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# Representations of Graph $C^*$-algebras

**submitted by**
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## Introduction

In this thesis, we examine the representation theory of graph $C^*$-algebras. Graph $C^*$-algebras were introduced in [8] in 1998 as a generalization of Cuntz-Krieger algebras introduced by Cuntz and Krieger in [5] in 1980, which in turn arose as a more generalized version of the Cuntz Algebra $\mathcal{O}_n$ introduced by Cuntz in [4] in 1977. A graph $E$ gives rise to a Cuntz-Krieger $E$-family $\{S, P\}$, via a set $P$ of mutually orthogonal projections and a set $S$ of partial isometries satisfying certain relations dependent on the graph $E$. The graph $C^*$-algebra $C^*(E)$ for the graph $E$ is then given by the universal $C^*$-algebra generated by the Cuntz-Krieger $E$-family $\{S, P\}$. As we will see, the class of graph-$C^*$-algebras is quite large and as such a useful one to understand.

One of the main achievements of this thesis is formulating the graph $C^*$-algebras as universal $C^*$-algebras in the sense of Definition 1.2.15 as in [3]. In the original paper by Kumjian, Pask and Raeburn [8], the authors proved explicitly that a $C^*$-algebra generated by a Cuntz-Krieger family exists and that this $C^*$-algebra has a universal property for concrete subalgebras of $B(H)$. In the main source for this thesis [11], Raeburn first defines a $C^*$-algebra $C^*(S, P)$ which is the subalgebra generated by a Cuntz-Krieger family $\{S, P\}$ in a $C^*$-algebra. He later also shows that there is a $C^*$-algebra $C^*(E)$ that has a universal property. However, many of the proofs in [11] are on the level of $C^*$-subalgebras and representations. We define $C^*(E)$ directly as a universal $C^*$-algebra and update the statements and proofs accordingly.

In Section 2.1 we formally define Cuntz-Krieger $E$-families and graph $C^*$-algebras. We will see that we get partial isometries $S_e$ associated to an edge $e$ in a graph. In Subsection 2.2, we extend this notion by associating paths $\mu$ in a graph to partial isometries $S_\mu$. This will allow us to classify graph $C^*$-algebras of graphs $E$, whenever $E$ is finite and has no cycles by finding an isomorphism of $C^*(E)$ onto a direct sum of matrix algebras. We then go to show examples of graphs that have interesting graph $C^*$-algebras. We see that there are graphs $E$ such that matrix algebras, the Toeplitz algebra, the continuous functions on a circle and the Cuntz algebras are isomorphic to $C^*(E)$.

In Section 3 we explore the connection between graph $C^*$-algebras and their precursors, the Cuntz-Krieger algebras. The Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal $C^*$-algebra generated by partial isometries $S_i$ whose range projections $S_i S_i^*$ are mutually orthogonal and whose relations are specified by a $\{0, 1\}$-matrix $A$ that has no zero-rows or zero-columns. We show that whenever we have a Cuntz-Krieger algebra $\mathcal{O}_A$, we can find a graph $E$ such that $\mathcal{O}_A \cong C^*(E)$. This graph $E$ is finite and has neither sinks nor sources. In a second step, we show that if we start with such a finite graph $E$ with no sinks and no sources, we can find a matrix $A$ such that again $C^*(E) \cong \mathcal{O}_A$. This shows that the class of Cuntz-Krieger algebras is equal to the class of graph $C^*$-algebras for finite graphs without sinks or sources. In the process of showing this we also prove generally that for $E$ without sources we get $C^*(E) \cong C^*(\widehat{E})$ for the line graph $\widehat{E}$ of $E$. Finally in this section we also name some limitations of the definitions for Cuntz-Krieger families given in [8] in the sense that we require the graphs $E$ to be row-finite. We refer to a more permissive definition given in [1] that allows for the graphs to be infinite in exchange for additional constraints on the partial isometries and projections forming a Cuntz-Krieger family.

Finally in Section 4.1 we find representations for the graph $C^*$-algebras we have used as examples throughout the thesis. We show different approaches to finding these representations. As a highlight, we present an algorithmic approach of constructing a Hilbert space $H$ for any row-finite directed graph such that there is a non-trivial representation of $C^*(E)$ on $H$. In the Section 4.2, we state two uniqueness theorems for graph $C^*$-algebras. For the first uniqueness theorem, we will introduce the gauge action $\gamma$ of $\mathbb{T}$ on $C^*(E)$. The *gauge-invariant uniqueness theorem* gives us an algebraic sufficient condition for a non-zero representation of $C^*(E)$ to be faithful. The second theorem, also known as *Cuntz-Krieger uniqueness theorem*, on the other hand is purely dependent on the structure of the graph $E$. If $E$ satisfies a condition (L) introduced in [8], then any non-zero representation of $C^*(E)$ is faithful.

## 1. Preliminaries

In this section we introduce the general terminology of graphs and some results from functional analysis on $C^*$-algebras which we will use in this thesis.

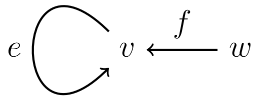
### 1.1. **Graphs.**

The definitions and concepts in this subsection are standard graph theoretical ones. We follow the conventions from [11].

1.1.1. **Definition.** A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets $E^0$, $E^1$ and functions $r, s : E^1 \to E^0$. The elements of $E^0$ are called *vertices* and the elements of $E^1$ are called *edges*. We call $s$ the *source map* and $r$ the *range map*. For each edge $e$, the *source* of $e$ is $s(e)$ and the *range* of $e$ is $r(e)$. If $s(e) = v$ and $r(e) = w$, then we also say that $v$ *emits* $e$ and that $w$ *receives* $e$, or that $e$ is *an edge from $v$ to $w$*.

Since all graphs in this thesis are directed, we may occasionally omit mentioning it explicitly. We also often omit writing down the vertex and edge set $E^0$ and $E^1$, as well as the range and source functions $r$ and $s$ of a graph $E$ unless it is needed to avoid ambiguities. Usually, we use a graphical notation to represent our graphs since it is easier to interpret and fits our natural understanding of what a graph should be, see the next example.

1.1.2. **Example.** Let $E = (E^0, E^1, r, s)$ be the directed graph with $E^0 = \{v, w\}$ and $E^1 = \{e, f\}$ such that $r(e) = s(e) = v$, $r(f) = v$ and $s(f) = w$. Then a possible graphical representation could be




We now introduce some additional graph-theoretical vocabulary.

1.1.3. **Definition.** Let $E = (E^0, E^1, r_E, s_E)$ be a graph, $e \in E^1$ and $v \in E^0$. Then we call

*(a)* $e$ a *loop based at $v$*, if $r_E(e) = s_E(e) = v$;
*(b)* $v$ a *source* if it receives no edges, i.e. if $r_E^{-1}(v) = \emptyset$ and we call it a *sink* if it emits no edges, i.e. if $s_E^{-1}(v) = \emptyset$;
*(c)* the graph $E$ *finite* if both $E^0$ and $E^1$ are finite.
*(d)* If there is another graph $F = (F^0, F^1, r_F, s_F)$, we say $E$ and $F$ are *isomorphic* if and only if there exist bijections $\varphi^0 : E^0 \to F^0$ and $\varphi^1 : E^1 \to F^1$ such that $r_F \circ \varphi^1 = \varphi^0 \circ r_E$ and $s_F \circ \varphi^1 = \varphi^0 \circ s_E$.

1.1.4. **Remark.** We now have two different uses for the word *source*, one being a vertex with no outgoing edges and the other being the image of the source map $s(e)$. However, there is little room for ambiguity, since it usually is apparent from both context and grammar which meaning is implied. Especially the grammatical component makes this obvious, as the image of the source map is always related to an edge, for example

"$v$ is the source of *[the edge]* $e$"

whereas a vertex without incoming edges may stand alone, for example "let $v$ be a source"

or is related to an entire graph, for example

"$v$ is a source of *[the graph]* $E$"

1.1.5. **Definition.** A *path of length* $n$ in a directed graph $E$ is a sequence $\mu = \mu_1\mu_2\cdots\mu_n$ of edges in $E$ such that $s(\mu_i) = r(\mu_{i+1})$ for all $i = 1, \cdots, n-1$. We write $|\mu| := n$ for the length of $\mu$ and regard vertices $v$ as paths of length 0. We denote by $E^n$ the set of all paths of length $n$ in $E$ and by $E^* := \bigcup_{n\geq 0} E^n$ the set of all paths of finite length. We extend the range and source maps $r$ and $s$ to $E^*$ by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$ for $|\mu| > 1$ and $r(v) = v = s(v)$ for $v \in E^0$. If $\mu$ and $\nu$ are paths with $r(\nu) = s(\mu)$, we write $\mu\nu$ for the path $\mu_1\cdots\mu_{|\mu|}\nu_1\cdots\nu_{|\nu|}$. A path $\mu$ is called a *cycle* if $|\mu| \geq 1$, $r(\mu) = s(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$.

1.1.6. **Remark.** The previous definition also explains the notational choice of calling the vertex set of a graph $E^0$ and the edge set $E^1$. The convention of defining the order of edges in a path and the range and source maps $r$ and $s$ in the way of the previous definition is not the most common one. It implies that in the path $\mu = \mu_1\cdots\mu_n$ the edge $\mu_n$ is the first edge of the path and $\mu_1$ is the last edge of the path. For example, in the graph

$$u \xrightarrow[\mu_2]{} v \xrightarrow[\mu_1]{} w$$

the path of length two is given by $\mu_1\mu_2$. However, as we will see later on in Definition 2.2.1, we want to associate operators to edges and more generally paths. Since the composition of operators $RT$ dictates that $T$ is applied before $R$, we defined the order of edges in a path to be consistent with this composition.

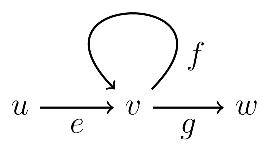
1.1.7. **Definition.** Let $E = (E^0, E^1, r, s)$ be a graph. The *adjacency matrix* or *vertex matrix* $A_E$ of the graph $E$ is the $E^0 \times E^0$-matrix defined by

$$A_E(v,w) = \#\{e \in E^1 \mid r(e) = v, s(e) = w\}$$

If every vertex in $E^0$ receives only finitely many edges (i.e. if $|r^{-1}(v)| < \infty$), we call $E$ *row-finite*. $E$ is row-finite if and only if every row in the adjacency matrix $A_E$ has finite sum.

It should be noted that we allow multiple edges between vertices possibly including infinitely many edges. As such, a graph $E$ with a finite vertex set might still not necessarily be row-finite. We visualize the notions introduced in this subsection in an example.

1.1.8. **Example.** Let $E$ be the following graph:




This graph has

- a loop $f$ based at $v$,
- a source $u$,
- a sink $w$ and
- four paths of length 2, namely $fe$, $ge$, $gf$ and $ff$.

Its adjacency matrix is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

In the adjacency matrix, we can recognize the source by the zero row and the sink by the zero column.

1.2. **Functional analysis.** In this subsection, we remind ourselves of some functional analytic terminology and results. Again, these concepts are standard and can be found in [3].

1.2.1. **Definition.** Let $H$ be a Hilbert space. Then

$$B(H) := \{A : H \to H \mid A \text{ is linear and bounded with respect to the operator norm}\}$$

is a Banach algebra with the operator norm given by

$$\|T\| := \inf \{C > 0 \mid \|Tx\|_H \leq C\|x\|_H \, \forall x \in H\}.$$

1.2.2. **Definition.** *(a)* A *$C^*$-algebra* is a complex Banach algebra $A$ with an involution, i.e. an anti-linear map

$$^* : A \to A, \qquad x \mapsto x^*$$

such that $x^{**} = x$, $(xy)^* = y^* x^*$ and $\|x^* x\| = \|x\|^2$. We call the last condition the *$C^*$-property*.

*(b)* We say $A$ is *unital* as an algebra, if $1 \in A$.

*(c)* Let $A$ and $B$ be $*$-Banach algebras. A map $\varphi : A \to B$ is called *$*$-homomorphism* if it is linear, multiplicative and $\varphi(a^*) = \varphi(a)^*$ holds.

1.2.3. **Remark.** If $H$ is a Hilbert space, then $B(H)$, the space of bounded operators acting on $H$, is a unital $C^*$-algebra where the existence of the involution follows as a result of the representation theorem by Riesz. This motivates the following definition.

1.2.4. **Definition.** Let $H$ be a Hilbert space and let $A$ be a $C^*$-algebra. A *representation* of $A$ on $H$ is a $*$-homomorphism $\pi : A \to B(H)$. We call a representation *faithful* if it is injective. If we have two representations $\pi_i : A \to B(H_i)$ for $i = 1, 2$ and there is a unitary $U : H_1 \to H_2$ (i.e. $U^*U = 1_{H_1}$ and $UU^* = 1_{H_2}$) such that $\pi_2(x) = U\pi_1(x)U^*$ for each $x \in A$, then the representations are *(unitary) equivalent.*

The next theorem is very one of the central ones for $C^*$-algebras and their representations. It has been proven by Gelfand and Naimark [7] in 1943. For a more modern wording we refer to [3].

1.2.5. **Theorem** (Second Gelfand-Naimark Theorem)**.** *Every $C^*$-algebra $A$ admits a faithful representation $\pi : A \hookrightarrow B(H)$ on a Hilbert space $H$. Hence, $A$ is isomorphic to a $C^*$-subalgebra of $B(H)$.*

Let us now define some general notions for elements of $C^*$-algebras.

1.2.6. **Definition.** Let $A$ be a $C^*$-algebra.

*(a)* $a \in A$ is called *selfadjoint* if $a = a^*$,

*(b)* $n \in A$ is called *normal* if $n^*n = nn^*$,

*(c)* $p \in A$ is called *(orthogonal) projection* if $p = p^2 = p^*$ and

*(d)* $s \in A$ is called *partial isometry* if $s = ss^*s$.

If $A$ is unital, then we also define

*(e)* $u \in A$ is called *unitary* if $u^*u = uu^* = 1$,
*(f)* $v \in A$ is called an *isometry* if $v^*v = 1$.

In this thesis we will especially need projections and partial isometries. Thus, we will take a closer look at some of their properties now.

1.2.7. **Remark.** The name *orthogonal projection* implies a geometrical origin. Indeed, if $M$ is a closed subspace of a Hilbert space $H$, then the bounded linear operator $P : H \to H$ with $Ph \in M$ and $h - Ph \perp M$ for all $h \in H$ is called the *orthogonal projection* of $H$ on $M$. These relations yield the algebraic statement $P = P^2$ and the second one explains the statement $P = P^*$. While this statement is made for bounded operators on a Hilbert space, Theorem 1.2.5 justifies lifting the terminology to $C^*$-algebras.

The following three propositions in this section and Corollary 1.2.11 come from [11] Appendix A.1.

1.2.8. **Proposition.** *Let $P$ and $Q$ be orthogonal projections onto closed subspaces of a Hilbert space $H$. Then the following statements are equivalent:*

(a) $PH \subseteq QH$;
(b) $QP = P = PQ$;
(c) $Q - P$ *is a projection;*
(d) $P \leq Q$ *in the sense that* $\langle Ph \,|\, h\rangle \leq \langle Qh \,|\, h\rangle$ *for all* $h \in H$.

*Proof.* (a) ⇒ (b): For $h \in H$, we have $Ph \in PH \subseteq QH$, so $Q(Ph) = Ph$ and thus $QP = P$. Taking adjoints yields $PQ = P$.

(b) ⇒ (c): We can directly calculate

$$(Q - P)^2 = Q^2 - QP - PQ + P^2 = Q - P - P + P = Q - P$$

and

$$(Q - P)^* = Q^* - P^* = Q - P.$$

Thus, $Q - P$ is a projection.

(c) ⇒ (d): We can directly calculate

$$\begin{aligned} \langle Qh \,|\, h\rangle - \langle Ph \,|\, h\rangle &= \langle (Q-P)h \,|\, h\rangle \\ &= \langle (Q-P)^2 h \,|\, h\rangle \\ &= \langle (Q-P)h \,|\, (Q-P)^* h\rangle \\ &= \langle (Q-P)h \,|\, (Q-P)h\rangle \\ &\geq 0. \end{aligned}$$

(d) ⇒ (a): Suppose $h \in PH$, so that $h = Ph$. Then $P \leq Q$ implies

$$\|Qh\|^2 = \langle Qh \,|\, Qh\rangle = \langle Qh \,|\, h\rangle \geq \langle Ph \,|\, h\rangle = \|h\|^2.$$

Since $\|h\|^2 = \|Qh\|^2 + \|(1 - Q)h\|^2$, this implies that $\|(1 - Q)h\|^2 = 0$ and thus $h = Qh \in QH$. □

1.2.9. **Proposition.** *Let $P$ and $Q$ be orthogonal projections onto closed subspaces of a Hilbert space $H$. Then the following statements are equivalent:*

(a) $PH \perp QH$;
(b) $QP = 0 = PQ$;
(c) $P + Q$ *is a projection.*

*Proof.* (a) $\Rightarrow$ (b): For $h \in H$, we have that $Ph \in PH$ is orthogonal to $QH$, so $QPh = 0$ and thus $QP = 0$. Taking adjoints yields $PQ = 0$.

