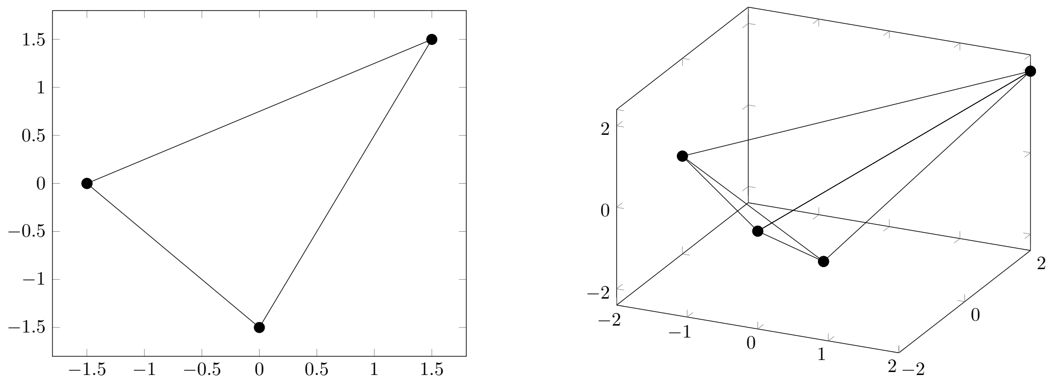
$$x_1 = -2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = -2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x_3 = -2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad x_4 = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The objects $\mathcal{D}_{\diamondsuit,3}(1)$ and $\mathcal{D}_{\diamondsuit,4}(1)$ are visualized in Figure IV.1.

We can ask how the first level of the matrix jewel base differs when considering $\mathbf{k} \in \mathbb{N}^g$ $(g \in \mathbb{N})$ instead of $k \in \mathbb{N}$.

Lemma IV.17. *For $g \in \mathbb{N}$ and $\boldsymbol{k} = (k_1, ..., k_g)$ we get*

$$\mathcal{D}_{\diamondsuit,\boldsymbol{k}}(1) = \mathcal{D}_{\diamondsuit,k_1}(1) \oplus ... \oplus \mathcal{D}_{\diamondsuit,k_g}(1).$$



(a) $\mathcal{D}_{\diamondsuit,3}(1)$ (b) $\mathcal{D}_{\diamondsuit,4}(1)$

Figure IV.1.: $\mathcal{D}_{\diamondsuit,k}(1)$ for $k \in [2,3]$.

Proof. Since by Lemma IV.14 $\mathcal{D}_{\diamondsuit,k_i}(1)$ is a polytope for all $i \in [g]$ and $\mathcal{D}_{\diamondsuit,k_i}(1) = \mathcal{W}_{\max}(\mathcal{D}_{\diamondsuit,k_i}(1))$ it follows from Lemma III.8

$$\begin{aligned}\mathcal{D}_{\diamondsuit,\mathbf{k}}(1) &= \mathcal{W}_{\max}(\mathcal{D}_{\diamondsuit,k_1}(1) \oplus \ldots \oplus \mathcal{D}_{\diamondsuit,k_g}(1)) \\ &= \mathcal{W}_{\max}(\mathcal{D}_{\diamondsuit,k_1}(1)) \hat{\oplus} \ldots \hat{\oplus} \mathcal{W}_{\max}(\mathcal{D}_{\diamondsuit,k_g}(1)) = \mathcal{D}_{\diamondsuit,k_1}(1) \oplus \ldots \oplus \mathcal{D}_{\diamondsuit,k_g}(1).\end{aligned}$$

□

With the preceding lemma we can also express $\mathcal{D}_{\diamondsuit,\mathbf{k}}(1)$ for $\mathbf{k} \in \mathbb{N}^g$ $(g \in \mathbb{N})$ by extreme points.

Example IV.18. Let $g \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}^g$, then $\mathcal{D}_{\diamondsuit,\mathbf{k}}(1)$ can be represented by a direct sum. We recall with Lemma I.10 that we can use the extreme points of $\mathcal{D}_{\diamondsuit,k_i(1)}$ $(i \in [g])$ to calculate the extreme points of $\mathcal{D}_{\diamondsuit,\mathbf{k}}(1)$. For

$$\mathcal{D}_{\diamondsuit,(2,2)}(1) = \mathcal{D}_{\diamondsuit,2}(1) \oplus \mathcal{D}_{\diamondsuit,2}(1)$$

we can use the results of Example IV.16 to calculate the extreme points

$$\begin{pmatrix}-1\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\-1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

to get a square. Similar we get for $\mathcal{D}_{\diamondsuit,(2,2,2)}(1) = \mathcal{D}_{\diamondsuit,2}(1) \oplus \mathcal{D}_{\diamondsuit,2}(1) \oplus \mathcal{D}_{\diamondsuit,2}(1)$ an octahedron with extreme points

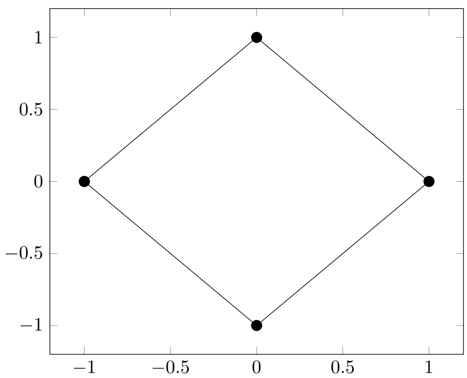
$$\begin{pmatrix}-1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\-1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\-1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

For $\mathcal{D}_{\diamondsuit,(2,3)}(1) = \mathcal{D}_{\diamondsuit,2}(1) \oplus \mathcal{D}_{\diamondsuit,3}(1)$ we get the extreme points

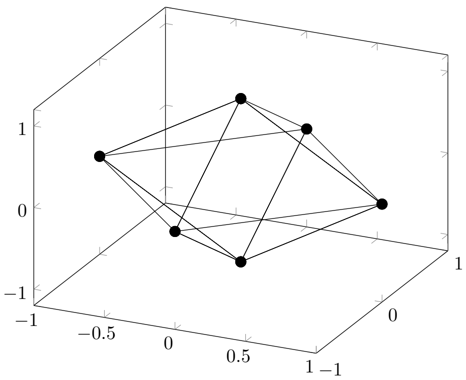
$$\begin{pmatrix}-1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\-3/2\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\-3/2\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}0\\3/2\\3/2\end{pmatrix}$$

and for $\mathcal{D}_{\diamondsuit,(3,2)}(1) = \mathcal{D}_{\diamondsuit,3}(1) \oplus \mathcal{D}_{\diamondsuit,2}(1)$

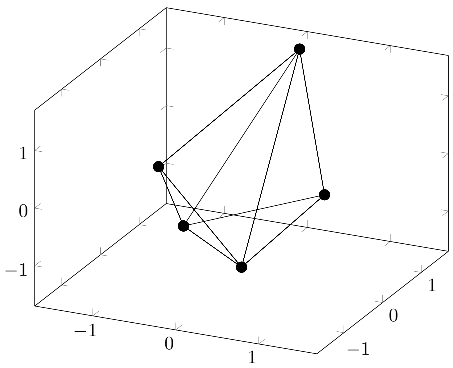
$$\begin{pmatrix}-3/2\\0\\0\end{pmatrix}, \quad \begin{pmatrix}3/2\\0\\0\end{pmatrix}, \quad \begin{pmatrix}3/2\\3/2\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\-1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}0\\0\\1\end{pmatrix}.$$



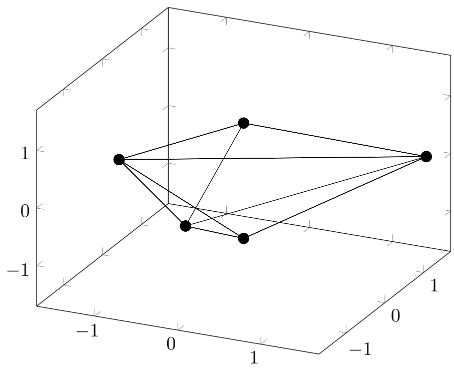
(a) $\mathcal{D}_{\diamondsuit,(2,2)}(1)$



(b) $\mathcal{D}_{\diamondsuit,(2,2,2)}(1)$



(c) $\mathcal{D}_{\diamondsuit,(2,3)}(1)$



(d) $\mathcal{D}_{\diamondsuit,(3,2)}(1)$

Figure IV.2.: $\mathcal{D}_{\diamondsuit,\mathbf{k}}(1)$ for different $\mathbf{k}$.

These objects are visualized in Figure IV.2. We recognize that the objects are still polytopes but no longer simplexes.

3 Inclusion of Free Spectrahedra and Matrix Jewel

We use this section to get a little summary and outlook to Chapter VI. Our main goal is to connect the Quantum Information Theory with the inclusion of free sets by employing the matrix jewel. To get more concrete, we show that we can express POVMs with the inclusion

$$\mathcal{D}_{\diamondsuit,k}(1) \subseteq \mathcal{D}_E(1)$$

for $k \in \mathbb{N}$ and given d-dimensional tuple $E \in (\mathcal{M}_d^{sa})^{k-1}$. Therefore, it is useful to express the polytope $\mathcal{D}_{\diamondsuit,k}$ by its extreme points. Analogously, we can express compatible POVMs with the inclusion

$$\mathcal{D}_{\diamondsuit,\mathbf{k}} \subseteq \mathcal{D}_E$$

for $\mathbf{k} = (k_1, ..., k_g) \in \mathbb{N}^g$, $g \in \mathbb{N}$ and given d-dimensional tuple $(E^{(1)}, ..., E^{(g)}) \in (\mathcal{M}_d^{sa})^{k_i-1}$. Therefore, we need the connection to the unital, completely positive maps. Furthermore, we connect the scalar $s = (s_1, ..., s_g) \in [0,1]^g$ which solves

$$\mathcal{D}_{\diamondsuit,\mathbf{k}}(1) \subseteq \mathcal{D}_E(1) \qquad \Rightarrow \qquad s\mathcal{D}_{\diamondsuit,\mathbf{k}} \subseteq \mathcal{D}_E$$

with the amount to make POVMs compatible, where again $\mathbf{k} = (k_1, ..., k_g) \in \mathbb{N}^g$, $g \in \mathbb{N}$ and given a d-dimensional tuple $(E^{(1)}, ..., E^{(g)}) \in (\mathcal{M}_d^{sa})^{k_i - 1}$. This scaling problem can be solved by an SDP ([HKM12, section 4.4]).

Notation IV.19. Let $d \in \mathbb{N}$ the dimension, $g \in \mathbb{N}$ and $\mathbf{k} = (k_1, ..., k_g)$. We notate the short-cut

$$\Delta(g, d, \mathbf{k}) := \Delta_{\mathcal{D}_{\diamondsuit,\mathbf{k}}}(g, d, \mathbf{k}).$$

Now, we have all preliminaries from the free spectrahedra and focus on Quantum Information Theory before combining the two topics in Chapter VI.

Chapter V.

Introduction to Quantum Information

1 Introduction in the Two-state System and Density Matrices

Many objects in mathematics can be easily ‘translated’ in the quantum information theory which is the theory about states in a quantum system. In this section, we will give an idea of quantum systems by looking at one of the simplest ones, the two-state system. This theory and the generalization can be found in [Wil16, page 72-76]. The two states are just the standard base normally notated in the ‘ket’ notation:

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can represent a general qubit by

$$|\phi\rangle := \alpha\,|0\rangle + \beta\,|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. Thus, we can also write a qubit as

$$|\phi\rangle := \cos(\theta)\,|0\rangle + e^{i\varphi}\sin(\theta)\,|1\rangle$$

where $0 \leq \theta \leq \pi/2$ and $0 \leq \varphi \leq 2\pi$. This is the common representation of the Bloch sphere (see Figure V.1a). We can also represent the Bloch sphere by the Pauli matrices ([Wil16, Page 84]).

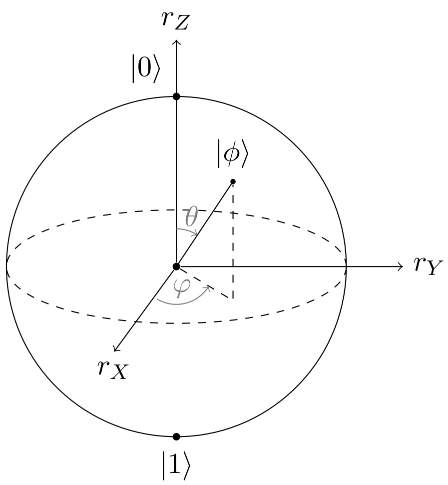
Definition V.1 ([Wil16, Page 82])**.** We define the Pauli matrices as

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

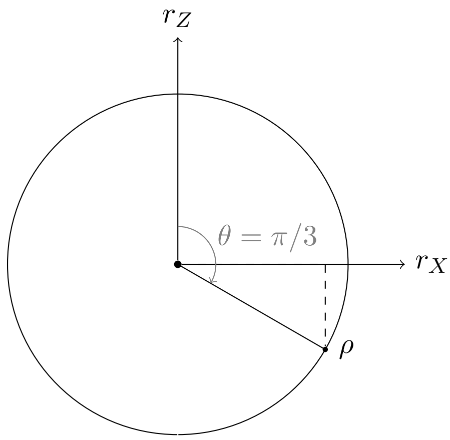
A qubit then has the same representation as a density operator.

