Saarland University
Faculty of Mathematics and Computer Science
Department of Mathematics

Bachelor's thesis

Quantum nonlocal games and Quantum commuting quantum no-signalling correlations

submitted by
Lars Lauer

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Reviewer 1: Prof. Dr. Moritz Weber
Reviewer 2: Prof. Dr. Michael Hartz

Erklärung

Ich erkläre hiermit, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Statement

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis

Saarbrücken, 07.08.2023 (Lars Lauer)

Contents

1 Introduction

In this thesis we study strategies for nonlocal games and quantum nonlocal games. Our main sources are [19], [25] and [4]. The strategies studied in this thesis for quantum nonlocal games are called quantum no-signalling correlations and quantum commuting quantum no-signalling correlations. Quantum no-signalling correlations were first defined by Duan and Winter in 2016 [9] in a different setting from quantum nonlocal games. Quantum commuting quantum no-signalling correlations and quantum nonlocal games were first defined by Todorov and Turowska in 2020 [25].

Nonlocal games are tuples $\mathcal{G} = (X, Y, A, B, \lambda)$, where X and Y are the questions for the two players Alice and Bob. These two players have to give an answer from the answer sets A and B to their questions without communicating to the other player. The referee then decides based on the function $\lambda : X \times Y \times A \times B \to \{0, 1\}$ whether Alice and Bob win. As Alice and Bob play cooperatively, they have to agree on a strategy beforehand to maximize their chance to win. There are different classes of strategies that limit the resources Alice and Bob can access. The two classes of strategies mainly studied in this thesis are no-signalling strategies and quantum commuting strategies. No-signalling strategies only limit Alice and Bob to the rule that they cannot communicate. This means Alice's answer is not dependent on Bob's question and vice versa. Quantum commuting strategies are a subclass of no-signalling strategies in which Alice and Bob share a quantum system that they can partially measure.

Quantum nonlocal games are generalizations of nonlocal games, where Alice and Bob get "quantum" questions and "quantum" answers. This is modeled by join-continuous, zero-preserving maps from the projections of a matrix algebra onto the projections of another matrix algebra. The strategies for quantum nonlocal games are given by quantum channels, which are maps that map quantum states onto quantum states, that also prevent direct communication between Alice and Bob.

In Section 2, we give a brief introduction to C^*-algebras and define positive elements and maps of C^*-algebras. We also define universal C^*-algebras.

In Section 3, we introduce traceclass operators which are a subclass of bounded operators on a Hilbert space. We then show that the traceclass operators are the predual of the bounded operators.

In Section 4, we introduce operator systems as these are needed to study nonlocal games. An operator system is a selfadjoint subspace of a unital C^*-algebra that contains the unit. Operator systems can also be defined as an abstract notion that we need in order to introduce their tensor products.

In Section 5, we give basic concepts of quantum information as these are needed to define the different strategies for nonlocal games and quantum nonlocal games.

In Section 6, we give an introduction to nonlocal games and study no-signalling and quantum commuting strategies. Then we classify perfect strategies, which are strategies that always win, by the state space of an operator system in a C^*-algebra. For mirror games, which are nonlocal games where for some questions Alice's answer is determined by Bob's answer and vice versa, we can classify the perfect quantum commuting strategies by traces from a given C^*-algebra. These classification results were shown in [19].

In Section 7, we generalize nonlocal games to quantum nonlocal games and show that nonlocal games are a subclass of quantum nonlocal games. Then we define quantum output mirror games, an analogue of mirror games for nonlocal games with classical questions but quantum answers. After this, we introduce two sets of strategies for quantum nonlocal games: quantum no-signalling correlations and quantum commuting quantum no-signalling correlations. These strategies are given by quantum channels which fulfill the no-signalling condition introduced in [9], preventing communication through this channel. Lastly, we get classification results similar to the ones, we have for nonlocal games: We can classify the strategies by the state space of an operator system in a C^*-algebra. These results were presented in [25]. For quantum output mirror games, we can classify perfect quantum commuting no-signalling strategies by traces from a given C^*-algebra. This result was presented in [4].

2 C^*-algebras

In this section, we introduce some basics on C^*-algebras. Most of these results can be found in [2].

Definition 2.1. (i) Let A be an algebra. A map $^*\colon A \to A$ is called involution if for all $v, w \in A$ and $\lambda \in \mathbb{C}$ it holds that:

(a) $(v^*)^* = v$,

(b) $(\lambda v + w)^* = \overline{\lambda} v^* + w^*$,

(c) $(vw)^* = w^* v^*$.

(ii) A C^*-algebra is an algebra A with an involution $^*\colon A \to A$ and a norm $\|\cdot\|$ on A such that

(a) $(A, \|\cdot\|)$ is complete,

(b) $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$,

(c) $\|\cdot\|$ fulfills the C^*-identity, i.e. $\|xx^*\| = \|x\|^2$ for all $x \in A$.

A C^*-algebra is called unital if there exist a neutral element with respect to the multiplication.

(iii) Let A be a C^*-algebra and $M \subseteq A$, then $C^*(M)$ is the C^*-algebra such that for all C^*-algebras $B \subseteq A$, we have $C^*(M) \subseteq B$.

(iv) Let A and B be C^*-algebras. An algebra homomorphism $\varphi\colon A \to B$ is called a *-homomorphism if $\varphi(x^*) = \varphi(x)^*$.

Example 2.2. (i) Let A be a finite set and define $\ell^\infty(A) = \{(\lambda_a)_{a\in A}; \lambda \in \mathbb{C}\}$ with pointwise addition, multiplication and $(\lambda_a)^*_{a\in A} = (\overline{\lambda_a})_{a\in A}$. If we equip $\ell^\infty(A)$ with the norm $\|(\lambda_a)_{a\in A}\| = \max_{a\in A}|\lambda_a|$, then $\ell^\infty(A)$ is a C^*-algebra.

(ii) Let H be a Hilbert space. Then

$$B(H) = \{A\colon H \to H; A \text{ is a bounded linear map}\}$$

with the operator norm $\|B\| = \sup_{h\in H, \|h\|\leq 1} \|B(h)\|$ for $B \in B(H)$ and the adjoint map $T \mapsto T^*$ as involution, defines a C^*-algebra.

(iii) Let X be a finite set and equip $\mathbb{C}^X = \oplus_{x\in X}\mathbb{C}$ with $\|x\| = \sqrt{\sum_{i\in X} x_i^2}$. The matrices M_X, equipped with operator norm $\|A\| = \sup_{x\in\mathbb{C}^X, \|x\|\le 1} \|Ax\|$ for $A \in M_X$ and $*\colon M_X \to M_X, A \mapsto A^* = \overline{A^t}$, form a C^*-algebra. Also, the diagonal matrices $D_X \subseteq M_X$ form a C^*-algebra. If $X = \{1, ..., n\}$ for a natural number n, we write M_n.

(iv) Let X be a compact Hausdorff space. Then

$$C(X) = \{f\colon X \to \mathbb{C}; f \text{ is continuous}\},\ \|f\|_\infty = \sup\{|f(x)|; x \in X\}$$

with pointwise multiplication and addition and $f^* = \overline{f}$, is a C^*-algebra.

The next theorem will give a classification of unital commutative C^*-algebras:

Proposition 2.3 (Gelfand-Naimark). *Let A be a commutative unital C^*-algebra. Then there exists a compact Hausdorff space X such that there is an isometric *-isomorphism $\phi\colon A \to C(X)$.*

Definition 2.4. Let A be a C^*-algebra and $x \in A$:

(i) x is called selfadjoint if $x = x^*$
(ii) x is called normal if $x^*x = xx^*$.
(iii) x is called a projection if $x = x^* = x^2$.
(iv) x is called a partial isometry if $x = xx^*x$.

The following lemma will give a nice characterization of partial isometries:

Lemma 2.5. *Let A be a C^*-algebra and $x \in A$. Then the following are equivalent:*

(i) *x is a partial isometry,*
(ii) *xx^* is a projection,*
(iii) *x^*x is a projection.*

Now we define positive elements.

Definition 2.6. Let A be a unital C^*-algebra and $x \in A$.

(i) We define $\mathrm{Sp}(x) = \{\lambda \in \mathbb{C}; \lambda 1 - x \text{ is not invertible in } A\}$. The set $\mathrm{Sp}(x)$ is called the spectrum of x.
(ii) An element $x \in A$ is called positive if $x = x^*$ and $\mathrm{Sp}(x) \subseteq [0, \infty)$ and we denote it by $x \geq 0$. We write A^+ for the set of positive elements in A.
(iii) Let x, y be elements in A. We write $x \geq y$ if $x - y \geq 0$.

Positive elements have multiple characterizations.

Lemma 2.7. *Let A be a C^*-algebra and $x \in A$. The following are equivalent:*

(i) *$x \geq 0$,*
(ii) *There is a selfadjoint element $y \in A$ such that $y^2 = x$,*
(iii) *There is a unique positive element $\sqrt{x} \in A$ such that $\sqrt{x}\sqrt{x} = x$,*
(iv) *There exists an element $z \in A$ such that $x = z^*z$.*

Lemma 2.8. *Let A be a C^*-algebra and $x_1, ..., x_n \in A$ be positive elements. Then $\sum_{i=1}^n x_i \geq 0$.*

The following characterization of the set of positive elements is given in [18].

Lemma 2.9. *Let A be a C^*-algebra and let $A_{sa} = \{x \in A; x = x^*\}$. Then A^+ is a convex cone, i.e.*

(i) $\lambda x \in A^+$ *for all* $\lambda \in \mathbb{C}, x \in A^+$,
(ii) $x + y \in A^+$ *for all* $x, y \in A^+$.

Moreover, $A_{sa} = \{x - y; x, y \in A^+\}$.

Definition 2.10. Let A be a C^*-algebra and H be a Hilbert space. A representation of A on H is a *-homomorphism $\pi\colon A \to B(H)$.

Proposition 2.11 (GNS-representation)**.** *Let A be a C^*-algebra and $s : A \to \mathbb{C}$ be a state. There exists a Hilbert space H_s, a representation $\pi_s : A \to B(H_s)$ and a vector $\xi_s \in H_s$ such that $s(a) = \langle \pi_s(a)\xi_s, \xi_s\rangle$ for all $a \in A$. Moreover, if A is unital, we have that π_s is unital and thus ξ_s is a unit vector.*

The next proposition follows from the GNS-construction and gives a classification of C^*-algebras as norm-closed subalgebras of bounded operators:

Proposition 2.12. *Let A be C^*-algebra. There exists a Hilbert space H and an injective representation $\pi\colon A \to B(H)$. Therefore A is isomorphic to C^*-subalgebra of $B(H)$. If A is unital the representation is unital, i.e. $\pi(1) = 1$.*

The next concept, we introduce, is universal C^*-algebras. This is just a brief introduction. A more complete introduction to the construction of universal C^*-algebras is for example presented in [18]. Universal C^*-algebras allow to define certain C^*-algebras in a more abstract way and this can be used to describe many known C^*-algebras like M_n and also lets us define tensor products and free products of C^*-algebras.

Definition 2.13. Let I be an index set and $E = \{x_i; i \in I\}$ be a set of generators. Let $P(E)$ be the set of *-polynomials in E and $R \subseteq P(E)$ be a set of relations. Denote $J(R)$ for the two sided ideal generated by R. Define $A(E,R) = P(E)/J(R)$ and

$$\|x\| = \sup\{p(x); p \text{ is a } C^*\text{-seminorm on } A(E,R)\}.$$

If for all $x \in A(E,R)$ we have $\|x\| < \infty$, we call

$$C^*(E|R) = \overline{A(E,R)/\{x \in A(E,R); \|x\| = 0\}}^{\|\cdot\|}$$

the universal C^*-algebra generated by E with relations R.

There exists a universal property for universal C^*-algebras. This is one of the most important properties as this gives us the connection between universal C^*-algebras and C^*-algebras in general.

Proposition 2.14. *Let I be an index set and $E = \{x_i; i \in I\}$ be generators and $R \subseteq P(E)$ be relations. Let B be a C^*-algebra containing $E' = \{y_i; i \in I\}$ satisfying the relations R. Then there exists a unique *-homomorphism $\varphi\colon C^*(E|R) \to B$ such that $\varphi(x_i) = y_i$ for all $i \in I$.*

Definition 2.15. Let A, B be unital C^*-algebras.

(i) Let I be a finite set, we define the free product as

$$A_{i_1} *_1 ... *_1 A_{i_n} = C^*(a \in \bigcup_{i\in I} A_i | 1_{A_i} = 1_{A_j},\text{ relations from } A_i \text{ for all } i,j \in I)$$

(ii) Let R_A, R_B be sets of relations such that $C^*(A|R_A) \cong A$ and $C^*(B|R_B) \cong B$. Then we define

$$A \otimes_{C^*\max} B = C^*(a \in A, b \in B | ab = ba, 1_A = 1_B, R_A, R_B).$$

Now we will give some examples of C^*-algebras that will be used later in this thesis. For this, we need the free product of groups. A reference for the free product of groups is for example [23].

Example 2.16. Let A be a finite set.

(i) Let G be a discrete group. We define

$$C^*(G) = C^*((u_g)_{g\in G} | u_g \text{ is a unitary}, u_g u_h = u_{gh}, u_g^* = u_{g^{-1}}).$$

Let n be a natural number and $(G_i)_{i\in\{1,\dots,n\}}$ be finite groups. Then we have $C^*(G_1) *_1 .. *_1 C^*(G_n) = C^*(G_1 * ... * G_n)$. A reference for this identity is for example [2].

(ii) Let $\mathbb{Z}_n$ be the cyclic group with n elements. Then $C^*(\mathbb{Z}_n) \cong \ell^\infty(A)$ if $|A| = n$ by the isomorphism

$$u_k \mapsto \sum_{a=1}^{n} \exp(2\pi i \frac{a}{n}) e_a$$

where e_a denotes the standard basis of $\ell^\infty(A)$. This isomorphism is taken from Remark 3.2 of [12].

(iii) Recall $\ell^\infty(A)$ from Example 2.2. Let X be another finite set. We can define $\mathcal{A}(X,A) = \ell^\infty(A) *_1 ... *_1 \ell^\infty(A)$ for $|X|$ copies of $\ell^\infty(A)$. Thus we get that $\mathcal{A}(X,A) \cong C^*(\mathbb{Z}_{|A|} * ... * \mathbb{Z}_{|A|})$ with the free product containing $|X|$ copies of $\mathbb{Z}_{|A|}$. We denote $(e_{x,a})_{x\in X, a\in A}$ for the standard basis in the x-th copy of $\ell^\infty(A)$. Since $(e_a)_{a\in A}$ generates $\ell^\infty(A)$, we get that $(e_{x,a})_{x\in X,a\in A}$ generates $\mathcal{A}(X,A)$.

(iv) Let X be another finite set. We can also define $\mathcal{B}_{X,A} = M_A *_1 ... *_1 M_A$ for $|X|$ copies of M_A. Note that $\{e_a e_{a'}^*; a, a' \in A\}$ is a basis of M_A where is the matrix $e_a e_{a'}^* \colon \mathbb{C}^A \to \mathbb{C}^A, x \mapsto e_a e_{a'}^* x$. Therefore we get that $(e_{x,a,a'})_{x\in X, a,a'\in A}$, where $e_{x,a,a'}$ is $e_a e_{a'}^*$ in the x-th copy of M_A, generates $\mathcal{B}_{X,A}$.

(v) We define

$$\tilde{M}_A = C^*(e_{a,a'}, a, a' \in A | e_{a,a'}^* = e_{a',a}, e_{a,a'} e_{a'',a'''} = \delta_{a',a''} e_{a,a'''} \text{ for all } a, a', a'', a''').$$

Now we define positive maps as these will be used in Section 6.4 to classify strategies for nonlocal games as states which are positive maps. They also play an important role in Section 7 as they are both used to define strategies for quantum nonlocal games as well as to classify these strategies.

Definition 2.17. Let A, B be C^*-algebras.

(i) A linear map $\phi\colon A \to B$ is called positive if $\phi(x) \geq 0$ for all $x \geq 0$.

(ii) Let $s\colon A \to \mathbb{C}$ be a positive map. The map s is called a state if $\|s\| = 1$.

(iii) Let $\tau\colon A \to \mathbb{C}$ be a state. τ is called a trace (or a tracial state) if $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

The following Lemma collects some useful results about positive maps:

Lemma 2.18. *Let A be a unital C^*-algebra.*

(i) *A positive map $s\colon A \to \mathbb{C}$ is a state if and only if $s(1) = 1$.*
(ii) *Let $\phi\colon A \to \mathbb{C}$ be a positive map. Then ϕ is involutive, i.e. $\phi(x^*) = \overline{\phi(x)}$ for all $x \in A$.*
(iii) *Let $\phi\colon A \to \mathbb{C}$ be a positive map. Then ϕ is continuous.*

3 Traceclass operators

This section introduces traceclass operators which are an important class of operators as the set of traceclass operators is the predual of the bounded operators. This identification will be used later on in this thesis to define quantum commuting quantum no-signalling correlations in Section 7.3. The following approach to traceclass operators is mainly taken from [7]. Although in the book it is restricted to separable Hilbert spaces, the proofs from this book that we used in this thesis either pass verbatim or just need slight adaptation for the non separable case. In this section, we will use some results from functional analysis like the polar decomposition of an operator and spectral decomposition of selfadjoint compact operators. These results can be found in [8].

Lemma 3.1. *Let H be a Hilbert space and let $A \in B(H)$ be an operator such that there exists an index set I and an orthonormal basis $(e_i)_{i\in I}$ such that the sum satisfies $\sum_{i\in I} \|Ae_i\|^2 < \infty$. Then for every orthonormal basis $(f_j)_{j\in J}$, it follows that*

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{i\in I} \|A^*e_i\|^2 = \sum_{i\in J} \|Af_j\|^2.$$

Proof. From Parsevals Identity, it follows that:

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{i\in I}\sum_{j\in J} |\langle Ae_i, f_j\rangle|^2 = \sum_{i\in I}\sum_{j\in J} |\langle e_i, A^*f_j\rangle|^2 = \sum_{j\in J} \|A^*f_j\|^2$$

For $(f_j)_{j\in J} = (e_i)_{i\in I}$ it follows that $\sum_{i\in I} \|Ae_i\|^2 = \sum_{i\in I} \|A^*e_i\|^2$ and therefore also $\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in J} \|Af_j\|^2$ for an arbitrary orthonormal basis $(f_j)_{j\in J}$. □

Definition 3.2. Let H be a Hilbert space. An operator $A \in B(H)$ is called Hilbert-Schmidt operator if there exists an orthornomal basis $(e_i)_{i\in I}$ such that $\sum_{i\in I} \|Ae_i\|^2 < \infty$. Define $\mathrm{B_{HS}}$ to be the space of Hilbert-Schmidt operators equipped with the norm $\|A\|_{\mathrm{HS}} = \sqrt{\sum_{i\in I} \|Ae_i\|^2}$ for an orthonormal basis $(e_i)_{i\in I} \in H$.

Although it is not done in this thesis, it is straightforward to check that $\|\cdot\|_{\mathrm{HS}}$ actually defines a norm.

Definition 3.3. Let H be a Hilbert space. The set $T(H) = \{AB; A, B \in \mathrm{B_{HS}}(H)\}$ is called traceclass. An operator $T \in T(H)$ is called traceclass operator.

Lemma 3.4. *Let $T \in T(H)$. Then the sum*

$$\sum_{i\in I}\langle Te_i, e_i\rangle = \sum_{j\in J}\langle Tf_j, f_j\rangle < \infty$$

for any two orthonormal basis $(e_i)_{i\in I}, (f_j)_{j\in J} \subseteq H$.