(b) $\Rightarrow$ (a): We can directly calculate $\langle Ph \,|\, Qk \rangle = \langle QPh \,|\, k \rangle = 0$ for every $h, k \in H$.

(b) $\Rightarrow$ (c): Again, we can directly calculate

$$(P+Q)^2 = P^2 + PQ + QP + Q^2 = P + 0 + 0 + Q = P + Q$$

and

$$(P+Q)^* = P^* + Q^* = P + Q.$$

Thus, $P + Q$ is a projection.

(c) $\Rightarrow$ (b): $(P+Q)^2 = P^2 + Q^2$ implies that $PQ = -QP$ which implies that $PQPQ = (PQ)^2 = (QP)^2 = QPQP$. This implies that

$$-PQ = P(-PQ)Q = P(QP)Q = QPQP = Q(-QP)P = -QP = PQ$$

which finally implies $PQ = 0$. Again, taking adjoints yields $QP = 0$. □

1.2.10. **Remark.** The statements of Proposition 1.2.8(a) and (d) as well as of Proposition 1.2.9(a) make sense only in the concrete case of bounded operators on a Hilbert space. However, the remaining statements also make sense in a general $C^*$-algebra. By interpreting the relation "$\leq$" in the context of positive elements in a $C^*$-algebra, even the statement of Proposition 1.2.8(d) can be understood. In fact, if we take projections $p$ and $q$ in a $C^*$-algebra $A$ and a faithful representation $\pi : A \to B(H)$ then $\pi(p)$ and $\pi(q)$ are projections on $H$ again. If these now fulfill any and thus all of the statements of either Proposition 1.2.8 or Proposition 1.2.9, the projections $p$ and $q$ also fulfill the corresponding statements (excluding the respective statement (a)), since $\pi$ is an injective $*$-homomorphism. Thus, we can conclude the following corollary.

1.2.11. **Corollary.** *Let $\{p_i \,|\, 1 \leq i \leq n\}$ be projections in a $C^*$-algebra $A$. Then $\sum_{i=1}^n p_i$ is a projection if and only if $p_i p_j = 0$ for all $i \neq j$, in which case we say that the projections are mutually orthogonal.*

*Proof.* If the projections $p_i$ are mutually orthogonal, the result follows directly from Proposition 1.2.9(c). The converse is proven by induction. For $n = 1$ it is true. Suppose now, the statement is true for $n = k$, and that $\sum_{i=1}^{k+1} p_i$ is a projection. We assume $p_{k+1} \neq 0$ since the statement would otherwise be true trivially. Since each $p_i = p_i^* p_i$ is a positive element in $A$, we have $\sum_{i=1}^{k+1} p_i \geq p_{k+1}$. Proposition 1.2.8 then gives us that

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{k+1} p_i - p_{k+1}$$

is a projection in $A$. Now, by induction hypothesis, the $p_i$ for $1 \leq i \leq k$ are mutually orthogonal, and Proposition 1.2.9 implies that $\sum_{i=1}^k p_i$ and $p_{k+1}$ are orthogonal. Thus for $i \leq k$ we have

$$0 \leq p_{k+1} p_i p_{k+1} \leq p_{k+1} \Big( \sum_{j=1}^{k} p_j \Big) p_{k+1} = 0.$$

This means $p_{k+1} p_i = 0$ and thus all $p_i$ for $1 \leq i \leq k+1$ are mutually orthogonal. □

Next, we will take a close look at partial isometries. If you first encounter partial isometries in the setting of Hilbert spaces rather than in the setting of $C^*$-algebras, you might have seen a different definition for them than the one given in Definition 1.2.6(d).

1.2.12. **Definition.** An operator $S$ on a Hilbert space $H$ is called a *partial isometry* if the restriction of $S$ to $(\ker S)^{\perp}$ is an isometry.

However, as we will see in the following proposition, these definitions are equivalent and the algebraic one given before is more general since it also makes sense in a $C^*$-algebraic setting.

1.2.13. **Proposition.** *Let $S$ be a bounded linear operator on a Hilbert space $H$. Then the following statements are equivalent:*

(a) *$S$ is a partial isometry in the sense of Definition 1.2.12;*
(b) *$S^*S$ is a projection;*
(c) *$S$ is a partial isometry in the sense of Definition 1.2.6(f), that is $S = SS^*S$;*
(d) *$SS^*$ is a projection;*
(e) *$S^* = S^*SS^*$.*

*If so, $S^*S$ is the projection on $(\ker S)^{\perp}$ and $SS^*$ is the projection on $\operatorname{ran} S$.*

*Proof.* We first show (b) ⇔ (c), as this gives (d) ⇔ (e) and (e) ⇔ (c) follows from conjugation. Obviously, (c) ⇒ (b) follows directly, so we now show (b) ⇒ (c). Since we assume $S^*S$ to be a projection, we can compute

$$\begin{aligned} \|S - SS^*S\|^2 &= \|S^*S - S^*SS^*S - S^*SS^*S + S^*SS^*SS^*S\| \\ &= \|S^*S - S^*S - S^*S + S^*S\| \\ &= 0 \end{aligned}$$

and as such $S = SS^*S$. Now we only need to show that (a) is also equivalent to the other statements.

(a) ⇒ (b): Let $P$ be the projection onto the space $(\ker S)^{\perp}$. We want to show, that $P = S^*S$. First, we check that for every $h \in (\ker S)^{\perp}$, we get

$$\langle S^*Sh \,|\, h\rangle = \langle Sh \,|\, Sh\rangle = \langle h \,|\, h\rangle = \langle Ph \,|\, h\rangle.$$

Together with the polarization identity, this tells us $S^*S = P$ if we restrict ourselves to $h \in (\ker S)^{\perp}$. From the well-known fact that $\operatorname{ran} S^* = (\ker S)^{\perp}$, we get $S^* = PS^*$, and since we may write $h \in H$ as $h = h_1 + h_2 \in (\ker S)^{\perp} \oplus \ker S$, we also get $S = SP$. Now we see that for general $h \in H$, we get

$$\langle S^*Sh \,|\, h\rangle = \langle (PS^*)(SP)h \,|\, h\rangle = \langle S^*S(Ph) \,|\, Ph\rangle = \langle Ph \,|\, h\rangle$$

where we were able to apply the previous computation for the last equality since $Ph \in (\ker S)^{\perp}$ and as such $S^*S = P$.

(c) ⇒ (a): First, we note that $S^*S$ is the projection onto $(\ker S)^{\perp}$. Indeed, let $k \in \ker S$ and $h \in H$. Then

$$\langle S^*Sh \,|\, k\rangle = \langle Sh \,|\, Sk\rangle = 0$$

and thus $S^*Sh \in (\ker S)^{\perp}$. On the other hand, $(S - SS^*S)h = 0$ implies that $h - S^*Sh \in \ker S$ for every $h \in H$. Thus, Remark 1.2.7 tells us, that $S^*S$ is the projection onto $(\ker S)^{\perp}$. Now, let $h \in (\ker S)^{\perp}$. Then

$$\langle Sh \,|\, Sh\rangle = \langle S^*Sh \,|\, h\rangle = \langle h \,|\, h\rangle$$

and thus $S$ is a partial isometry in the sense of Definition 1.2.12.

To see the final statement of the proposition, we note, that we have already shown that $S^*S$ is the projection onto $(\ker S)^{\perp}$ in (a) ⇒ (b). For $SS^*$, we directly see that $SS^*H \subseteq SH$. Conversely, from (c) we get $SH = SS^*SH \subseteq SS^*H$. □

1.2.14. **Remark.** Analogously to Remark 1.2.10, all statements from the previous proposition apart from (a) make sense in a general $C^*$-algebraic setting. Again, by using a representation on a Hilbert space, we may lift the equivalences to the $C^*$-algebraic level. For a partial isometry $S$, we call $S^*S$ the *initial projection of* $S$ and $SS^*$ the *final projection of* $S$.

Finally in this section, we want to remind ourselves of universal $C^*$-algebras.

1.2.15. **Definition.** Let $E = \{x_i \,|\, i \in I\}$ be a set of generators, $I$ an index set. Let $P(E)$ be the involutive $\mathbb{C}$-algebra of non-commutative $*$-polynomials in $E$ and let $R \subseteq P(E)$ be a set of relations. Let $J(R) \subseteq P(E)$ be the two-sided ideal in $P(E)$ generated by $R$. Define

$$A(E,R) := P(E)/J(R)$$

as the *universal involutive algebra with generators* $E$ *and relations* $R$. For $x \in A(E,R)$, put

$$\|x\| := \sup\{p(x) \,|\, p \text{ is a } C^*\text{-seminorm on } A(E,R)\}.$$

If now $\|x\| < \infty$ for all $x \in A(E,R)$, define

$$C^*(E|R) := \overline{A(E,R)/\{x \in A(E,R) \,|\, \|x\| = 0\}}^{\|\cdot\|}$$

as the *universal $C^*$-algebra with generators* $E$ *and relations* $R$.

We list a few well-known results without giving a proof. First, the universal $C^*$-algebras have a universal property.

1.2.16. **Proposition.** *Let $B$ be a $C^*$-algebra and $E' := \{y_i \in B \,|\, i \in I\} \subseteq B$ be a subset satisfying the relations $R$. Then there is a unique $*$-homomorphism $\varphi : C^*(E|R) \to B$ with $\varphi(x_i) = y_i$.*

We also remind ourselves of a lemma that simplifies showing the existence of $C^*(E|R)$.

1.2.17. **Lemma.** *If there is a constant $C > 0$ such that $p(x_i) < C$ for all $C^*$-seminorms $p$ on $A(E,R)$ and all $i \in I$, then $C^*(E|R)$ exists.*

1.2.18. **Remark.** This shows in particular that every universal $C^*$-algebra generated by orthogonal projections must exist, since for every $C^*$-seminorm $p$ and projection $x$ we have

$$p(x)^2 = p(x^*x) = p(x) \in \{0,1\}.$$

The same holds true for partial isometries, since for every partial isometry $s$ Proposition 1.2.13 tells us that $s^*s$ is a projection and thus

$$p(s)^2 = p(s^*s) \in \{0,1\}.$$

The remaining statements in this section come from [11] Appendix A.2.

1.2.19. **Example.** A special example of $C^*$-algebras is given by the set of complex-valued $n \times n$-matrices $M_n(\mathbb{C})$ acting as linear operators on the Hilbert space $\mathbb{C}^n$ through matrix-vector-multiplication. For $i, j \in \{1, \ldots, n\}$ we define $E_{i,j} \in M_n(\mathbb{C})$ as

$$(E_{i,j})_{k,l} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise} \end{cases}$$

the matrix with exactly one entry 1 at position $(i,j)$ and 0 everywhere else. The set $\{E_{i,j}\,|\,i,j\in\{1,\ldots,n\}\}$ forms a vector space basis for $M_n(\mathbb{C})$, since each $a=(a_{i,j})\in M_n(\mathbb{C})$ can be written as $\sum_{i,j\in\{1,\ldots,n\}} a_{i,j}E_{i,j}$. Furthermore they satisfy

$$E_{i,j}^* = E_{j,i} \qquad \text{and} \qquad E_{i,j}E_{k,l} = \begin{cases} E_{i,l} & \text{if } j=k \\ 0 & \textit{otherwise.} \end{cases}$$

We extend this concept to general $*$-algebras. Whenever we have a family $\{e_{i,j}\}$ in a $*$-algebra $B$ satisfying the relations from above, we call this family a set of *matrix units*. In fact, the universal $C^*$-algebra generated by $n^2$-many matrix units

$$A = C^*(e_{i,j}, i,j=1,\ldots,n\,|\,e_{i,j}^* = e_{j,i}, e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}\,\forall i,j,k,l)$$

is isomorphic to $M_n(\mathbb{C})$. To see this isomorphism, we consider the homomorphism given to us by the universal property of $A$ that maps $e_{i,j}$ to $E_{i,j}$. This homomorphism is obviously surjective. Since both $M_n(\mathbb{C})$ and $A$ are of dimension $n^2$, this homomorphism is also injective and thus an isomorphism.

1.2.20. **Lemma.** *Let $B$ be a $*$-algebra and let $\{e_{i,j}\}\subseteq B$ be a set of matrix units. If one of the $e_{i,j}$ is non-zero, they all are.*

*Proof.* Assume $e_{i,j}$ is non-zero. Then

$$e_{i,j} = e_{i,k}e_{k,l}e_{l,j}$$

forces $e_{k,l}$ to be non-zero for all $k,l$. □

## 2. Graph $C^*$-algebras

In this section we define the main objects we want to study in this thesis, namely Cuntz-Krieger families and graph $C^*$-algebras. We extend the notion of partial isometries associated to an edge of a graph to one associated to a path in a graph. This will ulitmately allow us to get a complete characterisation for graph $C^*$-algebras of finite graphs with no cycles in Proposition 2.2.9. Our primary source for the results in this section is [11], again. Finally in this section, we will show some examples of graph $C^*$-algebras and isomorphisms from them to well known $C^*$-algebras.

2.1. **Cuntz-Krieger families.** The central object needed to define graph $C^*$-algebras is a Cuntz-Krieger $E$-family. The definition we give here has first been given by Kumjian, Pask and Raeburn [8] in 1998.

2.1.1. **Definition** (Cuntz-Krieger $E$-family)**.** Let $E$ be a row-finite directed graph. A *Cuntz-Krieger $E$-family* $\{S,P\}$ consists of

*(a)* a set $P=\{P_v\,|\,v\in E^0\}$ of mutually orthogonal projections and
*(b)* a set $S=\{S_e\,|\,e\in E^1\}$ of partial isometries

such that

**(CK1):** $S_e^*S_e = P_{s(e)}$ for all $e\in E^1$ and
**(CK2):** $P_v = \sum_{\{e\in E^1\,|\,r(e)=v\}} S_eS_e^*$ whenever $v$ is not a source.

The conditions (CK1) and (CK2) are called *Cuntz-Krieger relations* and (CK2) is also called the *Cuntz-Krieger relation at $v$*.

2.1.2. **Remark.** At this point, it is important to mention that the direction of the edges associated to the partial isometries varies in different sources. In their original papers, Cuntz and Krieger defined their family in the opposite direction. This means, they stated (CK1) as

$$S_e^*S_e = P_{r(e)} \text{ for all } e\in E^1$$

and they stated (CK2) as

$$P_v = \sum_{\{e \in E^1 \mid s(e)=v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

We have opted to go with the convention used in [11], since, together with the way we have defined paths, it is better suited to encode operator composition, as we have mentioned in Remark 1.1.6. Furthermore, in higher-level graphs, when edges represent morphisms in a category, this notion of edge concatenation being the same as morphism or operator composition is even more intrinsically linked. However, when reading a publication on this topic such as the sources for this thesis, it is crucial to check the conventions used and to adjust accordingly. Luckily, this usually mostly amounts to exchanging the sources and sinks as well as changing the definition for paths.

2.1.3. **Remark.** We also would like to point to the fact, that Definition 2.1.1 is stated as broadly as possible by using only algebraic relations. To be more precise, we consider the objects in Definition 2.1.1 as elements of the involutive $\mathbb{C}$-algebra of non-commutative $*$-polynomials generated by the symbols in the sets $S$ and $P$. However, this very general definition also allows us to talk of a Cuntz-Krieger $E$-family in a $C^*$-algebra $A$, if we find a family $\{Q, R\}$ of mutually orthogonal projections $R$ and partial isometries $Q$ in $A$ that satisfies (CK1) and (CK2).

We can now define the central object of this thesis, graph $C^*$-algebras.

2.1.4. **Definition** (Graph $C^*$-algebras)**.** Let $E$ be a row-finite directed graph and let $\{S, P\}$ be a Cuntz-Krieger $E$-family. We call the universal $C^*$-algebra generated by $\{S, P\}$

$$C^*(E) := C^*(S \cup P | (\text{CK1}), (\text{CK2}))$$

the *graph $C^*$-algebra* for the graph $E$.