Definition V.2 ([Wil16, Definition 4.13])**.** A **density operator** is a positive semi-definite operator with trace equal to one.

The next example is a density matrix that we use as a standard example in this thesis.



(a) Bloch sphere in $\mathbb{R}^3$.



(b) Bloch sphere of Example V.4.

Figure V.1.: Bloch sphere.

Example V.3. Let

$$\rho := \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix}$$

with $\theta \in [0, \pi/2]$. This is an example of a density matrix since the $\operatorname{tr}(\rho) = 1$ and $\det(\rho) = 0 \geq 0$.

Every density matrix ρ in the Bloch sphere can be represented by the Pauli matrices by finding radii r_X, r_Y and r_Z such that

$$\rho = \frac{1}{2}\left(I_2 + r_X\sigma_X + r_Y\sigma_Y + r_Z\sigma_Z\right)$$

while r_i is the radius in the σ_i direction for $i \in \{X, Y, Z\}$ with $r_X^2 + r_Y^2 + r_Z^2 \leq 1$. If $r_X^2 + r_Y^2 + r_Z^2 = 1$ then the point is on the boundary of the Bloch sphere.

Example V.4. We can represent ρ from the last example as

$$\rho = \frac{1}{2}\Big(I_2 + 2\sin(\theta)\cos(\theta)\sigma_X + \Big(\cos^2(\theta) - \sin^2(\theta)\Big)\sigma_Z\Big)$$

where $\theta \in [0, \pi/2]$. With this, we can represent ρ in the Bloch sphere. We recognize that we are not using σ_Y such that we can represent our example in two dimensional pictures. This example is visualized in Figure V.1b for $\theta = \pi/3$.

I would like to thank Dr. Andreas Buchheit for highlighting the helpful connection to the Pauli matrices and the density matrices.

2 POVMs

Quantum measurements are fundamental aspects in the QIT. An often used type of measurement is the positive operator-valued measure (POVM).

Definition V.5. Let $d \in \mathbb{N}$. We define the **effect operators** by

$$\mathrm{Eff}_d = \{E \in \mathcal{M}_d^{sa} : 0 \leq E \leq I_d\}.$$

We say E is an **effect** if $E \in \mathrm{Eff}_d$. For $k \in \mathbb{N}$, we define the corresponding **POVM** as a set of effects $\{E_j\}_{j\in[k]}, E_j \in \mathrm{Eff}_d$ for all $j \in [k]$, such that

$$\sum_{j\in[k]} E_j = I_d.$$

Remark V.6. In much of the literature POVMs are introduced by a collection of positive operators which sum up to the identity matrix. This is enough, since it follows that the the eigenvalues are also less than 1. Thus, the operators are effects as defined above.

Example V.7. For fixed $\theta \in [0, \pi/2]$ let

$$E = \left\{ \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \right\}.$$

Since E_1 and E_2 are self-adjoint matrices with eigenvalues 0 and 1, they are effects of dimension 2. They also sum up to the identity matrix. Thus, this set of effects is a POVM.

3 Jointly Measurable POVMs

For given POVMs one can ask, if one can present them by using only one POVM. There is an equivalent definition of jointly measurable POVMs by consider these POVMs as marginals of one other POVM (see [HMZ16, 2.1 and 2.2] for an introduction). The connection of the two different definitions is described for example in [GHK$^+$23, II.B].

Definition V.8 ([BN20, Definition 3.30])**.** Let $g \in \mathbb{N}$, $k_i \in \mathbb{N}$ $(i \in [g])$ and $d \in \mathbb{N}$. Let $\{\{E_j^{(i)}\}_{j\in[k_i]}\}_{i\in[g]}$ a collection of d-dimensional POVMs. The POVMs are **jointly measurable** or **compatible**, if there exists a d-dimensional **joint POVM** $\{R_{j_1,\dots j_g}\}_{j_i\in[k_i]}$ such that

$$E_v^{(u)} = \sum_{j_i\in[k_i], i\in[g]\setminus\{u\}} R_{j_1,\dots,j_{u-1},v,j_{u+1},\dots,j_g}$$

for all $u \in [g]$ and $v \in [k_u]$.

A collection of POVMs which is not compatible is called **not jointly measurable** or **incompatible**.

Notation V.9. Let $g \in \mathbb{N}$, $i \in [g]$ and $k_i \in \mathbb{N}$. Let $E_j^{(i)}$ be the j-th effect in the i-th POVM. In this thesis, we generally notate by $E^{(i)}$ the collection of the k_i effects of the i-th POVM, which means $E^{(i)} := \{E_j^{(i)}\}_{j\in k_i}$.

Remark V.10. To gain an understanding of the sum, consider only two compatible POVMs $E^{(1)}$ and $E^{(2)}$. We can reduce the sum to

$$E_v^{(1)} = \sum_{j_2\in[k_2]} R_{v,j_2} \qquad \text{and} \qquad E_v^{(2)} = \sum_{j_1\in[k_1]} R_{j_1,v}$$

while $\{R_{i,j}\}$ is a POVM. Then we can also visualize the sum in a rectangle as follows:

$$\begin{array}{ccccccccc} R_{1,1} & + & R_{1,2} & + & \dots & + & R_{1,k_2} & = & E_1^{(1)} \\ + & & + & & & & + & & \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ + & & + & & & & + & & \\ R_{k_1,1} & + & R_{k_2,2} & + & \dots & + & R_{k_1,k_2} & = & E_{k_1}^{(1)} \\ \parallel & & \parallel & & & & \parallel & & \\ E_1^{(2)} & & E_2^{(2)} & & \dots & & E_{k_2}^{(2)} & & \end{array}$$

Remark V.11. We simplify the preceding remark to the case $g = 2$ and $\mathbf{k} = (2, 2)$. To know if the POVMs $E^{(1)} = \{E_j^{(1)}\}_{j\in[2]}$ and $E^{(2)} = \{E_j^{(2)}\}_{j\in[2]}$ are compatible, we try to find a POVM which solve the equations

$$\begin{array}{ccccc} R_{1,1} & + & R_{1,2} & = & E_1^{(1)} \\ + & & + & & \\ R_{2,1} & + & R_{2,2} & = & E_2^{(1)} \\ \parallel & & \parallel & & \\ E_1^{(2)} & & E_2^{(2)} & & \end{array} \tag{V.1}$$

Since we use such examples in this thesis, we try to simplify these equations. Therefore, we recognize that for $i \in [2]$ the sets $E^{(i)}$ are POVMs and thus $E_2^{(i)} = I - E_1^{(i)}$. Furthermore, we can replace $R_{1,2} = E_1^{(1)} - R_{1,1}$ and $R_{2,1} = E_1^{(2)} - R_{1,1}$. For $R_{2,2}$ we calculate

$$R_{2,2} = E_2^{(2)} - R_{1,2} = I - E_1^{(2)} - E_1^{(1)} + R_{1,1}.$$

We can visualize this by

$$\begin{array}{ccccc} R_{1,1} & + & (E_1^{(1)} - R_{1,1}) & = & E_1^{(1)} \\ + & & + & & \\ (E_1^{(2)} - R_{1,1}) & + & (I - E_1^{(2)} - E_1^{(1)} + R_{1,1}) & = & I - E_1^{(1)} \\ \parallel & & \parallel & & \\ E_1^{(2)} & & I - E_1^{(2)} & & \end{array}$$

Now, we can reformulate the problem to: We try to find a self-adjoint matrix $R_{1,1}$ such that

$$R_{1,1}, \quad E_1^{(1)} - R_{1,1}, \quad E_1^{(2)} - R_{1,1} \quad \text{and} \quad I - E_1^{(2)} - E_1^{(1)} + R_{1,1}$$

are simultaneously positive semi-definite. An equivalent formulation using a block diagonal matrix is: We try to find a self-adjoint matrix $R_{1,1}$ such that

$$\operatorname{diag}\left(R_{1,1},\ E_1^{(1)} - R_{1,1},\ E_1^{(2)} - R_{1,1},\ I - E_1^{(2)} - E_1^{(1)} + R_{1,1}\right) \geq 0.$$

In the preceding remark we have seen how we can understand the definition of compatible POVMs. We have also seen very detailed the case where we have two effects with two POVMs respectively. In this thesis, we take this case as a guiding example since it is still easy enough to be accessible to a detailed analysis and at the same time illustrative but in general not trivial. However, if we choose one of the two POVMs to have all effects be proportional to the identity, we get a trivial case. You can verify this in the next example.

Example V.12. Take

$$E^{(1)} = \left\{ \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and the arbitrary POVM $E^{(2)} = \{E_1^{(2)}, E_2^{(2)}\}$. Then $\{E^{(1)}, E^{(2)}\}$ are compatible, since we can choose

$$R_{1,1} = R_{2,1} = \frac{1}{2}E_1^{(2)} \qquad \text{and} \qquad R_{1,2} = R_{2,2} = \frac{1}{2}E_2^{(2)}$$

to get

$$\begin{array}{ccccc} \frac{1}{2}E_1^{(2)} & + & \frac{1}{2}E_2^{(2)} & = & \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ + & & + & & \\ \frac{1}{2}E_1^{(2)} & + & \frac{1}{2}E_2^{(2)} & = & \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \parallel & & \parallel & & \\ E_1^{(2)} & & E_2^{(2)} & & \end{array}$$

Since the $E_j^{(2)}$ $(j \in [2])$ are self-adjoint and positive semi-definite, so are the $R_{i,j}$ $(i, j \in [2])$. Since the $E_j^{(2)}$ $(j \in [2])$ sum up to the identity matrix, the $R_{i,j}$ $(j \in [2])$ sum up to $E_i^{(1)}$ for a fixed $i \in [2]$.

Remark V.13. We recognize in Example V.12 that the POVM $\{1/2I_2, 1/2I_2\}$ is a good choice to get a compatible collection of POVMs. The reason is that it is independent of the state of the system. You can see this in the Bloch sphere (Figure V.1) since the matrix $1/2I_2$ is the origin. Actually, you can generalize the 2-dimensional case to a d-dimensional case. We say that we have a **trivial measurement** of dimension d, if the collection of POVMs are of the form $\{E^{(i)}\}_{i\in[g]} = \{\{E_j^{(i)}\}_{j\in[k_i]}\}_{i\in[g]}$ where $E_j^{(i)} := 1/k_i I_d$ and $k_i \in \mathbb{N}$ is the number of effects in each POVM of a collection of $g \in \mathbb{N}$ POVMs.

In the next example we see that not every choice of a collection of POVMs is compatible.