Proof. As $T \in T(H)$, there exist $A, B \in \mathrm{B_{HS}}(H)$ such that $T = AB$. Lemma 3.1 shows that $C = A^* \in \mathrm{B_{HS}}(H)$. Therefore $T = C^*B$. Let $(e_j)_{j\in J}$ be an orthonormal basis of H.

$$\begin{aligned}
&4\langle Be_j, Ce_j\rangle \\
=&(\|Be_j + Ce_j\|^2 - \|Be_j - Ce_\|^2_j) + i(\|Be_j + iCe_j\|^2 - \|Be_j - iCe_j\|^2) \\
=&(\|Be_j + Ce_j\|^2 + \|Be_j - Ce_j\|^2) + i(\|Be_j + iCe_j\|^2 + \|Be_j - iCe_j\|^2) \\
=&2(\|Be_j\|^2 + \|Ce_j\|^2) + 2i(\|Be_j\|^2 + \|iCe_j\|^2)
\end{aligned}$$

Using this identity and the triangle inequality, it follows that:

$$\sum_{j\in J}|\langle Te_j, e_j\rangle| = \sum_{j\in J}|\langle Be_j, Ce_j\rangle| \leq \sum_{j\in J}(\|Be_j\|^2 + \|Ce_j\|^2) < \infty$$

Now we show that $\sum_{i\in I}\langle Te_i, e_i\rangle$ is independent of the choice of the orthonormal basis, by using the identity at the start of the proof again:

$$\begin{aligned}
&\sum_{j\in J}\langle Te_j, e_j\rangle \\
=&\sum_{j\in J}\langle Be_j, Ce_j\rangle \\
=&\sum_{j\in J}2(\|Be_j\|^2 + \|Ce_j\|^2) + 2i(\|Be_j\|^2 + \|iCe_j\|^2) \\
=&2\sum_{j\in J}\|Be_j\|^2 + 2\sum_{j\in J}\|Ce_j\|^2 + 2i\sum_{j\in J}\|Be_j\|^2 + 2i\sum_{j\in J}\|iCe_j\|^2
\end{aligned}$$

And from Lemma 3.1 it follows that all four sums are independent of the basis. □

Lemma 3.4 shows that the following definition is well defined:

Definition 3.5. Let H be a Hilbert space and let $(e_i)_{i\in I} \subseteq H$ be an orthonormal basis. The map $\mathrm{Tr}\colon T(H) \to \mathbb{C}, T \mapsto \sum_{i\in I}\langle Te_i, e_i\rangle$ is called the trace functional.

Proposition 3.6. *Let H be a Hilbert space and let $A \in B(H)$ be an operator. The following statements are equivalent:*

(i) $\mathrm{Tr}\,|A| < \infty$
(ii) $A \in T(H)$
(iii) $|A| \in T(H)$

Proof. *(i) $\Rightarrow$ (ii):*
$|A|$ is positive, therefore it follows from Lemma 2.7 that there exists a unique

positive operator $\sqrt{|A|}$ such that $|A| = \sqrt{|A|}\sqrt{|A|}$. And for any orthonormal basis $(e_i)_{i\in I} \in H$:

$$\sum_{i\in I} \|\sqrt{|A|}e_i\|^2 = \sum_{i\in I}\langle\sqrt{|A|}e_i, \sqrt{|A|}e_i\rangle = \sum_{i\in I}\langle|A|e_i, e_i\rangle < \infty$$

So $\sqrt{|A|} \in \mathrm{B_{HS}}(H)$. Consider the polar decomposition of $A = W|A| = (W\sqrt{|A|})(\sqrt{|A|})$. So it suffices to show that $W\sqrt{|A|} \in \mathrm{B_{HS}}(H)$ to show (ii).

$$\sum_{i\in I} \|W\sqrt{|A|}e_i\|^2 \leq \sum_{i\in I} \|W\|^2\|\sqrt{|A|}e_i\|^2 < \infty \tag{1}$$

$(ii) \Rightarrow (iii)$:
$A \in T(H)$ implies that there exist $B, C \in \mathrm{B_{HS}}(H)$ such that $A = BC$. And from the polar decomposition, it also follows that $|A| = W^*A = (W^*B)C$ and analogous to (1), it follows that $W^*B \in \mathrm{B_{HS}}(H)$ and therefore (iii).
$(iii) \Rightarrow (i)$:
Follows from Lemma 3.4. □

Remark 3.7. Proposition 3.6 explains the name traceclass operator for $T(H)$ as those are exactly the operators $T \in B(H)$ such that $\mathrm{Tr}\,|T| < \infty$.

Example 3.8. Let $n \in \mathbb{N}$ be a natural number and $\mathbb{C}^n$ be the inner product $\langle u, v\rangle = \overline{v^t}u$ for $u, v \in \mathbb{C}^n$. The trace $\mathrm{Tr}(M) = \sum_{i=1}^n M_{i,i}$ for matrices $M \in M_n$, is equal to $\sum_{i\in I}\langle Me_i, e_i\rangle$ for the standard basis $e_i \in \mathbb{C}^n$ and since we can identify M_n as $B(\mathbb{C}^n)$, this shows that both definitions coincide.

Proposition 3.9. *Let H be a Hilbert space. The set $T(H)$ forms a normed vector space equipped with the norm $\|T\|_{\mathrm{Tr}} = \mathrm{Tr}(|T|)$.*

Proof. It is straightforward to check that $T(H)$ is a vector space and that $\|T\|_{\mathrm{Tr}}$ defines a norm is Theorem 1.11 (a) in [7]. □

Lemma 3.10. *Let H be a Hilbert space. For all $A \in B(H)$, $B \in \mathrm{B_{HS}}(H)$ and $T \in T(H)$, it holds:*

(i) $|\,\mathrm{Tr}(A|T|)| \leq \|A\|\|T\|_{\mathrm{Tr}}$
(ii) $\|AB\|_{HS} \leq \|A\|\|B\|_{HS}$ *and* $\|BA\|_{HS} \leq \|B\|_{HS}\|A\|$
(iii) $\|AT\|_{\mathrm{Tr}} \leq \|A\|\|T\|_{\mathrm{Tr}}$

Proof. Let $(e_i)_{i\in I} \subseteq H$ be an orthonormal basis.

(i)

$$\begin{aligned}
|\operatorname{Tr}(A|T|)| &= \left|\sum_{i\in I}\langle A|T|e_i, e_i\rangle\right| \\
&\le \sum_{i\in I} |\langle \sqrt{|T|}e_i, \sqrt{|T|}A^*e_i\rangle| \\
&\le \sum_{i\in I} \|\sqrt{|T|}e_i\|\|\sqrt{|T|}A^*e_i\| \\
&\le \sqrt{\sum_{i\in I}\|\sqrt{|T|}e_i\|^2}\sqrt{\sum_{i\in I}\|\sqrt{|T|}A^*e_i\|^2} \\
&= \|\sqrt{T}\|_{\mathrm{HS}}\|\sqrt{T}A^*\|_{\mathrm{HS}} \\
&\le \|\sqrt{T}\|_{\mathrm{HS}}^2\|A\| \\
&\le \left(\sum_{i\in I}\|\sqrt{T}e_i\|\right)\|A\| \\
&\le \left(\sum_{i\in I}\langle\sqrt{|T|}e_i, \sqrt{|T|}e_i\rangle\right)\|A\| \\
&\le \left(\sum_{i\in I}\langle|T|e_i, e_i\rangle\right)\|A\| \\
&\le \|T\|_{\mathrm{Tr}}\|A\|
\end{aligned}$$

(ii)

$$\|AB\|_{\mathrm{HS}} = \sqrt{\sum_{i\in I}\|ABe_i\|^2} \le \sqrt{\sum_{i\in I}(\|A\|\|Be_i\|)^2} = \|A\|\|B\|_{\mathrm{HS}}$$

$$\|BA\|_{\mathrm{HS}} = \|(A^*B^*)^*\|_{\mathrm{HS}} = \|A^*B^*\|_{\mathrm{HS}} \le \|A^*\|\|B^*\|_{\mathrm{HS}} = \|A\|\|B\|_{\mathrm{HS}}$$

(iii) Let $W_1|T| = T$ be the polar decomposition of T and let $W_2|AT| = AT$ be the polar decomposition of AT. Define $S = W_2^*AW_1$, then

$$\begin{aligned}
\|AT\|_{\mathrm{Tr}} &= \operatorname{Tr}(W_2^*AT) \\
&= \operatorname{Tr}(W_2^*AW_1|T|) \\
&= \operatorname{Tr}(S|T|) \\
&\le \|T\|_{\mathrm{Tr}}\|S\| \\
&\le \|T\|_{\mathrm{Tr}}\|W_2\|\|A\|\|W_1\| \\
&= \|T\|_{\mathrm{Tr}}\|A\|
\end{aligned}$$

□

Lemma 3.11. *Let H be a Hilbert space. The map $\phi_B\colon T(H) \to \mathbb{C}, T \mapsto \operatorname{Tr}(BT)$ defines a well defined continous linear function with $\|\phi_B\| \le \|B\|$ for every $B \in B(H)$.*

Proof. Let $S, T \in T(H)$, let $\lambda \in \mathbb{C}$ and let $(e_i)_{i\in I} \subseteq H$ be an orthonormal basis. The linearity follows from:

$$\begin{aligned}\phi_B(S+\lambda T) &= \sum_{i\in I}\langle B(S+\lambda T)e_i, e_i\rangle \\ &= \sum_{i\in I}\langle BSe_i, e_i\rangle + \sum_{i\in I}\lambda\langle BTe_i, e_i\rangle \\ &= \phi_B(S) + \lambda\phi_B(T)\end{aligned}$$

Let $WT = |T|$ be the polar decomposition of T. Then $\|\phi_B\| \leq \|B\|$ and the continuity follow from:

$$\begin{aligned}|\phi_B(T)| &= |\operatorname{Tr}(BW^*|T|)| \\ &\leq \|B\|\|W\|\|T\|_{\mathrm{Tr}} \\ &= \|B\|\|T\|_{\mathrm{Tr}}\end{aligned}$$

□

Let H be a Hilbert space. Then we denote the set of operators with finite dimensional range by $F(H) = \{F \in B(H); \dim(\operatorname{Im}(F)) < \infty\}$.

Lemma 3.12. *Let H be a Hilbert space. Then it holds:*

(i) $\forall B \in \mathrm{B}_{\mathrm{HS}}(H)$: $\|B\| \leq \|B\|_{HS}$
(ii) $\mathrm{B}_{\mathrm{HS}}(H) \subseteq K(H)$
(iii) $T(H) \subseteq K(H)$
(iv) $F(H) \subseteq T(H)$ *and* $F(H)$ *is dense in* $T(H)$.

Proof. (i) Let e be a unit vector. As there exists an orthonormal basis containing e, it follows that

$$\begin{aligned}&\|Ae\| \leq \|A\|_{\mathrm{HS}}\ \forall e \in H, \|e\| = 1 \\ \Rightarrow &\|A\| \leq \|A\|_{\mathrm{HS}}\end{aligned}$$

(ii) Let $A \in \mathrm{B}_{\mathrm{HS}}(H)$ and let $(e_i)_{i\in I} \subseteq H$. For $n \in \mathbb{N}$, there exists a finite set $I_n \subseteq I$ such that

$$\Big|\sum_{i\in I}\|Ae_i\|^2 - \sum_{i\in I_n}\|Ae_i\|^2\Big| < \frac{1}{n}$$

Let $A_n \in F(H)$ such that $A_n(h) = \sum_{i\in I_n}\langle h, e_i\rangle Ae_i$ for all $h \in H$.

$$\|A_n - A\| \leq \|A_n - A\|_{\mathrm{HS}} = \sqrt{\sum_{i\in I\setminus I_n}\|Ae_i\|^2} < \sqrt{\frac{1}{n}}$$

(iii) Since $T(H) = \{AB, A, B \in \mathrm{B}_{\mathrm{HS}}(H)\}$ and the compact operators form an ideal, the statement follows directly from (ii).
(iv) Let $F \in F(H)$ be a finite rank operator. Then there exists a set of $\lambda_i \in \mathbb{C}$ and orthogonal sets of unit vectors $(e_i)_{i=1}^n, (e_i)_{i=1}^n \subseteq H$ such that we can write

$F(h) = \sum_{i=1}^{n} \lambda_i \langle h, e_i \rangle f_i$ for all $h \in H$. Extend $(e_i)_{i=1}^{n}$ to an orthonormal basis $(e_i)_{i \in I}$. Let $W|F| = F$ be the polar decomposition of F.

$$\|F\|_{\mathrm{Tr}} = \sum_{i \in I} \langle |F| e_i, e_i \rangle = \sum_{i \in I} \langle W F e_i, e_i \rangle = \sum_{i=1}^{n} \langle W \lambda_i f_i, e_i \rangle < \infty$$

Let $A \in T(H)$ be a traceclass operator. We first show that $|A|$ can be approximated by operators of finite rank. Since $|A|$ is selfadjoint and also compact, it follows that there exists a decomposition of $(\lambda)_{i=1}^{\infty} \subseteq \mathbb{C}$ and an orthogonal set of unit vectors $(e_i)_{i=1}^{\infty}$ such that $Ah = \sum_{i=1}^{\infty} \lambda_i \langle h, e_i \rangle e_i$ for all $h \in H$. Define A_n as $\sum_{i=1}^{n} \lambda_i \langle h, e_i \rangle e_i$. Then extend $(e_i)_{i=1}^{\infty}$ to an orthogonal basis $(e_i)_{i \in I}$ and use $|A| - A_n = \sum_{i=k+1}^{\infty} \lambda_n \langle h, e_n \rangle e_n$ to get:

$$\||A| - A_n\|_{\mathrm{Tr}} = \sum_{i=k+1}^{\infty} \langle \lambda_i e_i, e_i \rangle = \sum_{i=k+1}^{\infty} \lambda_i \overset{n \to \infty}{\to} 0$$

since $\|A\|_{\mathrm{Tr}} = \sum_{i=1}^{\infty} \lambda_i < \infty$. Let $W|A| = A$ be the polar decomposition of A. Then consider the sequence $WA_n \in F(H)$

$$\|(A - WA_n)\|_{\mathrm{Tr}} = \|W(|A| - A_n)\|_{\mathrm{Tr}} \leq \|W\| \||A| - A_n\|_{\mathrm{Tr}} \overset{n \to \infty}{\to} 0$$

Therefore A can also be approximated by operators of finite rank and therefore $F(H)$ is dense in $T(H)$.

□

Proposition 3.13. *Let H be a Hilbert space and let*

$$T(H)^d = \{f : T(H) \to \mathbb{C}; f \textit{ is continuous and linear}\}$$

be the topological dual space of $T(H)$. The map $D \colon B(H) \to T(H)^d, B \mapsto \phi_B$ is an isometric isomorphism from $(B(H), \|\cdot\|)$ to $(T(H), \|\cdot\|_{\mathrm{Tr}})$ and therefore $T(H)$ is the predual of $B(H)$.

Proof. From Lemma 3.11 it follows that D is well defined and the linearity is easy to check. $\|D(B)\| \leq 1$ for all $B \in B(H)$ follows directly from Lemma 3.11.
Now we show $\|D(B)\| \geq 1$ for all $B \in B(H)$:
Let $0 \neq B \in B(H)$ be an operator and let $\varepsilon > 0$. Then there exists a unit vector $g \in H$ such that $\|Bg\| > \|B\| - \varepsilon$. Then consider the map $F \colon H \to H, h \mapsto \left\langle h, \frac{Bg}{\|Bg\|} \right\rangle g$. Let $(e_i)_{i \in I} \subseteq H$ be an orthonormal basis containing $\frac{Bg}{\|Bg\|}$. Then $\|F\|_{Tr} = 1$.

$$\|D(B)\| = \|\phi_B\| \geq |\operatorname{Tr}(BF)| = \left| \sum_{i \in I} \langle B \langle e_i, \frac{Bg}{\|Bg\|} \rangle g, e_i \right| = \left\langle Bg, \frac{Bg}{\|Bg\|} \right\rangle = \|Bg\|$$

As $\|Bg\| = \|B\| - \varepsilon$ and $\varepsilon > 0$ arbitrary, it follows that $\|D(B)\| \geq \|B\|$ Therefore D is an Isometry and thus also injective.
So it remains to show that D is surjective. The operators of finite rank are dense in $T(H)$ and therefore it is sufficient to show that for every $\phi \in T(H)^d$ there exists an operator $A_\phi \in B(H)$ such that $\operatorname{Tr}(A_\phi F) = \phi(F)$ for all $F \in F(H)$ with $\operatorname{rank}(F) = 1$. Let $f, h \in H$ be vectors then all rank 1 operators can be written as

$f \otimes g \colon H \to H, h \mapsto \langle h, g\rangle f$. Now consider the map $[f,g] = \phi(f \otimes g)$. It is easy to check that this is a sesquilinear form. From the continuity of ϕ and the Lax-Milgram theorem, it follows that the exists A_ϕ such that $\phi(f\otimes g) = [f,g] = \langle A_\phi f, g\rangle$. Let $(e_i)_{i\in I}$ be an orthonormal basis of H containing $\frac{f}{\|f\|}$.

$$\mathrm{Tr}(A_\phi(f\otimes g)) = \left\langle A_\phi \left\langle \frac{f}{\|f\|}, f\right\rangle g, \frac{f}{\|f\|}\right\rangle = \langle A_\phi g, f\rangle = \phi(f\otimes g)$$

This shows that D is surjective and therefore that D is an isometric isomorphism. $\square$

This Proposition essentially states that $T(H)$ is the predual of $B(H)$ and defines a map that identifies each element of $B(H)$ with an element in $T(H)^d$.

4 Operator systems

4.1 Basics on operator systems

Although the following theory was first introduced in [5], this section follows more closely the corresponding sections in [15] and [20]. We will introduce abstract operator systems and concrete operator systems and compare these two structures.

Definition 4.1. (i) A $*$-vector space is a complex vector space V and a map $^*\colon V \to V$ that is involutive (i.e. $(v^*)^* = v$ for all $v \in V$) and conjugate linear. We denote $V_h = \{v \in V; v^* = v\}$ for the set of hermitian elements of V.

(ii) An ordered $*$-vector space is a pair (V, V^+) consisting of a $*$-vector space v and a subset $V^+ \subseteq V_h$ satisfying the following conditions:

   (a) V^+ is a cone in V_h, i.e for all $u, v \in V^+, \lambda \in \mathbb{C}, \lambda \geq 0$, we have that $u + v \in V^+, \lambda v \in V^+$.

   (b) Denote $-V^+ = \{-v; v \in V^+\}$, then $V^+ \cap (-V^+) = \{0\}$.

   The elements $v \in V^+$ are called positive elements.

Lemma 4.2. *Let (V, V^+) be an ordered $*$-vector space. Then $\geq$ defines a partial order on V_h by $v \geq w \Leftrightarrow v - w \in V^+$ for $v, w \in V_h$.*

As 0 is hermitian, we can see that for $v \in V_h$, it holds that $v \geq 0 \Leftrightarrow v \in V^+$. This explains the name positive elements for $v \in V^+$.

Definition 4.3. Let (V, V^+) be an ordered $*$-vector space.

(i) An element $e \in V_h$ is called order unit if for all $v \in V_h$, there exists a real number $r > 0$ such that $re \geq v$. e is called Archimedean order unit if it also fulfills for all $v \in V$:

$$(\forall r > 0 : re + v \geq 0) \Leftrightarrow v \in V^+.$$

(ii) A triple (V, V^+, e) is Archimedean ordered $*$-vector space (AOU space) if e is an Archimedean order unit on (V, V^+).

We denote by $M_{m,n}(V)$ the $m \times n$ matrices with entries in V. Let $X \in M_{l,m}$ be a scalar matrix. Then we define XA as $(XA)_{i,j} = \sum_{k=1}^m x_{i,k} a_{k,j}$ and BX analogous for $B \in M_{n,l}$. Denote $M_n(V) = M_{n,n}(V)$. Then we can also define an involution $^*\colon M_n(V) \to M_n(V), (a_{i,j})_{i,j} \mapsto (a^*_{j,i})_{i,j}$. With this operation $M_n(V)$ becomes a $*$-vector space.

Definition 4.4. Let V be a $*$-vector space. $\{C_n\}_{n=1}^{\infty}$ is a matrix ordering on V if

(i) C_n is a cone in $M_n(V)_h$ for all $n \in \mathbb{N}$,
(ii) $C_n \cap (-C_n) = \{0\}$ for all $n \in \mathbb{N}$,
(iii) for all $n, m \in \mathbb{N}$ and $X \in M_{n,m}$, we have $X^* C_n X \subseteq C_m$.

If $\{C_n\}_{n=1}^{\infty}$ is a matrix ordering on V, we call $(V, \{C_n\}_{n=1}^{\infty})$ a matrix ordered $*$-vector space.

Lemma 4.5. *Let $(V, \{C_n\}_{n=1}^{\infty})$ be a matrix ordered $*$-vector space. Then $(M_n(V), C_n)$ is an ordered $*$-vector space for all $n \in \mathbb{N}$.*

Proof. $M_n(V)$ is a $*$-vector space with the involution defined above. If we set $M_n(V)^+ = C_n$ conditions (i) and (ii) in the definition above are exactly the conditions from Definition 4.1. □

Definition 4.6. Let $(V, C_{n=1}^{\infty})$ be a matrix ordered $*$-vector space.

(i) Let $e \in V_h$ be a selfadjoint element, we define the diagonal matrix

$$e_n \colon = \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix} \in M_n(V).$$

e is called an archimedean matrix order unit if for all $n \in \mathbb{N}$ e_n is an archimedean order unit for the ordered $*$-vector space $(M_n(V), C_n)$.

(ii) A triple $(V, \{C_n\}_{n=1}^{\infty}, e)$ is called abstract operator system if V is a $*$-vector space, $\{C_n\}_{n=1}^{\infty}$ is a matrix ordering on V and $e \in V_h$ is an Archimedean matrix order unit.