2.1.5. **Remark.** Due to Remark 1.2.18, we know that $C^*(E)$ does indeed exist. In Proposition 4.1.7, we will also see, that for a row-finite directed *non-empty* graph $E$, the graph $C^*$-algebra $C^*(E)$ is non-zero.

2.1.6. **Remark.** From this point forward, we always assume a Cuntz-Krieger family $\{S, P\}$ to be in a $C^*$-algebra. We can do this due to Definition 2.1.4 and Remark 2.1.5.

2.1.7. **Remark.** In Definition 2.1.1 and subsequently in Definition 2.1.4, we required the graph $E$ to be row-finite. However, we can also define Cuntz-Krieger families for arbitrary directed graphs. In this case, we have to add additional constraints to the definition of a Cuntz-Krieger family, see Definition 3.3.5. Several constructions have been proposed to reduce the case of arbitrary (countable) graphs to the row-finite case. One of them is the *Drinen-Tomforde desingularisation* that adds a tail of infinitely-many vertices to a vertex receiving infinitely-many edges such that each of these additional vertices receives only finitely-many edges. In [6], Theorem 2.11, the authors show that this desingularisation yields a graph $C^*$-algebra that includes the original graph $C^*$-algebra as a full corner. See also [11], Chapter 5 for more information on the Drinen-Tomforde desingularisation.

We will now show some general results for Cuntz-Krieger families as well as see some first examples.

2.1.8. **Lemma** ([11] Remark 1.6). *Let $\{S, P\}$ be a Cuntz-Krieger family for a row-finite directed graph $E$. The relation*

$$S_e = S_e P_{s(e)} = P_{r(e)} S_e$$

*holds for each edge $e \in E^1$.*

*Proof.* If $S_e = 0$, the statement is trivially true. Assume it is not 0 in the following. Since $S_e$ is a partial isometry, We immediately get $S_e = S_e P_{s(e)}$ from (CK1) and Proposition 1.2.13. For the second equation, (CK2) and Corollary 1.2.11 tell us that the projections $S_f S_f^*$ for $f \in E^1$ such that $r(f) = r(e)$ are mutually orthogonal. Thus, for $f \neq e$ we get $S_f S_f^* S_e = 0$. Hence we get $P_{r(e)} S_e = (\sum_{\{f \in E^1 \,|\, r(f) = r(e)\}} S_f S_f^*) S_e = S_e S_e^* S_e = S_e$. □

We can get some computational rules from the Cuntz-Krieger conditions which will prove to be very useful when working with them.

2.1.9. **Proposition** ([11] Proposition 1.12). *Let $E$ be a row-finite graph and $\{S, P\}$ a Cuntz-Krieger $E$-family in a $C^*$-algebra $B$. Then*

(a) *the final projections $\{S_e S_e^* \,|\, e \in E^1\}$ are mutually orthogonal;*
(b) $S_e^* S_f \neq 0 \Longrightarrow e = f$;
(c) $S_e S_f \neq 0 \Longrightarrow s(e) = r(f)$;
(d) $S_e S_f^* \neq 0 \Longrightarrow s(e) = s(f)$.

*Proof.* To show (a), assume that $r(e) = r(f) = v$. The Cuntz-Krieger relation at $v$ then implies, that the projection $P_v$ is a sum of $S_e S_e^*$, $S_f S_f^*$ and other projections and thus $S_e S_e^*$ and $S_f S_f^*$ must be mutually orthogonal due to Corollary 1.2.11. Conversely, if $r(e) \neq r(f)$ then the first equation in Lemma 2.1.8 implies

$$\begin{aligned}(S_e S_e^*)(S_f S_f^*) &= (S_e (P_{r(e)} S_e)^*)(P_{r(f)} S_f S_f^*) \\ &= (S_e S_e^* P_{r(e)})(P_{r(f)} S_f S_f^*) \\ &= (S_e S_e^*) 0 (S_f S_f^*) \\ &= 0\end{aligned}$$

since the $\{P_v \,|\, v \in E^0\}$ are mutually orthogonal.

Statement (b) follows directly from (a), since $S_e^* S_f = S_e^* (S_e S_e^*)(S_f S_f^*) S_f = 0$ if $e \neq f$.

Using Lemma 2.1.8 again, we see that $S_e S_f = (S_e P_{s(e)})(P_{r(f)} S_f) = 0$ if $s(e) \neq r(f)$ which shows (c).

Analogously for (d), we get $S_e S_f^* = (S_e P_{s(e)})(P_{s(f)} S_f^*) = 0$ if $s(e) \neq s(f)$. □

For a finite graph $E$, we can also show that $C^*(E)$ is unital.

2.1.10. **Lemma.** *Let $E$ be a finite graph and let $\{S, P\}$ be a Cuntz-Krieger $E$-family. Then $C^*(E)$ is unital with unit $\sum_{\{v \in E^0\}} P_v$.*

*Proof.* We check that $\sum_{\{v \in E^0\}} P_v$ acts as the unit for each of generator. First, let $w \in E^0$. By Definition 2.1.1 (a), $P_w$ is orthogonal to each $P_u$ with $u \neq w$. Thus we get

$$\Big( \sum_{\{v \in E^0\}} P_v \Big) P_w = P_w P_w = P_w$$

and analogously

$$P_w \Big( \sum_{\{v \in E^0\}} P_v \Big) = P_w P_w = P_w.$$

Second, let $e \in E^1$. By Lemma 2.1.8, we get

$$\Big(\sum_{\{v\in E^0\}} P_v\Big)S_e = \Big(\sum_{\{v\in E^0\}} P_v\Big)P_{r(e)}S_e = P_{r(e)}P_{r(e)}S_e = S_e$$

and

$$S_e\Big(\sum_{\{v\in E^0\}} P_v\Big) = S_eP_{s(e)}\Big(\sum_{\{v\in E^0\}} P_v\Big) = S_eP_{s(e)}P_{s(e)} = S_e.$$

So, $\sum_{\{v\in E^0\}} P_v$ acts as the unit for each generator of $C^*(E)$ and thus $\sum_{\{v\in E^0\}} P_v = 1_{C^*(E)}$. □

2.1.11. **Remark.** The previous lemma can easily be extended to (countably) infinite graphs by switching to an approximate unit for $C^*(E)$. If we order the vertices of $E$ in an arbitrary fashion, then the net $(u_n)_{n\in\mathbb{N}}$ with $u_n := \sum_{i=1}^n P_{v_i}$ is an approximate unit for $C^*(E)$.

2.2. **Partial isometries associated to paths.** We will now show how to extend the notion of partial isometries associated to an edge $e$ of a graph to a partial isometry associated to a path $\mu$ in the graph. Using this, we can show that all mixed monomials of the partial isometries $S_\mu$ and their adjoints can be written in a fixed form. This will allow us to completely classify any graph $C^*$-algebra for a finite graph without cycles.

2.2.1. **Definition.** Let $\mu \in \prod_{i=1}^n E^1$ and $v \in E^0$. We define

$$S_\mu := S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n}$$

and

$$S_v := P_v.$$

In the previous definition we did not require $\mu$ to be a path. Rather, $\mu$ is simply a concatenation of edges. The following proposition however shows that $S_\mu$ acts as we would expect in case $\mu$ is not a path.

2.2.2. **Proposition** ([11] Remark 1.13)**.** *Let $\mu \in \prod_{i=1}^n E^1$ be a concatenation of edges. We have $S_\mu = 0$ unless $\mu$ is a path in $E$. If $\mu$ is a path, $S_\mu$ is a partial isometry with initial projection $P_{s(\mu)}$ whose range is dominated by $P_{r(\mu)}$,*

*Proof.* If $\mu$ is not a path, there are some consecutive edges $\mu_i$ and $\mu_{i+1}$ in $\mu$ with $s(\mu_i) \neq r(\mu_{i+1})$. Proposition 2.1.9(c) then tells us that $S_{\mu_i}S_{\mu_{i+1}} = 0$ and thus $S_\mu = 0$. Assume now that $\mu$ is a path in $E$. We show the second statement by induction over the length $n$ of the path. For $n = 1$ the statement is trivially true since $S_\mu = S_{\mu_1}$ is a partial isometry by definition and $S_\mu = S_\mu P_{s(\mu)} = P_{r(\mu)}S_\mu$ follows directly from Lemma 2.1.8. Now, let $n > 1$. Using (CK1) and Lemma 2.1.8, we get

$$\begin{aligned}
S_\mu^*S_\mu &= (S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n})^*S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*(S_{\mu_1}^*S_{\mu_1})S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*P_{s(\mu_1)}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*P_{r(\mu_2)}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*S_{\mu_2}\cdots S_{\mu_n} \\
&= (S_{\mu_2}\cdots S_{\mu_n})^*S_{\mu_2}\cdots S_{\mu_n} \\
&= P_{s(\mu)}
\end{aligned}$$

where we applied the induction hypothesis to the path $\mu_2\cdots\mu_n$ of length $n-1$. By Proposition 1.2.13, $S_\mu$ is a partial isometry. The equation above already shows that its initial projection is $P_{s(\mu)}$ By Lemma 2.1.8 we get $P_{r(\mu)}S_\mu = P_{r(\mu_1)}S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n} = S_\mu$ and thus its range is dominated by $P_{r(\mu)}$. □

The previous proposition required the length of the path $\mu$ to be at least 1. However, the statement trivially extends to paths of length zero to an analogue of Proposition 2.1.8.

2.2.3. **Corollary.** *Let $\{S, P\}$ be a Cuntz-Krieger family for a row-finite directed graph $E$. The relation*

$$S_\mu = S_\mu P_{s(\mu)} = P_{r(\mu)}S_\mu$$

*holds for each path $\mu \in E^*$.*

*Proof.* For $|\mu| \geq 1$ the statement follows directly from Propositon 2.2.2. For $\mu = v \in E^0$ we have $S_\mu = S_v = P_v$ by Definition 2.2.1 and thus the statement also holds. □

We can now check, how the calculations from Proposition 2.1.9 extend to the partial isometries $S_\mu$.

2.2.4. **Proposition** ([11] Corollary 1.14). *Let $E$ be a row-finite graph, $\{S, P\}$ a Cuntz-Krieger $E$-family in a $C^*$-algebra $B$ and $\mu, \nu \in E^*$. Then*

(a) *if $|\mu| = |\nu|$ and $\mu \neq \nu$, then $(S_\mu S_\mu^*)(S_\nu S_\nu^*) = 0$;*

(b) $S_\mu^* S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu\mu' \text{ for some } \mu' \in E^* \\ S_{\nu'} & \text{if } \nu = \mu\nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise;} \end{cases}$

(c) *if $S_\mu S_\nu \neq 0$, then $\mu\nu$ is a path in $E$ and $S_\mu S_\nu = S_{\mu\nu}$;*

(d) *if $S_\mu S_\nu^* \neq 0$, then $s(\mu) = s(\nu)$.*

*Proof.* For (a), let $i$ be the smallest integer such that $\mu_i \neq \nu_i$. Then, by applying Proposition 2.2.2 to $S_{\nu_1\cdots\nu_{i-1}}$, we get

$$\begin{aligned} S_\mu^* S_\nu &= (S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n})^*(S_{\nu_1}S_{\nu_2}\cdots S_{\nu_n}) \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^*(S_{\mu_{i-1}}^*\cdots S_{\mu_1}^*)(S_{\nu_1}\cdots S_{\nu_{i-1}})S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^*(S_{\mu_1\cdots\mu_{i-1}})^*(S_{\nu_1\cdots\nu_{i-1}})S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* P_{s(\mu_{i-1})}S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* P_{r(\mu_i)}S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* S_{\nu_i}\cdots S_{\nu_n} \end{aligned}$$

where the middle term $S_{\mu_i}^* S_{\nu_i}$ is equal to 0 by Proposition 2.1.9(b). Thus we get $(S_\mu S_\mu^*)(S_\nu S_\nu^*) = 0$.

In order to show (b), we differentiate between three cases. First, assume $n := |\mu| < |\nu|$ and write $\nu = \alpha\nu'$ with $|\alpha| = n$. Then

$$S_\mu^* S_\nu = S_\mu^*(S_\alpha S_{\nu'}) = (S_\mu^* S_\alpha)S_{\nu'}$$

If $\mu = \alpha$, Proposition 2.2.2 yields

$$S_\mu^* S_\nu = P_{s(\alpha)}S_{\nu'} = P_{r(\nu')}S_{\nu'} = S_{\nu'}$$

If on the other hand $\mu \neq \alpha$, we can apply the same calculation as in the proof to (a) to see that

$$S_\mu^* S_\nu = (S_\mu^* S_\alpha)S_\nu = 0.$$

For the second case of $|\mu| > |\nu|$, we can simply take the adjoint $(S_\mu^* S_\nu)^* = S_\nu^* S_\mu$ and arrive in the first case with the result being non-zero if and only if $\mu = \nu\mu'$. In this case, $S_\nu^* S_\mu = S_{\mu'}$ and taking the adjoint again to return to our original question, we get $S_\mu^* S_\nu = S_{\mu'}^*$. Lastly, if $|\mu| = |\nu|$ we are either in the same situation as in the proof to (a), where $\mu \neq \nu$ in which case the product $S_\mu^* S_\nu$ is 0, or we are in the situation of Proposition 2.2.2, where we see $S_\mu^* S_\nu = S_\mu^* S_\mu = P_{s(\mu)}$. This is already enough to prove (b), as we have $S_{s(\mu)} = P_{s(\mu)}$ because $s(\mu) \in E^0 \subseteq E^*$.

The proof for (c) is exactly the same proof as in Proposition 2.1.9: $S_\mu S_\nu = (S_\mu P_{s(\mu)})(P_{r(\nu)} S_\nu)$ which is 0 unless $s(\mu) = r(\nu)$ which by definition means $\mu\nu$ is a path and $S_\mu S_\nu = S_{\mu\nu}$.

Again analogously for (d), we see $S_\mu S_\nu^* = (S_\mu P_{s(\mu)})(P_{s(\nu)} S_\nu^*)$ which is 0 unless $s(\mu) = s(\nu)$. □

The previous proposition allows us to infer a crucial corollary. It tells us that we are able to simplify all mixed monomials in the set of partial isometries associated to paths in a graph $E$ and their adjoints, i.e. terms of the form $\prod_{i=1}^n S_{\mu_i}^{\varepsilon_i}$ where $\mu_i \in E^*$ and $\varepsilon_i \in \{1, *\}$, to a term of the form $S_\mu S_\nu^*$ or 0.

2.2.5. **Corollary** ([11] Corollary 1.15). *Let $E$ be a row-finite graph and let $\{S, P\}$ be a Cuntz-Krieger $E$-family in a $C^*$-algebra $B$. For $\mu, \nu, \alpha, \beta \in E^*$, we have*

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'} S_\beta^* & \text{if } \alpha = \nu\alpha' \\ S_\mu S_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

*In particular, every non-zero finite product of partial isometries $S_e$ and $S_f^*$ has the form $S_\mu S_\nu^*$ for some $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$.*

*Proof.* The formula follows directly from Proposition 2.2.4(b) and (c). For the last statement, let $S := \prod_{i=1}^n S_{e_i}^{\varepsilon_i}$ where $e_i \in E^1$ and $\varepsilon_i \in \{1, *\}$ be a non-zero monomial. Any adjacent $S_{e_i}$'s can be combined into a single term $S_\mu$ and since $S$ is non-zero, Proposition 2.2.2 tells us that $\mu$ must be a path. Analogously, any adjacent $S_{e_i}^*$'s can be collected in a single term $S_\nu^*$. Hence, $S$ is almost a product of terms of the form $S_{\mu'} S_{\nu'}^*$, however it might still be preceded by a term $S_\alpha^*$ or succeeded by a term $S_\beta$. In this case, we may add a prefix of the form $S_{s(\alpha)} = P_{s(\alpha)}$ or a suffix of the form $S_{s(\beta)}^* = P_{s(\beta)}^* = P_{s(\beta)}$ due to Lemma 2.1.8. Now, $S$ truly is a product of terms $S_{\mu'} S_{\nu'}^*$ and we may apply the formula from the corollary iteratively to get a single term $S_\mu S_\nu^*$. Since $S$ is not zero, Proposition 2.2.4(d) tells us, that $s(\mu)$ must be equal to $s(\nu)$. □

Since we are now able to simplify monomials in $C^*(E)$, the $C^*$-algebra generated by a Cuntz-Krieger family $\{S, P\}$, we get another characterization for it, that uses only monomials of the form $S_\mu S_\nu^*$.