Example V.14. Recall Example V.7 for the angles 0 and fixed $\theta \in (0, \pi/2)$ to get the two POVMs

$$E^{(1)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$E^{(2)} = \left\{ \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \right\}.$$

We could solve the problem by solving systems of linear equations. We want to show that $E^{(1)}$ and $E^{(2)}$ are not compatible and use a proof of contradiction. We assume that $E^{(1)}$ and $E^{(2)}$ are compatible. Thus, there is a joint POVM $\{R_1, R_2, R_3, R_4\} := \{R_{1,1}, R_{1,2}, R_{2,1}, R_{2,2}\}$ such that the equations in (V.1) are fulfilled. Since $E_j^{(i)}$ only has real entries for all $i, j \in [2]$, we may suppose that R_k only has real entries for all

$k \in [4]$. Furthermore, we recall that every R_k ($k \in [4]$) is self-adjoint. Thus, every matrix R_k ($k \in [4]$) has three unknown, real variables $r_{1,1}^{(k)}, r_{1,2}^{(k)}, r_{2,2}^{(k)}$ such that

$$R_k = \begin{pmatrix} r_{1,1}^{(k)} & r_{1,2}^{(k)} \\ r_{1,2}^{(k)} & r_{2,2}^{(k)} \end{pmatrix}$$

for $k \in [4]$. With these, we can set up three systems of linear equations. We put the k-th variable in the k-th row. For the entries $r_{1,1}^{(k)}$, $k \in [4]$ extract from the equations in (V.1) the following system of linear equations:

$$\begin{array}{ccccccccc} r_{1,1}^{(1)} & + & r_{1,1}^{(2)} & + & 0 & + & 0 & = & 1 \\ 0 & + & 0 & + & r_{1,1}^{(3)} & + & r_{1,1}^{(4)} & = & 0 \\ r_{1,1}^{(1)} & + & 0 & + & r_{1,1}^{(3)} & + & 0 & = & \cos^2(\theta) \\ 0 & + & r_{1,1}^{(2)} & + & 0 & + & r_{1,1}^{(4)} & = & \sin^2(\theta). \end{array}$$

By using elementary row operations, we calculate

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & \cos^2(\theta) \\ 0 & 1 & 0 & 1 & \sin^2(\theta) \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & \cos^2(\theta) \\ 0 & 1 & 0 & 1 & \sin^2(\theta) \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Analogously, we calculate for the entries $r_{1,2}^{(k)}$ ($k \in [4]$)

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & \sin(\theta)\cos(\theta) \\ 0 & 1 & 0 & 1 & -\sin(\theta)\cos(\theta) \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & \sin(\theta)\cos(\theta) \\ 0 & 1 & 0 & 1 & -\sin(\theta)\cos(\theta) \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The case $r_{2,2}^{(k)}$ ($k \in [4]$) is very similar to $r_{1,1}^{(k)}$. For brevity, we notate $r_{1,1} := r_{1,1}^{(4)}$, $r_{1,2} := r_{1,2}^{(4)}$ and $r_{2,2} := r_{2,2}^{(1)}$. For R_k ($k \in [4]$), we get the representations

$$R_1 = \begin{pmatrix} \cos^2(\theta) + r_{1,1} & \sin(\theta)\cos(\theta) + r_{1,2} \\ \sin(\theta)\cos(\theta) + r_{1,2} & r_{2,2} \end{pmatrix},$$
$$R_2 = \begin{pmatrix} \sin^2(\theta) - r_{1,1} & -\sin(\theta)\cos(\theta) - r_{1,2} \\ -\sin(\theta)\cos(\theta) - r_{1,2} & -r_{2,2} \end{pmatrix},$$
$$R_3 = \begin{pmatrix} -r_{1,1} & -r_{1,2} \\ -r_{1,2} & \sin^2(\theta) - r_{2,2} \end{pmatrix} \quad \text{and} \quad R_4 = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{1,2} & \cos^2(\theta) + r_{2,2} \end{pmatrix}.$$

Since R_k ($k \in [4]$) are positive semi-definite, we know that the leading principal minors are non-negative. Thus, the first entry of each matrix has to be non-negative. From R_3 and R_4 it follows that $r_{1,1} = 0$. By the leading principal minors, the determinants of R_3 and R_4 have to be positive. With $r_{1,1} = 0$ we calculate $-r_{1,2} \geq 0$ such that follows $r_{1,2} = 0$. So far, we have

$$R_1 = \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & r_{2,2} \end{pmatrix}, \qquad R_2 = \begin{pmatrix} \sin^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & -r_{2,2} \end{pmatrix}$$

We use again that the determinants have to be non-negative. We calculate

$$\det(R_1) = \cos^2(\theta)r_{2,2} - \sin^2(\theta)\cos^2(\theta) = \cos^2(\theta)(r_{2,2} - \sin^2(\theta))$$

and

$$\det(R_2) = -\sin^2(\theta)r_{2,2} - \sin^2(\theta)\cos^2(\theta) = \sin^2(\theta)(-r_{2,2} - \cos^2(\theta)).$$

Since $\sin^2(\theta) \geq 0$ and $\cos^2(\theta) \geq 0$ we get $r_{2,2} = 0$. In conclusion, we have that $r_{1,1} = r_{1,2} = r_{2,2} = 0$ and

$$R_1 = \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & 0 \end{pmatrix},$$
$$R_3 = \begin{pmatrix} 0 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} \quad \text{and} \quad R_4 = \begin{pmatrix} 0 & 0 \\ 0 & \cos^2(\theta) \end{pmatrix}.$$

Again we use that the determinant of R_k ($k \in [4]$) has to be non-negative. But since

$$\det(R_1) = -\sin^2(\theta)\cos^2(\theta) < 0$$

for $\theta \in (0, \pi/2)$ we have a contradiction. Consequently, $\{E^{(1)}, E^{(2)}\}$ are not compatible.

Remark V.15. We see that the calculation in the second part of Example V.14 is very laborious. Thus, in general you would use a semi-definite programming (SDP). We will present this in Chapter VII.

4 Adding Noise to make POVMs Compatible

As you can see in Example V.14 that there are collections of POVMs which are not compatible. But they can make compatible by adding enough noise. Therefore, we take a convex combination of a POVM and a trivial measurement which we discussed in Remark V.13.

Definition V.16. Let $g, d \in \mathbb{N}$ and $k_i \in \mathbb{N}$ for $i \in [g]$. By **adding noise** of a noise level $s \in [0,1]^g$ to a collection of d-dimensional POVMs $E = \{E^{(i)}\}_{i \in [g]}$ we mean to replace E by a new collection of d-dimensional POVMs

$$\left\{\left\{s_i E_j^{(i)} + (1 - s_i) I_d / k_i\right\}_{j \in [k_i]}\right\}_{i \in [g]}.$$

Furthermore, for a collection of d-dimensional POVMs $E = \{E^{(i)}\}_{i \in [g]}$ we define the set

$$\Gamma_{\{E^{(i)}\}_{i \in [g]}} := \left\{s \in [0,1]^g : \{s_i E_j^{(i)} + (1 - s_i) I_d / k_i\}_{i \in [g]} \text{ is compatible}\right\}.$$

The set $\Gamma_{\{E^{(i)}\}_{i \in [g]}}$, defined as above, is similar to the compatibility region. This set has the advantage that it presents the noise level for a given collection of POVMs. We will calculate the set explicitly in Chapter VII. One remarkable property of the set is its convexity. This statement is similar to [BN18, Proposition 3.2].

Lemma V.17. *Let $g \in \mathbb{N}$, $i \in [g]$, $k_i \in \mathbb{N}$ and $d \in \mathbb{N}$. Let $\{E^{(i)}\}_{i \in [g]}$ a collection of d-dimensional POVMs. Then $\Gamma_{\{E^{(i)}\}_{i \in [g]}}$ is a convex set.*

Proof. Take $s, t \in \Gamma_{\{E^{(i)}\}_{i\in[g]}}$ and $\lambda \in [0,1]$. We want to show that $\lambda s + (1-\lambda)t \in \Gamma_{\{E^{(i)}\}_{i\in[g]}}$, that means

$$\left\{(\lambda s_i + (1-\lambda)t_i)E^{(i)} + (1 - (\lambda s_i + (1-\lambda)t_i))I_d/k_i\right\}_{i\in[g]}$$

is compatible. Since $s, t \in \Gamma_{\{E^{(i)}\}_{i\in[g]}}$, we know that

$$\left\{s_i E^{(i)} + (1-s_i)I_d/k_i\right\}_{i\in[g]} \qquad \text{and} \qquad \left\{t_i E^{(i)} + (1-t_i)I_d/k_i\right\}_{i\in[g]}$$

are both compatible. By definition this means that there are POVMs $\{R_{j_1,\ldots,j_g}\}_{j_i\in[k_i]}$ and $\{\widetilde{R}_{j_1,\ldots,j_g}\}_{j_i\in[k_i]}$ such that

$$s_u E_v^{(u)} + (1-s_u)I_d/k_u = \sum_{j_i\in[k_i], i\in[g]\setminus\{u\}} R_{j_1,\ldots,j_{u-1},v,j_{u+1},\ldots,j_g}$$

and

$$t_u E_v^{(u)} + (1-t_u)I_d/k_u = \sum_{j_i\in[k_i], i\in[g]\setminus\{u\}} \widetilde{R}_{j_1,\ldots,j_{u-1},v,j_{u+1},\ldots,j_g}.$$

By using this representation, we can calculate

$$\begin{aligned}
&(\lambda s_u + (1-\lambda)t_u)E^{(u)} + (1 - (\lambda s_u + (1-\lambda)t_u))I_d/k_u \\
&= \lambda s_u E^{(u)} + \lambda(1-s_u)I_d/k_u + (1-\lambda)t_u E^{(u)} + (1-\lambda)(1-t_u)I_d/k_u \\
&= \left(\sum_{j_i\in[k_i], i\in[g]\setminus\{u\}} \lambda R_{j_1,\ldots,j_{u-1},v,j_{u+1},\ldots,j_g} + (1-\lambda)\widetilde{R}_{j_1,\ldots,j_{u-1},v,j_{u+1},\ldots,j_g}\right).
\end{aligned}$$

But since $\{R_{j_1,\ldots,j_g}\}_{j_i\in[k_i]}$ and $\{\widetilde{R}_{j_1,\ldots,j_g}\}_{j_i\in[k_i]}$ are POVMs, also

$$\{\lambda R_{j_1} + (1-\lambda)\widetilde{R}_{j_1}, \ldots, \lambda R_{j_g} + (1-\lambda)\widetilde{R}_{j_g}\}_{j_i\in[k_i]}$$

is a POVM, so we see that this is a joint POVM for

$$\left\{(\lambda s_i + (1-\lambda)t_i)E^{(i)} + (1 - (\lambda s_i + (1-\lambda)t_i))I_d/k_i\right\}_{i\in[g]}$$

and thus the claim holds true. □

Example V.18. We take two POVMs from Example V.7 for $\theta = 0$ and $\theta = \pi/3$

$$E^{(1)} = \left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\} \quad \text{and} \quad E^{(2)} = \left\{\frac{1}{4}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & 3\end{pmatrix}, \frac{1}{4}\begin{pmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}\right\}$$

which are by Example V.14 not compatible. We are interested in $\Gamma_{\{E^{(1)},E^{(2)}\}}$. Therefore, we are writing an SDP in Mathematica. You can convince the idea and the program in Chapter A. With this we visualize the problem for several $s_1, s_2 \in [0,1]$ in Figure V.2. The curve is the boundary which separates $s = (s_1, s_2)$ for which s we can make $\{E^{(1)}, E^{(2)}\}$ compatible and for which we cannot. We call this curve **critical curve**. The set $\Gamma_{\{E^{(1)},E^{(2)}\}}$ is everything below the curve. We see that $\Gamma_{\{E^{(1)},E^{(2)}\}}$ is a convex set.

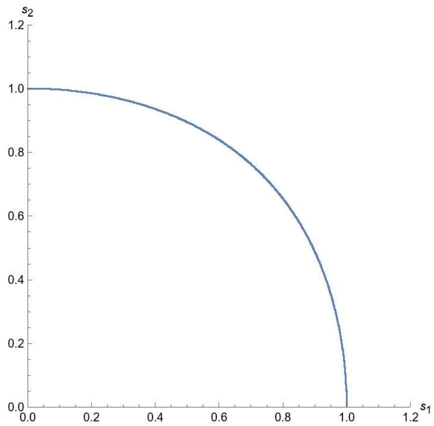


Figure V.2.: Visualization of Example V.18

Often we are not just interested in making a given collection of POVMs compatible but we want to know which noise level we need to make arbitrary collections of POVMs compatible. We only set the size and dimension of the collection.

Definition V.19 ([BN20, Definition 3.32])**.** Let $\mathbf{k} \in \mathbb{N}^g$, $d, g \in \mathbb{N}$. Then, we call

$$\Gamma(g,d,\mathbf{k}) := \\ \left\{ s \in [0,1]^g : s_i E^{(i)} + (1-s_i) I_d/k_i \text{ compatible } \forall \text{ POVM } E^{(i)} \in (\mathcal{M}_d^{sa})^{k_i} \right\}$$

the **(balanced) compatibility region** for g POVMs in dimension d with k_i outcomes, $i \in [g]$.

Remark V.20. Let $\mathbf{k} \in \mathbb{N}^g$, $d, g \in \mathbb{N}$. We can write $\Gamma(g,d,\mathbf{k})$ as the intersection of all $\Gamma_{\{E^{(i)}\}_{i\in[g]}}$ where $\{E^{(i)}\}_{i\in[g]}$ has dimension d and g POVMs with k_i effects in the i-th POVM, that means

$$\Gamma(g,d,\mathbf{k}) = \bigcap_{\{E^{(i)}\}_{i\in[g]}} \Gamma_{\{E^{(i)}\}_{i\in[g]}}.$$

Like in Lemma V.17 we can see that the compatibility region is convex.

Lemma V.21. *Let* $\boldsymbol{k} \in \mathbb{N}^g$*,* $d, g \in \mathbb{N}$*. Then* $\Gamma(g,d,\boldsymbol{k})$ *is convex.*

Proof. This statement follows directly from Lemma V.17, since we can choose the POVMs arbitrary. □

There are several options to define noise. In [BN18, Page 5] you can find some more types of compatibility regions. In this thesis we just look at the balanced compatibility region. Thus, when we speak about the compatibility region, we always mean the balanced one.