(iii) Let H be a Hilbert space. A concrete operator system $S \subseteq B(H)$ is a subspace such that $I \in S$ and $S = S^* = \{A^*; A \in S\}$.

Remark 4.7. Let $(V, \{C_n\}_{n=1}^{\infty}, e)$ be an abstract operator system, then we get that $(M_n(V), C_n, e_n)$ is an AOU space, because $(M_n(V), C_n)$ is an ordered $*$-vector space by Lemma 4.5 and e_n is an archimedean order unit for $M_n(V)$.

Definition 4.8. (i) Let $(V, \{C_n\}_{n=1}^{\infty}), (V', \{C'_n\}_{n=1}^{\infty})$ be matrix ordered $*$-vector spaces and $\phi \colon V \to V'$ be a linear map. For each $n \in \mathbb{N}$, define the map $\phi_n \colon M_n(V) \to M_n(V'), (v_{i,j})_{i,j} \mapsto (\phi(v_{i,j}))_{i,j}$. The map ϕ is called completely positive if $\phi_n(C_n) \subseteq \phi(C'_n)$ for all $n \in \mathbb{N}$. ϕ is called a complete order isomorphism if ϕ is invertible and both ϕ and ϕ^{-1} are completely positive.

(ii) Let $(V, \{C_n\}_{n=1}^{\infty}, e), (V', \{C'_n, e'\}_{n=1}^{\infty})$ be abstract operator systems such that $V \subseteq V'$. We write $V \subseteq_{\text{c.o.i.}} V'$ if the identity map $V \to V'$ is a complete order isomorphism on its image.

(iii) Let $(V, \{C_n\}_{n=1}^{\infty}, e), (V', \{C'_n, e'\}_{n=1}^{\infty})$ be abstract operator systems. A linear map $\phi \colon V \to V'$ is called unital if $\phi(e) = e'$.

Remark 4.9. Let $(V, \{C_n\}_{n=1}^{\infty}), (V', \{C'_n\}_{n=1}^{\infty})$ be matrix ordered $*$-vector spaces and $\phi \colon V \to V'$ be a completely positive map. In Lemma 4.2, we gave a partial order on $M_n(V)_h$. Now let $n \in \mathbb{N}$ be a natural number and $v, w \in M_n(V)_h$ such that $\phi(v), \phi(w) \in M_n(V')_h$ and $v \geq w$, we have that $\phi(v) \geq \phi(w)$ because

$$\phi(v) - \phi(w) = \phi(v - w) \in M_n(V')^+.$$

Therefore a completely positive map preserves the order on all matrix levels.

Let H be a Hilbert space and $S \subseteq B(H)$ be a concrete operator system. We can identify $M_n(B(H)) = B(H^n)$ where $H^n = \bigoplus_{k=1}^n H$. By doing this we get an involution on $M_n(B(H))$ and an order structure from $B(H^n)$. So $M_n(S)$ is an ordered $*$-vector space. One can also show that (S, S^+, I) defines an abstract operator system and therefore every concrete operator system is an abstract operator system. The converse is true as well. The following result was first shown in [5] and it is also shown in [20].

Proposition 4.10 (Choi-Effros)**.** *Let H be a Hilbert space and $S \subseteq B(H)$ be a concrete operator system. Then $(S, \{M_n(S)^*\}_{n=1}^\infty, I)$ is an abstract operator system. Conversely let $(V, \{C_n\}_{n=1}^\infty, e)$ be an abstract operator system. Then there exists a Hilbert space $\tilde{H}$, a concrete operator system $\tilde{S} \subseteq B(\tilde{H})$ and a complete order isomorphism $\varphi\colon V \to \tilde{S}$ such that $\varphi(e) = I_{\tilde{H}}$.*

From this proposition, we get a connection between abstract operator systems and concrete operator systems similar to the the connection we get from the GNS-construction between C^*-algebras and C^*-subalgebras of bounded operators on a Hilbert space.

From Proposition 4.10 follows that we get that we just need the term operator system and we can use either use the abstract or concrete operator system description, depending on what is needed in that case.

In Section 3 of [22] the concept of Archimedeanization of an matrix ordered $*$-vector space $(V, \{C_n\}_{n=1}^\infty$ with an matrix order unit e was introduced. The Archimedeanization $(V, \{C_n\}_{n=1}^\infty, e)$ turns into an operator system but in general the Archimedeanization of V does not have V as the underlying space but V/N, where N is a subspace that is closed under the $*$-operation. In this thesis, it is sufficient to consider the Archimedeanization of a matrix ordered $*$-vector space $(V, \{C_n\}_{n=1}^\infty)$ with a matrix order unit e such that (V, C_1, e) is an AOU space. By Proposition 3.20 in [22], we get the result:

Lemma 4.11. *Let $(V, \{C_n\}_{n=1}^\infty, e)$ be a matrix ordered $*$-vector space with matrix order unit e and (V, C_1, e) is an AOU space. Then there exists an operator system $V_{arch} = (V, C_n^{arch}, e)$, where the matrix ordering C_n^{arch} is given by $C_1^{arch} = C_1$ and*

$$C_n^{arch} = \{A \in M_n(V); re_n + A \in C_n \ \forall r > 0\}, n \geq 2.$$

4.2 Operator systems in C^*-algebras

In the previous subsection, we established a connection between abstract and concrete operator systems from Proposition 4.10. In this section, we will introduce operator systems in C^*-algebras and identify these as operator systems from Section 4.1.

Definition 4.12. Let A be a unital C^*-algebra and $S \subseteq A$ be a subspace such that $S = S^*$ and $1 \in S$. Then we call S an operator system.

Lemma 4.13. *Let A be a unital C^*-Algebra and $S \subseteq A$ be an operator system. Then there exists a concrete operator system $\tilde{S} \subseteq B(H)$ and a representation $\pi\colon A \to B(H)$ such that $\tilde{S} \cong S$.*

Proof. By Proposition 2.12, we get that there exists a unital, injective representation $\pi\colon A \to B(H)$. Consider $\tilde{S} = \pi(S)$, then $1 = \pi(1) \in \tilde{S}$ and $\pi(x)^* = \pi(x^*) \in \tilde{S}$. Therefore $\tilde{S}$ is a concrete operator system and since π is faithful $S \cong \tilde{S}$. □

From this lemma, we get a canonical way to identify operator systems of a unital C^*-Algebra A as operator systems from Section 4.1. $B(H)$ is a unital C^*-Algebra and therefore we also get a way to identify the operator systems from Section 4.1 in the C^*-algebraic way.

Let $S \subseteq A$ be an operator system in a unital C^*-algebra A. Then we can also show that $(S, S^+, 1)$ where S^+ are the positive elements of A that belong to S, form an abstract operator system. Hence all three notions are equivalent.

Now we want to define completely positive maps in C^*-algebras and introduce some of their properties. A good reference for this is [20]. For this we need to see that $M_n(A)$ is also a C^*-algebra:

Let H be a Hilbert space and let A be a C^*-algebra. Now let $\pi\colon A \to B(H)$ be an injective representation of A. We can identify $M_n(B(H)) = B(H^n)$ where $H^n = \bigoplus_{k=1}^n H$. Thus we can define

$$\pi_n\colon M_n(A) \to B(H^n) \cong M_n(B(H)), (a_{i,j}) \mapsto (\pi(a_{i,j})).$$

From this we get a C^*-Norm by $\|(a_{i,j})\| = \|\pi(a_{i,j})\|$.

Definition 4.14. Let A, B be unital C^*-algebras, $S \subseteq A$ be an operator system and $\phi\colon S \to B$ be a linear map. Let $M_n(S) \subseteq M_n(A)$ be the matrices with entries in S. And for a linear map $\phi\colon S \to B$, we define

$$\phi_n\colon M_n(S) \to M_n(B), (a_{i,j}) \mapsto (\phi(a_{i,j})).$$

In this situation, we get:

(i) ϕ is called positive if $\phi(x) \geq 0$ for all $x \in S^+ = S \cap A^+$.
(ii) ϕ is called a state if ϕ is positive and $\phi(1) = 1$.
(iii) ϕ is called n-positive if ϕ_n is positive.
(iv) ϕ is called completely positive if ϕ_n is positive for all $n \in \mathbb{N}$.

Remark 4.15. In Definition 4.8, we already defined the completely positive for maps between abstract operator systems. These two definitions coincide if we choose the abstract operator space $(S, S^+, 1)$ for the operator system S.

Analogous to Lemma 2.18 we get:

Lemma 4.16. *Let A be a unital C^*-algebra and $S \subseteq A$ an operator system.*

(i) *Let $\phi\colon S \to \mathbb{C}$ be a positive map. Then ϕ is involutive, i.e. $\phi(x^*) = \overline{\phi(x)}$ for all $x \in A$.*
(ii) *Let $\phi\colon S \to \mathbb{C}$ be a positive map. Then ϕ is continuous.*

A useful characterization of completely positive maps is given by Choi's Theorem which is proven in [20, Theorem 3.14]:

Proposition 4.17 (Choi's Theorem)**.** *Let $\mathcal{B}$ be a C^*-algebra, let $\phi\colon M_n \to B$. The following are equivalent:*

(i) *ϕ is completely positive,*
(ii) *ϕ is n-positive*
(iii) *The Choi Matrix of ϕ, which is given by $(\phi(e_i e_j^*))_{i,j=1}^n$, is positive in $M_n(B)$.*

Proposition 4.18 (Krein's Theorem). *Let A be a unital C^*-algebra, $S \subseteq A$ be an operator system and $\phi\colon S \to \mathbb{C}$ be positive. Then there exists a positive extension $\Phi\colon A \to \mathbb{C}$ such that $\Phi_{|S} = \phi$.*

In general, there are positive maps that are not completely positive. One example of this is the transpose map and this is shown in Chapter 1 of [20]. The following result is Theorem 3.9 and Theorem 3.11 in [20].

Proposition 4.19. *Let A be a commutative C^*-algebra, B be a unital C^*-algebra and $S \subseteq B$ be an operator system*

(i) *Let $\phi\colon A \to B$ be a positive map. Then ϕ is completely positive.*
(ii) *Let $\psi\colon S \to A$ be a positive map. Then ψ is completely positive.*

The following lemma is Theorem 12.8 in [20] and will be useful in a proof in Section 6.4 as it gives an extension of completely positive maps to the maximal tensor product:

Lemma 4.20. *Let A_1, A_2, B be unital C^*-algebras and $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ be completely positive maps with commuting ranges. Then there exists a completely positive map*

$$\phi_1 \otimes_{C^*max} \phi_2 : A_1 \otimes_{C^*max} A_2 \to B \ s.t. \ \phi_2 : A_1 \otimes_{C^*max} A_2(a_1 \otimes a_2) = \phi_1(a_1)\phi(a_2).$$

Lastly in this subsection, we define the coproduct of operator systems and give an example of this which will be used in Section 6.4.

Lemma 4.21. *Let $A_1, ..., A_n$ be unital C^*-Algebras and $S_1 \subseteq A_1, ..., S_n \subseteq A_n$ be operator systems. Then the set*

$$S = \operatorname{span}(\{s_1 + ... + s_n; s_i \in S_i, i = 1, ..., n\}) \subseteq A_1 *_1 ... *_1 A_n$$

is an operator system.

Proof. Since $1_{A_1} \in S_i$ and in $A_1 *_1 ... *_1 A_n$, we have $1 = 1_{A_1}$, we get $1 \in S$. That S is a linear subspace follows from the definition and since $S_1, ..., S_n$ are selfadjoint and sums of selfadjoint elements are selfadjoint, we get that S is an operator system. □

Definition 4.22. Let $A_1, ..., A_n$ be unital C^*-Algebras and $S_1 \subseteq A_1, ..., S_n \subseteq A_n$ be operator systems. Then we call the operator system S, defined in Lemma 4.21, coproduct of the operator systems $S_1, ..., S_n$.

Now we give an example of a coproduct of operator systems that will be used in Section 6.4:

Example 4.23. Let X, A be finite sets.

(i) Recall $\ell^\infty(A)$ from Example 2.2 and $\mathcal{A}(X, A)$ from Example 2.16. If we choose the operator systems to be $|X|$ copies of $\ell^\infty(A)$, we get that the coproduct of these operator systems is

$$\mathcal{S}_{X,A} = \operatorname{span}\{e_{x,a}; x \in X, a \in A\} \subseteq \mathcal{A}(X, A)$$

where $e_{x,a}$ is the a-th element of the standard basis of the x-th copy of $\ell^\infty(A)$.

(ii) Recall $\mathcal{B}_{X,A}$ from Example 2.16. Similar to the last example, if we choose the operator system to be $|X|$ copies of M_A, we get that the coproduct of these operator systems is

$$\mathcal{R}_{X,A} = \operatorname{span}\{e_{x,a,a'}; x \in X, a, a' \in A\} \subseteq \mathcal{B}(X, A)$$

where $e_{x,a,a'}$ is $e_a e_{a'}^*$ in the x-th copy of M_A.

In the literature, the coproduct of operator systems is usually defined by its universal property. For example in Section 8 in [14] or Section 5 in [10]. From the results in Section 5 in [10], we get that these two definitions coincide. Thus we get the following lemma:

Lemma 4.24. *Let $A_1, ..., A_n, B$ be unital C^*-algebras and $S_1 \subseteq A_1, ..., S_n \subseteq A_n$ be operator systems and let S be the coproduct of $S_1, ..., S_n$. Let $T \subseteq B$ be an operator system and $\phi_1 \colon S_1 \to T, ..., \phi_n \colon S_n \to T$ be unital completely positive maps. For $m \in \{1, ..., n\}$, denote by $i_m \colon S_m \to S$ the inclusion map of S_m into the coproduct. Then there exists a unique unital completely positive map $\phi \colon S \to T$ such that $\phi_m = \phi \circ i_m$ for all $m \in \{1, ..., n\}$.*

4.3 Tensor products of operator systems

The theory from this section is mainly taken from [15]. Let S and T be operator systems. We denote by $S \otimes T$ the algebraic tensor product. Let $n, m \in \mathbb{N}$ be natural numbers, then we can use the Kronecker identification of $M_n \otimes M_m \cong M_{nm}$ by identifying $(x_{i,j})_{i,j} \otimes (y_{k,l})_{k,l}$ with $(x_{i,j} y_{k,l})_{(i,k),(j,l)}$. Using this we can identify $M_n(S) \otimes M_m(T)$ by $M_{nm}(S \otimes T)$ by identifying $(s_{i,j})_{i,j} \otimes (t_{k,l})_{k,l}$ with $(s_{i,j} \otimes t_{k,l})_{(i,k),(j,l)}$. With this, we can define tensor products of operator systems that keep the operator system structure:

Definition 4.25. (i) Let $(S, \{P_n\}_{n=1}^\infty, e_1)$ and $(T, \{Q_n\}_{n=1}^\infty, e_2)$ be operator systems and let $\tau = \{C_n\}_{n=1}^\infty$ be a family of cones with $C_n \subseteq M_n(S \otimes T)$. We say $S \otimes_\tau T$ has operator system tensor structure on $S \otimes T$ if it satisfies:

(a) $(S \otimes T, \{C_n\}_{n=1}^\infty, e_1 \otimes e_2)$ is an operator system denoted $S \otimes_\tau T$,

(b) $P_n \otimes Q_m \subseteq C_{nm}$ for all $n, m \in \mathbb{N}$,

(c) If $\phi \colon S \to M_n$ and $\psi \colon T \to M_m$ are unital completely positive maps, then $\phi \otimes \psi \colon S \otimes_\tau T \to M_{mn}$ is a unital completely positive map.

We will write $M_n^+(S \otimes_\tau T) = C_n$.

(ii) Let $\mathcal{O}$ be the class of operator systems. An operator system tensor product is a map $\tau \colon \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ such that for all operator systems $S, T \in \mathcal{O}$, we have that $S \otimes_\tau T = \tau(S, T)$ has operator system tensor structure on $S \otimes T$.

(iii) Let τ_1, τ_2 be operator system tensor structures on $S \otimes T$. τ_1 is greater than τ_2 if the identity map $\mathrm{id} \colon S \otimes_{\tau_1} T \to S \otimes_{\tau_2} T$ is completely positive.

Remark 4.26. Let S, T be two operator systems and τ_1, τ_2 be operator system tensor structures on $S \otimes T$.

(i) A difference to other tensor product constructions is that the elements of $S \otimes_{\tau_1} T$ and $S \otimes_{\tau_2} T$ are always the same, as well as order unit. The difference is that the tensor products classify the positive elements of $S \otimes T$ and $M_n(S \otimes T)$ for all $n \in \mathbb{N}$.

(ii) We defined τ_1 is greater than τ_2 if the if the identity map $\mathrm{id}\colon S \otimes_{\tau_1} T \to S \otimes_{\tau_2} T$ is completely positive. This is exactly the case if $M_n(S \otimes_{\tau_1} T) \subseteq M_n(S \otimes_{\tau_2} T)$ for all $n \in \mathbb{N}$.

Definition 4.27. Let $\mathcal{O}$ denote the class of operator systems and $\tau\colon \mathcal{O} \otimes \mathcal{O} \to \mathcal{O}$ be an operator system tensor product.

(i) We call τ functorial if for any four operator systems S_1, S_2, T_1, T_2 and unital completely positive maps $\phi\colon S_1 \to S_2, \psi\colon T_1 \to T_2$, it follows that the map $\phi \otimes \psi\colon S_1 \otimes T_1 \to S_2 \otimes T_2$ is unital completely positive.
(ii) We call τ symmetric if for all operator systems S, T the linear extension of the map $\theta\colon S \otimes_\tau T \to T \otimes_\tau S, x \otimes y \to y \otimes x$ is a complete order isomorphism.
(iii) We call τ associative if for any three operator systems R, S, T the natural isomorphism from $(R \otimes_\tau S) \otimes_\tau T$ to $R \otimes_\tau (S \otimes_\tau T)$ is a complete order isomorphism.
(iv) We call τ injective if for all operator systems $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$ the inclusion map $S_1 \otimes_\tau T_1 \subseteq S_2 \otimes_\tau T_2$ is a complete order isomorphism on its range.

Remark 4.28. Let $\mathcal{O}$ denote the class of operator systems and $\tau\colon \mathcal{O} \otimes \mathcal{O} \to \mathcal{O}$ be an operator system tensor product. τ is injective if and only if

$$M_n(S_1 \otimes T_1) \cap M_n(S_2 \otimes_\tau T_2)^+ = M_n(S_1 \otimes_\tau T_1)^+.$$

Now we give three examples of operator system tensor products that were constructed in [15]. We will omit to prove that all these define operator system tensor products as this was shown in [15].

Definition 4.29. Let S, T be two operator systems and let e_S and e_T denote the Archimedean order unit of S and T respectively. Define

$$\begin{aligned} C_n^{\min}(S,T) = \{(p_{i,j}) \in M_n(S \otimes T); ((\phi \otimes \psi)(p_{i,j}))_{i,j} \in M_{nkm}^+, &\text{for all unital} \\ \text{completely positive maps } \phi\colon S \to M_k, \psi\colon T \to M_m &\forall\, k, m \in \mathbb{N}\}. \end{aligned}$$

The minimal tensor product is the mapping

$$\min\colon \mathcal{O} \times \mathcal{O} \to \mathcal{O}, (S,T) \mapsto (S \otimes T, \{C_n^{\min}(S,T)\}_{n=1}^\infty, e_S \otimes e_T)$$

and we denote it by $S \otimes_{\min} T$.

The following proposition is Theorem 4.5 in [15] and gives the reason for the name of this tensor product construction:

Proposition 4.30. *Let $\mathcal{O}$ denote the class of operator systems. The mapping min: $\mathcal{O} \times \mathcal{O} \to \mathcal{O}$ is an injective, associative, symmetric, functorial operator system tensor product and for all operator systems S, T with an operator system tensor structure $S \otimes_\tau T$, we have τ is larger than min.*

The next construction from [15] is the maximal tensor product. Let S, T be operator systems with units e_S, e_T and define the cones

$$D_n^{\max}(S,T) = \{\alpha(P \otimes Q)\alpha^*; P \in M_k(S)^+, Q \in M_m(T)^+, \alpha \in M_{n,km}, k, m \in \mathbb{N}\}. \quad (2)$$

Proposition 5.3 in [15] shows that $\{D_n^{\max}(S,T)\}_{n=1}^\infty$ is a matrix ordering on $S \otimes T$ with order unit $e_S \otimes e_T$. So if $(S \otimes T, D_1^{\max}(S,T), e_S \otimes e_T)$ is an AOU space, there exists an Archimedeanization of $\{D_n^{\max}(S,T)\}_{n=1}^\infty$ with underlying space $S \otimes T$. For this it remains to show that:

Lemma 4.31. *Let S, T be operator systems with units e_S, e_T. Then $e_S \otimes e_T$ is an Archimedean order unit on $(S \otimes T, D_1^{max}(S,T))$.*

This lemma is needed such that the next definition is well defined and the following definition is Definition 5.4 in [15]. This lemma was neither proven in the literature nor were we able to prove it in this thesis.