2.2.6. **Corollary** ([11] Corollary 1.16). *Let $E$ be a row-finite graph and let $\{S, P\}$ be a Cuntz-Krieger $E$-family in a $C^*$-algebra $B$. Then*

$$C^*(E) = \overline{\operatorname{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu)\}$$

*Proof.* Due to Corollary 2.2.5 $\operatorname{span}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu)\}$ is a subalgebra of $C^*(E)$ and due to $(S_\mu S_\nu^*)^* = S_\nu S_\mu^*$, it is also a $*$-subalgebra. Hence, its closure is a $C^*$-subalgebra and since the generators are included via $S_e = S_e P_{s(e)} = S_e P_{s(e)}^* = S_e S_{s(e)}^*$ and $P_v = P_v P_v^* = S_v S_v^*$ it is already all of $C^*(E)$. □

This corollary allows us to completely classify all graph $C^*$-algebras associated to finite directed graphs without cycles. First we use the additional constraints on the graphs to formulate the following two lemmas.

2.2.7. **Lemma.** *Let $E$ be a finite directed graph with no cycles and let $w_1, \dots, w_n$ be the sources in $E$. Then*

$$C^*(E) = \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu) = w_i \textit{ for some } i\}$$

*Proof.* By Corollary 2.2.6, we get $C^*(E) = \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu)\}$. Let $S_\mu S_\nu^*$ be one of these spanning elements with $|\mu| = k$ and $|\nu| = l$. If $s(\mu)$ is already a source $w_i$, we are done. Otherwise, Corollary 2.2.3 allows us to insert $P_{s(\mu)}$ to get $S_\mu P_{s(\mu)} S_\nu^*$. Since $s(\mu)$ is not a source by assumption, we may apply (CK2) at $s(\mu)$ to get $S_\mu S_\nu^* = \sum_{\{e \in E^1 \mid r(e) = s(\mu)\}} S_\mu S_e S_e^* S_\nu^*$. Since by construction $s(\mu) = r(e)$ and $s(\nu) = s(\mu) = r(e)$, the concatenations $\mu e$ and $\nu e$ are paths again and our term becomes $\sum_{\{e \in E^1 \mid r(e) = s(\mu)\}} S_{\mu e} S_{\nu e}^*$. Now, we have $|\mu e| = k + 1$ and $|\nu e| = l + 1$. Since the graph is finite and has no loops, each path is of finite length and we can repeat this inductively with each summand $S_{\mu e} S_{\nu e}^*$ until every path starts at a source $w_i$. Thus, we get the desired result

$$C^*(E) = \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu) = w_i \text{ for some } i\}.$$

□

2.2.8. **Lemma.** *Let $E$ be a finite directed graph with no cycles and let $w_1, \dots, w_n$ be the sources in $E$. Also, let $s^{-1}(w_i) = \{\mu \in E^* \mid s(\mu) = w_i\}$ be the set of paths whose source is $w_i$ and let $A_i = \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu) = w_i\}$ be the subalgebra of $C^*(E)$ generated by paths in $s^{-1}(w_i)$. Then we get*

$$A_i \cong M_{|s^{-1}(w_i)|}(\mathbb{C})$$

*Proof.* Let $\mu, \nu \in s^{-1}(w_i)$ be two paths with source $w_i$. Since $w_i$ is a source, neither of the two paths can be a true suffix of the other one. More precisely, whenever $\nu = \mu\nu'$ holds we already have $\nu = \mu$. For $\mu, \nu, \alpha, \beta \in s^{-1}(w_i)$, the formula from Corollary 2.2.5 can then be simplified to

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_\mu S_\beta^* & \text{if } \alpha = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu) = w_i\}$ is a set of matrix units which span $A_i$. Since the graph is finite and has no loops, the set $s^{-1}(w_i)$ is finite. Then, by Example 1.2.19, $A_i$ is isomorphic to $M_{|s^{-1}(w_i)|}(\mathbb{C})$. □

With these two lemmas we can now prove the classification of graph $C^*$-algebras associated to finite directed graphs without cycles.

2.2.9. **Proposition** ([11] Proposition 1.18)**.** *Let $E$ be a finite directed graph with no cycles and let $w_1, \dots, w_n$ be the sources in $E$. Then*

$$C^*(E) \cong \bigoplus_{i=1}^n M_{|s^{-1}(w_i)|}(\mathbb{C})$$

*where $s^{-1}(w_i) = \{\mu \in E^* \mid s(\mu) = w_i\}$ is the set of paths whose source is $w_i$.*

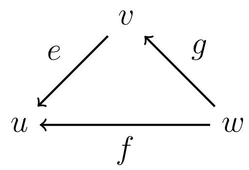
*Proof.* By Lemma 2.2.7 we get

$$C^*(E) = \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu) = w_i \text{ for some } i\}.$$

We define $A_i := \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu) = w_i\}$. By Lemma 2.2.8, $A_i$ is isomorphic to $M_{|s^{-1}(w_i)|}(\mathbb{C})$. For $i \neq j$, two paths $\nu \in s^{-1}(w_i)$ and $\alpha \in s^{-1}(w_j)$ can not be suffixes for each other, and hence for $S_\mu S_\nu^* \in A_i$ and $S_\alpha S_\beta^* \in A_j$, Proposition 2.2.4(b) gives us $(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = 0$ and thus $A_i A_j = 0$ where the product of the sets is meant in the sense of all products of elements. Since these subalgebras are pairwise orthogonal, we get $C^*(S, P) \cong \bigoplus_{i=1}^n A_i$. □

2.3. **Examples of graph $C^*$-algebras.** With the tools presented so far, we now show some examples of graph $C^*$-algebras. The first example will be exemplary for the class of graphs we have classified in Proposition 2.2.9.

2.3.1. **Example** ([11] Example 1.17)**.** Let $E$ be the following graph:



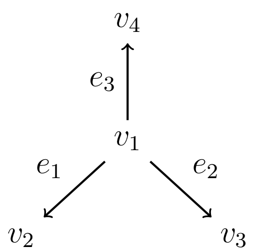

Since $E$ is finite and has no loops, we can apply Proposition 2.2.9. In this graph, the only source is $w$ and the set $s^{-1}(w)$ of paths originating in $w$ is given by $\{w, g, f, eg\}$. Thus, we get

$$C^*(E) \cong M_4(\mathbb{C}).$$

2.3.2. **Remark.** Proposition 2.2.9 lets us easily find other graphs whose associated $C^*$-algebra is also isomorphic to $M_4(\mathbb{C})$. The only requirements we have is for the graph to be finite and without loops such that there is exactly one source and 4 different paths originating from that source (including the path of length 0 that is the vertex itself). Thus both the graph $F$ given by

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} v_4$$

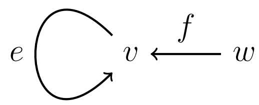
and the graph $G$ given by




yield graph $C^*$-algebras that are isomorphic to $M_4(\mathbb{C})$. The graphs $E$, $F$ and $G$ are however clearly non-isomorphic.

We now revisit the graph from Example 1.1.2 because it will turn out to induce a famous $C^*$-algebra.

2.3.3. **Proposition** ([11] Example 1.19)**.** *Let $E$ be the following graph:*




*Then $C^*(E)$ is isomorphic to the* Toeplitz algebra $\mathcal{T}$*, the universal $C^*$-algebra generated by a non-unitary isometry $V$.*

*Proof.* The Cuntz-Krieger relations give us the following equations for the Cuntz-Krieger $E$-family $\{S, P\}$:

- $S_e^* S_e = P_v$
- $S_f^* S_f = P_w$
- $P_v = S_e S_e^* + S_f S_f^*$

By Proposition 2.1.10, we get that $P_v + P_w$ is the identity on $C^*(E)$. Meanwhile, the element $S_e + S_f$ satisfies

$$\begin{aligned}(S_e + S_f)^*(S_e + S_f) &= S_e^* S_e + S_e^* S_f + S_f^* S_e + S_f^* S_f \\ &= P_v + 0 + 0 + P_w \\ &= 1_{C^*(E)}\end{aligned}$$

where we used Proposition 2.1.9. Hence, $S_e + S_f$ is an isometry on $C^*(S, P)$. Furthermore, we get

$$\begin{aligned}(S_e + S_f)(S_e + S_f)^* &= S_e S_e^* + S_e S_f^* + S_f S_e^* + S_f S_f^* \\ &= S_e S_e^* + 0 + 0 + S_f S_f^* \\ &= P_v\end{aligned}$$

where we used Proposition 2.1.9(d). Thus, $S_e + S_f$ is a non-unitary isometry. By the universal property of the Toeplitz algebra $\mathcal{T}$, we get a homomorphism $\psi : \mathcal{T} \to C^*(E)$ mapping $V$ to $S_e + S_f$ and $1_{\mathcal{T}}$ to $1_{C^*(E)} = P_v + P_w$. On the other hand, we want to find a Cuntz-Krieger $E$-family in $\mathcal{T}$. We notice that

$$(VV^*)(VV^*) = V(V^*V)V^* = V1_{\mathcal{T}}V^* = VV^* = (VV^*)^*$$

and thus $P'_v := VV^* \in \mathcal{T}$ is an orthogonal projection. By

$$1_{\mathcal{T}}(VV^*) = VV^* = (VV^*)1_{\mathcal{T}}$$

and Proposition 1.2.8, $P'_w := 1_{\mathcal{T}} - VV^*$ is also a projection. By

$$P'_w + P'_v = (1_{\mathcal{T}} - VV^*) + VV^* = 1_{\mathcal{T}}$$

and Proposition 1.2.9 they are orthogonal. Due to

$$\begin{aligned}(VVV^*)^*(VVV^*) &= VV^*V^*VVV^* \\ &= VV^* \\ &= P'_v\end{aligned}$$

and Proposition 1.2.13, $S'_e := VVV^*$ is a partial isometry that satisfies (CK1). Analogously,

$$\begin{aligned}(V(1_{\mathcal{T}} - VV^*))^*(V(1_{\mathcal{T}} - VV^*)) &= (1_{\mathcal{T}} - VV^*)^*V^*V(1_{\mathcal{T}} - VV^*) \\ &= (1_{\mathcal{T}} - VV^*)^*(1_{\mathcal{T}} - VV^*) \\ &= 1_{\mathcal{T}} - VV^* = P'_w\end{aligned}$$

shows that $S'_f := V(1_{\mathcal{T}} - VV^*)$ is also a partial isometry that satisfies (CK1). Finally, we compute

$$\begin{aligned}S'_e S'^*_e + S'_f S'^*_f &= VVV^*(VVV^*)^* + V(1_{\mathcal{T}} - VV^*)(V(1_{\mathcal{T}} - VV^*))^* \\ &= V(VV^*)(VV^*)V^* + V(1_{\mathcal{T}} - VV^*)(1_{\mathcal{T}} - VV^*)V^* \\ &= V(VV^*)V^* + V(1_{\mathcal{T}} - VV^*)V^* \\ &= V(VV^* + 1_{\mathcal{T}} - VV^*)V^* \\ &= VV^* \\ &= P'_v.\end{aligned}$$

Thus $\{S', P'\}$ with $S' = \{S'_e, S'_f\}$ and $P' = \{P'_v, P'_w\}$ is a Cuntz-Krieger $E$-family in $\mathcal{T}$. The universal property of $C^*(E)$ then gives us a homomorphism $\varphi : C^*(E) \to \mathcal{T}$ specified by

$$\begin{aligned}
\varphi(P_v) &= P'_v = VV^* \\
\varphi(P_w) &= P'_w = 1_{\mathcal{T}} - VV^* \\
\varphi(S_e) &= S'_e = VVV^* \\
\varphi(S_f) &= S'_f = V(1_{\mathcal{T}} - VV^*).
\end{aligned}$$

To conclude, we show that the homomorphisms $\varphi$ and $\psi$ are inverse to each other. It suffices to show this on the generators of the universal $C^*$-algebras. First, we check

$$\begin{aligned}
(\varphi \circ \psi)(V) &= \varphi(\psi(V)) \\
&= \varphi(S_e + S_f) \\
&= VVV^* + V(1_{\mathcal{T}} - VV^*) \\
&= V(VV^* + 1_{\mathcal{T}} - VV^*) \\
&= V \\
&= \mathrm{id}_{\mathcal{T}}(V).
\end{aligned}$$

and

$$\begin{aligned}
(\varphi \circ \psi)(1_{\mathcal{T}}) &= \varphi(\psi(1_{\mathcal{T}})) \\
&= \varphi(P_v + P_w) \\
&= VV^* + 1_{\mathcal{T}} - VV^* \\
&= 1_{\mathcal{T}} \\
&= \mathrm{id}_{\mathcal{T}}(1_{\mathcal{T}})
\end{aligned}$$

For the other direction, we have already computed $(S_e + S_f)^*(S_e + S_f) = P_v + P_w = 1_{C^*(E)}$ and $(S_e + S_f)(S_e + S_f)^* = P_v$ above and thus

$$\begin{aligned}
(\psi \circ \varphi)(P_v) &= \psi(\varphi(P_v)) \\
&= \psi(VV^*) \\
&= (S_e + S_f)(S_e + S_f)^* \\
&= P_v \\
&= \mathrm{id}_{C^*(E)}(P_v)
\end{aligned}$$

and analogously

$$\begin{aligned}
(\psi \circ \varphi)(P_w) &= \psi(\varphi(P_w)) \\
&= \psi(1_{\mathcal{T}} - VV^*) \\
&= 1_{C^*(E)} - (S_e + S_f)(S_e + S_f)^* \\
&= P_v + P_w - P_v \\
&= P_w \\
&= \mathrm{id}_{C^*(E)}(P_w).
\end{aligned}$$

For the partial isometries we compute

$$\begin{aligned}(\psi\circ\varphi)(S_e) &= \psi(\varphi(S_e))\\ &= \psi(VVV^*)\\ &= (S_e+S_f)P_v\\ &= S_eP_{s(e)}+S_fP_{s(e)}\\ &= S_e\\ &= \mathrm{id}_{C^*(E)}(S_e)\end{aligned}$$

and

$$\begin{aligned}(\psi\circ\varphi)(S_f) &= \psi(\varphi(S_f))\\ &= \psi(V(1-VV^*))\\ &= (S_e+S_f)P_w\\ &= S_eP_{s(f)}+S_fP_{s(f)}\\ &= S_f\\ &= \mathrm{id}_{C^*(E)}(S_f).\end{aligned}$$

Thus, $\varphi$ and $\psi$ are inverse to each other and we get

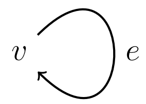
$$C^*(E)\cong\mathcal{T}.$$

□

While we still followed [11] here, we want to note that the proof of the previous proposition differs from the one given by Raeburn, since ours uses the universal properties of the universal $C^*$-algebras $C^*(E)$ and $\mathcal{T}$ whereas Raeburn relies on Coburn's Theorem ([10], Theorem 3.5.18) for the isomorphism.

If we remove the incident edge $f$ and its source $w$ from the example above, we get another interesting example.

2.3.4. **Example.** Let $E$ be the graph with one vertex and a loop based at that vertex:




The Cuntz-Krieger relations give us $S_e^*S_e = P_v = S_eS_e^*$. Again by Lemma 2.1.10, $P_v$ acts as the identity for $C^*(E)$ and thus $S_e$ is a unitary. Thus, $C^*(E)$ is generated by 1 and a unitary and is thus isomorphic to the continuous functions on the circle $C(\mathcal{S}^1)$ ([3], Example II.8.3.2 (ii)). Note that just removing the edge $f$ from the graph from Proposition 2.3.3 would not yield the same result since then $P_v + P_w$ would be the unit for the graph $C^*$-algebra rather than $P_v$.

## 3. Cuntz-Krieger algebras

In this section we show the origin for graph $C^*$-algebras. Recall that the definition we used for Cuntz-Krieger $E$-families and graph $C^*$-algebras comes from [8] from

1998. In fact, these $C^*$-algebras generated by Cuntz-Krieger families are a generalization of a class of $C^*$-algebras first described in [5] in 1980. Here, the authors studied $C^*$-algebras $\mathcal{O}_A$ generated by partial isometries whose relations were largely given by a matrix $A$ of a certain type. Those $C^*$-algebras in turn were a generalization of the *Cuntz algebra* $O_n$, the universal $C^*$-algebra generated by $n \geq 2$ isometries $S_1, \ldots, S_n$ with $\sum_{i=1}^{n} S_i S_i^* = 1$, introduced in [4] in 1977. We will show the extent of this generalization from Cuntz-Krieger algebras to graph $C^*$-algebras and make the correspondence clear by showing how the Cuntz algebra $\mathcal{O}_n$ can be seen as a representative of either class. Unfortunately, the nomenclature is not unambiguous with several objects sharing similar names. There are sources that call the object $C^*(E)$ the *Cuntz-Krieger algebra for the graph* $E$. We will speak of Cuntz-Krieger algebras associated to a matrix $A$ when we mean objects as defined in [5]. It is worth noting, that the statements in this section come from [11] again, we have however provided our own proofs using the modern language of universal $C^*$-algebras.