5 Commutativity and Compatibility of two POVMs

Commutativity is often discussed in Quantum Information. We will bring commutativity in connection with compatibility of two POVMs.

Definition V.22. Let $d \in \mathbb{N}$ and $k_i \in \mathbb{N}$ for $i \in [2]$. We say the two d-dimensional POVMs $\{E_1^{(1)}, ..., E_{k_1}^{(1)}\}$ and $\{E_{k_1}^{(2)}, ..., E_{k_2}^{(2)}\}$ **commute** if $[E_{j_1}^{(1)}, E_{j_2}^{(2)}] = 0$ for all $j_1 \in [k_1]$ and $j_2 \in [k_2]$.

Remark V.23. We do not consider that effects from the same POVM commute, that means it is possible that $[E_j^{(1)}, E_{j'}^{(1)}] \neq 0$ for $j, j' \in [k_1]$.

The next lemma described the connection between commutativity and compatibility of POVMs.

Lemma V.24 ([HW10, Section 2.4])**.** *Two commutative POVMs are compatible.*

Example V.25. We take the same matrices as in Example V.12 and Example V.14

1. Take $E^{(1)} = \{1/2 I_2, 1/2 I_2\}$ and the arbitrary POVM $E^{(2)} = \{E_1^{(2)}, E_2^{(2)}\}$. Since $1/2 I_2$ is commuting with every 2-dimensional matrix, we know with Lemma V.24 that $E^{(1)}$ and $E^{(2)}$ are compatible.
2. Let

$$E^{(1)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$E^{(2)} = \left\{ \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \right\}$$

for $\theta \in (0, \pi/2)$. These are not jointly measurable by Example V.14. With Lemma V.24 we know that they can't be commutative. We can verify this by calculate the commutator for the first tuple entries. The commutator is

$$\left[E_1^{(1)}, E_1^{(2)}\right] = E_1^{(1)} \cdot E_1^{(2)} - E_1^{(2)} \cdot E_1^{(1)} = \begin{pmatrix} 0 & \sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & 0 \end{pmatrix} \neq 0$$

for all $\theta \in (0, \pi/2)$ and thus, $E^{(1)}$ and $E^{(2)}$ are not commutative.

The converse direction of Lemma V.24 is not always true. In the next example we see a collection of POVMs which is compatible but not commutative.

Example V.26. Let

$$E^{(1)} = \left\{ \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } E^{(2)} = \left\{ \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \right\}.$$

We try to find an $s = (s_1, s_2) \in [0,1]^2$ such that $\widetilde{E}^{(i)} = s_i E^{(i)} + (1 - s_i) I_2 / k_i$ is compatible for $i \in [2]$. Fix $s = (s_1, s_2) \in [0,1]^2$ and calculate

$$\widetilde{E}_1^{(1)} := s_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1 - s_1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + s_1 & 0 \\ 0 & 1 - s_1 \end{pmatrix}$$

and

$$\widetilde{E}_1^{(2)} := s_2 \cdot \frac{1}{4} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} + \frac{1 - s_2}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 - s_2 & \sqrt{3}s_2 \\ \sqrt{3}s_2 & 2 + s_2 \end{pmatrix}.$$

The commutator

$$\left[\widetilde{E}_1^{(1)}, \widetilde{E}_1^{(2)}\right] = \frac{1}{4}\begin{pmatrix} 0 & \sqrt{3}s_1 s_2 \\ -\sqrt{3}s_1 s_2 & 0 \end{pmatrix}$$

is zero if and only $s_1 = 0$ or $s_2 = 0$. But by Lemma V.17 we know that $\Gamma_{\{E^{(1)},E^{(2)}\}}$ is a convex set, such that at least every convex combination of $(s_1, s_2) = (0, 1)$ and $(s_1, s_2) = (1, 0)$ is compatible.

Chapter VI.

Connection between POVMs and the Inclusion of Free Spectrahedra

1 Connection between a Single POVM and the Inclusion of Spectrahedra at the first Level

Now, we have all required preliminaries to see and show the connection between of POVMs and the inclusion of free spectrahedra. We start with the simplest case which is the inclusion of spectrahedra by using the first jewel base with $k \in \mathbb{N}$. We compare this inclusion to a single POVM.

Theorem VI.1 ([BN20, Proposition 5.1])**.** *For dimension $d \in \mathbb{N}$ and $k \in \mathbb{N}$, let $E = (E_1, .., E_{k-1}) \in (\mathcal{M}_d^{sa})^{k-1}$ and define $E_k := I_d - E_1 - ... - E_{k-1}$. Then, $\{E_1, ..., E_k\}$ is a POVM if and only if*

$$\mathcal{D}_{\diamond,k}(1) \subseteq \mathcal{D}_{2E-\frac{2}{k}I_d}(1).$$

Proof. By Lemma IV.14 the matrix jewel base $\mathcal{D}_{\diamond,k}(1)$ is a polytope. Thus, we only need to check the assertion on the extreme points as given by Lemma IV.15. For a fixed $i \in [k-1]$ we take

$$-\frac{k}{2}e_i \in \mathcal{D}_{2E-\frac{2}{k}I_d}(1).$$

This holds by definition if and only if

$$-\frac{k}{2}\left(2E_i - \frac{2}{k}I_d\right) \leq I_d$$

but this is the same as $E_i \leq 0$. Analogously, it holds

$$\frac{k}{2}(1, ..., 1)^T \in \mathcal{D}_{2E-\frac{2}{k}I_d}(1)$$

if and only if

$$\frac{k}{2}\sum_{i=1}^{k-1}\left(2E_i - \frac{2}{k}I_d\right) \leq I_d.$$

We calculate

$$\frac{k}{2}\sum_{i=1}^{k-1}\left(2E_i-\frac{2}{k}I_d\right)\leq I_d \qquad \Leftrightarrow \qquad k\sum_{i=1}^{k-1}E_i-(k-1)I_d\leq I_d$$

which is nothing else than $\sum_{i=1}^{k-1}E_i\leq I_d$. Since we defined $E_k=I_d-\sum_{i=1}^{k-1}E_i$ we get $\sum_{i=1}^{k}E_i=I_d$, so the conditions for a POVM (Definition V.5) is fulfilled. $\square$

Example VI.2. We recall Example V.7. For a fixed $\theta\in[0,\pi/2]$ we saw that

$$\{E,I_2-E\}=\left\{\begin{pmatrix}\cos^2(\theta) & \sin(\theta)\cos(\theta)\\ \sin(\theta)\cos(\theta) & \sin^2(\theta)\end{pmatrix},\begin{pmatrix}\sin^2(\theta) & -\sin(\theta)\cos(\theta)\\ -\sin(\theta)\cos(\theta) & \cos^2(\theta)\end{pmatrix}\right\}$$

is a POVM. Now, we want to prove it by using Theorem VI.1. We want to show that the extreme points of $\mathcal{D}_{\text{💎},2}(1)$ are elements of $\mathcal{D}_{2E-I_2}(1)$. By Example IV.16, we know that the extreme points of $\mathcal{D}_{\text{💎},2}(1)$ are $x=-1$ and $x=1$. We calculate

$$\mathcal{D}_{2E-I_2}(1)=\left\{x\in\mathbb{R}:I_2-x\begin{pmatrix}2\cos^2(\theta)-1 & 2\sin(\theta)\cos(\theta)\\ 2\sin(\theta)\cos(\theta) & 2\sin^2(\theta)-1\end{pmatrix}\geq 0\right\}.$$

Since the matrices

$$I_2\pm 1\begin{pmatrix}2\cos^2(\theta)-1 & 2\sin(\theta)\cos(\theta)\\ 2\sin(\theta)\cos(\theta) & 2\sin^2(\theta)-1\end{pmatrix}$$

have the eigenvalues 0 and 2, the matrices are positive semi-definite. Thus, the inclusion $\mathcal{D}_{\text{💎},2}(1)\subseteq\mathcal{D}_{2E-I_2}(1)$ holds and $\{E,I_2-E\}$ is a POVM.

2 Connection between POVMs and the Inclusion of Spectrahedra at the first Level

Normally, we do not have one single POVM, but a collection of POVMs. Once again, we compare this collection again with the inclusion of free spectrahedra, this time by using the first jewel base with $\mathbf{k}=(k_1,...,k_g)\in\mathbb{N}$ where $g\in\mathbb{N}$ instead of just a $k\in\mathbb{N}$.

Theorem VI.3 ([BN20, Theorem 5.2(1)])**.** *Let $d\in\mathbb{N}$ and $g\in\mathbb{N}$. For $i\in[g]$, take $E^{(i)}=(E_1^{(i)},...,E_{k_i-1}^{(i)})\in(\mathcal{M}_d^{sa})^{k_i-1}$ and set $E_{k_i}^{(i)}:=I_d-E_1^{(i)}...-E_{k_i-1}^{(i)}$. Let $\boldsymbol{k}=(k_1,...,k_g)$ and define*

$$E:=\left(2E^{(1)}-\frac{2}{k_1}I_d,...,2E^{(g)}-\frac{2}{k_g}I_d\right).$$

It holds $\mathcal{D}_{\text{💎},\boldsymbol{k}}(1)\subseteq\mathcal{D}_E(1)$ if and only if $\{E_1^{(i)},...,E_{k_i}^{(i)}\}$, $i\in[g]$, are POVMs.

Proof. We know with Lemma IV.17

$$\mathcal{D}_{\text{💎},k}(1)=\mathcal{D}_{\text{💎},k_1}(1)\oplus...\oplus\mathcal{D}_{\text{💎},k_g}(1).$$

Since $\mathcal{D}_{\text{💎},k_i}(1)$ is a polytope for all $i\in[g]$ (Lemma IV.14), we know with Lemma III.9 that

$$\mathcal{D}_{\text{💎},k_1}(1)\oplus...\oplus\mathcal{D}_{\text{💎},k_g}(1)\subseteq\mathcal{D}_E(1)$$

if and only if $\mathcal{D}_{\text{💎},k_i}(1)\subseteq\mathcal{D}_{E_{k_i}}$ for all $i\in[g]$. But with Proposition VI.1 this holds if and only if $\{E_1^{(i)},...,E_{k_i}^{(i)}\}$ are POVMs. $\square$

Example VI.4. We recall again Example V.7 for $\theta = 0$ and $\theta = \pi/3$. This means, let

$$E^{(1)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \qquad \text{and} \qquad E^{(2)} = \left\{ \frac{1}{4} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \right\}.$$

In Example V.7 we showed that they are POVMs. The inclusion $\mathcal{D}_{\diamondsuit,(2,2)}(1) \subseteq \mathcal{D}_E(1)$ is visualized in Figure VI.1. You can see that $\mathcal{D}_{\diamondsuit,(2,2)}(1)$ is completely contained in $\mathcal{D}_E(1)$. The program of the visualization is contained in the Chapter A.

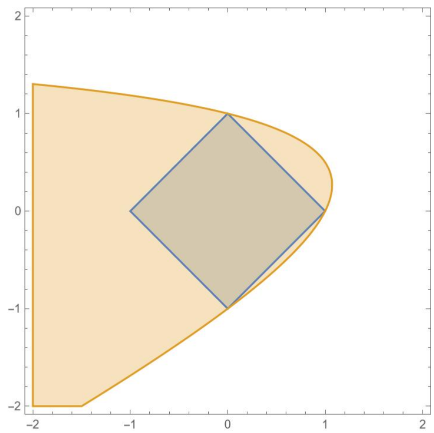


Figure VI.1.: Visualization of Example VI.4: The set $\mathcal{D}_E(1)$ is visualized in orange and $\mathcal{D}_{\diamondsuit,(2,2)}(1)$ in blue.

3 Connection between the Compatibility of POVMs and the Inclusion of Spectrahedra

Theorem VI.5 ([BN20, Theorem 5.2(2)])**.** *Let $d \in \mathbb{N}$ and $g \in \mathbb{N}$. For $i \in [g]$, take $E^{(i)} = (E_1^{(i)}, ..., E_{k_{i-1}}^{(i)}) \in (\mathcal{M}_d^{sa})^{k_i - 1}$ and set $E_{k_i}^{(i)} := I_d - E_1^{(i)} - ... - E_{k_i-1}^{(i)}$. Let $\boldsymbol{k} = (k_1, ..., k_g)$ and define*

$$E := \left(2E^{(1)} - \frac{2}{k_1} I_d, ..., 2E^{(g)} - \frac{2}{k_g} I_d \right).$$

It holds $\mathcal{D}_{\diamondsuit,\boldsymbol{k}} \subseteq \mathcal{D}_E$ if and only if $\{E_1^{(i)}, ..., E_{k_i}^{(i)}\}$, $i \in [g]$, are jointly measurable POVMs.