Definition 4.32. Let S, T be two operator systems and let e_S and e_T denote the archimedean order unit of S and T respectively. Let $\{C_n\}_{n=1}^{\infty}$ be the Archimedeanization of the matrix ordering defined in (2). The maximal tensor product is the mapping

$$\max\colon \mathcal{O} \times \mathcal{O} \to \mathcal{O}, (S,T) \mapsto (S \otimes T, \{C_n^{\max}(S,T)\}_{n=1}^{\infty}, e_S \otimes e_T)$$

and we denote it by $S \otimes_{\max} T$.

This is Proposition 5.5 in [15] and shows that max is actually maximal:

Proposition 4.33. *Let $\mathcal{O}$ denote the class of operator systems. The mapping max: $\mathcal{O} \times \mathcal{O} \to \mathcal{O}$ is an associative, symmetric, functorial operator system tensor product and for all operator systems S, T with an operator system tensor structure $S \otimes_\tau T$, we have max is larger than τ.*

Another nice property of the maximal tensor product is that it behaves well with the maximal tensor product for C^*-algebras. Recall that the notation $\subseteq_{\text{c.o.i.}}$ was defined in Definition 4.8. This property is given in Theorem 5.12 of [15].

Proposition 4.34. *Let A and B be C^*-Algebras. Then $A \otimes_{max} B \subseteq_{c.o.i.} A \otimes_{C^* max} B$.*

Now we will briefly introduce the notion of jointly completely positive as this will be needed in the proof of Theorem 6.24.

Definition 4.35. Let H be a Hilbert space and S, T be operator systems. A bilinear map $\phi\colon S \times T \to B(H)$ is called jointly completely positive if for all $n, m \in \mathbb{N}$, we have that $\phi_{(n,m)}(P,Q)$ is a positive element of $M_{nm}(B(H))$, for all $P \in M_n(S)^+$ and $Q \in M_m(T)^+$.

Let S, T be operator systems and $\phi\colon S \times T \to \mathbb{C}$ be a bounded bilinear map. We can define

$$\begin{aligned} L(\phi)\colon S \to T^d, L(\phi)(s) \mapsto (t \mapsto \phi(s,t)), \\ R(\phi)\colon T \to S^d, L(\phi)(t) \mapsto (s \mapsto \phi(s,t)). \end{aligned}$$

The following two results are Lemma 5.7 and Theorem 5.8 in [15].

Lemma 4.36. *Let S, T be operator systems and let $\phi\colon S \times T \to \mathbb{C}$ be a bilinear map. Then the following are equivalent:*

(i) *ϕ is jointly completely positive.*
(ii) *$L(\phi)\colon S \to T^d$ is completely positive.*
(iii) *$R(\phi)\colon T \to S^d$ is completely positive.*

Lemma 4.37. *Let S and T be operator systems and $\phi\colon S \times T \to \mathbb{C}$ be a jointly completely positive map, then its linearization $\phi\colon S \otimes T \to \mathbb{C}$ is completely positive on $S \otimes_{max} T$.*

The last example of an operator system tensor product will be the commuting tensor product. In [15], it was shown that on the level of C^*-algebras, the commuting tensor product coincides with the maximal tensor product. The commuting tensor product is, therefore, a different extension of the maximal tensor product for C^*-algebras. The commuting tensor is characterized by the property for unital completely positive maps ϕ, ψ that we get a unital completely positive map $\phi \cdot \psi$ from the commuting tensor product. For the construction of the commuting tensor product define for two operator systems S, T:

$$\begin{aligned}\mathrm{cp}(S,T) = \{(\phi,\psi); H \text{ is a Hilbert space}, \phi\colon S \to B(H), \psi\colon T \to B(H) \text{ are}\\ \text{completely positive maps such that } \phi(S) \text{ and } \psi(T) \text{ commute}\}\end{aligned}$$

and for $\phi\colon S \to B(H), \psi\colon T \to B(H)$ and $\phi\colon S \to B(H), \psi\colon T \to B(H)$ such that $(\phi,\psi) \in \mathrm{cp}(S,T)$, define $\phi \cdot \psi\colon S \otimes T \to B(H)$ as the linear extension of the map $x \otimes y \mapsto \phi(x)\psi(y)$.

Definition 4.38. Let S, T be two operator systems and let e_S and e_T denote the archimedean order unit of S and T respectively. Define

$$P_n = \{u \in M_n(S \otimes T); (\phi \cdot \psi)_n(u) \geq 0 \ \forall (\phi.\psi) \in \mathrm{cp}(S,T)\}.$$

The commuting tensor product is the mapping

$$\mathrm{c}\colon \mathcal{O} \times \mathcal{O} \to \mathcal{O}, (S,T) \mapsto (S \otimes T, \{P_n(S,T)\}_{n=1}^{\infty}, e_S \otimes e_T)$$

and we denote it by $S \otimes_{\mathrm{c}} T$.

Proposition 4.39. *Let $\mathcal{O}$ denote the class of operator systems. The mapping $c\colon \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ is a symmetric, functorial operator system tensor product.*

Although this was not shown yet, we will see later in this thesis that the maximal tensor product is not injective and thus for two C^*-algebras A, B and two operator systems $S \subseteq A, T \subseteq B$, the inclusion map $S \otimes_{\max} T \subseteq A \otimes_{\max} B$ is not always completely order isomorphic on its range. For the commuting tensor product, we have two inclusions into the maximal tensor product on the level of C^*-algebras.

First, we need to introduce the maximal C^*-algebra of an operator system. The maximal C^*-algebra of an operator system was first introduced in [17] as the universal C^*-algebra of an operator system. In this article, the authors also prove that this object exists for every operator system and is unique up to *-isomorphism.

Definition 4.40. Let S be an operator system. The unital C^*-algebra $C_u^*(S)$ is called maximal C^*-algebra of S if there exists a unital completely positive map $\iota\colon S \to C_u^*(S)$ such that $\iota(S)$ generates $C_u^*(S)$ as a C^*-algebra and for all unital completely positive maps $\phi\colon S \to B(H)$, there exists a unique *-homomorphism $\pi\colon C_u^*(S) \to B(H)$ such that $\pi \circ \iota = \phi$.

Proposition 4.41. *Let S, T be operator systems and A, B be unital C^*-algebras such that $C_u^*(S) = A$ and $C_u^*(T) = B$. Then*

$$S \otimes_c T \subseteq_{c.o.i.} A \otimes_{max} B \subseteq_{c.o.i.} A \otimes_{C^*max} B.$$

Proof. The first inclusion follows from Theorem 6.4 in [15]. The second inclusion then follows from Proposition 4.34. □

The next Proposition is Lemma 2.8 in [21] and an application of Proposition 4.34 and it is the second inclusion into the maximal tensor product.

Proposition 4.42. *Let $A_1, ..., A_n$ and $B_1, ..., B_m$ be unital C^*-algebras, S be the coproduct of $A_1, ..., A_n$ and T be the coproduct of $B_1, ..., B_m$. Then*

$$\begin{aligned} S \otimes_c T &\subseteq_{c.o.i.} (A_1 *_1 ... *_1 A_n) \otimes_{max} (B_1 *_1 ... *_1 B_m) \\ &\subseteq_{c.o.i.} (A_1 *_1 ... *_1 A_n) \otimes_{C^*max} (B_1 *_1 ... *_1 B_m) \end{aligned}$$

The following Lemma is Theorem 6.6 on [15]:

Lemma 4.43. *Let A, B be unital C^*-algebras, then $A \otimes_c B = A \otimes_{max} B$.*

4.4 Dual of an operator system

In this subsection, all operator systems are assumed to be finite dimensional. One way to show that the dual of a finite dimensional operator system is an operator system is given in [11]:

Let S be an operator system. Since S is a $*$-vector space, we can turn the dual $S^d = \{f : S \to \mathbb{C}; f \text{ is a linear}\}$ into a $*$-vector space with the involution $^*\colon S^d \to S^d$ such that $f^*(s) = \overline{f(s^*)}$. $M_n(S^d) \cong M_n(S)^d$ by the isomorphism

$$\Phi\colon M_n(S^d) \to M_n(S)^d, (g_{i,j})_{i,j} \mapsto ((s_{i,j})_{i,j} \mapsto \sum_{i,j=1}^{n} g_{i,j}(s_{i,j}))$$

From this we can define $M_n^+(S^d) = \{G \in M_n(S^d); \phi(G) \geq 0 \ \forall x \in M_n(S)^+\}$. These define a matrix ordering on S^d. If we choose e to be a faithful state in S^d(i.e. for all $a \in S^+$ $e(a) = 0 \Rightarrow a = 0$), we get that $(S^d, \{M_n(S^d)\}_{n=1}^{\infty}, e)$ is an operator system.

This statement is Corollary 4.5 in [5]:

Proposition 4.44. *Let S be a finite dimensional operator system, then S^d is completely order isomorphic to an operator system.*

Proposition 4.45. *Let S be a finite dimensional operator system. Then there exists a complete order isomorphism such that $(S^d)^d \cong S$.*

Proof. Since S is finite dimensional, we have that $\dim(S) = \dim(S^d) = \dim((S^d)^d)$. The rest follows from Proposition 6.2 in [16]. □

For two operator systems S, T, there is also a useful connection between $S^d \otimes_{\max} T^d$ and $(S \otimes_{\min} T)^d$ given in Proposition 1.16 in [11]:

Proposition 4.46. *Let S, T be finite dimensional operator systems, then there is a complete order isomorphism such that $S^d \otimes_{max} T^d \cong (S \otimes_{min} T)^d$.*

5 Foundations of quantum information

This section is a brief introduction to some of the most basic concepts of quantum information: quantum states, channels and measurements. Most of this section is taken from [26].

Let X, Y be finite index sets. In this section $\mathbb{C}^X$ will be the Hilbert space equipped with the inner product $\langle v, w\rangle = w^* v$ with being $w^* = \overline{w^t}$ for $v, w \in \mathbb{C}^X$. Note that w^* is also the corresponding element to w in $(\mathbb{C}^X)^d$. Also, recall from Example 3.8 that the trace can be defined in multiple ways. We denote $(e_x)_{x \in X}$ for the standard basis of $\mathbb{C}^X$ and we denote the matrix units of M_X by $e_x e_{x'}^*$.

5.1 Quantum States

Quantum states can be generally described in two ways. One is vectors of a finite dimensional Hilbert space and the other, more general, approach are density matrices. This is also the approach of this thesis.

Definition 5.1. An euclidean space $\mathcal{X}$ is one of the two following:

1. $\mathcal{X} = \mathbb{C}^X$ for some finite set X.

2. $\mathcal{X} = \mathcal{X}_1 \otimes ... \otimes \mathcal{X}_n$ for some $n \in \mathbb{N}$ and euclidean spaces $\mathcal{X}_1, ...\mathcal{X}_n$.

Definition 5.2. Let $\mathcal{X}$ be an euclidean space.

(i) A quantum state is a matrix $\rho \in S_Q(\mathcal{X}) = \{\rho \in \mathcal{L}(\mathcal{X}); \rho \geq 0 \wedge \mathrm{Tr}(\rho) = 1\}$.
(ii) A quantum state ρ is called a pure state if there exists a vector $v \in \mathcal{X}$ such that $vv^* = \rho$.

Remark 5.3. Recall that the definition of a state (Definition 2.17) and the trace norm from Proposition 3.9. The quantum states are then exactly the states in the trace norm as $\mathcal{X}$ is always finite dimensional and therefore $T(\mathcal{X}) = B(\mathcal{X}) = \mathcal{L}(\mathcal{X})$.

The next proposition gives the connection to the aforementioned other description of quantum states:

Proposition 5.4. *Let $\mathcal{X}$ be a euclidean space. Then $\{vv^*; v \in \mathcal{X},\ \|v\| = 1\}$ is the set of all pure states. Also for two unit vectors $u, v \in \mathcal{X}$, the associated pure states uu^* and vv^* are equal iff there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $u = \alpha v$.*

Proof. Let $\rho \in S_Q(\mathcal{X})$ be a pure state. Therefore there exists a vector $v \in \mathcal{X}$ such that $\rho = vv^*$.

$$\mathrm{Tr}(vv^*) = \sum_{i \in I} \langle vv^* e_i, e_i \rangle = \sum_{i \in I} \langle v, e_i \rangle \langle e_i, v \rangle = \|v\|^2 \tag{3}$$

shows that v is a unit vector. Therefore $vv^* \in \{uu^*; u \in \mathbb{C}^X,\ \|u\| = 1\}$. Now let $u \in \mathcal{X}$ be a unit vector. Then analogous to (3) follows that $\mathrm{Tr}(uu^*) = 1$. Since uu^* is a projection, it is also positive and therefore a pure state.
Let $u, v \in \mathcal{X}$ be unit vectors such that $uu^* = vv^*$. Then

$$u = uu^* u = vv^* u = \langle v, u \rangle v$$

and since v and u are unit vectors $|\langle v, u \rangle| = 1$. Now let α be in $\mathbb{C}$ and let $u, v \in \mathbb{C}^X$ be unit vectors such that $u = \alpha v$.

$$uu^* = \alpha v (\alpha v)^* = |\alpha|^2 vv^*$$

This shows the second statement of the proposition. □

Now we give some examples of quantum states and the connection to probabilistic states:

Example 5.5. Let X be a finite set.

(i) Let $\rho \in D_X$ be a quantum state in M_X. As ρ is positive semidefinite and its eigenvalues are its diagonal entries, it follows that the diagonal entries are all ≥ 0. Also $\operatorname{Tr} \rho = 1$ so the function $p\colon X \to [0,1], x \mapsto \rho_{x,x}$ gives a probability distribution over X. Analogous a we can define a probability distribution $p\colon X \to [0,1], x \mapsto \rho_{x,x}$ which gives rise to a density matrix $\rho = \sum_{x \in X} p(x) e_x e_x^*$. Therefore the quantum states in D_X can be interpreted as the classical states.

(ii) Define $J_X^{\text{cl}} = \sum_{x \in X} e_x e_x^* \otimes e_x e_x^* \in M_X \otimes M_X$. Then the normalisation of J_X^{cl}, being $\frac{1}{|X|} J_X^{\text{cl}}$, is also a state that is not a pure state.

A concept in quantum information is that we are also interested in the state in one of the two situations:

1. The resulting state after removing a part of the system.
2. The situation that the states is actually just part of a bigger system.

For the first situation, we will define the mapping that will define the resulting state:

Definition 5.6. Let $k, n \in \mathbb{N}$ be a natural number and $\mathcal{X}_1, ..., \mathcal{X}_n$ be euclidean spaces. The linear extension of the map

$$\begin{aligned} \operatorname{Tr}_{\mathcal{X}_k}\colon \mathcal{L}(\mathcal{X}_1) \otimes ... \otimes \mathcal{L}(\mathcal{X}_n) &\to \mathcal{L}(\mathcal{X}_1) \otimes ... \otimes \mathcal{L}(\mathcal{X}_{k-1}) \otimes \mathcal{L}(\mathcal{X}_{k+1}) \otimes ... \otimes \mathcal{L}(\mathcal{X}_n), \\ A_1 \otimes ... \otimes A_n &\mapsto \operatorname{Tr}(A_k) A_1 \otimes ... \otimes A_{k-1} \otimes A_{k+1} \otimes ... \otimes A_n \end{aligned}$$

is called partial trace.

For the second situation, we can show that there exists a bigger system for every state such that the state in the bigger system is a pure state. This follows from the following proposition that is Theorem 2.10 in [26]:

Proposition 5.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be finite sets and let $P \in \mathcal{L}(\mathcal{X})$ be a positive semidefinite matrix. There exists a vector $u \in \mathcal{X} \otimes \mathcal{Y}$ such that $\operatorname{Tr}_{\mathcal{Y}}(uu^*) = P$ iff $\dim \mathcal{Y} \geq \operatorname{rank}(P)$.*

Corollary 5.8. *Let X be a finite set and let $\rho \in S_Q(\mathcal{X})$ be a quantum state. There exists a euclidean space $\mathcal{Y}$ such that there exists a unit vector $u \in \mathcal{X} \otimes \mathcal{Y}$ such that uu^* is a pure state and $\operatorname{Tr}_{\mathcal{Y}}(uu^*) = \rho$.*

Proof. Let Y be a finite set with $|Y| \geq \dim(\mathcal{Y})$. Then it follows from Proposition 5.8 that there exists a vector $u \in \mathcal{X} \otimes \mathbb{C}^Y$ such that $\operatorname{Tr}_{\mathbb{C}^Y}(uu^*) = \rho$.

$$\begin{aligned} 1 = \operatorname{Tr}(\rho) = \operatorname{Tr}(\operatorname{Tr}_{\mathbb{C}^Y}(uu^*)) &= \operatorname{Tr}\left(\operatorname{Tr}_{\mathbb{C}^Y}\left(\sum_{i \in I} \lambda_i A_i\right)\right) \\ &= \sum_{i \in I} \operatorname{Tr}(\operatorname{Tr}_{\mathbb{C}^Y}(\lambda_i A_i)) = \sum_{i \in I} \operatorname{Tr}(\lambda_i A_i)) = \operatorname{Tr}(uu^*) = \|u\|^2 \end{aligned}$$

□

Another important concept of quantum information is entanglement of states. Without entanglement, a lot of the surprising phenomena would not be possible. We give the mathematical description of this concept as it also plays an important role in the mathematical theory:

Definition 5.9. Let $\mathcal{X}_1, ..., \mathcal{X}_n$ be euclidean spaces and $u \in \mathcal{X}_1 \otimes ... \otimes \mathcal{X}_n$ be a unit vector such that uu^* is a state. The state uu^* is called separable if there exist $u_1 \in \mathcal{X}_1, ..., u_n \in \mathcal{X}_n$ such that $u = u_1 \otimes ... \otimes u_n$. States that are not separable are called entangled.

Example 5.10. Let X be a finite set and define

$$J_X = \sum_{x,x'} e_x e_{x'}^* \otimes e_x e_{x'}^* \in M_X \otimes M_X.$$

The pure state

$$\frac{1}{|X|} J_X = \frac{1}{\sqrt{|X|}} \left(\sum_{x \in X} e_x \otimes e_x \right) \left(\sum_{x \in X} e_x \otimes e_x \right)^*$$

is called the maximally entangled state.

5.2 Quantum Channel

In the last section, we defined quantum states which describe a state of a quantum system. Quantum channels describe the possible changes of a system as these are the maps that quantum states to quantum states.

Let $X_1, ..., X_n$ be finite sets, we write $M_{X_1 ... X_n}$ for $M_{X_1} \otimes ... \otimes M_{X_n}$ and $D_{X_1 ... X_n}$ for $D_{X_1} \otimes ... \otimes D_{X_n}$

Definition 5.11. Let $\mathcal{X}, \mathcal{Y}$ be euclidean spaces. Let $\phi \colon \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{Y})$ be a linear map.

(i) ϕ is called trace preserving if

$$\forall M \in \mathcal{L}(\mathcal{X}) \colon \operatorname{Tr}(\phi(M)) = \operatorname{Tr}(M)$$

(ii) The map ϕ is called a quantum channel if it is trace preserving and completely positive.

(iii) A positive trace preserving map $\mathcal{N} \colon D_{X_1 ... X_n} \to D_{Y_1 ... Y_m}$ is called a classical channel.

(iv) A positive trace preserving map $\mathcal{E} \colon D_{X_1 ... X_n} \to M_{Y_1 ... Y_m}$ is called a classical-to-quantum channel.

Both classical and classical-to-quantum channels are just required to be positive instead of completely positive. But in this case, positive maps and completely positive maps coincide which can be seen in the following remark:

Remark 5.12. (i) Let $\mathcal{N} \colon D_{X_1 ... X_n} \to D_{Y_1 ... Y_m}$ be a classical channel. Since $D_{X_1 ... X_n}$ is commutative, it follows from Proposition 4.19 that $\mathcal{N}$ is also completely positive.

(ii) Let $\mathcal{E} \colon D_{X_1 ... X_n} \to M_{Y_1 ... Y_m}$ be a classical-to-quantum channel. Since $D_{X_1 ... X_n}$ is commutative, it follows from Proposition 4.19 that $\mathcal{E}$ is also completely positive.

(iii) In [26] Example 2.7, it is shown that there exists a basis of the matrices in the density matrices. Therefore by linear extension of a map $\phi \colon S_Q(\mathcal{X}) \to S_Q(\mathcal{X})$ it is uniquely defined on $\mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{Y})$.