We start by stating the definition of Cuntz-Krieger algebras associated to a matrix $A$ as given in [5].

3.0.1. **Definition** (Cuntz-Krieger algebra)**.** Let $A = (A_{ij})_{i,j \in \Sigma}$ be a matrix where $\Sigma$ is a finite index set, the entries $A_{ij}$ are in $\{0, 1\}$ and every row and column is non-zero. The *Cuntz-Krieger algebra* $\mathcal{O}_A$ *associated to the matrix* $A$ is the universal $C^*$-algebra generated by partial isometries $\{S_i\}_{i \in \Sigma}$ such that their initial projections $Q_i := S_i^* S_i$ and final projections $P_i := S_i S_i^*$ satisfy the relations

**(A):** $P_i P_j = 0$ for $i \neq j$ and $Q_i = \sum_{\{j \in \Sigma\}} A_{ij} P_j$.

We claim that the class of graph $C^*$-algebras contains the class of Cuntz-Krieger algebras associated to matrices. We will now show in two steps, how this is indeed the case. In Section 3.1, we will show that whenever we have a matrix $A$ as in Definition 3.0.1, we find a graph $E$ such that we again have an isomorphism between $C^*(E)$ and $\mathcal{O}_A$. In Section 3.2, we will see that if we have a graph $E$ that underlies some limitations, we can find a matrix $A$ such that we have an isomorphism between the $C^*$-algebras again. We want to note that Cuntz-Krieger algebras originally had no intrinsic relation to graphs at all. The first connection was made by Kumjian, Pask, Raeburn and Renault [9] in 1997 who associated to a graph $G$ a groupoid $C^*$-algebra $C^*(\mathcal{G})$ and showed that this groupoid $C^*$-algebra is the universal $C^*$-algebra generated by partial isometries underlying Cuntz-Krieger relations dependent on the graph $G$. This construction is already very close to the definition of graph $C^*$-algebras in [8] published only one year later in 1998 also by Kumjian, Pask and Raeburn.

3.1. **Finding $C^*(E)$ isomorphic to $\mathcal{O}_A$.** We start with a Cuntz-Krieger algebra $\mathcal{O}_A$ and look for a graph $E$ such that $\mathcal{O}_A \cong C^*(E)$. We outline the procedure to make it more transparent. In Definition 3.1.1, we find a graph $E_A$ dependent on the matrix $A$. In Lemma 3.1.3, we then find a family of partial isometries satisfying the relations (A) from Definition 3.0.1 in $C^*(E)$. The universal property of $\mathcal{O}_A$ then yields a $*$-homomorphism $\varphi : \mathcal{O}_A \to C^*(E)$. Next, in Lemma 3.1.4, we find a Cuntz-Krieger $E$-family in $\mathcal{O}_A$. The universal property of $C^*(E)$ then yields a $*$-homomorphism $\psi : C^*(E) \to \mathcal{O}_A$. Finally, in Proposition 3.1.5, we show that the $*$-homomorphisms are inverse to each other.

3.1.1. **Definition.** Let $A = (A_{ij})_{i,j \in \Sigma}$ as in Definition 3.0.1. We define by $E_A$ the graph whose adjacency matrix is $A$. More precisely, we have $E_A^0 = \Sigma$ and $ij \in E_A^1$ if and only if $A_{ij} = 1$. For the edge $ij$ its source and range are defined by $s_A(ij) = j$ and $r_A(ij) = i$ respectively.

3.1.2. **Remark.** In the previous definition, the conditions imposed on $A$ from Definition 3.0.1 give us some characteristics of the graph $E_A$. Since $A$ is finite, so is $E_A$ and since every entry in $A$ is either 0 or 1, there is at most one edge between any two vertices in $E_A$. Most importantly, since $A$ has no zero rows or columns, $E_A$ has neither sinks nor sources. Notationwise, writing edges as $ij$ lets us immediately see their source and range. This allows us for instance to write $\{j \in \Sigma \,|\, ij \in E_A^1\}$ rather than $\{j \in \Sigma \,|\, \exists e \in E_A^1$ such that $r_A(e) = i$ and $s_A(e) = j\}$.

We can now proceed along the outline given above to prove $\mathcal{O}_A \cong C^*(E_A)$.

3.1.3. **Lemma.** *Let $A = (A_{ij})_{i,j\in\Sigma}$ as in Definition 3.0.1 and let $E_A$ be the graph whose adjacency matrix is $A$. Denote by $\{S_i\}_{i\in\Sigma}$ the partial isometries generating $\mathcal{O}_A$. Further, denote by $\{T_{ij}\}_{ij\in E_A^1}$ the partial isometries and by $\{R_i\}_{i\in\Sigma}$ the projections generating $C^*(E_A)$. Then, there is a $*$-homomorphism $\varphi : \mathcal{O}_A \to C^*(E)$ mapping $S_i$ to $\sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij} = \sum_{\{j\in\Sigma\}} A_{ij}T_{ij}$.*

*Proof.* We define $s_i := \sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij} \in C^*(E_A)$. We want to show that $\{s_i\}_{i\in\Sigma}$ is a family of partial isometries satisfying the relations (A) from Definition 3.0.1. For better readability, we define $q_i := s_i^* s_i$ and $p_i := s_i s_i^*$. First we need to check that $s_i$ is actually a partial isometry. We compute

$$\begin{aligned} p_i &= s_i s_i^* \\ &= \Big(\sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij}\Big)\Big(\sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij}\Big)^* \\ &= \sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij}T_{ij}^* + \sum_{\{j,k\in\Sigma \,|\, ij,ik\in E_A^1, k\neq j\}} T_{ij}T_{ik}^* \\ &= \sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij}T_{ij}^* + 0 \\ &= R_i \end{aligned}$$

where we have used Proposition 2.1.9 (d) for the second to last equality and (CK2) for the last equality. By Proposition 1.2.13, $s_i$ is a partial isometry indeed and $q_i$ and $p_i$ are its initial and final projection respectively. Now we need to check if the relations (A) hold. Let $i \neq j \in \Sigma$. Then we get

$$p_i p_j = R_i R_j = 0$$

since the projections $R_i$ and $R_j$ are mutually orthogonal by assumption. Finally we compute

$$\begin{aligned} q_i &= s_i^* s_i \\ &= \Big(\sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij}\Big)^* \Big(\sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij}\Big) \\ &= \sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij}^* T_{ij} + \sum_{\{j,k\in\Sigma \,|\, ij,ik\in E_A^1, k\neq j\}} T_{ij}^* T_{ik} \\ &= \sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} T_{ij}^* T_{ij} + 0 \\ &= \sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} R_j \\ &= \sum_{\{j\in\Sigma \,|\, ij\in E_A^1\}} p_j \\ &= \sum_{\{j\in\Sigma\}} A_{ij} p_j \end{aligned}$$

where we have used Proposition 2.1.9 (b) for the fourth equality and (CK1) for the fifth equality. Thus $\{s_i\}_{i\in\Sigma}$ satisfies the relations (A). The universal property of $\mathcal{O}_A$ then gives us a $*$-homomorphism $\varphi : \mathcal{O}_A \to C^*(E)$ mapping $S_i$ to $s_i$. □

3.1.4. **Lemma.** *Let $A = (A_{ij})_{i,j\in\Sigma}$ as in Definition 3.0.1 and let $E_A$ be the graph whose adjacency matrix is $A$. Denote by $\{S_i\}_{i\in\Sigma}$ the partial isometries generating $\mathcal{O}_A$. Further, denote by $\{T_{ij}\}_{ij\in E_A^1}$ the partial isometries and by $\{R_i\}_{i\in\Sigma}$ the projections generating $C^*(E_A)$. Then, there is a $*$-homomorphism $\psi : C^*(E) \to \mathcal{O}_A$ mapping $T_{ij}$ to $S_iS_jS_j^*$ and $R_i$ to $S_iS_i^*$.*

*Proof.* We proceed analogously to the proof of Lemma 3.1.3. Recall that in Definition 3.0.1 we have defined the initial and final projections of $S_i$ as $Q_i = S_i^*S_i$ and $P_i = S_iS_i^*$ respectively. We define $t_{ij} := S_iS_jS_j^*$ and $r_i := S_iS_i^*$. We want to show that $\{t, r\}$ with $t = \{t_{ij}\}_{ij\in E_A^1}$ and $r = \{r_i\}_{i\in\Sigma}$ forms a Cuntz-Krieger $E_A$-family. First, let $ij$ be an edge in $E_A$. We check that

$$\begin{aligned} t_{ij}^* t_{ij} &= (S_iS_jS_j^*)^*(S_iS_jS_j^*) \\ &= (S_jS_j^*)(S_i^*S_i)(S_jS_j^*) \\ &= P_jQ_iP_j \\ &= \sum_{\{k\in\Sigma\}} (P_jA_{ik}P_kP_j) \\ &= P_j + \sum_{\{k\in\Sigma, k\neq j\}} P_jA_{ik}P_kP_j \\ &= P_j + 0 \\ &= P_j \end{aligned}$$

where we used the condition (A) from Definition 3.0.1 to split $Q_i$ up as a sum for the fourth equality and to get the pairwise orthogonality of the projections $P_j$ and $P_k$ for $k \neq j$ for the second to last equality. Note, that since $ij$ is an edge in $E_A$, we especially have $A_{ij} = 1$ and as such $P_j$ is indeed present in the summation

$\sum_{\{k\in\Sigma\}} A_{ik}P_k$. Since $P_j$ is a projection, by Proposition 1.2.13 $t_{ij}$ is a partial isometry indeed. The projections $r_i = P_i$ are also pairwise orthogonal by condition (A).

It remains to show that $\{t, r\}$ satisfy (CK1) and (CK2). In fact, the calculation from above already shows (CK1), since

$$t_{ij}^* t_{ij} = P_j = r_j = r_{s_{E_A}(ij)}$$

where $s_{E_A}$ is the source map in the graph $E_A$. For (CK2), let $i \in \Sigma$. As explained in Remark 3.1.2, $i$ is not a source and we get

$$\begin{aligned} r_i &= r_i r_i \\ &= S_i(S_i^* S_i)S_i^* \\ &= S_i Q_i S_i^* \\ &= S_i\Big(\sum_{\{j\in\Sigma\}} A_{ij} S_j S_j^*\Big)S_i^* \\ &= S_i\Big(\sum_{\{j\in\Sigma\,|\,ij\in E^1\}} S_j S_j^*\Big)S_i^* \\ &= S_i\Big(\sum_{\{j\in\Sigma\,|\,ij\in E^1\}} S_j S_j^* S_j S_j^*\Big)S_i^* \\ &= \sum_{\{j\in\Sigma\,|\,ij\in E^1\}} S_i S_j S_j^* S_j S_j^* S_i^* \\ &= \sum_{\{j\in\Sigma\,|\,ij\in E^1\}} (S_i S_j S_j^*)(S_i S_j S_j^*)^* \\ &= \sum_{\{j\in\Sigma\,|\,ij\in E^1\}} t_{ij} t_{ij}^* \end{aligned}$$

where we rewrote (CK2) in the sense of Remark 3.1.2. Thus $\{t, r\}$ is a Cuntz-Krieger $E_A$-family in $\mathcal{O}_A$. The universal property of $C^*(E)$ then gives us a $*$-homomorphism $\psi : C^*(E) \to \mathcal{O}_A$ mapping $T_{ij}$ to $t_{ij}$ and $R_i$ to $r_i$. □

We can now conclude the proof for the claim we made.

3.1.5. **Proposition.** *Let $A = (A_{ij})_{i,j\in\Sigma}$ as in Definition 3.0.1 and let $E_A$ be the graph whose adjacency matrix is $A$. Then we have*

$$\mathcal{O}_A \cong C^*(E_A).$$

*Proof.* As before, we denote by $\{S_i\}_{i\in\Sigma}$ the partial isometries generating $\mathcal{O}_A$ and we denote by $\{T_{ij}\}_{ij\in E_A^1}$ the partial isometries and by $\{R_i\}_{i\in\Sigma}$ the projections generating $C^*(E_A)$.

By Lemma 3.1.3, there is a $*$-homomorphism $\varphi : \mathcal{O}_A \to C^*(E)$ mapping $S_i$ to $\sum_{\{j\in\Sigma\,|\,ij\in E_A^1\}} T_{ij} = \sum_{\{j\in\Sigma\}} A_{ij}T_{ij}$. By Lemma 3.1.4, there is a $*$-homomorphism $\psi : C^*(E) \to \mathcal{O}_A$ mapping $T_{ij}$ to $S_iS_jS_j^*$ and $R_i$ to $S_iS_i^*$.

We check that the two $*$-homomorphisms are inverse to each other on the generators of the $C^*$-algebras, since they are then inverse everywhere. First, for $S_i \in \mathcal{O}_A$, we compute

$$\begin{aligned}(\psi\circ\varphi)(S_i) &= \psi(\varphi(S_i))\\ &= \psi\Big(\sum_{\{j\in\Sigma\,|\,ij\in E_A^1\}} T_{ij}\Big)\\ &= \sum_{\{j\in\Sigma\,|\,ij\in E_A^1\}} \psi(T_{ij})\\ &= \sum_{\{j\in\Sigma\,|\,ij\in E_A^1\}} S_iS_jS_j^*\\ &= S_i\sum_{\{j\in\Sigma\,|\,ij\in E_A^1\}} S_jS_j^*\\ &= S_i\sum_{\{j\in\Sigma\}} A_{ij}P_j\\ &= S_iQ_i\\ &= S_iS_i^*S_i\\ &= S_i\end{aligned}$$

where we used the relation (A) for the third to last equation. Conversely, let $T_{ij}$ and $R_i \in C^*(E_A)$. Recall that we have shown

$$\Big(\sum_{\{j\in\Sigma\,|\,ij\in E_A^1\}} T_{ij}\Big)\Big(\sum_{\{j\in\Sigma\,|\,ij\in E_A^1\}} T_{ij}\Big)^* = R_i$$

in the proof of Lemma 3.1.3. We compute

$$\begin{aligned}(\varphi\circ\psi)(T_{ij}) &= \varphi(\psi(T_{ij}))\\ &= \varphi(S_iS_jS_j^*)\\ &= \Big(\sum_{\{k\in\Sigma\,|\,ik\in E_A^1\}} T_{ik}\Big)\Big(\sum_{\{k\in\Sigma\,|\,jk\in E_A^1\}} T_{jk}\Big)\Big(\sum_{\{k\in\Sigma\,|\,jk\in E_A^1\}} T_{jk}\Big)^*\\ &= \Big(\sum_{\{k\in\Sigma\,|\,ik\in E_A^1\}} T_{ik}\Big)R_j\\ &= \Big(\sum_{\{k\in\Sigma\,|\,ik\in E_A^1\}} T_{ik}R_k\Big)R_j\\ &= T_{ij}\end{aligned}$$

where we used Lemma 2.1.8 for the second to last equality and the fact that the projections $R_i$ are mutually orthogonal for the last one. Finally, using the equation from the proof of Lemma 3.1.3 again, we compute

$$\begin{aligned}(\varphi\circ\psi)(R_i) &= \varphi(\psi(R_i))\\ &= \varphi(S_iS_i^*)\\ &= \Big(\sum_{\{k\in\Sigma\,|\,ik\in E_A^1\}} T_{ik}\Big)\Big(\sum_{\{k\in\Sigma\,|\,ik\in E_A^1\}} T_{ik}\Big)^*\\ &= R_i.\end{aligned}$$

We have now shown $(\psi\circ\varphi) = \mathrm{id}_{\mathcal{O}_A}$ and $(\varphi\circ\psi) = \mathrm{id}_{C^*(E)}$ on the generators of $\mathcal{O}_A$ and $C^*(E)$ respectively and thus already on the entire $C^*$-algebras. Hence, $\varphi$ and

$\psi$ are inverse to each other and we get

$$\mathcal{O}_A \cong C^*(E).$$

□

3.2. **Finding $\mathcal{O}_A$ isomorphic to $C^*(E)$.** We have now seen that we can find a graph $E$ for a Cuntz-Krieger algebra $\mathcal{O}_A$ such that $C^*(E)$ is isomorphic to $\mathcal{O}_A$. In Remark 3.1.2 we have noted that this graph $E$ is finite, has no multiple edges and neither sinks nor sources. This raises the question if we can also show a similar result when starting with a graph $E$ underlying the same limitations. We will show that this is indeed the case, but we need to introduce the edge matrix of a graph first.