Proof. Denotate $V^{(k)} = (V_1^{(k)}, ..., V_{k-1}^{(k))})$ the diagonal matrices defining $\mathcal{D}_{\diamondsuit,k} = \mathcal{D}_{V^{(k)}}$. We show the equivalence by using the unital function

$$\Phi : \mathcal{OS}_{\left\{W_j^{(i)}\right\}_{i\in[g], j\in[k_i-1]}} \to \mathcal{M}_d, \qquad W_j^{(i)} \mapsto 2E_j^{(i)} - \frac{2}{k_i} I_d$$

where

$$W_j^{(i)} := I_{k_1} \otimes ... \otimes I_{k_{i-1}} \otimes V_j^{(k_i)} \otimes I_{k_{i+1}} \otimes ... \otimes I_{k_g}$$

for $i \in [g]$ and $j \in [k_i - 1]$. Since we can see the diagonal matrices as a subalgebra (recall Remark IV.9), we can write Φ as

$$\Phi : \mathcal{OS}_{\left\{w_j^{(i)}\right\}_{i\in[g],j\in[k_i-1]}} \to \mathcal{M}_d$$

where $w_j^{(i)} \subseteq \mathbb{C}^{k_i}$ is the diagonal of $W_j^{(i)}$ for $i \in [g]$ and $j \in [k_i - 1]$. This map Φ can express the inclusion $\mathcal{D}_{\diamondsuit,\mathbf{k}} \subseteq \mathcal{D}_E$ and is important for the translation to the jointly measurable POVMs.

We recognize that the $(W_j^{(i)})_{i\in[g],j\in[k_i-1]} = (I_{k_i}^{\otimes(i-1)} \otimes V_j^{(k_i)} \otimes I_{k_i}^{\otimes(g-i)})_{i\in[g],j\in[k_i-1]}$ and $I_{\prod_{i=1}^g k_i}$ are linearly independent. Furthermore, $\mathcal{D}_{\diamondsuit,\mathbf{k}}(1)$ is bounded. Thus, we can use Lemma IV.6 to deduce that $\mathcal{D}_{\diamondsuit,\mathbf{k}} \subseteq \mathcal{D}_E$ if and only if Φ is completely positive. It holds

$$\mathcal{OS}_{\left\{w_j^{(i)}\right\}_{i\in[g],j\in[k_i-1]}} \subseteq \mathbb{C}^{k_1\cdot\ldots\cdot k_g}.$$

By Arveson's extension theorem ([Pau02, Theorem 6.2]) we can extend Φ to a (unital) completely positive map $\widetilde{\Phi} : \mathbb{C}^{k_1\cdot\ldots\cdot k_g} \to \mathcal{M}_d$. With the theorem of Stinespring [Pau02, Theorem 3.11] the map $\widetilde{\Phi}$ is completely positive if and only if $\widetilde{\Phi}$ is positive. Thus, it remains to show that $\widetilde{\Phi}$ is a positive extension if and only if the $E^{(i)}$ are jointly measurable POVMs.

For $i \in [g]$ and $j \in [k_i - 1]$ we define $[\mathbf{k}] := \times_{i=1}^g [k_i]$ as Cartesian product to write $\{w_j^{(i)}\}_{i\in[g],j\in[k_i-1]}$ as the function

$$w_j^{(i)} : [\mathbf{k}] \to \mathcal{OS}_{\left\{w_j^{(i)}\right\}_{i\in[g],j\in[k_i-1]}}, \qquad \varepsilon \mapsto -\frac{2}{k_i} + 2\delta_{\varepsilon(i),j}.$$

We denote by $\varepsilon(i)$ the i-th tuple entry in $[\mathbf{k}]$. We define $g_\eta \in \mathbb{C}^{k_1\cdot\ldots\cdot k_g}$ by $\eta \in [\mathbf{k}]$ and $g_\eta(\varepsilon) = \delta_{\varepsilon,\eta}$. These vectors form a basis of $\mathbb{C}^{k_1\cdot\ldots\cdot k_g}$ Thus, we can rewrite the map as

$$w_j^{(i)}(\varepsilon) = -\frac{2}{k_i} \sum_{\eta\in[\mathbf{k}]} g_\eta(\varepsilon) + 2 \sum_{\eta\in[\mathbf{k}],\eta(i)=j} g_\eta(\varepsilon).$$

We recognize the following properties of g_η

$$\sum_{\eta\in[\mathbf{k}]} g_\eta(\varepsilon) = 1 \qquad \text{and} \qquad \sum_{\eta\in[\mathbf{k}],\,\eta(i)=j} g_\eta(\varepsilon) = \delta_{\varepsilon(i),j}. \tag{VI.1}$$

Now, let us define $R_\eta := \widetilde{\Phi}(g_\eta)$ and reformulate the remaining assertion of the proof to: The map $\widetilde{\Phi}$ is a positive extension of Φ if and only if the collection of matrices R_η is a joint POVM for $E^{(i)}$. For the remaining proof, we show each direction separately.

Let us assume that $\widetilde{\Phi}$ is a positive extension of Φ. We want to show that $R_{\eta\in[\mathbf{k}]}$ is a joint POVM. Since $\widetilde{\Phi}$ is positive, it follows that $R_\eta \geq 0$ for all $\eta \in [\mathbf{k}]$. With the property that $\widetilde{\Phi}$ is unital we can calculate

$$I_d = \widetilde{\Phi}(1) = \widetilde{\Phi}\left(\sum_{\eta\in[\mathbf{k}]} g_\eta\right) = \sum_{\eta\in[\mathbf{k}]} R_\eta.$$

to see that $\{R_\eta\}_{\eta\in[\mathbf{k}]}$ sum up to the identity matrix. Thus, the collection $\{R_\eta\}_{\eta\in[\mathbf{k}]}$ is a POVM. It remains to show that this is the joint POVM which can represent the POVMs $\{E_1^{(i)}, ..., E_{k_i}^{(i)}\}$. Therefor, we start with

$$\widetilde{\Phi}(w_j^{(i)}) = 2E_j^{(i)} - \frac{2}{k_i} I_d. \tag{VI.2}$$

On the other side we can write $w_j^{(i)}$ as $-2/k_i + 2\delta_{\varepsilon(i),j}$ to rewrite $w_j^{(i)}$ as the sum of $g_\eta(\varepsilon)$ like in (VI.1). That means

$$\widetilde{\Phi}\left(w_j^{(i)}\right) = \widetilde{\Phi}\left(-\frac{2}{k_i}\sum_{\eta\in[\mathbf{k}]} g_\eta(\varepsilon) + 2\sum_{\eta\in[\mathbf{k}],\,\eta(i)=j} g_\eta(\varepsilon)\right).$$

Now, we can use the linearity of $\widetilde{\Phi}$ and that R_η is defined as $\widetilde{\Phi}(g_\eta)$ to get

$$\widetilde{\Phi}\left(w_j^{(i)}\right) = -\frac{2}{k_i}\sum_{\eta\in[\mathbf{k}]} R_\eta + 2\sum_{\eta\in[\mathbf{k}],\,\eta(i)=j} R_\eta.$$

We already know that R_η sum up to the identity matrix, so in conclusion we get

$$2E_j^{(i)} - \frac{2}{k_i}I_d = \widetilde{\Phi}(w_j^{(i)}) = -\frac{2}{k_i}I_d + 2\sum_{\eta\in[\mathbf{k}],\,\eta(i)=j} R_\eta.$$

Thus, we have a joint POVM for $\{E^{(i)}\}_{i\in[g]}$, that is

$$E_j^{(i)} = \sum_{\eta\in[\mathbf{k}],\,\eta(i)=j} R_\eta.$$

Now, let $\{R_\eta\}_{\eta\in[\mathbf{k}]}$ be a joint POVM for $\{E_1^{(i)}, ..., E_{k_i}^{(i)}\}$. First we use the fact, that $\{R_\eta\}_{\eta\in[\mathbf{k}]}$ is a POVM. This tells us that $\widetilde{\Phi}$ is positive and unital. It remains to show that $\widetilde{\Phi}$ is an extension of Φ. By similar calculations as above we get

$$\widetilde{\Phi}(w_j^{(i)}) = -\frac{2}{k_i}I_d + 2\sum_{\eta\in[\mathbf{k}],\eta(i)=j} R_\eta.$$

But since

$$\sum_{\eta\in[\mathbf{k}],\eta(i)=j} R_\eta = E_j^{(i)}$$

we get

$$\widetilde{\Phi}(w_j^{(i)}) = 2E_j^{(i)} - \frac{2}{k_i}I_d$$

which means that $\widetilde{\Phi}$ is an extension of Φ. □

4 Equivalence between the Inclusion Set and the Compatibility Region

For a given collection of POVMs we may want to determine how large the noise level must be to make then compatible. For this problem, we again want to give a connection to the inclusion of free spectrahedra. In this setting we are asking how much we have to shrink the matrix jewel. Before we show the connection we need an auxiliary lemma.

Lemma VI.6. *Let $d, g \in \mathbb{N}$, $\boldsymbol{k} = (k_1, ..., k_g) \in \mathbb{N}^g$ and $s \in [0,1]^g$. For the d-dimensional POVM $\left\{E_1^{(i)}, ..., E_{k_i}^{(i)}\right\}$, $i \in [g]$, it holds*

$$(s_1^{\times(k_1-1)}, ..., s_g^{\times(k_g-1)})\mathcal{D}_{(2s_1E^{(1)}-\frac{2s_1}{k_1}I_d,...,2s_gE^{(g)}-\frac{2s_g}{k_g}I_d)} = \mathcal{D}_{(2E^{(1)}-\frac{2}{k_1}I_d,...,2E^{(g)}-\frac{2}{k_g}I_d)}.$$

Proof. With the Definition I.21 of the free spectrahedron we get:

$$\begin{aligned}
&\mathcal{D}_{(2s_1E^{(1)}-\frac{2s_1}{k_1}I_d,...,2s_gE^{(g)}-\frac{2s_g}{k_g}I_d)} \\
&= \bigcup_{n=1}^{\infty}\left\{X \in (\mathcal{M}_n^{sa})^{\sum_{i=1}^{g}(k_i-1)} : \sum_{i=1}^{g}\sum_{j=1}^{k_i-1}\left(2s_iE^{(i)} - \frac{2s_i}{k_i}I_d\right) \otimes X_{i,j} \leq I_{dn}\right\} \\
&= \bigcup_{n=1}^{\infty}\left\{X \in (\mathcal{M}_n^{sa})^{\sum_{i=1}^{g}(k_i-1)} : \sum_{i=1}^{g}\sum_{j=1}^{k_i-1}\left(2E_j^{(i)} - \frac{2}{k_i}I_d\right) \otimes s_iX_{i,j} \leq I_{dn}\right\}.
\end{aligned}$$

When we now multiply with $(s_1^{\times(k_1-1)}, ..., s_g^{\times(k_g-1)})$, we get

$$\bigcup_{n=1}^{\infty}\left\{X \in (\mathcal{M}_n^{sa})^{\sum_{i=1}^{g}(k_i-1)} : \sum_{i=1}^{g}\sum_{j=1}^{k_i-1}\left(2E_j^{(i)} - \frac{2}{k_i}I_d\right) \otimes X_{i,j} \leq I_{dn}\right\}$$

for the right-hand side, which is nothing else than

$$\mathcal{D}_{(2E^{(1)}-\frac{2}{k_1}I_d,...,2E^{(g)}-\frac{2}{k_g}I_d)}.$$

Thus, the assertion holds true. □

You can find the statement of the next Theorem in the proof of [BN20, Theorem 5.5]. The advantage of this theorem is, that we have the connection not only for an arbitrary collection of POVMs but for a given collection of POVMs.

Theorem VI.7. *Let $d \in \mathbb{N}$, $g \in \mathbb{N}$, $\boldsymbol{k} = (k_1, ..., k_g) \in \mathbb{N}^g$ and $s \in [0,1]^g$. For the given d-dimensional POVMs $\left\{E_1^{(i)}, ..., E_{k_i}^{(i)}\right\}$, $i \in [g]$ we have, that $s \in \Gamma_{\{E^{(i)}\}_{i\in[g]}}$ if and only if*

$$(s_1^{\times(k_1-1)}, ..., s_g^{\times(k_g-1)})\mathcal{D}_{\diamondsuit,\boldsymbol{k}} \subseteq \mathcal{D}_{(2E^{(1)}-\frac{2}{k_1}I_d,...,2E^{(g)}-\frac{2}{k_g}I_d)}$$

holds true.