Quantum channels are maps that map quantum states to quantum states. This requires the map to be positive and trace preserving. It does not have to be completely positive. The requirement of complete positivity comes from the fact that one often wants to look at a state in a bigger system. This is shown in the following Remark:

Remark 5.13. Let $\mathcal{X}, \mathcal{Y}$ be euclidean spaces, let $\phi\colon \mathcal{L}(\mathcal{X}) \to \mathcal{L}(\mathcal{Y})$ be a channel and let $\rho \in S_Q(\mathcal{X})$ be a state. Consider another state $\sigma \in S_Q(\mathbb{C}^k)$. By Lemma 2.7, there exists $M \in M_X$ such that $\rho = M^*M$ and $N \in M_k$ such that $\sigma = N^*N$. Then the $\rho \otimes \sigma = (M \otimes N)(M \otimes N)^*$ is also a state. Then $\phi(\rho) \otimes \sigma$ is also a state because $\phi \otimes I(\rho \otimes \sigma) = \phi(\rho) \otimes \sigma$ and ϕ is completely positive.

Now we give examples of some quantum channels that will be used in Section 7 of this thesis:

Example 5.14. (i) Let X be a finite set and define $\Delta_X\colon M_x \to M_x, (a_{x,x'}) \mapsto \sum_{x \in X} a_{x,x'} e_x e_x^*$. This map is trace preserving and also positive as the eigenvalues of Δ are its diagonal entries and $A_{x,x} \geq 0$ for all $x \in X$. Since the image of Δ is in D_X, we get by Proposition 4.19 that Δ is completely positive. Therefore Δ_X is a channel. It is called the completely dephasing channel. For finite sets $X_1, ..., X_n$, we will write $\Delta_{X_1 ... X_n}$ for $\Delta_{X_1} \otimes ... \otimes \Delta_{X_n}$.

(ii) This identification is taken from [25]. Let $X_1, ..., X_n, A_1, ..., A_n$ be finite sets, $X = X_1 \times ... \times X_n, A = A_1 \times ... \times A_n$ and $\phi\colon M_{X_1 ... X_n} \to M_{A_1 ... A_n}$ be a quantum channel. Then ϕ is called a (X, A)-classical (quantum) channel if $\phi = \Delta_{A_1 ... A_n} \circ \phi \circ \Delta_{X_1 ... X_n}$. A (X, A)-classical channel ϕ gives rise to a classical channel $\mathcal{N}_\phi\colon D_{X_1 ... X_n} \to D_{A_1 ... A_n}, \mathcal{N}_\phi = \Delta_{A_1 ... A_n} \circ \phi_{|D_{X_1 ... X_n}}$. Conversely a classical channel $\mathcal{N}\colon D_{X_1 ... X_n} \to D_{A_1, ... A_n}$ gives rise to a (X, A)-classical channel $\phi_{\mathcal{N}} = \mathcal{N} \circ \Delta_{X_1 ... X_n}$.

5.3 Measurements

Definition 5.15. (i) Let $m \in \mathbb{N}$ be natural number and H be a Hilbert space. A positive operator valued measurement (POVM) is a finite set of positive operators $\{A_i \in B(H)^+; i \in \{1, ..., m\}\}$ such that $\sum_{i=1}^m A_i = I$.

(ii) Let $n \in \mathbb{N}$ be a natural number and $(A_i)_{i=1}^n$ be a POVM. $(A_i)_{i=1}^n$ is called projective valued measurement (PVM) if A_i is a projection for all $i \in \{1, ..., n\}$.

Let $\{A_i \in M_n^+; i \in \{1, ..., m\}\}$ be a POVM, $\rho \in S(\mathcal{X})$ be a state and $B, C \in \mathcal{L}(\mathcal{X})$ then we can turn $\mathcal{L}(\mathcal{X})$ into a Hilbert space by using $\langle B, C \rangle = \mathrm{Tr}(C^*B)$ as inner product. With this, we get:

$$\sum_{i=1}^m \langle \rho, A_i \rangle = \langle \rho, \sum_{i=1}^m A_i \rangle = \langle \rho, I \rangle = 1$$

and from the spectral theorem follows that there exist vectors $\phi_i \in \mathcal{X}, \lambda_i \in \mathbb{R}^*$ for $i \in \{1, ..., d\}$ such that $\rho = \sum_{i=1}^d \lambda_i \phi_i \phi_i^*$ for $d = \dim(\mathcal{X})$.

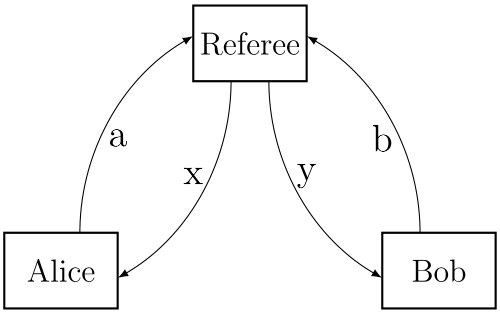
$$\begin{aligned} \langle \rho, A_i \rangle = \left\langle \sum_{k=1}^d \lambda_i \phi_k \phi_k^*, A_i \right\rangle &= \sum_{k=1}^d \langle \lambda_i \phi_k \phi_k^*, A_i \rangle \\ &= \sum_{k=1}^d \lambda_i \,\mathrm{Tr}(A_i \phi_k \phi_k^*) = \sum_{k=1}^d \lambda_i \phi_k^* A_i \phi_k \geq 0 \end{aligned}$$

because $A_i \geq 0$. Therefore $p\colon \{1, ..., m\} \to [0,1], i \mapsto \langle \rho, A_i \rangle$ is a well defined probability distribution. And therefore POVMs "measure" a state in the form that there is an outcome i with probability $p(i)$.

6 Nonlocal Games

6.1 Basics on nonlocal games

The contents of this subsection are mainly inspired by Lecture 6 of [13] and [1]. A nonlocal game is a two player game where two cooperating players Alice and Bob each receive a question from a referee and they each give an answer to their question but they cannot communicate with each other after they received their question. The referee then evaluates whether the given answers to the questions are correct.



Definition 6.1. A nonlocal game is a 5-tuple (A, B, X, Y, λ) where

(i) A, B, X, Y are finite nonempty sets,
(ii) $\lambda\colon X \times Y \times A \times B \to \{0, 1\}$ is a function.

In this definition X and Y are the question sets for Alice and Bob respectively, A and B are the answer sets for Alice and Bob respectively and λ is the function that evaluates whether the answers are correct.

Remark 6.2. In some literature (e.g. [1]) the definition of a nonlocal game includes a probability distribution over the question sets $\pi\colon X \times Y \to \{0, 1\}$. This is not needed, in the context of this thesis, as we try to find perfect strategies so a question that has probability greater than 0 needs to be answered correctly and questions whose probability is 0 can be excluded from the question set.

Definition 6.3. Let $\mathcal{G} = (A, B, X, Y, \lambda)$ be a non local game. $\mathcal{G}$ is called a mirror game if there exist functions $f\colon X \to Y, g\colon Y \to X$ such that for every $x \in X, y \in Y$ the sets $\{(a, b) \in A \times B; \lambda(x, f(x), a, b) = 1\}$ and $\{(a, b) \in A \times B; \lambda(g(y), y, a, b) = 1\}$ are graphs of bijections.

Example 6.4. (i) Graph homomorphism game:
Let G, H be graphs, let V_G, V_H be the vertex set of G and H respectively and let E_G, E_H be the edge set of G and H respectively. Let $X = Y = V_G$ be the question sets and let $A, B = V_H$ be the answer sets.

$$\lambda(x, y, a, b) = \begin{cases} 0, & \text{if } x = y \wedge a \neq b \\ 0, & \text{if } x \sim y \wedge a \nsim b \\ 1, & \text{else} \end{cases}$$

The game $\mathcal{G} = (X, Y, A, B, \lambda)$ is called graph homomorphism game. The winning conditions can be reformulated to $x = y \Rightarrow a = b$ and $x \sim y \Rightarrow a \sim b$. By taking the function to be the identity map we can see that $\mathcal{G}$ is a mirror game.

(ii) Graph isomorphism game:
Let G, H be graphs such that $|V_G| = |V_H|$, where V_G, V_H be the vertex set of G and H respectively. Let E_G, E_H be the edge set of G and H respectively. Let $X = Y = V_G$ be the question sets and let $A, B = V_H$ be the answer sets.

$$\lambda(x, y, a, b) = \begin{cases} 0, & \text{if } x = y \wedge a \neq b \\ 0, & \text{if } x \neq y \wedge a = b \\ 0, & \text{if } x \sim y \wedge a \nsim b \\ 0, & \text{if } x \nsim y \wedge a \sim b \\ 1, & \text{else} \end{cases}$$

The game $\mathcal{G} = (X, Y, A, B, \lambda)$ is called graph isomorphism game. The winning conditions can be reformulated to $x = y \Leftrightarrow a = b$ and $x \sim y \Leftrightarrow a \sim b$. By taking the function to be the identity map we can see that $\mathcal{G}$ is a mirror game.

(iii) CHSH game:
Let (X, Y, A, B, λ) be the nonlocal game with $X = Y = A = B = \{0, 1\}$ and $\lambda\colon \{0, 1\}^4 \to \{0, 1\}, (x, y, a, b) \mapsto x \oplus y = a \wedge b$. This means that for the pair of questions $(1, 1)$, Alice and Bob have to answer either $(0, 1), (1, 0)$ and if the pair of questions is not $(1, 1)$, Alice and Bob need to answer $(0, 0)$ or $(1, 1)$. This game is a mirror game as the sets $\{(a, b) \in \{0, 1\}^2; \lambda(0, 0, a, b)\} = 1\} = \{(0, 0), (1, 1)\}$ and $\{(a, b) \in \{0, 1\}^2; \lambda(1, 1, a, b)\} = 1\} = \{(0, 1), (1, 0)\}$ are graph bijections.

6.2 Strategies for nonlocal games

As mentioned before the aim of Alice and Bob is to win in a given nonlocal game. Thus they are allowed to form a strategy beforehand. These strategies need some restrictions as Alice and Bob are unable to communicate. We will now give some of the common strategies:

Definition 6.5. Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game.

(i) A deterministic strategy for $\mathcal{G}$ are two functions $f\colon X \to A$ and $g\colon Y \to B$. The winning probability of the strategy is defined as

$$\frac{1}{|X||Y|} \sum_{x \in X, y \in Y} \lambda(x, y, f(x), g(y)).$$

(ii) A classical strategy for $\mathcal{G}$ is a finite set R and two probability distributions $p_1\colon X \times R \to [0, 1], p_2\colon Y \times R \to [0, 1]$ and two functions $f\colon X \times R \to A$ and $g\colon Y \times R \to B$. The winning probability of the strategy is defined as

$$\frac{1}{|X||Y|} \sum_{x \in X, y \in Y, r \in R} p_1(x, r) p_2(y, r) \lambda(x, y, f(x, r), g(y, r)).$$

(iii) A quantum strategy for $\mathcal{G}$ consists of sets of POVM $\{E_{x,a} \in M_d; a \in A\}_{x\in X}$ with $d \in \mathbb{N}$ and $\{F_{y,b} \in M_d; b \in B\}_{y\in Y}$, a quantum state $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$. The winning probability of the strategy is defined as

$$\frac{1}{|X||Y|} \sum_{x\in X, y\in Y, a\in A, b\in B} (\psi^*(E_{x,a} \otimes F_{y,b})\psi)\lambda(x, y, a, b).$$

(iv) Let H be a Hilbert space. A quantum commuting strategy $\mathcal{G}$ consists of a unit vector $\xi \in H$ and POVMs of operators $(E_{x,a})_{a\in A} \subseteq B(H)$ and $(F_{y,b})_{b\in B} \subseteq B(H)$ such that for all $x \in X, y \in Y, a \in A, b \in B$: $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$. The winning probability of the strategy is given as

$$\frac{1}{|X||Y|} \sum_{x\in X, y\in Y, a\in A, b\in B} \langle E_{x,a}F_{y,b}\xi, \xi\rangle\lambda(x, y, a, b).$$

For classical, deterministic and quantum strategies, one can immediately see that the summands of the winning probability are all non-negative. For quantum commuting strategies, it is not that immediate but will be proven in the following lemma:

Lemma 6.6. *Let $E, F \subseteq B(H)$ be positive operators such that $EF = FE$. Then EF is also positive and in particular $\langle EF\xi, \xi\rangle \geq 0$ for all $\xi \in H$.*

Proof. Consider the C^*-Algebra $C^*(E, F, 1)$. Since $E, F, 1$ are all selfadjoint and commute and $(EF)^* = F^*E^* = EF$, we get that $C^*(E, F, 1)$ is commutative. Thus by Proposition 2.3 there exists a *-isomorphism $\phi\colon C^*(E, F, 1) \to C(X)$ for some compact Hausdorff space X. Since F is positive, $\phi(F) \geq 0$. If we define $\sqrt{F} = \phi^{-1}(\sqrt{\phi(F)})$, then $\sqrt{F} \geq 0$ since *-homomorphisms are positive and

$$\sqrt{F}^2 = \phi^{-1}(\sqrt{\phi(F)})\phi^{-1}(\sqrt{\phi(F)}) = \phi^{-1}(\sqrt{\phi(F)}\sqrt{\phi(F)}) = F.$$

So $\sqrt{F}$ is the unique positive square root of F and it commutes with E since $\sqrt{F} \in C^*(E, F, 1)$. For $\xi \in H$, we get:

$$\langle EF\xi, \xi\rangle = \langle \sqrt{F}E\sqrt{F}\xi, \xi\rangle = \langle E(\sqrt{F}\xi), (\sqrt{F}\xi)\rangle \geq 0.$$

□

Now we look at some possible strategies for the CHSH game we defined earlier:

Example 6.7. Let be $\mathcal{G}$ be the CHSH-game from Example 6.4. Consider the deterministic strategy $f = g = 0$ or the strategy $f = g \equiv 1$. As $\lambda(x, y, 0, 0) = 0$ iff $x = 1 = y$, the winning probability is 0.75. Let f, g now be any deterministic strategy, then to answer the questions $(0, 1), (1, 1), (1, 0)$ correctly, it has to hold that $f(0) = g(1) = f(1) = g(0)$, so deterministic strategies can not reach a winning probability of 1. As classical strategies are just convex combinations of deterministic strategies, they can not achieve 1 either. Now consider the quantum strategy, which is given in [6]. Alice and Bob share the state $\psi = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1)$ and the POVM are

$$\begin{aligned} X_{0,a} &= \phi_a(0)\phi_a(0)^* \\ X_{1,a} &= \phi_a(\tfrac{\pi}{4})\phi_a(\tfrac{\pi}{4})^* \\ Y_{0,b} &= \phi_b(\tfrac{\pi}{8})\phi_b(\tfrac{\pi}{8})^* \\ Y_{1,b} &= \phi_b(-\tfrac{\pi}{8})\phi_b(-\tfrac{\pi}{8})^* \end{aligned}$$

for $\phi_0(\theta) = \cos(\theta)e_0 + \sin(\theta)e_1$ and $\phi_0(\theta) = -\sin(\theta)e_0 + \cos(\theta)e_1$. This strategy achieves a winning probability of $\cos^2(\frac{\pi}{8}) > 0.85$. So the quantum strategies contain strategies that are not included in the classical strategies and those can also have a higher winning probability.

Remark 6.8. (i) Sometimes quantum strategies are defined by sets of PVMs instead of POVMs. Obviously, every PVM is a POVM so these strategies are still quantum strategies in the sense of Definition 6.5. The converse is not true but from Naimark's theorem ([26, 2.42,2.43]) follows that there exists a strategy that has the same winning probability. A similar statement holds for quantum commuting strategies. We will show this later in this thesis.

(ii) Consider a quantum strategy $\{E_{x,a} \in M_d; a \in A\}_{x\in X}$ with a separable state $\psi = \psi_1 \otimes \psi_2 \in \mathbb{C}^d \otimes \mathbb{C}^d$. Define the classical strategy

$$p_1 : X \times A \times B \to [0,1], (x,a,b) \mapsto (\psi_1^* E_{x,a}\psi_1),$$
$$p_2 \colon Y \times A \times B, (y,a,b) \mapsto (\psi_2^* F_{y,b}\psi_2)$$

and $f\colon X \times A \times B \to A, (x,a,b) \mapsto a, g\colon Y \times A \times B \to A, (x,a,b) \mapsto b.$

$$\begin{aligned}
&\frac{1}{|X||Y|} \sum_{x\in X, y\in Y, a\in A, b\in B} (\psi^* E_{x,a} \otimes F_{y,b}\psi)\lambda(x,y,a,b)\\
=&\frac{1}{|X||Y|} \sum_{x\in X, y\in Y, a\in A, b\in B} (\psi_1^* E_{x,a}\psi_1) \otimes (\psi_2^* F_{y,b}\psi_2)\lambda(x,y,a,b)\\
=&\frac{1}{|X||Y|} \sum_{x\in X, y\in Y, a\in A, b\in B} p_1(x,a,b)p_2(y,a,b)\lambda(x,y,a,b)\\
=&\frac{1}{|X||Y|} \sum_{x\in X, y\in Y, a\in A, b\in B} p_1(x,a,b)p_2(y,a,b)\lambda(x,y,f(x,a,b).g(x,a,b)).
\end{aligned}$$

So just quantum strategies with an entangled state are interesting as the quantum strategies with a separable state are just classical strategies.

6.3 Strategies as Correlations

Strategies for nonlocal games are more commonly expressed as correlations. The definitions of multiple classes of correlations can be found in [19]. The most general set of strategies, that is usually studied, are no-signalling correlations. These are always defined through correlations.

Definition 6.9. Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game and let p be a set $\{p(a,b|x,y); x \in X, y \in Y, a \in A, b \in B\}$.

(i) p is called a correlation if $p(\cdot,\cdot|x,y)$ is a probability distribution over $A \times B$ for every $(x,y) \in X \times Y$.

(ii) A correlation p is called no-signalling if

$$\sum_{b\in B} p(a,b|x,y) = \sum_{b\in B} p(a,b|x,y') \ \forall x \in X, y, y' \in Y, a \in A,$$

and

$$\sum_{a\in A} p(a,b|x,y) = \sum_{a\in A} p(a,b|x',y) \ \forall x, x' \in X, y \in Y, b \in B.$$

The set of quantum correlations will be denoted as $C_{\mathrm{ns}}(\mathcal{G})$.

(iii) A correlation p is called a quantum correlation if there exists a quantum strategy $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d, \{E_{x,a} \in M_d; a \in A\}_{x \in X}, \{F_{y,b} \in M_d; b \in B\}_{y \in Y}$ such that

$$p(a, b|x, y) = \psi^*(E_{x,a} \otimes F_{y,b})\psi.$$

The set of quantum correlations will be denoted as $C_{\mathrm{q}}(\mathcal{G})$.

(iv) A correlation p is called a quantum commuting correlation if there exists a quantum commuting strategy consisting of a Hilbert space H, a unit vector $\xi \in H$ and POVMs of operators $(E_{x,a})_{a \in A} \subseteq B(H)$ and $(F_{y,b})_{b \in B} \subseteq B(H)$ such that

$$p(a, b|x, y) = \langle E_{x,a} F_{y,b} \xi, \xi \rangle.$$

The set of quantum commuting correlations will be denoted as $C_{\mathrm{qc}}(\mathcal{G})$.

Remark 6.10. Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game

(i) Let $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d, \{E_{x,a} \in M_d; a \in A\}_{x \in X}, \{F_{y,b} \in M_d; b \in B\}_{y \in Y}$ be a quantum strategy for $\mathcal{G}$. Now consider the Hilbert space $H = \mathbb{C}^d \otimes \mathbb{C}^d$ with the canonical inner product and define the sets of commuting POVMs

$$\{E_{x,a} \otimes \mathrm{id}_{\mathbb{C}^d} \in M_d \otimes M_d; a \in A\}_{x \in X}, \{\mathrm{id}_{\mathbb{C}^d} \otimes F_{y,b} \in M_d \otimes M_d; b \in B\}_{y \in Y}.$$

This is a quantum commuting strategy as

$$\begin{aligned} p(a, b|x, y) &= \psi^*(E_{x,a} \otimes F_{y,b})\psi \\ &= \langle (E_{x,a} \otimes F_{y,b})\psi, \psi \rangle \\ &= \langle (E_{x,a} \otimes \mathrm{id}_{\mathbb{C}^d})(\mathrm{id}_{\mathbb{C}^d} \otimes F_{y,b})\psi, \psi \rangle. \end{aligned}$$

Therefore $C_q(\mathcal{G}) \subseteq C_{qc}(\mathcal{G})$.