3.2.1. **Definition.** Let $E = (E^0, E^1, r_E, s_E)$ be a finite directed graph. We denote by $M_E$ the *edge matrix* of $E$, the $|E^1| \times |E^1|$-matrix given by

$$M_E(e,f) = \begin{cases} 1 & \text{if } s_E(e) = r_E(f) \\ 0 & \text{otherwise.} \end{cases}$$

This matrix has an entry of 1 at position $(e, f)$ if and only if the concatenation of edges $ef$ is a path of length 2 in $E$. As such, it coincides with the adjacency matrix of the *line graph* of $E$, given by $\widehat{E} = (E^1, E^2, r_{\widehat{E}}, s_{\widehat{E}})$, where $r_{\widehat{E}}(ef) = e$ and $s_{\widehat{E}}(ef) = f$.

3.2.2. **Lemma.** *Let $E$ be a finite directed graph. The line graph $\widehat{E}$ of $E$ has no multiple edges between vertices and thus its adjacency matrix $A_{\widehat{E}} = M_E$ only has entries in $\{0,1\}$. Further, if $E$ has no sinks, neither has $\widehat{E}$ and the edge matrix $M_E$ has no zero-columns. If $E$ has no sources, neither has $\widehat{E}$ and $M_E$ has no zero-rows.*

*Proof.* The first statement follows directly from the definition of the edge matrix $M_E$ of the graph $E$. It is clearly a matrix with entries only in $\{0,1\}$. Since the edge matrix of $E$ is the adjacency matrix of the line graph $\widehat{E}$, it has at most one edge connecting any two vertices.

For the second statement, assume first that $E$ has no sinks. Then, for any edge $e \in E^1$ its range $r(e) \in E^0$ must admit an edge $f \in E^1$ leaving it. This means $fe$ is in $E^2$ and thus the vertex $e$ in $\widehat{E}$ has the outgoing edge $fe$ and is not a sink. Since $fe$ in a path of length 2, the entry $(f, e)$ of $M_E$ is 1 and thus the $e$-column is non-zero. Analogously, if $E$ has no sources, for any edge $e \in E^1$ its source $s(e) \in E^0$ must admit an edge $f \in E^1$ incident to it. Then the vertex $e$ in $\widehat{E}$ has the incident edge $ef$ and $e$ is not a source. This also means that the entry $(e, f)$ of $M_E$ is 1 and thus the $e$-row is non-zero. □

3.2.3. **Remark.** The previous lemma makes apparent why we introduced the edge matrix. For a finite directed graph $E$, it is not a true in general that the adjacency matrix $A_E$ of $E$ has only entries in $\{0,1\}$. However, the statement is true for the edge matrix. If we also require $E$ to have neither sinks nor sources, then $M_E$ is of the form required in Definition 3.0.1. We can thus give the following corollary to Proposition 3.1.5.

3.2.4. **Corollary.** *Let $E$ be a finite directed graph without sinks or sources. Then $\mathcal{O}_{M_E}$, the Cuntz-Krieger algebra associated to the edge matrix of $E$, is isomorphic to $C^*(\widehat{E})$, the graph $C^*$-algebra of the line graph $\widehat{E}$ of $E$.*

*Proof.* By Lemma 3.2.2, the edge matrix $M_E$ is of the form specified in Definition 3.0.1 and thus the Cuntz-Krieger algebra $\mathcal{O}_{M_E}$ does indeed exist. By Definition 3.2.1, the edge matrix $M_E$ is also the adjacency matrix $A_{\widehat{E}}$ of the line graph $\widehat{E}$ of $E$. By Proposition 3.1.5 we then directly get

$$\mathcal{O}_{M_E} \cong C^*(\widehat{E}).$$

□

To conclude showing that for each finite graph $E$ without sinks or sources we can find a matrix $A$ such that $C^*(E)$ is isomorphic to $\mathcal{O}_A$, it remains to show that $C^*(E)$ is isomorphic to $C^*(\widehat{E})$. To show this, we will again find $*$-homomorphisms inverse to each other as we did with Proposition 3.1.5. In fact, the $*$-homomorphisms will be almost identical to the previous case but we need to change the proofs of their existence to reflect the different relations in the universal $C^*$-algebras. We can relax some of the requirements for the graph $E$, giving us a more general statement. We need the following lemma to allow us to consider a graph that is only row-finite rather than finite.

3.2.5. **Lemma.** *Let $E$ be a row-finite directed graph. Then its line graph $\widehat{E}$ is also row-finite.*

*Proof.* We prove the lemma by contraposition. Assume the line graph $\widehat{E}$ is not row-finite. Then there exists a vertex $e \in E^1 = \widehat{E}^0$ in the line graph such that $r_{\widehat{E}}^{-1}(e)$ is infinite. This means there are infinitely many edges $f \in E^1$ in the graph $E$, such that $ef$ is a path of length 2 or in other words an edge in $\widehat{E}$. This again means that there are infinitely many edges $f$ in the graph $E$ with $r_E(f) = s_E(e)$ and thus the vertex $s_E(e)$ breaks the row-finiteness of $E$. □

The previous lemma allows us to take the graph $C^*$-algebra for the line graph $\widehat{E}$ even when $E$ is only row-finite and not finite. We now show the existence of the two $*$-isomorphisms analogously to Lemma 3.1.3 and Lemma 3.1.4.

3.2.6. **Lemma.** *Let $E$ be a row-finite directed graph with no sources. Denote by $\{S_e\}_{e\in E^1}$ the partial isometries and by $\{P_v\}_{v\in E^0}$ the projections generating $C^*(E)$. Let $\widehat{E}$ be the line graph of $E$ and denote by $\{\hat{S}_{fe}\}_{fe\in E^2}$ the partial isometries and by $\{\hat{P}_e\}_{e\in E^1}$ the projections generating $C^*(\widehat{E})$. Then there is a $*$-homomorphism $\varphi : C^*(\widehat{E}) \to C^*(E)$ mapping $\hat{S}_{fe}$ to $S_f S_e S_e^*$ and $\hat{P}_e$ to $S_e S_e^*$.*

*Proof.* We define $\hat{s}_{fe} := S_f S_e S_e^*$ and $\hat{p}_e := S_e S_e^*$. We want to show that $\{\hat{s}, \hat{p}\}$ with $\hat{s} = \{\hat{s}_{fe}\}_{fe\in E^2}$ and $\hat{p} = \{\hat{p}_e\}_{e\in E^1}$ forms a Cuntz-Krieger $\widehat{E}$-family in $C^*(E)$. First, let $fe \in E^2$ be an edge in $\widehat{E}$. We check that

$$\begin{aligned}
\hat{s}_{fe}^*\hat{s}_{fe} &= (S_f S_e S_e^*)^*(S_f S_e S_e^*) \\
&= (S_e S_e^*)(S_f^* S_f)(S_e S_e^*) \\
&= S_e S_e^* P_{s_E(f)} S_e S_e^* \\
&= S_e S_e^* P_{r_E(e)} S_e S_e^* \\
&= S_e S_e^* \\
&= \hat{p}_e
\end{aligned}$$

where we have used (CK1) for the third equality, the fact that $fe$ is a path in $E$ for the fourth equality and Lemma 2.1.8 for the fifth equality. $\hat{p}_e = S_e S_e^*$ is clearly a projection and thus $\hat{s}_{fe}$ is a partial isometry by Proposition 1.2.13. By Proposition

2.1.9(a), the projections $\hat{p}_e$ are also mutually orthogonal. The calculation from above also already shows that (CK1) holds for $\{\hat{s},\hat{p}\}$. For (CK2), let $e \in E^1$ be a vertex in $\widehat{E}$. By Lemma 3.2.2, it is not a source and we compute

$$\begin{aligned}
\hat{p}_e &= S_e S_e^* \\
&= S_e P_{s_E(e)} S_e^* \\
&= S_e \Big( \sum_{\{g \in E^1 \,|\, r_E(g) = s_E(f)\}} S_g S_g^* \Big) S_e^* \\
&= S_e \Big( \sum_{\{g \in E^1 \,|\, r_E(g) = s_E(f)\}} (S_g S_g^*)(S_g S_g^*) \Big) S_e^* \\
&= \sum_{\{g \in E^1 \,|\, r_E(g) = s_E(f)\}} S_e (S_g S_g^*)(S_g S_g^*) S_e^* \\
&= \sum_{\{g \in E^1 \,|\, r_E(g) = s_E(f)\}} \hat{s}_{eg}
\end{aligned}$$

where we applied (CK2) at the vertex $s_E(e)$ for the third equality. This is exactly (CK2) at the vertex $e \in \widehat{E}^0 = E^1$ rewritten in the sense of Remark 3.1.2. Thus $\{\hat{s},\hat{p}\}$ is a Cuntz-Krieger $\widehat{E}$-family in $C^*(E)$. The universal property of $C^*(\widehat{E})$ then gives us a $*$-homomorphism $\varphi : C^*(\widehat{E}) \to C^*(E)$ mapping $\hat{S}_{fe}$ to $S_f S_e S_e^*$ and $\hat{P}_e$ to $S_e S_e^*$. □

3.2.7. **Lemma.** *Let $E$ be a row-finite directed graph with no sources. Denote by $\{S_e\}_{e \in E^1}$ the partial isometries and by $\{P_v\}_{v \in E^0}$ the projections generating $C^*(E)$. Let $\widehat{E}$ be the line graph of $E$ and denote by $\{\hat{S}_{fe}\}_{fe \in E^2}$ the partial isometries and by $\{\hat{P}_e\}_{e \in E^1}$ the projections generating $C^*(\widehat{E})$. Then there is a $*$-homomorphism $\psi : C^*(E) \to C^*(\widehat{E})$ mapping $S_f$ to $\sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe}$ and $P_v$ to $\sum_{\{e \in E^1 \,|\, r_E(e) = v\}} \hat{P}_e$.*

*Proof.* We define $s_f := \sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe}$ and $p_v := \sum_{\{e \in E^1 \,|\, r_E(e) = v\}} \hat{P}_e$. Since $E$ has no sources, the $s_f$ and $p_v$ are non-zero, because this guarantees that the sums are non-empty. We want to show that $\{s,p\}$ with $s = \{s_f\}_{f \in E^1}$ and $p = \{p_v\}_{p \in E^0}$ forms a Cuntz-Krieger $E$-family in $C^*(\widehat{E})$. By Corollary 1.2.11, the $p_v$ are orthogonal projections. Let $v \neq w \in E^0$ be two vertices in $E$. For any two edges $e, f \in E^1$ such that $r_E(e) = v$ and $r_E(f) = w$, the edges can not be equal. Since the projections $\hat{P}_e$ are mutually orthogonal, we can then compute

$$p_v p_w = \Big( \sum_{\{e \in E^1 \,|\, r_E(e) = v\}} \hat{P}_e \Big) \Big( \sum_{\{e \in E^1 \,|\, r_E(e) = w\}} \hat{P}_e \Big) = \sum_{\{e,f \in E^1 \,|\, r_E(e) = v \neq w = r_E(f)\}} \hat{P}_e \hat{P}_f = 0$$

and thus the $p_v$ are mutually orthogonal. To see that the $s_f$ are partial isometries, let $f \in E^1$ be an edge in $E$. Then

$$\begin{aligned} s_f^* s_f &= \Big( \sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe} \Big)^* \Big( \sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe} \Big) \\ &= \sum_{\{e,g \in E^1 \,|\, r_E(g) = s_E(f) = r_E(e)\}} \hat{S}_{fe}^* \hat{S}_{fg} \\ &= \sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe}^* \hat{S}_{fe} + \sum_{\{e,g \in E^1 \,|\, r_E(g) = s_E(f) = r_E(e), e \neq g\}} \hat{S}_{fe}^* \hat{S}_{fg} \\ &= \sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe}^* \hat{S}_{fe} + 0 \\ &= \sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{P}_e \\ &= p_{s_E(f)} \end{aligned}$$

where we have used Proposition 2.1.9(b) for the fourth equality and (CK1) in $C^*(\widehat{E})$ for the fifth equality. Since we have already shown that $p_v$ is a projection, Proposition 1.2.13 tells us that the $s_f$ are partial isometries. Again, the calculation from above shows that (CK1) already holds for $\{s, p\}$. For (CK2), we first show a intermediary step. Let $e \in E^1$ be an edge. Then we compute

$$\begin{aligned} s_e s_e^* &= \Big( \sum_{\{f \in E^1 \,|\, s_E(e) = r_E(f)\}} \hat{S}_{ef} \Big) \Big( \sum_{\{f \in E^1 \,|\, s_E(e) = r_E(f)\}} \hat{S}_{ef} \Big)^* \\ &= \sum_{\{f \in E^1 \,|\, s_E(e) = r_E(f)\}} \hat{S}_{ef} \hat{S}_{ef}^* + \sum_{\{f,g \in E^1 \,|\, r_E(g) = s_E(e) = r_E(f), f \neq g\}} \hat{S}_{ef} \hat{S}_{eg}^* \\ &= \sum_{\{f \in E^1 \,|\, s_E(e) = r_E(f)\}} \hat{S}_{ef} \hat{S}_{ef}^* + 0 \\ &= \hat{P}_e \end{aligned}$$

where we used Proposition 2.1.9(d) for the third equality and (CK2) in $C^*(\widehat{E})$ for the fourth equality. Now, let $v \in E^0$ be a vertex in $E$. It is not a source by assumption and we compute

$$\sum_{\{e \in E^1 \,|\, r_E(e) = v\}} s_e s_e^* = \sum_{\{e \in E^1 \,|\, r_E(e) = v\}} \hat{P}_e = p_v.$$

Thus $\{s, p\}$ is a Cuntz-Krieger $E$-family in $C^*(\widehat{E})$. The universal property of $C^*(E)$ then gives us a $*$-homomorphism $\psi : C^*(E) \to C^*(\widehat{E})$ mapping $S_f$ to $\sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe}$ and $P_v$ to $\sum_{\{e \in E^1 \,|\, r_E(e) = v\}} \hat{P}_e$. □

We can now prove the following proposition.

3.2.8. **Proposition** ([11] Corollary 2.6). *Let $E$ be a row-finite directed graph with no sources and let $\widehat{E}$ be its line graph. Then we have*

$$C^*(E) \cong C^*(\widehat{E}).$$

*Proof.* As before, we denote by $\{S_e\}_{e \in E^1}$ the partial isometries and by $\{P_v\}_{v \in E^0}$ the projections generating $C^*(E)$ as well as by $\{\hat{S}_{fe}\}_{fe \in E^2}$ the partial isometries and by $\{\hat{P}_e\}_{e \in E^1}$ the projections generating $C^*(\widehat{E})$.

By Lemma 3.2.6, there is a $*$-homomorphism $\varphi : C^*(\widehat{E}) \to C^*(E)$ mapping $\hat{S}_{fe}$ to $S_f S_e S_e^*$ and $\hat{P}_e$ to $S_e S_e^*$. By Lemma 3.2.7, there is a $*$-homomorphism $\psi : C^*(E) \to C^*(\widehat{E})$ mapping $S_f$ to $\sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe}$ and $P_v$ to $\sum_{\{e \in E^1 \,|\, r_E(e) = v\}} \hat{P}_e$.