Proof. With Theorem VI.5 we know,

$$\left\{s_iE^{(i)} + (1-s_i)I_d/k_i\right\}_{i\in[g]}$$

are compatible POVMs if and only if

$$\begin{aligned}\mathcal{D}_{\diamond,\mathbf{k}} &\subseteq \mathcal{D}_{(2(s_1E_1^{(1)}+(1-s_1)I_d/k_1)-\frac{2}{k_1}I_d,\ldots,2(s_gE_g^{(g)}+(1-s_g)I_d/k_g)-\frac{2}{k_g}I)} \\ &= \mathcal{D}_{(2s_1E_1^{(1)}-\frac{2s_1}{k_1}I_d,\ldots,2s_gE_g^{(g)}-\frac{2s_g}{k_g}I_d)}\end{aligned}$$

holds true. By Lemma VI.6 the inclusion is equivalent to

$$(s_1^{\times(k_1-1)},\ldots,s_g^{\times(k_g-1)})\mathcal{D}_{\diamond,\mathbf{k}} \subseteq \mathcal{D}_{(2E^{(1)}-\frac{2}{k_1}I_d,\ldots,2E^{(g)}-\frac{2}{k_g}I_d)}.$$

□

We could also ask how large the noise level must be for arbitrary POVMs of fixed size and dimension. We see that the compatibility region (recall Definition V.19) is the same as the inclusion set (recall Definition IV.3 and Notation IV.19).

Theorem VI.8 ([BN20, Theorem 5.5])**.** *Let $d, g \in \mathbb{N}$ and $\boldsymbol{k} \in \mathbb{N}^g$. Then*

$$\Gamma(g,d,\boldsymbol{k}) = \Delta(g,d,\boldsymbol{k}).$$

Proof. For an $s \in \mathbb{R}_+^g$ it holds $s \in \Gamma(g,d,\mathbf{k})$ if and only if $s_iE^{(i)} + (1-s_i)I_d/k_i$ is compatible for any d-dimensional POVMs with k_i outcomes for the i-th POVM with $i \in [g]$. So, when we have a POVM $\{E_1^{(i)},\ldots,E_{k_i}^{(i)}\}$ then

$$\left\{s_iE_1^{(i)} + (1-s_i)I_d/k_i,\ldots,s_iE_{k_i}^{(i)} + (1-s_i)I_d/k_i\right\}$$

is compatible. Set $E = (2E^{(1)} - 2/k_1I_d,\ldots,2E^{(g)} - 2/k_gI_d)$, then we know with Theorem VI.7 and Theorem VI.3 this is true if and only if the inclusion

$$\mathcal{D}_{\diamond,\mathbf{k}}(1) \subseteq \mathcal{D}_E(1) \quad\Rightarrow\quad (s_1^{\times(k_1-1)},\ldots,s_g^{\times(k_g-1)})\mathcal{D}_{\diamond,\mathbf{k}} \subseteq \mathcal{D}_E \tag{VI.3}$$

holds true. Since $A \mapsto 2A - (2/k)I_d$, $k \in \mathbb{N}$ is a bijective map on $\mathcal{M}_d^{sa}$, we can write any $F_i \in (\mathcal{M}_d^{sa})^{k_i-1}$ as $F_i = 2E^{(i)} - (2/k_i)I_d$ for $E^{(i)} \in (\mathcal{M}_d^{sa})^{k_i-1}$ for all $i \in [g]$. Thus, the implication VI.3 holds true if and only if

$$\mathcal{D}_{\diamond,\mathbf{k}}(1) \subseteq \mathcal{D}_F(1) \quad\Rightarrow\quad (s_1^{\times(k_1-1)},\ldots,s_g^{\times(k_g-1)})\mathcal{D}_{\diamond,\mathbf{k}} \subseteq \mathcal{D}_{(F^{(1)},\ldots,F^{(g)})}$$

for all $F \in (\mathcal{M}_d^{sa})^{\sum_{i=1}^g(k_i-1)}$. □

Chapter VII.

Further Ideas for a given POVM

In this chapter we look at the two POVMs

$$E^{(1)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$E^{(2)}(\theta) = \left\{ \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos^2(\theta) & \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \right\}$$

for $\theta \in [0, \pi/2]$. We want to know, how the set $\Gamma_{\{E^{(1)}, E^{(2)}(\theta)\}}$ looks like for different angles $\theta \in [0, \pi/2]$.

1 Choice of angle

You may ask why the angle is only between $[0, \pi/2]$. First, we recognize that $\cos^2(\theta) = \cos^2(\theta + \pi)$, $\sin(\theta)\cos(\theta) = \sin(\theta + \pi)\cos(\theta + \pi)$, $\sin^2(\theta) = \sin^2(\theta + \pi)$ such that $E^{(2)}(\theta) = E^{(2)}(\theta + z\pi)$ for all $z \in \mathbb{Z}$. You also can see this in the Bloch sphere (see Figure V.1a). Furthermore, we recognize that

$$\begin{aligned} &E^{(2)}\left(\theta + \pi/2\right) \\ &= \left\{ \begin{pmatrix} \cos^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos^2(\theta) & \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \sin^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \right\}. \end{aligned}$$

Thus, we know that $\{E^{(1)}, E^{(2)}(\theta)\}$ is compatible if and only if $\{E^{(1)}, E^{(2)}(\theta + \pi/2)\}$ is compatible. Consequently, it is enough to look at $\theta \in [0, \pi/2]$.

2 Using SDP

In this section, we approximate the critical curve like in Example V.18 but for several angles $\theta \in [0, 2\pi]$. For that, we replace the effects of the POVMs by effects $E^{(1)}$ and $E^{(2)}(\theta)$ with noise level $(s_1, s_2) \in [0,1]^2$, that means

$$\widetilde{E}^{(1)} = s_1 E^{(1)} + (1 - s_1) I_2/2 \qquad \text{and} \qquad \widetilde{E}^{(2)}(\theta) = s_2 E^{(2)}(\theta) + (1 - s_2) I_2/2.$$

We use the idea of Remark V.11. For the POVMs $\{\widetilde{E}_1^{(1)}, \widetilde{E}_2^{(1)}\}$ and $\{\widetilde{E}_1^{(2)}(\theta), \widetilde{E}_2^{(2)}(\theta)\}$, we try to find an

$$R := \begin{pmatrix} a_0 + d_3 & d_1 + id_2 \\ d_1 - id_2 & a_0 - d_3 \end{pmatrix}, \qquad a_0, d_1, d_2, d_3 \in \mathbb{R}$$

such that the matrix

$$M := \operatorname{diag}\left(R, \widetilde{E}_1^{(1)}(\theta) - R, \widetilde{E}_1^{(2)}(\theta) - R, I - \widetilde{E}_1^{(2)}(\theta) - \widetilde{E}_1^{(1)}(\theta) + R\right)$$

is positive semi-definite. By an SDP we can get for given θ and $(s1, s_2)$ a representation of M such that we maximize the smallest eigenvalue of M. We explain in detail the idea in Chapter A. We are interested in the critical curve, which separates the set $\Gamma_{\{E^{(1)},E^{(2)}(\theta)\}}$ from the set $[0,1]^2 \setminus \Gamma_{\{E^{(1)},E^{(2)}(\theta)\}}$ for different angles $\theta \in [0, \pi/2]$. We recall Example V.18 where we showed the critical curve for $\theta = \pi/3$. In Figure VII.1 we see the critical curve for $\theta \in \{\pi/12, \pi/6, \pi/4\}$.

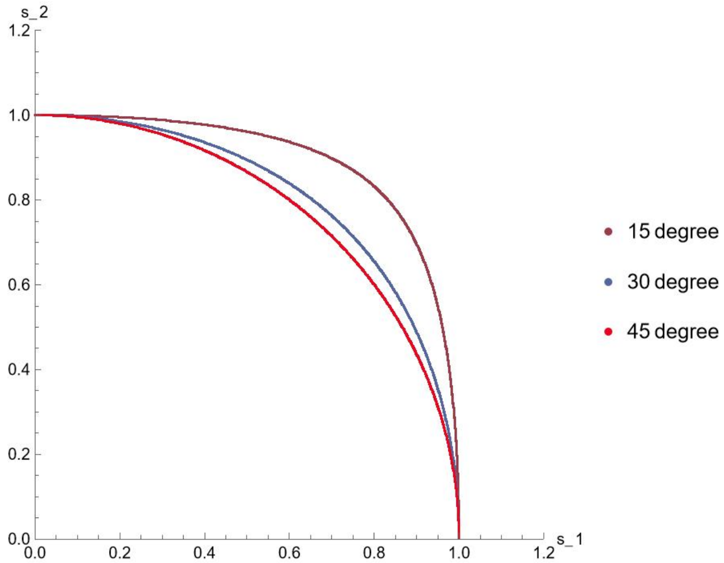


Figure VII.1.: Critical curve for $\theta \in \{\pi/12, \pi/6, \pi/4\}$.

We hypothesize that the set $\Gamma_{\{E^{(1)},E^{(2)}(\theta)\}}$ is for $\theta = \pi/4$ smaller than for $\theta \neq \pi/4$. To see it more directly, we plot for $\theta \in [0, \pi/2]$ the point of the critical curve where $s := s_1 = s_2$. In Figure VII.2 the value s is probably minimal in $\theta = \pi/4$. Since we can not be sure, we want to calculate the same problem analytically.

3 Using an Analytical Calculation

We recall the two POVMs

$$E^{(1)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$E^{(2)}(\theta) = \left\{ \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos^2(\theta) & \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \right\}$$

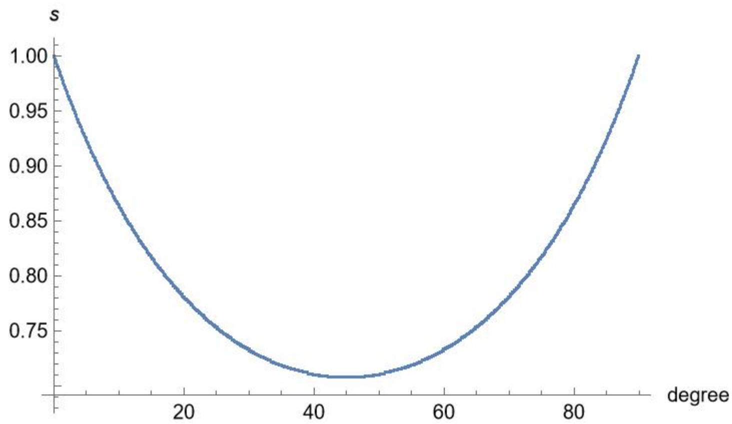


Figure VII.2.: The noise level s for different degree.

for $\theta \in [0, \pi/2]$. In this section we show that the critical curves which we showed approximated in Figure VII.1 can be calculated explicitly by

$$r_\theta(\phi) \cdot (\cos(\phi), \sin(\phi)) \qquad \text{for } \phi \in [0, \pi/2]$$

where

$$r_\theta : [0, \pi/2] \to [0, \infty), \qquad \phi \mapsto \frac{2}{\sqrt{1 + \sin(2\phi)\cos(2\theta)} + \sqrt{1 - \sin(2\phi)\cos(2\theta)}}$$

and $\theta \in [0, \pi/2]$.