(ii) Correlations of the set $C_{\mathrm{ns}}(\mathcal{G})$ are called no-signalling because the conditions imply that the probability distribution of

$$p_1 \colon A \times X \to [0, 1], (a, x) \mapsto \sum_{b \in B} p(a, b|x, y)$$

for some $y \in Y$ are well defined as they are independent of the chosen $y \in Y$. Analogous is the probability distribution

$$p_2 \colon B \times Y \to [0, 1], (b, y) \mapsto \sum_{a \in A} p(a, b|x, y)$$

for some $x \in X$ well defined by the second condition. Therefore Alice and Bob can not communicate classical information as their answers are independent on the question and answer of the other person.

Recall the CHSH-game defined in Example 6.4 and the strategies for the CHSH game from Example 6.7. In Lecture 7 of [13], it is actually shown that there cannot exist a quantum strategy that exceeds the winning probability of the one given in the previous example. But there exists a no-signalling strategy that wins with probability 1:

Remark 6.11. Let $\mathcal{G}$ be the CHSH game defined in Example 6.4. Now consider the strategy given by the correlation p such that $p(0, 1|1, 1) = p(1, 0|1, 1) = \frac{1}{2}$ and $p(a, a|x, y) = \frac{1}{2}$ for $a \in \{0, 1\}$ and $(x, y) \in \{(0, 0), (0, 1), (1, 0)\}$. It is easy to check that this is actually a no-signalling correlation and the probability to win is 1.

6.4 Perfect Strategies for nonlocal games

The objective of this section is to classify perfect strategies of a nonlocal game within certain classes of correlation, namely quantum commuting and no-signalling. The results in this section are taken from [19]. This article also contains the classification of more classes of correlations than quantum commuting and no-signalling.

Definition 6.12. Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game. A correlation p is called perfect if $\{(a, b, x, y); \lambda(a, b, x, y) = 0\} \subseteq \{(a, b, x, y); p(a, b|x, y) = 0\}$. And denote $C^p_{\mathrm{ns}}(\mathcal{G})$ for the set of perfect no-signalling correlations and $C^p_{\mathrm{qc}}(\mathcal{G})$ for the set of perfect quantum commuting correlations.

Remark 6.13. We can see that a correlation is perfect iff the corresponding strategy has winning probability 1. This follows from

$$\begin{aligned}&\frac{1}{|X||Y|}\sum_{a\in A,b\in B,X\in X,y\in Y} p(a,b|x,y)\lambda(x,y,a,b)\\ =&\frac{1}{|X||Y|}\sum_{(a,b,x,y)\in A\times B\times X\times Y,\lambda(x,y,a,b)\neq 0} p(a,b|x,y)\end{aligned}$$

Since $\frac{1}{|X||Y|}\sum_{(a,b,x,y)\in A\times B\times X\times Y} p(a,b|x,y) = 1$ and therefore p can just give rise to a perfect strategy if $\{(a, b, x, y); p(a, b|x, y) \neq 0\} \subseteq \{(a, b, x, y); \lambda(a, b, x, y) \neq 0\}$.

Let $X, Y, A, B.$ be finite sets. Recall in Definition 4.23, we defined the operator system $\mathcal{S}_{X,A} \subseteq \mathcal{A}(X, A)$ and the basis $(e_{x,a})_{x\in X,a\in A}$. Now for $\mathcal{S}_{X,A}$ and $\mathcal{S}_{Y,B}$, we denote this basis of $\mathcal{S}_{X,A}$ by $(e_{x,a})_{x\in X,a\in A}$ and for $\mathcal{S}_{Y,B}$ by $(f_{y,b})_{y\in Y,b\in B}$

Lemma 6.14. *Let X, Y, A, B be finite sets and $p\colon A \times B \times X \times Y \to \mathbb{C}$ be a no-signalling function, i.e.*

$$\sum_{b\in B} p(a,b,x,y) = \sum_{b\in B} p(a,b,x,y')\ \forall x\in X, y,y'\in Y, a\in A, \tag{4}$$

and

$$\sum_{a\in A} p(a,b,x,y) = \sum_{a\in A} p(a,b,x',y)\ \forall x,x'\in X, y\in Y, b\in B. \tag{5}$$

Then there exists a continuous linear bijection between

$$\{p\colon A\times B\times X\times Y\to\mathbb{C}; p \text{ is no-signalling}\} \text{ and } (\mathcal{S}_{X,A}\otimes\mathcal{S}_{Y,B})^d.$$

For a no-signalling $p\colon A \times B \times X \times Y \to \mathbb{C}$, the corresponding $s_p \in (\mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B})^d$ is given by

$$s_p(e_{x,a}\otimes f_{y,b}) = p(a,b,x,y)$$

and conversely for $s \in (\mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B})^d$ the corresponding no-signalling function p_s is given by

$$p_s(x,y,a,b) = s(e_{x,a}\otimes f_{y,b}).$$

Proof. Linearity is obvious and therefore since the spaces are finite dimensional we also get continuity.

First we show that for a given a no-signalling $p\colon A \times B \times X \times Y \to \mathbb{C}$ that $s_p \in (\mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B})^d$. Since $(e_{x,a})_{a\in A}$ are the basis of the x-th copy of $\ell^\infty(A)$ and respectively $(f_{y,b})_{b\in B}$ are the basis of the y-th copy of $\ell^\infty(B)$, it suffices to show that s_p that $s_p(e_{x,a}\otimes 1)$ and $s_p(1\otimes f_{y,b})$ are well defined for all $x\in X, a\in A, x\in X, y\in Y$. For any $x, x'\in X$:

$$\begin{aligned}
s_p(1\otimes e_{y,b}) &= s_p\left(\sum_{a\in A} e_{x,a}\otimes f_{y,b}\right)\\
&= \sum_{a\in A} p(x,y,a,b)\\
&= \sum_{a\in A} p(x',y,a,b)\\
&= s_p\left(\sum_{a\in A} e_{x',a}\otimes f_{y,b}\right)\\
&= s_p(1\otimes e_{y,b}).
\end{aligned}$$

Therefore $s_p(e_{x,a}\otimes 1)$ is well defined. The fact that $s_p(1\otimes f_{y,b})$ is well defined can be shown analogous from (4). Therefore s_p is well defined.

Now let $s\in (\mathcal{S}_{X,A}\otimes\mathcal{S}_{Y,B})^d$. Then we need to show that p_s is no-signalling. For any $x, x'\in X$:

$$\begin{aligned}
\sum_{a\in A} p_s(x,y,a,b) &= \sum_{a\in A} s(e_{x,a}\otimes f_{y,b})\\
&= s(1\otimes f_{y,b})\\
&= s\left(\sum_{a\in A} e_{x',a}\otimes f_{y,b}\right)\\
&= \sum_{a\in A} p_s(x',y,a,b).
\end{aligned}$$

Therefore condition (5) holds and (4) can be shown analogous by using

$$\sum_{b\in B} e_{x,a}\otimes f_{y,b} = \sum_{b\in B} e_{x,a}\otimes f_{y',b}$$

for any $y, y'\in Y, x\in X, a\in A$. $\square$

This Lemma gives a characterization of overall functions $p\colon A\times B\times X\times Y\to\mathbb{C}$. To characterize the correlations we still need these functions to be probability distributions. But if p is a probability distribution we get that s_p is a state in $(\mathcal{S}_{X,A}\otimes\mathcal{S}_{Y,B})^d$ as $s_p(e_{x,a}\otimes f_{y,b})\geq 0$ for all $x\in X, y\in Y, a\in A, b\in B$ and $s_p(1\otimes 1)=1$. The main objective is classifying the perfect correlations of a nonlocal game. Therefore define for a nonlocal game $\mathcal{G}=(X,Y,A,B,\lambda)$:

$$J(\mathcal{G}) = \operatorname{span}\{e_{x,a}\otimes f_{y,b};\lambda(x,y,a,b)=0\}$$

Also, define

$$\mathcal{P}_\tau(\mathcal{G}) = \{s\in(\mathcal{S}_{X,A}\otimes_\tau\mathcal{S}_{Y,B})^d; s \text{ is a state}\}$$

and

$$\mathcal{P}_\tau^p(\mathcal{G}) = \{s\in(\mathcal{S}_{X,A}\otimes_\tau\mathcal{S}_{Y,B})^d; s \text{ is a state with } J(\mathcal{G})\subseteq\ker(s)\}.$$

for $\tau \in \{c, \max\}$.

The following is Theorem 3.1 in [19] and gives the classification of the no-signalling and quantum commuting strategies as states:

Theorem 6.15. *Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game. The map defined in Lemma 6.14 defines a continuous map M such that for all $\lambda \in [0,1]$, we have*

$$M(\lambda a + (1-\lambda)b) = M(\lambda a) + M((1-\lambda)b) \text{ for all } a, b \in \operatorname{dom}(M) \tag{6}$$

and M is a bijection between

(i) $\mathcal{P}_{max}(\mathcal{G})$ *and* $C_{ns}(\mathcal{G})$,
(ii) $\mathcal{P}_c(\mathcal{G})$ *and* $C_{qc}(\mathcal{G})$.

Proof. (i) Define

$$\mathcal{R}_{X,A} = \left\{ (z_{x,a})_{x\in X, a\in A} \in \ell^\infty(X \times A); \sum_{a\in A} z_{x,a} = \sum_{a\in A} z_{x',a} \ \forall x, x' \in X \right\}.$$

$\mathcal{R}_{X,A}$ is a selfadjoint vector space and $1 = (1)_{x\in X, a\in A} \in \mathcal{R}_{X,A}$, therefore $\mathcal{R}_{X,A}$ is an operator system. From Proposition 4.44 we get that $\mathcal{R}^d_{X,A}$ is also an operator system. From Theorem 5.2 and Theorem 5.9 in [10] we get that $\mathcal{S}_{X,A} \cong \mathcal{R}^d_{X,A}$ by the complete order isomorphism

$$\phi\colon \mathcal{S}_{X,A} \to \mathcal{R}^d_{X,A}, e_{x,a} \mapsto ((z_{x',a'})_{x'\in X, a'\in A} \mapsto z_{x,a}).$$

$\mathcal{R}_{X,A}$ is finite dimensional, therefore we get $\mathcal{R}_{X,A} \cong (\mathcal{R}^d_{X,A})^d \cong \mathcal{S}^d_{X,A}$ by Proposition 4.45. By Proposition 4.46 we get,

$$(\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B})^d \cong \mathcal{S}^d_{X,A} \otimes_{\min} \mathcal{S}^d_{Y,B} \cong \mathcal{R}_{X,A} \otimes_{\min} \mathcal{R}_{Y,B}.$$

By the injectivity of the minimal tensor product, we get that the inclusion

$$\mathcal{R}_{X,A} \otimes_{\min} \mathcal{R}_{Y,B} \subseteq_{\text{c.o.i.}} \ell^\infty_{X,A} \otimes_{\min} \ell^\infty_{Y,B}$$

is a complete order isomorphism. Since $\ell^\infty_{X,A}$ is finite dimensional we get by Proposition 4.34

$$\ell^\infty_{X,A} \otimes \ell^\infty_{Y,B} = \ell^\infty_{X,A} \otimes_{\max} \ell^\infty_{Y,B} = \ell^\infty_{X,A,Y,B}.$$

From Proposition 4.30, we get that there exists a complete order isomorphism from $\ell^\infty_{X,A} \otimes_{\min} \ell^\infty_{Y,B}$ to $\ell^\infty_{X,A,Y,B}$. Therefore there exists an operator system $S \subseteq \ell^\infty_{X,A,Y,B}$ that is completely order isomorphic to $(\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B})^d$. From Lemma 6.14, we get that all elements in S will be no-signalling. As all $p(x,y,a,b) \geq 0$ for a correlation, $s \in (\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B})^d$ has to be positive as well. Since s has to be positive, the condition $\sum_{a\in A, b\in B} p(a,b,x,y) = 1$ is equivalent to $s(1) = 1$. Therefore s needs to be a state. Since $\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B}$ is greater than all other operator system tensor products, s_p is positive in $\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B}$. This shows statement (i).

(ii) Let $\tilde{s}\colon \mathcal{S}_{X,A} \otimes_c \mathcal{S}_{Y,B} \to \mathbb{C}$ be a state. By Proposition 4.42, we get

$$\mathcal{S}_{X,A} \otimes_c \mathcal{S}_{Y,B} \subseteq_{\text{c.o.i.}} \mathcal{A}_{X,A} \otimes_{C^*\max} \mathcal{A}_{Y,B}$$

because $C^*(\mathcal{S}_{X,A}) = \mathcal{A}(X,A)$ and $C^*(\mathcal{S}_{Y,B}) = \mathcal{A}(Y,B)$. Therefore it follows from Krein's theorem (Proposition 4.18) that $\tilde{s}$ can be extended to a state on $s\colon \mathcal{A}_{X,A} \otimes_{C^*\max} \mathcal{A}_{Y,B} \to \mathbb{C}$. Now this proof follows the arguments of the proof of 3.4 (b) in [12]:
Let (H_s, π_s, ξ_s) be the GNS-representation of s which was given in Lemma 2.11. Let $a \in A, b \in B, y \in Y, b \in B$ be any element of the corresponding set. Since $e_{x,a} \otimes 1$ and $1 \otimes f_{y,b}$ are projections and $\sum_{a\in A} e_{x,a} \otimes 1 = 1$ and $\sum_{b\in B} 1 \otimes f_{y,b} = 1$, we get that $E_{x,a} = \pi_s(e_{x,a} \otimes 1)$ and $F_{y,b} = \pi_s(1 \otimes f_{y,b})$ are projections such that $\sum_{a\in A} E_{x,a} = 1 = \sum_{b\in B} F_{y,b}$. Also $e_{x,a}$ and $f_{y,b}$ commute, therefore we get that $(E_{x,a})_{a\in A}$ and $(F_{y,b})_{b\in B}$ are commuting sets of PVMs. Since ξ_s is a unit vector, we get that $(E_{x,a})_{a\in A}$ and $(F_{y,b})_{b\in B}$ combined with ξ form a quantum commuting strategy such that

$$s(e_{x,a} \otimes f_{y,b}) = \langle E_{x,a} F_{y,b} \xi_s, \xi_s \rangle.$$

Now let $p \in C_{\text{qc}}(\mathcal{G})$ be a correlation such that there exists a Hilbert space H and POVMs $(E_{x,a})_{a\in A}(F_{y,b})_{b\in B}$ and a unit vector ξ such that $p(a,b|x,y) = \langle E_{x,a} F_{y,b} \xi, \xi \rangle$. The maps defined by

$$\phi_x\colon \ell^\infty(A) \to B(H), e_a \mapsto E_{x,a}$$
$$\phi_y\colon \ell^\infty(B) \to B(H), f_b \mapsto F_{y,b}$$

are positive, since $E_{x,a} \geq 0$ and $F_{y,b} \geq 0$. Because $\ell^\infty(A), \ell^\infty(B)$ are commutative, it follows from Proposition 4.19 that all ϕ_x, ϕ_y are completely positive. The maps ϕ_x are also unital as

$$\phi_x(1) = \sum_{a\in A} \phi_x(e_a) = \sum_a E_{x,a} = 1.$$

Analogous we get that the ϕ_y are unital. Thus by Section 3 in [3], we get that there exist unital completely positive maps $\Phi_X\colon A(X,A) \to B(H)$ and $\Phi_Y\colon A(X,A) \to B(H)$ that $\Phi_X(e_{x,a}) = E_{x,a}$ and $\Phi_Y(f_{y,b}) = F_{y,b}$. By Lemma 4.20, we get that $\Phi_X \otimes_{C^*\max} \Phi_Y$ is completely positive and it is unital as both Φ_X and Φ_Y are unital. Now define

$$s\colon A(X,A) \otimes A(Y,B) \to \mathbb{C}, a \mapsto \langle \Phi_X \otimes \Phi_Y(a)\xi, \xi \rangle.$$

Since $\Phi_X \otimes \Phi_Y$ is unital and completely positive and $\|\xi\| = 1$, s is also positive and unital. Thus s is a state and $s(e_{x,a} \otimes f_{y,b}) = p(a,b|x,y)$ for all $a \in A, b \in B, x \in X, y \in Y$. Since $\mathcal{S}_{X,A} \otimes_c \mathcal{S}_{Y,B} \subseteq_{\text{c.o.i.}} A(X,A) \otimes A(Y,B)$, we can just restrict $s_{|\mathcal{S}_{X,A} \otimes_c \mathcal{S}_{Y,B}}$. This shows the statement.

$\square$

In Remark 6.8, we could see that one can define quantum strategies by restricting to PVMs instead of POVMs. The following corollary shows that the same holds true for quantum commuting strategies.

Corollary 6.16. *Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game and let $p \in C_{qc}(\mathcal{G})$ be a quantum commuting correlation. Then there exists a Hilbert space H, a unit vector $\xi \in H$ and PVMs of operators $(E_{x,a})_{a\in A} \subseteq B(H)$ and $(F_{y,b})_{b\in B} \subseteq B(H)$ such that*

$$p(a, b|x, y) = \langle E_{x,a}F_{y,b}\xi, \xi\rangle.$$

Proof. In the proof of Theorem 6.15 (ii), we could see that the quantum commuting strategy that was constructed for a state already consisted of PVMs instead of POVMs. Thus follows the statement. □

Corollary 6.17. *Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game and let $p \in C_{ns}(\mathcal{G})$ be a no-signalling correlation. The following are equivalent:*

(i) $p \in C_{qc}(\mathcal{G})$
(ii) $p(a, b|x, y) = s(e_{x,a} \otimes f_{y,b})$ *for some state* $s \in S(\mathcal{A}(X, A) \otimes_{C^*max} \mathcal{A}(Y, B))$
(iii) $p(a, b|x, y) = s(e_{x,a} \otimes f_{y,b})$ *for some state* $s \in S(\mathcal{S}_{X,A} \otimes_c \mathcal{S}_{Y,B})$

Proof. This follows directly from the proof of Theorem 6.15 (ii). □

Proposition 6.18. *The set $C_{qc}(\mathcal{G})$ is closed and convex.*

Proof. This argument is taken from the proof of Theorem 3.4(b) of [12]: The state space of $\mathcal{S}(A(X, A) \otimes_{C^*\max} A(Y, B))$ is convex and bounded. The state space can be written as

$$\{s\colon A(X, A)\otimes_{C^*\max}A(Y, B); s(1\otimes1) = 1\}\cap\{s\colon A(X, A)\otimes_{C^*\max}A(Y, B) \to \mathbb{C}; s \geq 0\},$$

therefore it is weak* closed as it is an intersection of weak* closed sets. Thus by Banach–Alaoglu weak*-compact. Now consider the map

$$\mathcal{S}(A(X, A) \otimes_{C^*\max} A(Y, B)) \to C(A \times B \times X \times Y), s \mapsto ((a, b, x, y) \mapsto s(e_{x,a} \otimes f_{y,b})).$$

This map is linear and continuous and from Corollary 6.17, we get that the image is $C_{\mathrm{qc}}(\mathcal{G})$. Therefore $C_{\mathrm{qc}}(\mathcal{G})$ is convex and weak*-compact. From V.1.5 in [8], we get that $C_{\mathrm{qc}}(\mathcal{G})$ is closed.

□

Corollary 6.19. *In the situation of Theorem 6.15, the map is also a continuous bijection that fulfils* (6) *between*

(i) $\mathcal{P}^p_{max}(\mathcal{G})$ *and* $C^p_{ns}(\mathcal{G})$
(ii) $\mathcal{P}^p_c(\mathcal{G})$ *and* $C^p_{qc}(\mathcal{G})$

Proof. This follows from Theorem 6.15 and the fact that for a state s $J(\mathcal{G}) \subseteq \ker(s)$ is equivalent to $\{(a, b, x, y); \lambda(a, b, x, y) = 0\} \subseteq \{(a, b, x, y); p_s(a, b|x, y) = 0\}$. □

Remark 6.20. It is easy to see that the no-signalling strategies are a strictly larger class of strategies than the quantum commuting strategies. Thus we get from Theorem 6.15 and Corollary 6.17 that $\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B}$ does not sit completely order isomorphic in $\mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}$ as otherwise the no-signalling and quantum commuting strategies would coincide. Therefore this shows that the maximal operator system tensor product is not injective and shows that we actually need the construction of the commuting tensor product.

This theorem and corollary give a classification of the perfect no-signalling (quantum commuting) strategies of a nonlocal game in the form of states of an operator system. The following two propositions will further classify the quantum commuting strategies of a mirror game in the form of a trace. Recall that mirror games were defined in Definition 6.3.