We check that the two $*$-homomorphisms are inverse to each other on the generators of the $C^*$-algebras, since they are then inverse everywhere. First, let $\hat{S}_{fe}$ and $\hat{P}_e$ be in $C^*(\widehat{E})$. Recall, that we have shown

$$\Big( \sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe} \Big) \Big( \sum_{\{e \in E^1 \,|\, s_E(f) = r_E(e)\}} \hat{S}_{fe} \Big)^* = \hat{P}_f$$

in the proof of Lemma 3.2.7. Using this, we compute

$$\begin{aligned}
(\psi \circ \varphi)(\hat{S}_{fe}) &= \psi(\varphi(\hat{S}_{fe})) \\
&= \psi(S_f S_e S_e^*) \\
&= \Big( \sum_{\{g \in E^1 \,|\, s_E(f) = r_E(g)\}} \hat{S}_{fg} \Big) \Big( \sum_{\{g \in E^1 \,|\, s_E(e) = r_E(g)\}} \hat{S}_{eg} \Big) \Big( \sum_{\{g \in E^1 \,|\, s_E(e) = r_E(g)\}} \hat{S}_{eg} \Big)^* \\
&= \Big( \sum_{\{g \in E^1 \,|\, s_E(f) = r_E(g)\}} \hat{S}_{fg} \Big) \hat{P}_e \\
&= \hat{S}_{fe}
\end{aligned}$$

where we used Lemma 2.1.8 for the last equality. We also compute

$$\begin{aligned}
(\psi \circ \varphi)(\hat{P}_e) &= \psi(\varphi(\hat{P}_e)) \\
&= \psi(S_e S_e^*) \\
&= \Big( \sum_{\{g \in E^1 \,|\, s_E(e) = r_E(g)\}} \hat{S}_{eg} \Big) \Big( \sum_{\{g \in E^1 \,|\, s_E(e) = r_E(g)\}} \hat{S}_{eg} \Big)^* \\
&= \hat{P}_e.
\end{aligned}$$

Thus, $(\psi \circ \varphi)$ agrees with $\mathrm{id}_{C^*(\widehat{E})}$ on the generators of $C^*(\widehat{E})$ and therefore on the entire $C^*$-algebra. For the other direction, let $S_e$ and $P_v$ be in $C^*(E)$. We compute

$$\begin{aligned}
(\varphi \circ \psi)(S_e) &= \varphi(\psi(S_e)) \\
&= \varphi\Big( \sum_{\{f \in E^1 \,|\, s_E(e) = r_E(f)\}} \hat{S}_{ef} \Big) \\
&= \sum_{\{f \in E^1 \,|\, s_E(e) = r_E(f)\}} \varphi(\hat{S}_{ef}) \\
&= \sum_{\{f \in E^1 \,|\, s_E(e) = r_E(f)\}} S_e S_f S_f^* \\
&= S_e \sum_{\{f \in E^1 \,|\, s_E(e) = r_E(f)\}} S_f S_f^* \\
&= S_e P_{s_E(e)} \\
&= S_e
\end{aligned}$$

where we used (CK2) at the vertex $s_E(e)$ for the second to last equality and Lemma 2.1.8 for the last equality. Finally, we compute

$$\begin{aligned}(\varphi \circ \psi)(P_v) &= \varphi(\psi(P_v)) \\ &= \varphi\Big( \sum_{\{e \in E^1 \,|\, r_E(e)=v\}} \hat{P}_e \Big) \\ &= \sum_{\{e \in E^1 \,|\, r_E(e)=v\}} \varphi(\hat{P}_e) \\ &= \sum_{\{e \in E^1 \,|\, r_E(e)=v\}} S_e S_e^* \\ &= P_v\end{aligned}$$

where we used (CK2) at the vertex $v$ for the last equality. Thus, $(\varphi \circ \psi)$ agrees with $\mathrm{id}_{C^*(E)}$ on the generators of $C^*(E)$ and thus on the entire $C^*$-algebra. Hence, $\varphi$ and $\psi$ are inverse to each other and we get

$$C^*(E) \cong C^*(\widehat{E}).$$

□

We can now combine the result from Corollary 3.2.4 and Proposition 3.2.8 in the following way.

3.2.9. **Corollary.** *Let $E$ be a finite directed graph without sinks or sources and let $M_E$ be its edge matrix. Then we get*

$$C^*(M_E) \cong \mathcal{O}_A.$$

*Proof.* By Corollary 3.2.4, we have

$$\mathcal{O}_{M_E} \cong C^*(\widehat{E})$$

and by Proposition 3.2.8, we have

$$\mathcal{O}_{M_E} \cong C^*(\widehat{E}) \cong C^*(E).$$

□

3.2.10. **Remark.** We have seen that for any Cuntz-Krieger algebra $\mathcal{O}_A$, we can find a graph $E$ such that again $C^*(E) \cong \mathcal{O}_A$. This shows, that we have a subclass relationship

$$\{\text{Cuntz-Krieger algebras}\} \subseteq \{\text{Graph } C^*\text{-algebras}\}.$$

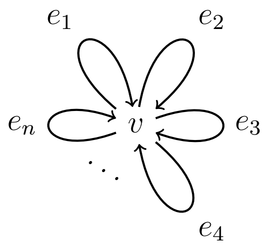
We have seen that the graph $E$ that we find is finite and has no sinks and no sources. Corollary 3.2.9 also tells us the converse: If we have a finite graph $E$ with no sinks and no sources then there is a matrix $A$ such that $C^*(E) \cong \mathcal{O}_A$. This tells us that the class of Cuntz-Krieger algebras and the class of graph $C^*$-algebras for finite graphs with no sinks and no sources are the same. To see, that graph $C^*$-algebras are actually a bigger class, consider the following. First, with graph $C^*$-algebras, we may add unrelated projections as generators by adding a disconnected vertex to the graph $E$ whereas in Cuntz-Krieger algebras the projections arise only as initial or final projections of the partial isometries. Second, and more importantly, Cuntz-Krieger algebras are limited by the finiteness of the matrix $A$, whereas graph $C^*$-algebras may be generated by infinitely many partial isometries and projections.

3.3. **The Cuntz algebra.** We want to conclude this section by presenting the Cuntz algebra $\mathcal{O}_n$ from [4] and interpret it both as a graph $C^*$-algebra and as a Cuntz-Krieger algebra.

3.3.1. **Definition.** Let $n \in \mathbb{N}$ with $n \geq 2$. The *Cuntz algebra* $\mathcal{O}_n$ is the universal $C^*$-algebra generated by $n$ isometries $S_i$ such that

$$\sum_{i=1}^{N} S_i S_i^* = 1.$$

3.3.2. **Proposition.** *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $E$ be the graph with 1 vertex and $n$-many loops at that vertex. Then $C^*(E)$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$.*




*Proof.* Let $\{S, P\}$ with $S = \{S_{e_i}\}_{i \in \{1,\dots,n\}}$ and $P = \{P_v\}$ be the Cuntz-Krieger $E$-family generating $C^*(E)$. By Lemma 2.1.10, we get that $P_v$ is the unit for $C^*(E)$. Then (CK1) yields

$$S_{e_i}^* S_{e_i} = P_v = 1_{C^*(E)}$$

and thus the partial isometries $S_{e_i}$ are actually isometries. Applying (CK2) at the vertex $v$ yields

$$1_{C^*(E)} = P_v = \sum_{i=1}^{n} S_{e_i} S_{e_i}^*.$$

and thus the isometries satisfy the relation from Definition 3.3.1. Thus the universal property of $\mathcal{O}_n$ gives us a $*$-homomorphism $\varphi : \mathcal{O}_n \to C^*(E)$ mapping $S_i$ to $S_{e_i}$. On the other hand, $\{s, p\}$ with $s = \{S_i\}_{i \in \{1,\dots,n\}}$ and $p = \{1_{\mathcal{O}_n}\}$ is a Cuntz-Krieger $E$-family which follows directly from the conditions stated in Definition 3.3.1. Thus the universal property of $C^*(E)$ gives us a $*$-homomorphism $\psi : C^*(E) \to \mathcal{O}_n$ mapping $S_{e_i}$ to $S_i$ and $P_v$ to $1_{\mathcal{O}_n}$. The $*$-homomorphisms map the partial isometries $S_{e_i}$ invertibly to the isometries $S_i$ and map $P_v = 1_{C^*(E)}$ invertibly to $1_{\mathcal{O}_n}$. They are inverse to each other on the generators of the $C^*$-algebras and thus everywhere and we get $C^*(E) \cong \mathcal{O}_n$. □

3.3.3. **Corollary.** *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $A = (A_{ij})_{i,j \in \{1,\dots,n\}}$ be the $n \times n$-matrix with $A_{ij} = 1$ for every $i, j \in \{1, \dots, n\}$. Then the Cuntz-Krieger algebra $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_n$.*

*Proof.* Let $E$ be the graph from Proposition 3.3.2. Since any two edges in $E$ form a path of length 2, the edge matrix $M_E$ is given by the matrix $A$. By Corollary 3.2.9, we then get

$$\mathcal{O}_A \cong C^*(E)$$

and by Proposition 3.3.2, we get

$$\mathcal{O}_A \cong C^*(E) \cong \mathcal{O}_n.$$

□

3.3.4. **Remark.** The class of Cuntz-Krieger algebras extends the class of Cuntz algebras by allowing for different relations between the generators. However, it should be noted that Cuntz in his original paper [4] also included the *infinite Cuntz algebra* $\mathcal{O}_\infty$, the universal $C^*$-algebra generated by countably infinitely many isometries $S_i$ such that $\sum_{i=1}^r S_i S_i^* \leq 1$ holds for every finite sum. Due to its infiniteness, this $C^*$-algebra is clearly not a Cuntz-Krieger algebra. The graph $E$ with one vertex and countably infinitely many edges is also not row-finite. Definition 2.1.1 requires the graph to be row-finite and thus there is no graph $C^*$-algebra for this graph $E$ in the sense of Definition 2.1.4. However, it is possible to define Cuntz-Krieger families also for non-row-finite graphs by adding additional constraints as in [1].

3.3.5. **Definition.** Let $E$ be a (countable) directed graph. A *Cuntz-Krieger $E$-family* $\{S, P\}$ consists of

*(a)* a set $P = \{P_v \mid v \in E^0\}$ of mutually orthogonal projections and
*(b)* a set $S = \{S_e \mid e \in E^1\}$ of partial isometries with mutually orthogonal ranges

such that

**(G1):** $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$,
**(G2):** $S_e S_e^* \leq P_{r(e)}$ and
**(G3):** $P_v = \sum_{\{e \in E^1 \mid r(e)=v\}} S_e S_e^*$ for every $v \in E^0$ such that $0 < |r^{-1}(v)| < \infty$.

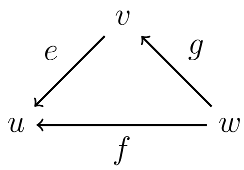
3.3.6. **Remark.** Note that this definition is consistent with Definition 2.1.1 for row-finite graphs: We have shown in Proposition 2.1.9(a) that in the row-finite case the range projections of the partial isometries of a Cuntz-Krieger family are mutually orthogonal. We see immediately that (G1) is the same as (CK1). If $E$ is row-finite, the condition $0 < |r^{-1}(v)| < \infty$ translates exactly to "$v$ is not a sink" and then (G3) is the same as (CK2). Finally, in the row-finite case, (G2) already follows from (G3) and is thus not axiomatic. For more information, we refer to [11] Chapter 5 and to [1].

## 4. Representations of graph $C^*$-algebras

In this section we study representations of graph $C^*$-algebras on Hilbert spaces. Recall from Definition 1.2.4 that a representation of a $C^*$-algebra $A$ on a Hilbert space $H$ is a $*$-homomorphism $\pi : A \to B(H)$. If $\pi$ is injective, we call the representation faithful. In this case, $A$ is isomorphic to a $C^*$-subalgebra of $B(H)$. By Theorem 1.2.5, every $C^*$-algebra admits a faithful representation. In this section we will revisit examples from this thesis and look at representations of them. Then we will give a general algorithmic approach to find a concrete representation for any graph $C^*$-algebra. Finally, we will state two uniqueness theorems for graph $C^*$-algebras that give conditions under which representations are faithful. For their proofs we refer to [2] and [11]. The first of these uniqueness theorems is purely algebraic and functional analytic in nature without concretely caring about the graph $E$. It requires the definition of the gauge action $\gamma$. The second uniqueness theorem on the other hand hinges on the graph's structure. If the graph satisfies a condition (L) introduced in [8], then any non-trivial representation is faithful.

4.1. **Examples of representations of graph $C^*$-algebras.** If we take a row-finite directed graph $E$ and a Hilbert space $H$, we can find a representation of $C^*(E)$ on $H$ by finding a Cuntz-Krieger family in $B(H)$. In this case, the universal property of $C^*(E)$ already provides the $*$-homomorphism required. We now revisit the graphs we have used as examples in Section 2.3.

4.1.1. **Example.** Let $E$ be the graph from Example 2.3.1:



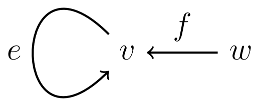

We have seen in Example 2.3.1, that $C^*(E)$ is isomorphic to $M_4(\mathbb{C})$. Since $M_4(\mathbb{C})$ is the $C^*$-algebra of linear bounded operators on the Hilbert space $\mathbb{C}^4$, this was already the first representation of a graph $C^*$-algebra we have seen. Since we have an isomorphism here, this representation is even faithful. However, we have not concretely shown how the generators are mapped. By retracing our construction, we can find this mapping. We show it exemplary for the partial isometry $S_e$. In the proof of Corollary 2.2.6, we find $S_e$ in $\overline{\operatorname{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu)\}$ as $S_e S_{s(e)}^*$. Then we can apply the iterative expansion mentioned in the proof of Lemma 2.2.7 to get $S_e = S_{eg} S_g^*$. By Lemma 2.2.8 and Example 1.2.19, this gets mapped to the matrix unit $E_{eg,g}$ under our isomorphism. If we repeat this for all our generators, we get the following correspondences

$$\begin{array}{lll} S_e \sim E_{eg,g}, & S_f \sim E_{f,w}, & S_g \sim E_{g,w}, \\ P_u \sim E_{eg,eg} + E_{f,f}, & P_v \sim E_{g,g}, & P_w \sim E_{w,w}. \end{array}$$

It is easy to see that these matrix units form a Cuntz-Krieger $E$-family and thus the universal property of $C^*(E)$ would also yield an equivalent representation. The representations might differ in the order of the basis $\{e_w, e_f, e_g, e_{eg}\}$ of $\mathbb{C}^4$. In this case there is a unitary matrix $U$ encoding this change of basis and hence the representations are equivalent.

In the previous example we already knew there was a representation of $C^*(E)$ on the Hilbert space $M_4(\mathbb{C})$. The next example is different because we have shown in Proposition 2.3.3 that the graph $C^*$-algebra is isomorphic to the Toeplitz algebra $\mathcal{T}$. We can thus pick a representation of $\mathcal{T}$ on a Hilbert space and get a representation of our graph $C^*$-algebra.

4.1.2. **Example.** Let $E$ be the graph from Proposition 2.3.3:




We have seen that $C^*(E)$ is isomorphic to $\mathcal{T}$. The classical example of a representation of the Toeplitz algebra is given on $H = \ell^2$ with the unilateral shift $S$ by Coburn's Theorem ([10], Theorem 3.5.18). Let $\pi_{\mathcal{T}} : \mathcal{T} \to B(\ell^2)$ be this representation and let $\varphi : C^*(E) \to \mathcal{T}$ be the isomorphism from Proposition 2.3.3. We then get a representation of $C^*(E)$ on $\ell^2$ by $\pi_{C^*(E)} := \pi_{\mathcal{T}} \circ \varphi$. This representation acts as follows:

- $\pi_{C^*(E)}(P_v)(x_0, x_1, x_2, \ldots) = SS^*(x_0, x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$
- $\pi_{C^*(E)}(P_w)(x_0, x_1, x_2, \ldots) = (1 - SS^*)(x_0, x_1, x_2, \ldots) = (x_0, 0, 0, \ldots)$
- $\pi_{C^*(E)}(S_e)(x_0, x_1, x_2, \ldots) = SSS^*(x_0, x_1, x_2, \ldots) = (0, 0, x_1, x_2, \ldots)$

- $\pi_{C^*(E)}(S_f)(x_0, x_1, x_2, \ldots) = S(1 - SS^*)(x_0, x_1, x_2, \ldots) = (0, x_0, 0, \ldots)$

In the previous example we saw that the subspace $\pi_{C^*(E)}(P_v)H$ is infinite-dimensional. It might seem intuitive for this to be the case, since there is a loop at the vertex $v$. This raises the question on whether the subspace will always be infinite-dimensional regardless of the choice of the representation and whether a vertex with a loop always yields a subspace that is infinite-dimensional. We will answer both of those questions. In order to do so, we first look at the following lemma.