3.1 Calculation of the Critical Curve

We replace the effects of the POVMs by effects with noise level $(s_1, s_2) \in [0, 1]^2$:

$$\begin{aligned}
\widetilde{E}_1^{(1)} &= \frac{1}{2} I_2 + \frac{1}{2} s_1 \sigma_Z, \\
\widetilde{E}_2^{(1)} &= \frac{1}{2} I_2 - \frac{1}{2} s_1 \sigma_Z, \\
\widetilde{E}_1^{(2)}(\theta) &= \frac{1}{2} I_2 + \frac{1}{2} s_2 \sin(2\theta)\sigma_X + \frac{1}{2} s_2 \cos(2\theta)\sigma_Z, \\
\widetilde{E}_2^{(2)}(\theta) &= \frac{1}{2} I_2 - \frac{1}{2} s_2 \sin(2\theta)\sigma_X - \frac{1}{2} s_2 \cos(2\theta)\sigma_Z,
\end{aligned}$$

where σ_i ($i \in \{X, Y, Z\}$) are the Pauli matrices (recall Definition V.1. For the POVMs $\{\widetilde{E}_1^{(1)}, \widetilde{E}_2^{(1)}\}$ and $\{\widetilde{E}_1^{(2)}(\theta), \widetilde{E}_2^{(2)}(\theta)\}$, let $\{R_{1,1}(\theta), R_{1,2}(\theta), R_{2,1}(\theta), R_{2,2}(\theta)\}$ be a set of effects such that

$$\begin{array}{ccccc}
R_{1,1}(\theta) & + & R_{1,2}(\theta) & = & \widetilde{E}_1^{(1)} \\
+ & & + & & \\
R_{2,1}(\theta) & + & R_{2,2}(\theta) & = & \widetilde{E}_2^{(1)} \\
\| & & \| & & \\
\widetilde{E}_1^{(2)}(\theta) & & \widetilde{E}_2^{(2)}(\theta) & &
\end{array}$$

is fulfilled. We choose to represent $R_{i,j}(\theta)$ ($i, j \in [2]$) with help of the Pauli matrices. For $R_{1,1}(\theta)$ there are a_0, d_1, d_2, d_3 such that

$$R_{1,1}(\theta) = a_0 I_2 + d_1\sigma_X + d_2\sigma_Y + d_3\sigma_Z.$$

With the representation of $R_{1,1}(\theta)$ we can calculate $R_{1,2}(\theta)$, $R_{2,1}(\theta)$ and $R_{2,2}(\theta)$ (recall V.11):

$$\begin{aligned} R_{1,2}(\theta) &= \left(\frac{1}{2} - a_0\right) I_2 - d_1\sigma_X - d_2\sigma_Y + \left(\frac{1}{2}s_1 - d_3\right)\sigma_Z, \\ R_{2,1}(\theta) &= \left(\frac{1}{2} - a_0\right) I_2 + \left(\frac{1}{2}s_2\sin(2\theta) - d_1\right)\sigma_X - d_2\sigma_Y + \left(\frac{1}{2}s_2\cos(2\theta) - d_3\right)\sigma_Z, \\ R_{2,2}(\theta) &= a_0 I_2 - \left(\frac{1}{2}s_2\sin(2\theta) - d_1\right)\sigma_X + d_2\sigma_Y - \left(\frac{1}{2}s_2\cos(2\theta) + \frac{1}{2}s_1 - d_3\right)\sigma_Z. \end{aligned}$$

The set $\{R_{1,1}(\theta), R_{1,2}(\theta), R_{2,1}(\theta), R_{2,2}(\theta)\}$ is then the joint POVM if the eigenvalues of all effects are non-negative. For $R_{1,1}(\theta)$ this means

$$a_0 \pm \sqrt{d_1^2 + d_2^2 + d_3^2} \geq 0.$$

We can follow that $a_0 \geq 0$. Furthermore, we can write this inequality as $\|d - w_{1,1}\| \leq a_0$ where $\|\cdot\|$ is the Euklidean norm and $d = (d_1, d_2, d_3)^T \in \mathbb{R}^3$. We use the same idea for the other effects. For a given vector $d = (d_1, d_2, d_3)^T \in \mathbb{R}^3$ we calculate

$$\begin{aligned} R_{1,1}(\theta) \geq 0 \quad &\Leftrightarrow \quad \|d - w_{1,1}\| \leq a_0 && \text{where} \quad w_{1,1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ R_{1,2}(\theta) \geq 0 \quad &\Leftrightarrow \quad \|d - w_{1,2}\| \leq \frac{1}{2} - a_0 && \text{where} \quad w_{1,2} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ s_1 \end{pmatrix}, \\ R_{2,1}(\theta) \geq 0 \quad &\Leftrightarrow \quad \|d - w_{2,1}\| \leq \frac{1}{2} - a_0 && \text{where} \quad w_{2,1} = \frac{1}{2}\begin{pmatrix} s_2\sin(2\theta) \\ 0 \\ s_2\cos(2\theta) \end{pmatrix}, \\ R_{2,2}(\theta) \geq 0 \quad &\Leftrightarrow \quad \|d - w_{2,2}\| \leq a_0 && \text{where} \quad w_{2,2} = \frac{1}{2}\begin{pmatrix} s_2\sin(2\theta) \\ 0 \\ s_2\cos(2\theta) + s_1 \end{pmatrix}. \end{aligned}$$

We recognize that $w_{2,1}$ and $w_{2,2}$ depends of θ. Furthermore, it follows that $a_0 \in [0, 1/2\pi]$.

Remark VII.1. We can interpret the conditions geometrically: We remark that we can consider $d_2 = 0$ since we are interested that the set (s_1, s_2) are as large as possible. Then we can interpret $w_{i,j}$ $(i, j \in [2])$ as vertices of a parallelogram in the d_1-d_3-plane. We can interpret the norm inequalities as circles around the vertices. The radius of the circles around $w_{1,1}$ and $w_{2,2}$ is a_0 and the radius of the circles around $w_{1,2}$ and $w_{2,1}$ is $1/2 - a_0$. There exists a point w in the intersection of the four circles

$$\|w_{2,2} - w_{1,1}\| \leq 2a_0 \qquad \text{and} \qquad \|w_{1,2} - w_{2,1}\| \leq 2\left(\frac{1}{2} - a_0\right).$$

Furthermore, we can calculate the point of intersection of diagonals

$$w = \frac{1}{2}\left(w_{2,2} + w_{1,1}\right) = \frac{1}{2}(w_{1,2} + w_{2,1}) = \frac{1}{4}\begin{pmatrix} s_2\sin(2\theta) \\ 0 \\ s_2\cos(2\theta) + s_1 \end{pmatrix}.$$

With the previous inequalities and the triangle inequality it follows

$$\|w_{2,2} - w_{1,1}\| \leq \|d - w_{1,1}\| + \|d - w_{2,2}\| \leq 2a_0$$

and

$$\|w_{1,2} - w_{2,1}\| \leq \|d - w_{1,2}\| + \|d - w_{2,1}\| \leq 2\left(\frac{1}{2} - a_0\right).$$

By insertion of $w_{i,j}$ $(i, j \in [2])$ we get

$$\|w_{2,2} - w_{1,1}\| = \frac{1}{2}\sqrt{s_2^2 \sin^2(2\theta) + s_2^2 \cos^2(2\theta) + 2s_2^2 \sin(2\theta)\cos(2\theta) + s_1^2} \leq 2a_0$$

or equivalently

$$s_1^2 + 2s_1 s_2 \cos(2\theta) + s_2^2 \leq 16a_0^2.$$

Analogously, we get

$$s_1^2 - 2s_1 s_2 \cos(2\theta) + s_2^2 \leq 16a_0^2.$$

We use the parameterization

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = r\begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} \qquad \text{where } r \geq 0 \text{ and } \phi \in [0, \pi/2]$$

to get the two inequalities

$$\begin{aligned} r^2(1 + \sin(2\phi)\cos(2\phi)) &\leq 16a_0^2 &&\Leftrightarrow && r \leq \frac{4a_0}{\sqrt{1 + \sin(2\phi)\cos(2\theta)}} \\ r^2(1 - \sin(2\phi)\cos(2\phi)) &\leq 16\left(\frac{1}{2} - a_0\right)^2 &&\Leftrightarrow && r \leq \frac{4(1/2 - a_0)}{\sqrt{1 - \sin(2\phi)\cos(2\theta)}}. \end{aligned}$$

We try to find the maximal value of r. Thus, we set

$$\frac{4a_0}{\sqrt{1 + \sin(2\phi)\cos(2\theta)}} = \frac{4(1/2 - a_0)}{\sqrt{1 - \sin(2\phi)\cos(2\theta)}}$$

and solve the equation for a_0. The equation is equivalent to

$$4a_0\left(\frac{1}{\sqrt{1 + \sin(2\phi)\cos(2\theta)}} + \frac{1}{\sqrt{1 - \sin(2\phi)\cos(2\theta)}}\right) = \frac{2}{\sqrt{1 - \sin(2\phi)\cos(2\theta)}}$$

which is

$$a_0 = \frac{1}{2}\frac{1}{1 + \frac{\sqrt{1 - \sin(2\phi)\cos(2\theta)}}{\sqrt{1 + \sin(2\phi)\cos(2\theta)}}}.$$

With this representation of a_0 we get the maximal value of r, that is when

$$r = \frac{4(1/2 - a_0)}{\sqrt{1 + \sin(2\phi)\cos(2\theta)} + \sqrt{1 - \sin(2\phi)\cos(2\theta)}}.$$

For a fixed $\phi \in [0, \pi/2]$, we conclude that the maximal value of r is given by

$$r_\theta := \frac{2}{\sqrt{1+\sin(2\phi)\cos(2\theta)} + \sqrt{1-\sin(2\phi)\cos(2\theta)}}$$

where $\theta \in [0, \pi/2]$. The critical curve is then given by

$$r_\theta(\phi) \cdot (\cos(\phi), \sin(\phi)) \qquad \text{for } \phi \in [0, \pi/2]$$

where

$$r_\theta : [0, \pi/2] \to [0, \infty), \qquad \phi \mapsto \frac{2}{\sqrt{1+\sin(2\phi)\cos(2\theta)} + \sqrt{1-\sin(2\phi)\cos(2\theta)}}$$

and $\theta \in [0, \pi/2]$. With this we get the exact solution

$$\Gamma_{\{E^{(1)}, E^{(2)}(\theta)\}} = \left\{ (s_1, s_2) \in (0, 1] : \sqrt{s_1^2 + s_2^2} \leq r_\theta \left(\cos^{-1} \left(\frac{s_1}{\sqrt{s_1^2 + s_2^2}} \right) \right) \right\} \cup \{0\}$$

for a fixed $\theta \in [0, \pi/4]$.

Remark VII.2. In the given example it was possible to calculate the set $\Gamma_{\{E^{(i)}\}_{i \in [g]}}$ for a given POVM $\{E^{(i)}\}_{i \in [g]}$ explicit. In general, this is not possible. Thus, one could use an SDP to get an approximated result.

3.2 Calculation of the Critical Points

Like previous, we set $s = s_1 = s_2$ for calculating how the maximal noise level s is changing for different angles. We use again the parametrization

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = r_\theta \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} \qquad \text{where } r_\theta \geq 0 \text{ and } \phi \in [0, \pi/2]$$

where $\theta \in [0, \pi/2]$. By setting $s_1 = s_2$ it follows that we have to set $\phi = \pi/4$. Then we get the map

$$[0, \pi/2] \to [0, 1], \qquad \theta \mapsto r_\theta \left(\frac{\pi}{4} \right) = \frac{\sqrt{2}}{\sqrt{1+\cos(2\theta)} + \sqrt{1-\cos(2\theta)}}$$

is the exact solution of Figure VII.2. We see that the minimum is given in the point $(\pi/4, \sqrt{2})$.

3.3 Connection to the Quarter Circle

We recall the exact solution

$$\Gamma_{\{E^{(1)}, E^{(2)}(\theta)\}} = \left\{ (s_1, s_2) \in (0, 1] : \sqrt{s_1^2 + s_2^2} \leq r_\theta \left(\cos^{-1} \left(\frac{s_1}{\sqrt{s_1^2 + s_2^2}} \right) \right) \right\} \cup \{0\}$$

where

$$r_\theta : [0, \pi/2] \to [0, \infty), \qquad \phi \mapsto \frac{2}{\sqrt{1+\sin(2\phi)\cos(2\theta)} + \sqrt{1-\sin(2\phi)\cos(2\theta)}}$$

and calculate

$$\Gamma_{\{E^{(1)}, E^{(2)}(\pi/4)\}} = \left\{ (s_1, s_2) \in [0, 1] : \sqrt{s_1^2 + s_2^2} \leq 1. \right\}$$

This set is the same as the quarter circle for $g = 2$:

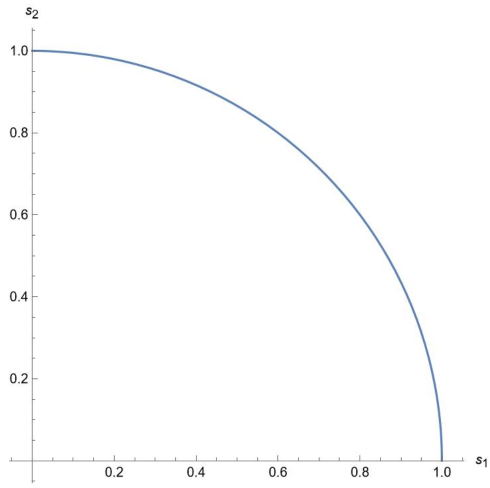


Figure VII.3.: The critical curve for $\theta = \pi/4$ is equal to the boundary of the Quarter Circle.

Definition VII.3. For $g \in \mathbb{N}$ we define the **quarter-circle** by

$$\mathrm{QC}_g := \left\{ s \in \mathbb{R}_+^g : \sum_{i=1}^{g} s_i^2 \leq 1 \right\}.$$

The curve is visualized in Figure VII.3. This is indeed the smallest set we could get for the compatibility set $\Gamma(2,2)$. In [BN18] Bluhm and Nechita have proven that the quarter-circle is a subset of the inclusion set. They generalize the assertion in [BN20].