Remark 6.21. Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game and define

$$E^a_{x,y} = \{b \in B; \lambda(x, y, a, b) = 1\}, E^b_{x,y} = \{a \in A; \lambda(x, y, a, b) = 1\}.$$

In [19] mirror games were defined differently from this thesis as they defined them as:

$\mathcal{G}$ is called a mirror game if there exist functions $f\colon X \to Y$ and $g\colon Y \to X$ such that for all $x \in X, y \in Y$:

$$E^a_{x,f(x)} \cap E^{a'}_{x,f(x)} = \emptyset,\ \forall a, a' \in A, a \neq a' \tag{7}$$

and

$$E^b_{g(y),y} \cap E^{b'}_{y,g(y)} = \emptyset,\ \forall b, b' \in B, b \neq b'. \tag{8}$$

It is easy to check that mirror games that are defined like 6.3 are still mirror games in the sense of the definition of [19].

Since the objective of this section is classifying perfect strategies, we are only interested in games which have a perfect strategy therefore λ should fulfill the following condition:

$$\forall x \in X, y \in Y\colon \exists a \in A, b \in B\colon \lambda(x, y, a, b) = 1$$

Therefore for all $x \in X, a \in A$, there exists $b \in B$ such that $\lambda(x, f(x), a, b) = 1$ and for all $y \in Y, b \in B$, there exists $a \in A$ such that $\lambda(g(y), y, a, b) = 1$. But (7) implies that for $a, a' \in A, x \in X$ the existing $b, b' \in B$, such that

$$\lambda(x, f(x), a, b) = 1 = \lambda(x, f(x), a', b'),$$

have to be different from each other. Therefore $|A| \leq |B|$. Analogous from (8) follows that $|B| \leq |A|$. Therefore $|A| = |B|$ and this shows that the sets

$$\{(a, b) \in A \times B; \lambda(x, f(x), a, b) = 1\} \text{ and } \{(a, b) \in A \times B; \lambda(g(y), y, a, b) = 1\}$$

have to be the graph of bijections for all $x \in X, y \in Y$ and thus $\mathcal{G}$ is also a mirror game in the sense of 6.3.

From the following theorem, we get that quantum commuting strategies for mirror games can be written as a trace of a C^*-algebra. This Proposition is Theorem 6.1 in [19].

Proposition 6.22. *Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a mirror game, $p \in C^p_{qc}(\mathcal{G})$ and $s \in S(\mathcal{A}(X, A) \otimes_{C^*max} \mathcal{A}(Y, B))$ such that $p = p_s$. Then*

(i) *the functional $\tau\colon \mathcal{A}(X, A) \to \mathbb{C}, z \mapsto s(z \otimes 1)$ is a trace,*
(ii) *there exists a unital *-homomorphism $\varphi\colon \mathcal{A}(Y, B) \to \mathcal{A}(X, A)$ such that*

$$p(a, b|x, y) = \tau(e_{x,a}\varphi(f_{y,b})),\ \forall x \in X, y \in Y, a \in A, b \in B$$

and

$$s(z \otimes f_{y_1,b_1} ... f_{y_k,b_k}) = \tau(z\varphi(f_{y_k,b_k} f_{y_1,b_1}))$$

for all $z \in \mathcal{A}(A, X), k \in \mathbb{N}, y_i \in Y, b_i \in B, i = 1, ...k$.

Proof. We first prove some helpful identities for this following theorem:

Since $\mathcal{G}$ is a mirror game, there exist functions $g\colon X \to Y$ and $h\colon Y \to X$ such that the sets $\{(a,b) \in A \times B; \lambda(x, g(x), a, b) = 1\}$ and $\{(a,b) \in A \times B; \lambda(h(y), y, a, b) = 1\}$ are the graph of a bijection. Denote $\xi_x\colon A \to B$ as the bijections corresponding to the first sets and $\eta_y\colon B \to A$ as the bijections of the second sets. Define $p_{x,a} = f_{g(x),\xi_x(a)}$ and $q_{y,b} = e_{h(y),\eta_y(a)}$. Since $\sum_{b \in B} q_{y,b} = 1$, the $q_{y,b}$ fulfill the relations of $f_{y,b} \in \mathcal{A}(Y,B)$ and since the $f_{y,b}$ generate $\mathcal{A}(Y,B)$, we get that the assignment $f_{y,b} \mapsto q_{y,b}$ extends to a *-homomorphism $\varphi\colon \mathcal{A}(X,A) \to \mathcal{A}(Y,B)$. For $u_1, u_2 \in \mathcal{A}(X,A) \otimes_{C^*\max} \mathcal{A}(Y,B)$, define the equivalence relation $u_1 \sim u_2$ if $s(u_1) = s(u_2) \Leftrightarrow s(u_1 - u_2) = 0$.

$$s(e_{x,a} \otimes 1) = \sum_{b \in B} s(e_{x,a} \otimes f_{g(x),b}) = s(e_{x,a} \otimes f_{g(x),\xi_x(a)}) = s(e_{x,a} \otimes p_{x,a})$$

On the other hand for $a \neq a'$

$$s(e_{x,a'} \otimes p_{x,a}) = p(x, g(x), a', \xi_x(a)) = 0.$$

It follows that

$$s(1 \otimes p_{x,a}) = \sum_{a' \in A} s(e_{x,a'} \otimes p_{x,a}) = s(e_{x,a} \otimes p_{x,a}).$$

Thus,

$$e_{x,a} \otimes 1 \sim e_{x,a} \otimes p_{x,a} \sim 1 \otimes p_{x,a} \tag{9}$$

Define $h_{x,a} = e_{x,a} \otimes 1 - 1 \otimes p_{x,a}$. $h_{x,a}$ is selfadjoint because $e_{x,a}, p_{x,a}$ and 1 are selfadjoint.

$$h_{x,a}^2 = e_{x,a} \otimes 1 - 2e_{x,a} \otimes p_{x,a} + 1 \otimes p_{x,a}$$

and (9) implies that $h_{x,a}^2 \sim 0$. Both

$$\langle u, v\rangle_1 = s(uv^*), \langle u, v\rangle_2 = s(v^*u)$$

are inner products on $\mathcal{A}(X,A) \otimes_{C^*\max} \mathcal{A}(Y,B)$, because s is a state. The Cauchy-Schwarz inequality now implies for $u \in \mathcal{A}(X,A) \otimes_{C^*\max} \mathcal{A}(Y,B)$ that

$$\begin{aligned} |s(uh_{x,a})|^2 &\leq s(uu^*)s(h_{x,a}^2) = 0, \\ |s(h_{x,a}u)|^2 &\leq s(h_{x,a}^2)s(u^*u) = 0. \end{aligned}$$

From this follows $uh_{x,a} \sim 0 \sim h_{x,a}u$ for all $u \in \mathcal{A}(X,A) \otimes_{C^*\max} \mathcal{A}(Y,B)$ and $x \in X, a \in A$. Let $z \in \mathcal{A}(X,A)$ and setting $u = z \otimes 1$ shows,

$$ze_{x,a} \otimes 1 \sim z \otimes p_{x,a} \sim e_{x,a}z \otimes 1. \tag{10}$$

Set $h_{y,b} = q_{y,b} \otimes 1 - 1 \otimes f_{y,b}$, then we get analogous to how it was shown for $h_{x,a}$, $uh_{x,a} \sim 0 \sim h_{x,a}u$ for all $u \in \mathcal{A}(X,A) \otimes_{C^*\max} \mathcal{A}(Y,B)$. Setting $u = z \otimes 1$ shows that

$$zq_{y,b} \otimes 1 \sim z \otimes f_{y,b} \sim q_{y,b}z \otimes 1$$

and setting $u = z \otimes w$ shows that

$$zq_{y,b} \otimes w \sim z \otimes wf_{y,b} \tag{11}$$

for all $y \in Y, b \in B, z \in \mathcal{A}(X,A), w \in \mathcal{A}(Y,B)$.

(i) Let $z \in \mathcal{A}(X, A)$ be positive. Then there exists $\tilde{z} \in \mathcal{A}(X, A)$ such that $\tilde{z}^*\tilde{z} = z$.

$$\tau(z) = \tau((\tilde{z} \otimes 1)^*(\tilde{z} \otimes 1)) \geq 0$$

Also $\tau(1) = s(1 \otimes 1) = 1$. Therefore τ is a state. It is sufficient to show that $\tau(zw) = \tau(wz)$ for words from the set $M = \{e_{x,a}, x \in X, a \in A\}$ because the set of linear combinations of words in M is dense in $\mathcal{A}(X, A)$, so this follows from the fact that τ is continuous. We will now prove by induction over the length $|w|$ of the word w that $zw \otimes 1 \sim wz \otimes 1$. In the case that $|w| = 1$, this follows from (10). Suppose the assumption holds if $|w| \leq n - 1$. Let $|w| = n$ and write $w = w'e$, where $e \in M$. Then using (10), we have

$$zw \otimes 1 = zw'e \otimes 1 \sim ezw' \otimes 1 \sim w'ez \otimes 1 = wz \otimes 1.$$

(ii) By (11),

$$p(a, b|x, y) = s(e_{x,a} \otimes f_{y,b}) = s(e_{x,a}q_{y,b} \otimes 1) = \tau(e_{x,a}q_{y,b}) = \tau(e_{x,a}\varphi(f_{y,b})),$$

The second claim of (ii) will be shown by induction on k. For $k = 1$, we get by (11)

$$s(z \otimes f_{y,b}) = s(zq_{y,b} \otimes 1) = \tau(z\varphi(f_{y,b})).$$

Suppose the assumption holds for $k - 1$ terms. Then using (11), we have

$$\begin{aligned} s(z \otimes f_{y_1,b_1}...f_{y_k,b_k}) &= s(zq_{y_k,b_k} \otimes f_{y_1,b_1}...f_{y_k,b_k}) \\ &= \tau(z\varphi(f_{y_k,b_k})\varphi(f_{b_1,k_1}...f_{y_{k-1},b_{k-1}})) \\ &= \tau(z\varphi(f_{y_k,b_k}f_{b_1,k_1}...f_{y_{k-1},b_{k-1}})). \end{aligned}$$

□

The following is Lemma 3.3 in [24] and will be needed for the following proposition:

Lemma 6.23. *There exists a* **-isomorphism* $\gamma\colon \mathcal{A}(X, A) \to \mathcal{A}(X, A)^{op}$ *such that*

$$\gamma(e_{x_1,a_1}...e_{x_k,a_k}) = (e_{x_k,a_k}...e_{x_1,a_1})^{op}, x_i \in X, a_i \in A, i = 1, ..., k, k \in \mathbb{N}$$

for any finite sets X, A.

From the following theorem, we get the converse of Proposition 6.22: if there exists a trace that fulfills some conditions and models the correlation that correlation is also a perfect quantum commuting strategy. This theorem is Theorem 6.3 in [19].

Theorem 6.24. *Let* $\mathcal{G} = (X, Y, A, B, \lambda)$ *be a mirror game,* $p \in C^p_{ns}(\mathcal{G})$*. The following are equivalent:*

(i) $p \in C^p_{qc}(\mathcal{G})$
(ii) *there exists a tracial state* $\tau\colon \mathcal{A}(X, A) \to \mathbb{C}$ *and a unital* **-homomorphism* $\varphi\colon \mathcal{A}(Y, B) \to \mathcal{A}(Y, B)$ *with* $\varphi(\mathcal{S}_{X,A}) \subseteq \mathcal{S}_{Y,B}$ *such that*

$$p(a, b|x, y) = \tau(e_{x,a}\varphi(f_{y,b})) \; \forall x \in X, y \in Y, a \in A, b \in B$$

Proof. *(i)* $\Rightarrow$ *(ii)*: Follows from Theorem 6.1 and its proof, using the fact that there exist $x \in X, a \in A$ $e_{x,a} = q_{y,b}$ for all $y \in Y, b \in B$.
(ii) $\Rightarrow$ *(i)*: Let $\phi \colon \mathcal{A}(X,A) \times \mathcal{A}(X,A)^{\mathrm{op}}, (z, w^{\mathrm{op}}) \mapsto \tau(zw)$. The map

$$R(\phi) \colon \mathcal{A}(X,A) \to (\mathcal{A}(X,A)^{\mathrm{op}})^d, z \mapsto \tau(z\ \cdot)$$

is completely positive because for a positive $z \in \mathcal{A}(X,A)$, there exists $\tilde{z} \in \mathcal{A}(X,A)$ such that $\tilde{z}^*\tilde{z} = z$ and since τ is a trace $\tau(zw) = \tau(\tilde{z}^* w \tilde{z})$ and Proposition 4.19 shows that a positive functional is already completely positive. Thus we get from Lemma 5.7 in [15] that ϕ is jointly positive and by Krein's Theorem(Proposition 4.18) and Theorem 5.8 and 5.12 in [15], we get that ϕ extends to a positive map on $\mathcal{A}(X,A) \otimes_{C^*\max} \mathcal{A}(X,A)^{\mathrm{op}}$. Since $\phi(1 \otimes 1) = \tau(1) = 1$, ϕ is a state. Let $\mathcal{A}(X,A) \times \mathcal{A}(Y,B) \to \mathbb{C}$ be the linear functional defined by

$$s = \phi(\mathrm{id} \otimes \gamma) \circ (\mathrm{id} \otimes \varphi)$$

Since *-homomorphisms are positive and φ is unital, it follows that s is a state and

$$s(e_{x,a} \otimes f_{y,b}) = \phi(e_{x,a}) \otimes \gamma(\varphi(f_{y,b}))) = \tau(e_{x,a}\varphi(f_{y,b}))$$

By Corollary 6.17, $p \in C_{\mathrm{qc}}(\mathcal{G})$. □

7 Quantum nonlocal games

In this section, we will introduce quantum nonlocal games, which are a generalization of nonlocal games. "General" quantum nonlocal games were first introduced in [25] and quantum output mirror games, which are a generalization of mirror games, were defined in [4]. We will introduce correlations for quantum nonlocal games. Lastly, we will also give a classification of these correlations similar to the classification in Section 6.4 for nonlocal games. These correlations for quantum nonlocal games were also defined and classified in [25].

Let B be a Banach space and let $B^d = \{f \colon B \to \mathbb{C}; f \text{ is linear and continuous}\}$ be its topological dual space. We can then identify two elements $x \in B, x' \in B^d$ by $\langle x, x' \rangle = x'(x)$. Let X be a finite set, then M_X is self dual and this is given by the complete order isomorphism $D : M_X \to M_X^d, B \mapsto (A \mapsto \mathrm{Tr}(AB^t))$. This was shown in [22, Theorem 6.2]. Thus we can write for $A, B \in M_x$, $\langle A, B \rangle = \mathrm{Tr}(AB^t)$ and can use M_X instead of M_X^d.

Let X be a finite set. Although we mainly use the Banach space identification for the remainder of this thesis, for the vector space $\mathbb{C}^X$ we will still use the inner product $\langle u, v \rangle = (\overline{v})^t u$ and its induced norm. And for tensor products $\mathbb{C}^{X_1} \otimes ... \otimes \mathbb{C}^{X_n}$, we still use the induced inner product.

7.1 Basics on Quantum nonlocal games

To define quantum nonlocal games, we first need to introduce projection lattices.

Definition 7.1. Let X, Y be finite sets.

(i) The projection lattice of M_{XY} is defined as $\mathcal{P}_{XY} = \{P \in M_{XY}; P \text{ is a projection}\}$. The projection lattice of D_{XY} is defined as $\mathcal{P}^{\mathrm{cl}}_{XY} = \{P \in D_{XY}; P \text{ is a projection}\}$.

(ii) Let I be an index set and $(P_i)_{i\in I} \in \mathcal{P}_{XY}$ and $S_{P_i} = \{v \in \mathbb{C}^{XY}; P_i v = v\}$ The join $\bigvee_{i\in I} P_i \in \mathcal{P}_{XY}$ is defined as

$$\bigvee_{i\in I} P_i(v) = \begin{cases} 1, & \text{if } \operatorname{span}(\bigcup_{i\in I} S_{P_i}) \\ 0, & \text{if } \operatorname{span}(\bigcup_{i\in I} S_{P_i})^{\perp} \end{cases}$$

Now we want to show that $\mathcal{P}_{XY}^{\mathrm{cl}}$ is closed under the join of elements in $\mathcal{P}_{XY}^{\mathrm{cl}}$. For this, we first need to a Lemma about the projections in D_{XY}. Let X be a finite set. Recall that we denote the matrix units of M_X by $e_x e_{x'}^*$ which we defined in Example 2.16.

Lemma 7.2. *Let X, Y be finite sets and let $P \in M_X \otimes M_Y$ be a projection. Then there exist $\lambda_{x,y} \in \{0,1\}$ such that*

$$P = \sum_{x\in X, y\in Y} \lambda_{x,y} e_x e_x^* \otimes e_y e_y^*.$$

Let $P(X \times Y)$ be the power set of $X \times Y$. Therefore the function

$$S\colon P(X \times Y) \to \mathcal{P}_{XY}, M \mapsto \sum_{(x,y)\in M} \lambda_{x,y} e_y e_y^* \otimes e_y e_y^*$$

is a bijection.

Proof. The set $\{e_x e_x^* \otimes e_y e_y^*; (x,y) \in X \times Y\}$ is a basis of $D_X \otimes D_Y$. Therefore there exist $\lambda_{x,y} \in \mathbb{C}$ such that $\sum_{x\in X, y\in Y} \lambda_{x,y} e_x e_x^* \otimes e_y e_y^*$. Consider $\lambda \in \mathbb{C}$ and

$$P - \lambda 1 = \sum_{x\in X, y\in Y} (\lambda_{x,y} - \lambda) e_x e_x^* \otimes e_y e_y^*$$

This is invertible iff all $(\lambda_{x,y} - \lambda) \neq 0$ for all $(x,y) \in X \times Y$ because if this holds then $\sum_{x\in X, y\in Y} \frac{1}{\lambda_{x,y}-\lambda} e_x e_x^* \otimes e_y e_y^*$ is the inverse and if there exists $(x,y) \in X \times Y$ such that $(\lambda_{x,y} - \lambda) = 0$ then $(P - \lambda 1)(e_x \otimes e_y) = 0$. Then $\lambda_{x,y} \in \{0,1\}$ for all $(x,y) \in X \times Y$ because all eigenvalues of projections are either 0 or 1. □

Lemma 7.3. *Let X, Y be finite sets, let I be an index set and $P_i \in \mathcal{P}_{XY}^{cl}$. Then $\bigvee_{i\in I} P_i \in \mathcal{P}_{XY}^{cl}$ and we can write*

$$\bigvee_{i\in I} P_i = \sum_{(x,y)\in Q} e_x e_x^* \otimes e_y e_y^* \text{ with } Q = \{(x,y); \exists i \in I\colon e_x \otimes e_y \in \operatorname{Im}(P_i)\}.$$

Proof. Since $\mathcal{P}_{XY}^{\mathrm{cl}}$ is a finite set by Lemma 7.2, we can assume that I is finite. From Lemma 7.2, it also follows that the rank 1 projections in $\mathcal{P}_{XY}^{\mathrm{cl}}$ are orthogonal to each other because $e_x \otimes e_y \perp e_{x'} \otimes e_{y'}$ if $(x,y) \neq (x',y')$. Therefore we have that $\bigvee_{i\in I} P_i = \sum_{(x,y)\in Q} e_x e_x^* \otimes e_y e_y^*$ for $Q = \{(x,y); \exists i \in I\colon e_x \otimes e_y \in \operatorname{Im}(P_i)\}$. Thus we have $\bigvee_{i\in I} P_i \in \mathcal{P}_{XY}^{\mathrm{cl}}$. □

Now we can define quantum nonlocal games and classical-to-quantum nonlocal games. The latter are nonlocal games where we have classical answers and only the questions can be quantum.

Definition 7.4. Let X, Y, A, B be finite sets, I an index set, $\mathcal{P}_{XY}$ the projection lattice of M_{XY}, $\mathcal{P}_{AB}$ the projection lattice of M_{AB} and $\mathcal{P}^{\mathrm{cl}}_{XY}$ the projection lattice of D_{XY}.

(i) A quantum nonlocal game is a map $\varphi\colon \mathcal{P}_{XY} \to \mathcal{P}_{AB}$ with $\varphi(0) = 0$ that is also join continuous, i.e. $\varphi(\bigvee_{i\in I} P_i) = \bigvee_{i\in I} \varphi(P_i)$ for all $P_i \in \mathcal{P}_{XY}$.
(ii) A classical-to-quantum nonlocal game is a join continuous map $\varphi\colon \mathcal{P}^{\mathrm{cl}}_{XY} \to \mathcal{P}_{AB}$ with $\varphi(0) = 0$.
(iii) A classical nonlocal game is a join continuous map $\varphi\colon \mathcal{P}^{\mathrm{cl}}_{XY} \to \mathcal{P}^{\mathrm{cl}}_{AB}$ with $\varphi(0) = 0$.

Recall nonlocal games were defined in Definition 6.1 and consist of four finite sets X, Y, A, B and a function $\lambda\colon X \times Y \times A \times B \to \{0, 1\}$. The following Lemma establishes the connection between classical nonlocal games defined in Definition 7.4 and nonlocal games from Definition 6.1.