4.1.3. **Lemma** ([11] Remark 1.6)**.** *Let $H$ be a Hilbert space and let $\{S, P\}$ be a Cuntz-Krieger family in $B(H)$ for a row-finite directed graph $E$. Then $S_e$ is an isometry from its initial space $P_{s(e)}H$ onto a closed subspace of $P_{r(e)}H$. Furthermore, in this case we get*

$$P_v H \cong \bigoplus_{\{e \in E^1 \mid r(e) = v\}} S_e H$$

*for each vertex $v \in E^0$ that is not a source.*

*Proof.* Proposition 1.2.13 tells us that $S_e^* S_e = P_{s(e)}$ is the initial projection onto $(\ker S)^\perp$ and the equation $S_e = P_{r(e)} S_e$ tells us that the range of $S_e$ is a subspace of $P_{r(e)}H$. Together with Definition 1.2.12, this tells us that $S_e$ is an isometry from $P_{s(e)}H$ onto a subspace of $P_{r(e)}H$. The second equation is an immediate consequence of Corollary 1.2.11 and (CK2). □

Now we can answer the first of the two questions we posed before.

4.1.4. **Proposition** ([11] Example 1.11)**.** *Let $E$ be the graph from Example 4.1.2 again and let $\{S, P\}$ be the Cuntz-Krieger $E$-family generating $C^*(E)$. Also, let $\pi$ be a representation of $C^*(E)$ on a Hilbert space $H$ such that $\pi(P_v)$ and $\pi(P_w)$ are non-zero. Then the subspace $\pi(P_v)H$ is infinite-dimensional.*

*Proof.* We show a more general statement first. Let $\{T, Q\}$ be a Cuntz-Krieger $E$-family acting on a Hilbert space $H$. By Lemma 4.1.3, $T_e$ is an isometry from its initial space $Q_{s(e)}H = Q_v H$ onto its range $T_e H$ and thus $\dim(Q_v H) = \dim(T_e H)$. Analogously we also see $\dim(Q_w H) = \dim(T_f H)$. The Cuntz-Krieger condition at $v$ then implies

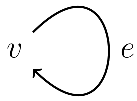
$$\dim(Q_v H) = \dim(T_f H) + \dim(T_e H) = \dim(Q_w H) + \dim(Q_v H)$$

Therefore, if $Q_v$ and $Q_w$ are both non-zero, $Q_v H$ must be infinite-dimensional.

Since $\pi$ is a $*$-homomorphism and $\{S, P\}$ is a Cuntz-Krieger $E$-family, $\{\pi(S), \pi(P)\}$ is a Cuntz-Krieger $E$-family in $B(H)$. By assumption $\pi(P_v)$ and $\pi(P_w)$ are non-zero. The previous general statement then directly proves the claim. □

4.1.5. **Remark.** The space $\pi(P_v)H$ associated to the vertex $v$ being infinite-dimensional is a result ”caused” by the additional incident edge $f$ at a vertex that also has the loop $e$. In fact, we can see that the above equation for the dimensionality must necessarily arise whenever we have a vertex with a loop and another incident edge. To answer the second question we asked, we look at another example now:

4.1.6. **Example** ([11] Example 1.10)**.** Let $E$ be the graph from Example 2.3.4:




We have seen that $C^*(E)$ is generated by $S_e$ which is a unitary element and $P_v = 1_{C^*(E)}$. We can thus find a representation $\pi$ on $H = \mathbb{C}$ with $\pi(S_e) = e^{i\vartheta}$. The subspace $\pi(P_v)H = \mathbb{C}$ is finite-dimensional despite being associated to a vertex with a loop. Thus, a loop is not enough to result in an infinite-dimensional subspace.

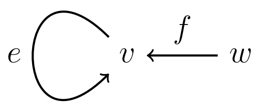
We have now seen some examples for representations of graph $C^*$-algebras. Our approaches to find these representations have been varied but we would like to have a straightforward approach that guarantees a non-trivial representation, where by non-trivial we mean that our generators will not be mapped to zero. Such a construction exists and we will present it now

4.1.7. **Proposition** ([11] Example 1.9)**.** *Let $E$ be a row-finite directed graph. Pick a separable infinite-dimensional Hilbert space $H_v$ for each $v \in E^0$ and set $H$ : $\bigoplus_{\{v \in E^0\}} H_v$. Next, decompose each $H_v$ as a direct sum $H_v = \bigoplus_{\{e \in E^1 \,|\, r(e)=v\}} H_{v,e}$ of infinite-dimensional subspaces. Then there is a non-trivial representation of $C^*(E)$ on $H$.*

*Proof.* We define $P_v$ as the orthogonal projection onto $H_v$ for each $v \in E^0$. Since the spaces $H_v$ are mutually orthogonal, so are the projections $P_v$. Next, we define $S_e$ as the unitary isomorphism of $H_{s(e)}$ onto $H_{r(e),e}$ for each $e \in E^1$. By Definition 1.2.12, $S_e$ is a partial isometry on $H$ with initial space $H_{s(e)}$. Hence we get $S_e^* S_e = P_{s(e)}$. By definition $P_v$ is the projection onto $H_v$ and by Proposition 1.2.13, $S_e S_e^*$ is the projection onto $H_{r(e),e}$. The decomposition $H_v = \bigoplus_{\{e \in E^1 \,|\, r(e)=v\}} H_{v,e}$ then yields $P_v = \sum_{\{e \in E^1 \,|\, r(e)=v\}} S_e S_e^*$. Thus, the $\{S, P\}$ with $S = \{S_e\}_{e \in E^1}$ and $P = \{P_v\}_{v \in E^0}$ is a Cuntz-Krieger $E$-family in $B(H)$ with each of the $S_e$ and $P_v$ non-zero. The universal property of $C^*(E)$ then yields a representation on $H$ mapping the generators of $C^*(E)$ to $\{S, P\}$. □

4.1.8. **Remark.** As we have already eluded to in Remark 2.1.5, the previous proposition shows that the universal $C^*$-algebra $C^*(E)$ is non-zero (for $E$ non-empty) because we can always find a non-zero representation on a Hilbert space. In order to visualize this construction will now apply it to the graph from Example 4.1.2.

4.1.9. **Example.** Let $E$ be the graph from Example 1.1.2.




According to the construction, we pick a separable infinite-dimensional Hilbert space for both $H_v$ and $H_w$. Since any such Hilbert space is isomorphic to $\ell^2$, we may as well pick it and arrive at

$$H = H_v \oplus H_w = \ell^2 \oplus \ell^2$$

and set $P_v$ as the canonical projection onto $H_v$ and $P_w$ as the canonical projection onto $H_w$. Next, we decompose both $H_v$ and $H_w$ into a direct sum of as many elements as they have incident edges. For $H_v$ we get $H_v = H_{v,e} \oplus H_{v,f}$ and since both subspaces are to be infinite-dimensional again, we again pick $\ell^2$ for both of them via

$$H_{v,e} = \overline{\operatorname{span}}\{e_{2n-1} \mid n \in \mathbb{N}\} \cong \ell^2$$

and

$$H_{v,f} = \overline{\operatorname{span}}\{e_{2n} \mid n \in \mathbb{N}\} \cong \ell^2$$

where $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis for $\ell^2$. As $w$ has no incident edges, we are done with this step. We now define the partial isometry corresponding to the edge $e$ as the unitary isomorphism from the space $H_{s(e)}$ onto $H_{r(e),e}$, that is

$$S_e : H_v \to H_{v,e}$$

and analogously for the edge $f$

$$S_f : H_w \to H_{v,f}.$$

As both the domain and the range of those maps are isomorphic to $\ell^2$, we naturally get unitary isomorphisms that we view as partial isometries on the entire space $H$. Overall, we get

$$H = H_v \oplus H_w = (H_{v,e} \oplus H_{v,f}) \oplus H_w.$$

To arrive at the result we have seen in Example 4.1.2, we can modify this algorithmically generated space. We notice that the unitary isometry $S_e : (H_{v,e} \oplus H_{v,f}) \to H_{v,e}$ does indeed force $H_{v,e}$ to be infinite-dimensional. However, the only restriction on the dimensionality of $H_{v,f}$ is the dimensionality of $H_w$ due to $S_f$, which in turn is also not required to be infinite-dimensional. Thus, rather than picking $\ell^2$ for $H_w$, we can also pick $\mathbb{C}$. The isomorphism $H = \ell^2 \oplus \mathbb{C} \to \ell^2$ acting via $((x_i)_{i\in\mathbb{N}}, \zeta) \mapsto (\zeta, x_1, x_2, \ldots)$ then gives us the same representation we have found in Example 4.1.2.

4.2. **Uniqueness theorems for graph $C^*$-algebras.** In this subsection we introduce the gauge action on $C^*(E)$ and the concept of an entry to a cycle in a graph $E$. With these concepts we can state two uniqueness theorems. They have different requirements but both give a tool to show faithfulness of a representation on a Hilbert space.

4.2.1. **Definition.** Let $G$ be a locally compact group and let $A$ be a $C^*$-algebra. An *action* of the group $G$ on $A$ is a group homomorphism $s \mapsto \alpha_s$ of $G$ into the automorphism group $\operatorname{Aut} A$ of $A$ such that $s \mapsto \alpha_s(a)$ is continuous for each fixed $a \in A$.

4.2.2. **Proposition** ([11] Proposition 2.1)**.** *Let $E$ be a row-finite directed graph and let $\{S, P\}$ be the Cuntz-Krieger $E$-family generating $C^*(E)$. Then there is an action $\gamma$ of the circle $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ on $C^*(E)$ such that $\gamma_z(S_e) = zS_e$ for every $e \in E^1$ and $\gamma_z(P_v) = P_v$ for every $v \in E^0$. This action is called the* gauge action *of $\mathbb{T}$ on $C^*(E)$*

*Proof.* First, fix $z \in \mathbb{T}$. Then $\{zS, P\}$ is a Cuntz-Krieger $E$-family for $zS = \{zS_e\}_{e\in E^1}$, since

$$(zS_e)^*(zS_e) = (\overline{z}S_e^*)(zS_e) = \overline{z}zS_e^*S_e = S_e^*S_e = P_{s(e)}$$

and

$$\begin{aligned} P_v &= \sum_{\{e \in E^1 \mid r(e)=v\}} S_e S_e^* \\ &= \sum_{\{e \in E^1 \mid r(e)=v\}} (z\overline{z}) S_e S_e^* \\ &= \sum_{\{e \in E^1 \mid r(e)=v\}} (zS_e)(\overline{z}S_e^*) \\ &= \sum_{\{e \in E^1 \mid r(e)=v\}} (zS_e)(zS_e)^*. \end{aligned}$$

By the universal property of $C^*(E)$, we thus get a $*$-homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(S_e) = zS_e$ and $\gamma_z(P_v) = P_v$. Since $\gamma_z$ only multiplies the generators with non-zero elements, it is an isomorphism. For $w \in \mathbb{T}$, we get

$$\gamma_z(\gamma_w(S_e)) = \gamma_z(wS_e) = zwS_e = \gamma_{zw}(S_e)$$

and

$$\gamma_z(\gamma_w(P_v)) = \gamma_z(P_v) = P_v = \gamma_{zw}(P_v)$$

for all generators $S_e$ and $P_v$ and thus $\gamma_z \circ \gamma_w = \gamma_{zw}$ on all of $C^*(E)$. Thus, $\gamma$ is a group homomorphism of $\mathbb{T}$ into $\operatorname{Aut} C^*(E)$.

For the continuity of $\gamma$, fix $z \in \mathbb{T}$, $A \in C^*(E)$ and $\varepsilon > 0$. Due to Corollary 2.2.6, we may choose $C := \sum \lambda_{\mu,\nu} S_\mu S_\nu^* \in C^*(E)$ such that $\|A - C\| < \varepsilon/3$. Since $\gamma_z$ is an automorphism of $C^*(E)$, we get $\gamma_z(S_e^*) = (\gamma_z(S_e))^* = \overline{z}S_e^*$ for each edge $e$ in $E$ and also $\gamma_z(S_\mu) = z^{|\mu|}S_\mu$ for each path $\mu \in E^*$. Recall, that $E^*$ is the space of all finite paths in $E$ and as such the length $|\mu|$ is well-defined. Taking both observations together, we get

$$\gamma_w(c) = \sum \lambda_{\mu,\nu} w^{|\mu|-|\nu|} S_\mu S_\nu^*$$

and since $\gamma_w$ acts merely by scalar multiplication on $C$, we get that $w \mapsto \gamma_w(C)$ is continuous. Hence, there exists a $\delta > 0$ such that for $|w - z| < \delta$ we get $\|\gamma_w(C) - \gamma_z(C)\| < \varepsilon/3$. Additionally, since $\gamma_z$ as an automorphism is in particular isometric, we get $\|\gamma_z(A - C)\| = \|A - C\| < \varepsilon/3$. Thus, for $|w - z| < \delta$, we have

$$\|\gamma_w(A) - \gamma_z(A)\| \leq \|\gamma_w(A - C)\| + \|\gamma_w(C) - \gamma_z(C)\| + \|\gamma_z(A - C)\| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

Since we arbitrarily chose $z \in \mathbb{T}$, we see that

$$z \mapsto \gamma_z(A)$$

is continuous for each $A \in C^*(E)$ and thus $\gamma$ is an action of $\mathbb{T}$ on $C^*(E)$. □

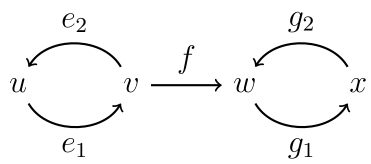
We can now state the first uniqueness theorem, also known as the gauge-invariant uniqueness theorem. It states that the gauge action already uniquely determines the graph $C^*$-algebra of $E$. More precisely, it is given as follows in [2], Theorem 2.1.

4.2.3. **Theorem** (The gauge-invariant uniqueness theorem)**.** *Let $E$ be a row-finite directed graph and let $\{S, P\}$ be the Cuntz-Krieger $E$-family generating $C^*(E)$. Also, let $\pi$ be a representation of $C^*(E)$ on a Hilbert space $H$. If each $\pi(P_v)$ is non-zero and if there is a continuous action $\beta$ of $\mathbb{T}$ on the subalgebra of $B(H)$ generated by $\{\pi(S), \pi(P)\}$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for $z \in \mathbb{T}$, then $\pi$ is faithful.*

For the proof we refer to [2]. Recall that there are two conventions used for the direction of the edges associated to partial isometries. The source above uses the other one. For the second uniquess theorem we introduce the concept of an entry to a cycle. Recall from Definition 1.1.5 that a cycle is a path $\mu = \mu_1 \cdots \mu_n$ with $n \geq 1$ such that $s(\mu_n) = r(\mu_1)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$.

4.2.4. **Definition.** Let $\mu$ be a cycle in a graph $E$. We say, the edge $e \in E^1$ is an *entry* to the cycle $\mu$ if there exists an index $i$ such that $r(e) = r(\mu_i)$ and $e \neq \mu_i$.

4.2.5. **Remark.** The condition $e \neq \mu_i$ means that $e$ is not part of the cycle itself. As such, an entry to a cycle makes sense visually. Consider the following example. Let $E$ be the graph given by




There are exactly four cycles in this graph, namely $e_2e_1, e_1e_2, g_2g_1$ and $g_1g_2$. However, only the $g$-cycles have an entry through the edge $f$ and the $e$-cycles have no entry. With this definition we can state the second uniqueness theorem as in [11] Theorem 2.4.

4.2.6. **Theorem** (The Cuntz-Krieger uniqueness theorem)**.** *Let $E$ be a row-finite graph in which every cycle has an entry and let $\{S, P\}$ be the Cuntz-Krieger $E$-family that generates $C^*(E)$. Also, let $\pi$ be a representation of $C^*(E)$ on a Hilbert space $H$. If each $\pi(P_v)$ is non-zero, then $\pi$ is faithful.*

We have changed the wording of the theorem to closer reflect the modern approach to universal $C^*$-algebras. Another reason for doing this, is making the statement relevant for representations more apparent and closer in spirit to Theorem 4.2.3. For the proof we refer to [11], Chapter 3. If every cycle in a graph $E$ has an entry, we say the graph $E$ satisfies the condition (L). This notation has been introduced in [8]. However, note that [8] uses the inverse convention of labeling the edges in a path and thus the condition (L) reads here as every cycle having an exit.

The first uniqueness theorem was mostly algebraic and functional analytic in nature. The second one however depends almost entirely on the structure of the graph. This makes it very useful to quickly show that a representation is faithful. If we consider the graphs from this section again, we quickly see, that the graph from Example 4.1.1 has no cycles at all and thus the representation given here is faithful. We had however already proven that. For the graph $E$ from Example 4.1.2, we have found two representations, one in Example 4.1.2 and one in Example 4.1.9. By the Cuntz-Krieger uniqueness theorem, both of them are faithful.

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