Theorem VII.4 ([BN20, Theorem 7.2])**.** *Let $g, d \in \mathbb{N}$ and $\boldsymbol{k} = (k_1, ..., k_g) \in \mathbb{N}^g$. Then,*

$$\left(\frac{1}{(k_1 - 1)^2}, ..., \frac{1}{(k_g - 1)^2} \right) \mathrm{QC}_{\sum_{i=1}^{g}(k_i - 1)} \subseteq \Delta(g, d, \boldsymbol{k}).$$

The proof of Theorem VII.4 is based on the theory about the inclusion of free spectrahedra but with the connection to the POVMs this gives us information about the inclusion set.

Chapter A.

Appendix

For computation and visualization I use the program 'Mathematica', [Inc]. Every command I used can be found on the website `https://reference.wolfram.com/language/`.

Example VI.4

We want to plot the two sets $\mathcal{D}_{\diamondsuit,(2,2)}(1)$ and $\mathcal{D}_E(1)$ to see that

$$\mathcal{D}_{\diamondsuit,(2,2)}(1) \subseteq \mathcal{D}_E(1).$$

We start to present the code for the set

$$\mathcal{D}_{\diamondsuit,(2,2)} = \{(x_1, x_2) : I_4 - x_1 V \otimes I_2 - x_2 I_2 \otimes V \geq 0\}$$

where

$$V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which was calculated in Example IV.12.

```
V = {{1, 0}, {0, −1}};
RegionPlot[Eigenvalues[IdentityMatrix[4] − x1∗KroneckerProduct[V, IdentityMatrix[2] ] −
    x2∗KroneckerProduct[IdentityMatrix[2], V] ] ⪰0, {x1, −2, 2}, {x2, −2, 2}]
```

In the algorithm, we fix $x_1, x_2 \in [-2, 2]$ since we can not visualize an infinite space. For the set

$$\mathcal{D}_E(1) = \mathcal{D}_{2E_1^{(1)} - I_2, 2E_1^{(2)} - I_2}(1)$$

we first have to the define the two matrices

$$E_1^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad E_1^{(2)} = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$

We rename them as $\rho_1 := E_1^{(1)} - 2E_1^{(1)}$ and $\rho_2 := E_1^{(2)}$ and plot the set

$$\left\{(x_1, x_2) \in \mathbb{R}^2 : I_2 - x_1\rho_1 - x_2 * (2 * \rho_2 - I_2) \geq 0\right\}.$$

```
ρ1 = {{1, 0}, {0, 0}};
ρ2 = 2*Transpose[{{Cos[Pi/3], Sin[Pi/3]}}] . {{Cos[Pi/3], Sin[Pi/3]}} − IdentityMatrix[2];
RegionPlot[Eigenvalues[IdentityMatrix[2] − x1*ρ1 −
    x2*(2*ρ2 − IdentityMatrix[2])] ⪰ 0, {x1, −2, 2}, {x2, −2, 2}]
```

To plot the sets together you could use the command **Show** or put the inequalities together in the command **RegionPlot**.

How to solve the compatibility problem by an SDP

The documentation of the semi-definite optimization can be found in `https://reference.wolfram.com/language/ref/SemidefiniteOptimization.html`.

Preliminaries

First, we take two POVMs from Example V.14

$$E^{(1)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$E^{(2)} = \left\{ \begin{pmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \sin^2(\theta) & -\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \cos^2(\theta) \end{pmatrix} \right\}$$

where $\theta \in (0, \pi/2)$. By adding noise we can make the POVMs compatible. For the program it is enough to look at $E_1^{(1)}$ and $E_1^{(2)}$. We set

$$\rho_1 = s_1 E_1^{(1)} + (1 - s_1) I_2 \qquad \text{and} \qquad \rho_2 = s_1 E_1^{(2)} + (1 - s_2) I_2$$

for arbitrary s_1, s_2.

```
ρ1[s1_] = s1*{{1, 0}, {0, 0}} + (1 − s1)*DiagonalMatrix[{1/2, 1/2}];
ρ2[θ_, s2_] = s2*Transpose[{{Cos[θ], Sin[θ]}}] . {{Cos[θ],Sin[θ]}}
    + (1 − s2)*DiagonalMatrix[{1/2, 1/2}];
```

We use the idea of Remark V.11. We recall that it is enough to find an

$$R := \begin{pmatrix} a_0 + d_3 & d_1 + id_2 \\ d_1 - id_2 & a_0 - d_3 \end{pmatrix}, \qquad a_0, d_1, d_2, d_3 \in \mathbb{R}$$

such that the matrix

$$M := \operatorname{diag}\Big(R,\ E_1^{(1)} - R,\ E_1^{(2)} - R,\ I - E_1^{(2)} - E_1^{(1)} + R\Big)$$

is positive semi-definite.

```
R[a0_, d1_, d2_, d3_] = {{a0 + d3, d1+ I d2}, {d1 − I d2, a0− d3}};
M[θ_, a0_, d1_, d2_, d3_, s1_, s2_] = ArrayFlatten[{{R[a0, d1, d2, d3], 0, 0, 0}, {0, ρ2[θ, s1]
    − R[a0, d1, d2, d3], 0, 0}, {0,0, ρ1[s2] − R[a0, d1, d2, d3], 0}, {0, 0, 0,IdentityMatrix[2]
    +R[a0, d1, d2, d3] − ρ2[θ, s1] − ρ1[s2]}}];
```

SDP

The idea is to find a_0, d_1, d_2, d_3 which for given θ, s_1, s_2 maximize the smallest eigenvalue λ of the matrix M. We reformulate the problem to $\lambda I_8 \leq M$. To maximize λ, we minimize $-\lambda$ using a semi-definite optimization.

For the matrix M we can choose θ, s_1 and s_2 to get λ, a_0, d_1, d_2 and d_3. The numbers a_0, d_1, d_2 and d_3 are the calculated best choice to get the smallest possible λ. If $\lambda \geq 0$ then the two POVMs are compatible, otherwise they are not. In the algorithm we decide to set for example $\theta = \pi/3$, $s_1 = s_2 = 1/2$ like we did in Example V.18.

```
SemidefiniteOptimization[−λ, λIdentityMatrix[8] ⪯_{S^8_+} M[Pi/3, a0, d1, d2, d3, 0.5, 0.5],
    {λ, a0, d1, d2, d3}]
```

The result of this example is

$$\{\lambda \to 0.0792468, \mathrm{a0} \to 0.204247, \mathrm{d1} \to 0.108253, \mathrm{d2} \to 0., \mathrm{d3} \to 0.0625\}.$$

We see that $\lambda \geq 0$ and thus, the POVMs are compatible. We also have a possible representation for

$$R = \begin{pmatrix} a_0 + d_3 & d_1 + id_2 \\ d_1 - id_2 & a_0 - d_2 \end{pmatrix}$$

such that set $\{R,\ E_1^{(1)} - R,\ E_1^{(2)} - R,\ I - E_1^{(2)} - E_1^{(1)} + R\}$ is a POVM. To verify this, we can output the set of eigenvalues of M

$$\{0.512259, 0.512259, 0.329247, 0.329247, 0.0792472, 0.0792468, 0.0792467, 0.0792466\}$$

to see that every eigenvalue is positive.

Graphic for the Critical Curve for Different Angles

We want to visualize the critical curve for the angles $\theta \in \{\pi/12, \pi/6, \pi/4\}$ like in Chapter VII. To using the angles in a loop, we set the starting angle $\theta_0 = \pi/12$, the ending angle $\theta_m = \pi/4$ and the number of all angles $m = 3$. Since we can not test the program for every $(s_1, s_2) \in [0, 1]$, we take a discretization, that means we test the program for $(s_1, s_2) \in \{(1/i, 1/i : i \in [n]\}$ where $n \in \mathbb{N}$ fixed. We are choosing $n = 1000$.

```
n = 1000;
θ0= Pi/12;
θm= Pi/4;
m = 3;
diff = (θm − θ0)/(m − 1);
```

We can use the SDP to find out, if two POVMs for given (s_1, s_2) and angle θ are compatible. Since we do not need the whole representation of the best choice of matrix M, we read out the value of λ with the command **Part**. Furthermore, we are not interested in the concrete value of λ but if λ is positive or negative. Thus, we use the **HeavisideTheta** command which returns $\widetilde{\lambda} := 0$ for $\lambda < 0$ and $\widetilde{\lambda} := 1$ for $\lambda > 0$. We let us give out a table T with the values of θ, s_1, s_2 and $\widetilde{\lambda}$.

```
T = ParallelTable[{θ, s₁, s₂, HeavisideTheta[λ /.
    Part[SemidefiniteOptimization[−λ, λIdentityMatrix[8] ⪯_{S^8_+} , M[θ, a₀, d₁, d₂, d₃, s₁, s₂],
    {λ, a₀, d₁, d₂, d₃}], 1] ]}, {θ, θ₀, θₘ, diff}, {s₁, 0, 1, 1/n}, {s₂, 0, 1, 1/n}];
```

To plot the critical curve, we search for each s_1 from 0 to 1 the first value of s_2 for which $\widetilde{\lambda} = 0$. We are saving the values of s_1 in the first $n+1$ entries of vector1 and the values of s_2 in vector2. We are using the same idea in the horizontal direction, that means: For each s_2 from 0 to 1 we are looking for the first $\widetilde{\lambda} = 0$. These date are saved in the last $n+1$ entries of vector1 and vector2. Then we can plot the vectors with the command **ListPlot**.

```
vector1 = ConstantArray[1, 2 n + 2];
vector2 = ConstantArray[0, 2 n + 2];
Show[Table[
   For[i₁ = 1, i₁≤n + 1, i₁++,
    For[i₂ = 1, i₂≤n + 1, i₂++,
     If [T⟦(θ − θ₀)/diff + 1⟧⟦i₁⟧⟦i₂⟧⟦4⟧ == 0,
      vector1 = ReplacePart[vector1, i₁→T⟦(θ − θ₀)/diff + 1⟧⟦i₁⟧⟦i₂⟧⟦2⟧];
      vector2 = ReplacePart[vector2, i₁→T⟦(θ − θ₀)/diff + 1⟧⟦i₁⟧⟦i₂⟧⟦3⟧];
      Break[] ] ] ];
   For[i₂ = 1, i₂≤n + 1, i₂++,
    For[i₁ = 1, i₁≤n + 1, i₁++,
     If [T⟦(θ − θ₀)/diff + 1⟧⟦i₁⟧⟦i₂⟧⟦4⟧ == 0,
      vector1 = ReplacePart[vector1, i₂+ n + 1 →T⟦(θ − θ₀)/diff + 1⟧⟦i₁⟧⟦i₂⟧⟦2⟧];
      vector2 = ReplacePart[vector2, i₂+ n + 1 →T⟦(θ − θ₀)/diff + 1⟧⟦i₁⟧⟦i₂⟧⟦3⟧];
      Break[] ] ] ];
   ListPlot [Transpose@{vector1, vector2}], {θ, θ₀, θₘ, diff}]]
```

Graphic for the Critical Points where $s_1 = s_2$

Let $s := s_1 = s_2$. Then we get for all angle θ a critical point s such that the POVM is compatible for every noise level less than s and incompatible for all noise level greater s. We can choose the number of the angles num between 0 and $\pi/4$. With this value we can calculate the difference radians and degrees.

```
num = 1 + 90*10;
 diff1 = 90*1/(num − 1);
 diff2 = 2 Pi/360*diff1;
```

Since we want to use **Listplot** again, we declare two vectors v_θ and v_s, both of the length num. In vector v_θ we are saving the angles. In the vector v_s we will overwrite the calculated values of s.

```
v_θ= Range[0, 90, diff1 ];
v_s= ConstantArray[1, num];
```

Now, we are ready to use the SDP. We are showing the values θ, s and $\widetilde{\lambda}$ in a table. Remark, that $\widetilde{\lambda} \in \{0, 1\}$ is explained previously.

```
T = ParallelTable[{θ, s, HeavisideTheta[λ /.
    Part[SemidefiniteOptimization[−λ, λIdentityMatrix[8] ⪯_{S^8_+} M[θ, a₀, d₁, d₂, d₃, s, s],
    {λ, a₀, d₁, d₂, d₃}], 1] ]}, {θ, 0, Pi/2, diff2 }, {s, 0, 1, 1/n}];
```

At least we want to save the values of s in the vector v_s. Then we can plot the vectors v_θ and v_s with **ListPlot**.

```
For[θ = 1, θ ≤num, θ++,
 For[i = 1, i ≤n + 1, i++,
   If [T⟦θ⟧⟦i⟧⟦3⟧ == 0,
     v_s= ReplacePart[v_s, θ→T⟦θ⟧⟦i⟧⟦2⟧];
  Break[] ] ] ];
ListPlot [Transpose@{v_θ, v_s}]
```

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