Proposition 7.5. *Let X, Y, A, B be finite sets. There is a bijection*

$$\{\lambda\colon X \times Y \times A \times B \to \{0,1\}\} \to \{\varphi\colon \mathcal{P}^{cl}_{XY} \to \mathcal{P}^{cl}_{AB}; \varphi \text{ is a classical nonlocal game}\}$$
$$\lambda \mapsto (\varphi_\lambda\colon \mathcal{P}^{cl}_{XY} \to \mathcal{P}^{cl}_{AB}, P \mapsto S(W_P))$$

where $W_P = \{(a, b); \exists (x, y) \in S^{-1}(P)$ such that $\lambda(x, y, a, b) = 1\}$ and S is the function defined in Lemma 7.2.

Proof. Let $\lambda : X \times Y \times A \times B \to \{0, 1\}$ be a function. Then we want to show that φ_λ is join continuous. Let I be an index set and $(P_i)_{i\in I} \subseteq \mathcal{P}^{\mathrm{cl}}_{XY}$. By Lemma 7.3, we can write

$$\bigvee_{i\in I} P_i = \sum_{(x,y)\in Q} e_x e_x^* \otimes e_y e_y^* \text{ with } Q = \{(x, y); \exists i \in I\colon e_x \otimes e_y \in \mathrm{Im}(P_i)\}.$$

Since $e_x e_x^* \otimes e_y e_y^* \perp e_{x'} e_{x'}^* \otimes e_{y'} e_{y'}^*$ for $x, x' \in X, y, y' \in Y$ and $x \neq x'$ or $y \neq y'$, we get that

$$\sum_{(x,y)\in Q} e_x e_x^* \otimes e_y e_y^* = \bigvee_{(x,y)\in Q} e_x e_x^* \otimes e_y e_y^*.$$

Define $Q'_{x,y} = \{(a, b) \in A \times B; (a, b) \in \mathrm{Im}(\varphi(e_x e_x^* \otimes e_y e_y^*))\}$. Then we get:

$$\begin{aligned}
\varphi_\lambda\left(\bigvee_{i\in I} P_i\right) &= \varphi_\lambda\left(\bigvee_{(x,y)\in Q} e_x e_x^* \otimes e_y e_y^*\right) \\
&= S(\{(a, b); \exists (x, y) \in Q \text{ such that } \lambda(a, b, x, y) = 1\}) \\
&= \bigvee_{(x,y)\in Q} S(\{(a, b); \lambda(a, b, x, y) = 1\}) \\
&= \bigvee_{(i\in I} \varphi_\lambda(P_i).
\end{aligned}$$

The fact that $\varphi_\lambda(0) = 0$ is obvious and thus the map is well defined. By Lemma 7.2 and 7.3, we can see that a join continuous map $\mathcal{P}^{\mathrm{cl}}_{XY} \to \mathcal{P}^{\mathrm{cl}}_{AB}$, that maps 0 to 0, is defined by its action on the rank one projections. From this it is easy to see that this map is bijective. ∎

This Proposition shows that classical nonlocal games and the nonlocal games from Definition 6.1 coincide by this bijection. We will later show that the strategies and perfect strategies also coincide. Thus we can see quantum nonlocal games as a generalization of nonlocal games.

In the following, we want to define quantum output mirror games which should generalize mirror games. These were introduced in [4]. Also, recall that mirror games were defined in 6.3. As motivation for the definition of bijective projections, we first show which classical quantum nonlocal games are mirror games:

Proposition 7.6.

Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game and φ_λ the "classical" quantum nonlocal game associated to λ. $\mathcal{G}$ is a mirror game iff there exist functions $f\colon X \to Y$, $g\colon Y \to X$ such that for every $x \in X$ and $y \in Y$, there exists bijections $\alpha_x, \beta_y\colon A \to B$ such that $\varphi_\lambda(e_x e_x^* \otimes e_{f(x)} e_{f(x)}^*) = P_{\alpha_x}$ and $\varphi_\lambda(e_{g(y)} e_{g(y)}^* \otimes e_y e_y^*) = P_{\beta_y^{-1}}$ where

$$P_{\alpha_x} = \sum_{a \in A} e_a e_a^* \otimes e_{\alpha_x(a)} e_{\alpha_x(a)}^* \text{ and } P_{\beta_y^{-1}} = \sum_{a \in A} e_a e_a^* \otimes e_{\beta_y^{-1}(a)} e_{\beta_y^{-1}(a)}^*.$$

Proof. "$\Rightarrow$": Fix $x \in X$. $\mathcal{G}$ is a mirror game, therefore there exists $f\colon X \to Y$ such that the set $\{(a,b) \in A \times B; \lambda(x, f(x), a, b) = 1\}$ is the graph of a bijection. Define $\alpha_x\colon A \to B$ as the bijection such that this set is its graph. Then it holds that:

$$\varphi(e_x e_x^* \otimes e_{f(x)} e_{f(x)}^*) = \{e_a e_a^* \otimes e_b e_b^*; \lambda(x, f(x), a, b) = 1\} = P_{\alpha_x}.$$

Fix $y \in Y$. Analogous there exists $g\colon Y \to X$ such that $\{(a,b); \lambda(g(y), y)\}$ is the graph of a bijection and define $\beta_y\colon Y \to X$ as the bijection to the given graph. Then it holds that:

$$\varphi(e_{g(y)} e_{g(y)}^* \otimes e_y e_y^*) = \{e_a e_a^* \otimes e_b e_b^*; \lambda(g(y), y, a, b) = 1\} = P_{\beta_y^{-1}}.$$

"$\Leftarrow$":
For $x \in X, y \in Y$, the sets $\{(a, \alpha_x(a)); a \in A\}$ and $\{(a, \beta_y(a)); a \in A\}$ are graphs of a bijection. Let $f\colon X \to Y, g\colon Y \to X$ be functions such that

$$\varphi_\lambda(e_x e_x^* \otimes e_{f(x)} e_{f(x)}^*) = P_{\alpha_x} \text{ and } \varphi_\lambda(e_{g(y)} e_{g(y)}^* \otimes e_y e_y^*) = P_{\beta_y^{-1}}.$$

From this follows that

$$\begin{aligned} &\{(a,b) \in A \times B; \lambda(a, b, x, f(x)) = 1\} = \{(a, \alpha_x(a)) \in A \times B; a \in A\} \\ &\{(a, \beta_y(a)); a \in A\} = \{(\beta_y^{-1}(b), b); b \in B\} = \{(a,b) \in A \times B; \lambda(a, b, g(y), y) = 1\} \end{aligned}$$

Therefore $\mathcal{G}$ is a mirror game. ☐

From this proposition, we get that we need some "quantum" version of bijections. To define these, we first need to introduce some maps.

Also, note that the tensor product of Banach spaces in this section is just the algebraic tensor product. The examples of tensor products of Banach spaces in this thesis do not contain tensor products with multiple Banach spaces of infinite dimensions and are only matrix algebras except for at most one space being traceclass or bounded operators. Thus also the C^*-algebraic tensor product would not contain different elements.

Definition 7.7. Let $B_1, ..., B_n$ be Banach spaces and $B_1^d, ..., B_n^d$ their respective topological dual spaces. Let $k \leq n$ and $i_1, ..., i_k \in \{1, ..., n\}$ such that $i_j \neq i_l$ for $j \neq l$. For $w_{i_1} \in B_{i_1}^d, ..., w_{i_k} \in B_{i_k}^d$, we define the slice map as the linear extension of

$$L_{w_{i_1},...,w_{i_k}} : B_1 \otimes ... \otimes B_n \to B_1 \otimes ... \otimes B_{i_1-1} \otimes B_{i_1+1} \otimes ... \otimes B_{i_k-1} \otimes B_{i_k+1} \otimes ... \otimes B_n,$$
$$b_1 \otimes ... \otimes b_n \mapsto \langle b_{i_1}, w_{i_1}\rangle ... \langle b_{i_k}, w_{i_k}\rangle b_1 \otimes ... \otimes b_{i_1-1} \otimes b_{i_1+1} \otimes ... \otimes b_{i_k-1} \otimes b_{i_k+1} \otimes ... \otimes b_n.$$

For the rest of this subsection, we assume that $A = B$. But we still write A and B as if those were different sets. This is not a very strong restriction as we could encode different sets with the same cardinality and we also had for a classical mirror game that has perfect strategies that $|A| = |B|$. Now we introduce the vec mapping. Let X be a finite set. Then we can define the vec mapping by

$$\mathrm{vec}\colon M_x \to \mathbb{C}^X \otimes \mathbb{C}^X, e_x e_{x'}^* \mapsto e_x \otimes e_{x'}.$$

Now we can define for an operator $U \in M_X$:

$$\zeta_U = \frac{\mathrm{vec}(U)}{\|\mathrm{vec}(U)\|} \in \mathbb{C}^X \otimes \mathbb{C}^X.$$

One way to achieve these “quantum” versions of bijections are “bijective” Projections. Recall that partial isometries were defined in 2.4.

Definition 7.8. Let A, B be finite sets and $P \in M_{AB}$ be a projection of rank r. We call P a bijective projection if there exist partial isometries $(U_i)_{i=1}^r \in M_A$ such that $P = \sum_{i=1}^r \zeta_{U_i}\zeta_{U_i}^*$ and

$$\sum_{i=1}^r U_i U_i^* = \sum_{i=1}^r U_i^* U_i = I.$$

Remark 7.9. Let A, B be finite sets and $\alpha\colon A \to B$ a bijection.

(i) If α is a bijection then P_α is a bijective projection of rank $|A|$ with corresponding partial isometries $e_{\alpha(a)}e_a^*$.
(ii) Let A, B be finite sets and $P \in M_{AB}$ be projection. In Lemma 2.2 in [4], it was shown that a projection of rank 1 is bijective iff for all $e, f \in \mathbb{C}^A, e \perp f$, we have $L_{ee^*}(P) \perp L_{ff^*}(P)$. For a projection $P_\alpha \in M_{AB}$, this is the case if e, f are chosen from the standard basis.

Now we can use these bijective projections to give a generalization of mirror games:

Definition 7.10. Let $X, Y, , A, B$ be finite sets and let $\varphi\colon \mathcal{P}_{XY}^{\mathrm{cl}} \to \mathcal{P}_{AB}$ be a classical-to-quantum nonlocal game. φ is called a quantum output mirror game if there exist functions $f\colon X \to Y, g\colon Y \to X$ such that for all $x \in X, y \in Y$ the projections $\varphi(e_x e_{x'}^* \otimes e_{f(x)}e_{f(x)}^*)$ and $\varphi(e_{g(y)}e_{g(y)}^* \otimes e_y e_y^*)$ are bijective.

It is easy to see from Proposition 7.5 and Remark 7.9 that mirror games are a subset of quantum output mirror games.

7.2 Quantum no-signalling correlations

Quantum no-signalling correlations were first introduced in [9]. Although the setting was different from nonlocal games the idea was as well that Alice and Bob can not communicate through this correlation. As strategies for quantum nonlocal games these correlations were introduced in [25]. Recall that quantum channels and classical channels were defined in Definition 5.11 and the partial trace was defined in Definition 5.6.

Definition 7.11. Let X, Y, A, B be finite sets and let $\Gamma\colon M_{XY} \to M_{AB}$ be a quantum channel.

(i) Γ is a quantum no-signalling (QNS) correlation iff for all $\rho_X \in M_X$ with $\mathrm{Tr}(\rho_X) = 0$

$$\mathrm{Tr}_A \Gamma(\rho_X \otimes \rho_Y) = 0 \ \forall \rho_Y \in M_Y,$$

and for all $\rho_Y \in M_Y$ with $\mathrm{Tr}(\rho_Y) = 0$

$$\mathrm{Tr}_B \Gamma(\rho_X \otimes \rho_Y) = 0 \ \forall \rho_X \in M_X.$$

(ii) Let $\varphi\colon P_{XY} \to P_{AB}$ be a quantum nonlocal game. Γ is called a perfect strategy for φ iff for all $P \in P_{XY}$ holds $\langle \Gamma(P), \varphi(P)_\perp \rangle = 0$.

In the previous section, we showed that quantum nonlocal games are a generalization of nonlocal games. Similarly, we can show now that a classical channel can be associated with a "classical" correlation and vice versa. First, we need to introduce a lemma that is needed in the proof.

Lemma 7.12. *Let X, Y be finte sets. Let $A, B \in M_{XY}$ be a positive elements, then we have* $\mathrm{Tr}(AB) \geq 0$ *and* $\mathrm{Tr}(AB^t) \geq 0$.

Proof. If B is positive B^t is still selfadjoint and $\mathrm{Sp}(B^t) \subseteq [0, \infty)$. Therefore B^t is positive and it is sufficient to show that $\mathrm{Tr}(AB) \geq 0$. From the spectral theorem follows that there exists a unitary $U_A \in M_{XY}$ and a diagonal matrix $D_A \in D_{XY}$ such that all entries of D_A are ≥ 0 and $A = U_A D_A U_A^*$. Similarly from the spectral theorem follows that there exists a unitary $U_B \in M_{XY}$ and a diagonal matrix $D_B \in D_{XY}$ such that all entries of D_B are ≥ 0 and $B = U_B D_B U_B^*$. Therefore we have

$$\mathrm{Tr}(AB) = \mathrm{Tr}(U_A D_A U_A^* U_B D_D U_B^*) = \mathrm{Tr}(U_A U_A^* U_B U_B^* D_A D_B) = \mathrm{Tr}(D_A D_B) \geq 0.$$

□

Recall that classical channels were defined in Definition 5.11.

Proposition 7.13. *Let X, Y, A, B be finite sets. Then there exists a bijection between the sets*

$$\{p\colon A \times B \times X \times Y \to [0,1]; p \textit{ is a correlation}\}$$

and

$$\{\mathcal{N}\colon D_{XY} \to D_{AB}; \mathcal{N} \textit{ is a classical channel}\}.$$

For a classical channel $\mathcal{N} : D_{XY} \to D_{AB}$ the corresponding correlation is given by

$$p_{\mathcal{N}}\colon X \times Y \times A \times B \to [0,1], (x, y, a, b) \mapsto \langle \mathcal{N}(e_x e_x^* \otimes e_y e_y^*), e_a e_a^* \otimes e_b e_b^* \rangle$$

and conversely for a correlation $p : A \times B \times X \times Y \to [0,1]$ *the corresponding classical channel is given by*

$$\mathcal{N}_p : D_{XY} \to D_{AB}, e_x e_x^* \otimes e_y e_y^* \mapsto \sum_{a \in A, b \in B} p(a,b|x,y) e_a e_a^* \otimes e_b e_b^*.$$

Proof. $\mathrm{Im}(p_{\mathcal{N}}) \subseteq [0,\infty)$, by Lemma 7.12 as the matrix units are positive and $\mathcal{N}$ is positive. Then we get that $p_{\mathcal{N}}$ is a correlation from the fact that $\mathcal{N}$ is trace-preserving. Conversely $\mathcal{N}_p$ is a positive because $\mathrm{Im}(p) \subseteq [0,1]$ and $\mathcal{N}_p$ is trace-preserving because for all $x \in X, y \in Y$, we have $\sum_{a \in A, b \in B} p(a,b|x,y) = 1$. Since it easy to see that $p = p_{\mathcal{N}_p}$ and $\mathcal{N} = \mathcal{N}_{p_{\mathcal{N}}}$, we get that this is a bijection. □

This proposition shows that there is a bijection between classical channels and correlations. But for a given nonlocal game they determine which strategies are perfect strategies. This is Proposition 10.14 in [25].

Proposition 7.14. *Let* $\mathcal{G} = (X, Y, A, B, \lambda)$ *be a nonlocal game and* p *be a no-signalling correlation for* G. *Then* p *is a perfect Strategy for* $\mathcal{G}$ *iff* $\mathcal{N}_p$ *is a perfect Strategy for* φ_λ, *i.e.*

$$\langle \mathcal{N}_p(P), \varphi_\lambda(P)_\perp \rangle = 0 \ \forall P \in P_{XY}^{cl}.$$

Proof. Let p be a perfect correlation. Then we have

$$\begin{aligned} &\{(a,b,x,y) \in A \times B \times X \times Y; \lambda(a,b,x,y) = 0\} \\ \subseteq &\{(a,b,x,y) \in A \times B \times X \times Y; p(a,b|x,y) = 0\}. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\{(a,b,x,y) \in A \times B \times X \times Y; p(a,b|x,y) = 1\} \\ \subseteq &\{(a,b,x,y) \in A \times B \times X \times Y; \lambda(a,b,x,y) = 1\} \end{aligned}$$

Thus follows that $\langle \mathcal{N}_p(e_x e_x^* \otimes e_y e_y^*), \varphi_\lambda(e_x e_x^* \otimes e_y e_y^*)_\perp \rangle = 0$. Every projection in D_{XY} can be written as $\sum_{x,y \in \tilde{X}, \tilde{Y}} e_x e_x^* \otimes e_y e_y^*$ for some subset $\tilde{X} \subseteq X, \tilde{Y} \subseteq Y$.

$$\begin{aligned} &\left\langle \mathcal{N}_p \left(\sum_{x,y \in \tilde{X}, \tilde{Y}} e_x e_x^* \otimes e_y e_y^* \right), \varphi_\lambda \left(\sum_{x,y \in \tilde{X}, \tilde{Y}} e_x e_x^* \otimes e_y e_y^* \right)_\perp \right\rangle \\ = &\sum_{x,y \in \tilde{X}, \tilde{Y}} \left\langle \mathcal{N}_p \left(e_x e_x^* \otimes e_y e_y^* \right), \left(\bigvee_{x,y \in \tilde{X}, \tilde{Y}} \varphi_\lambda \left(e_x e_x^* \otimes e_y e_y^* \right) \right)_\perp \right\rangle \\ \geq &\sum_{x,y \in \tilde{X}, \tilde{Y}} \left\langle \mathcal{N}_p \left(e_x e_x^* \otimes e_y e_y^* \right), \varphi_\lambda \left(e_x e_x^* \otimes e_y e_y^* \right)_\perp \right\rangle = 0. \end{aligned}$$

Let $\mathcal{N}_p$ be a perfect strategy for φ_λ. To show that p is a perfect strategy, it is sufficient to show that

$$\begin{aligned} &\{(a,b,x,y) \in A \times B \times X \times Y; \lambda(a,b,x,y) = 0\} \\ \subseteq &\{(a,b,x,y) \in A \times B \times X \times Y; p(a,b|x,y) = 0\}. \end{aligned}$$

Let $(a,b,x,y) \in \{(a,b,x,y) \in A \times B \times X \times Y; \lambda(a,b,x,y) = 0\}$.

$$\begin{aligned} p(a,b|x,y) =& \langle \mathcal{N}_p(e_xe_x^* \otimes e_ye_y^*), e_ae_a^* \otimes e_ye_y^* \rangle \\ \leq& \langle \mathcal{N}_p(e_xe_x^* \otimes e_ye_y^*), \varphi_\lambda(e_xe_x^* \otimes e_ye_y^*)_\perp \rangle \\ =& 0 \end{aligned}$$

Thus $(a,b,x,y) \in \{(a,b,x,y) \in A \times B \times X \times Y; p(a,b|x,y) = 0\}$. □

We can also directly embed "classical" correlations into the QNS correlations. For this, recall the completely dephasing from Definition 5.14.

Proposition 7.15. *Let X, Y, A, B be finite sets and $p\colon A \times B \times X \times Y \to [0,1]$ be a correlation. The map $\Gamma_p\colon M_{XY} \to M_{AB}$, such that $\Gamma_p = \mathcal{N}_p \circ \Delta_{XY}$, is a quantum channel. Moreover, p is a no-signalling correlation iff Γ_p is a QNS correlation.*

Proof. It is straight forward calculation to see that Γ_p is given by

$$\Gamma_p(A) = \sum_{x\in X, y\in Y} \sum_{a\in A, b\in B} p(a,b|x,y) \langle A(e_x \otimes e_y)e_x \otimes e_y, \rangle e_ae_a^* \otimes e_be_b^*.$$

So let p be a no-signalling correlation. Then Γ_p is a QNS correlation because for $\rho_x \in M_X, \rho_y \in M_Y$ with $\mathrm{Tr}(\rho_X) = 0$:

$$\begin{aligned} &\mathrm{Tr}_A \Gamma_p(\rho_x \otimes \rho_Y) \\ =& \sum_{x\in X, y\in Y} \mathrm{Tr}_A \left(\sum_{a\in A, b\in B} p(a,b|x,y) \langle p_X e_x, e_x \rangle \langle p_Y e_y, e_y \rangle e_ae_a^* \otimes e_be_b^* \right) \\ =& \sum_{x\in X, y\in Y} \sum_{a\in A, b\in B} p(a,b|x,y) \langle p_X e_x, e_x \rangle \langle p_Y e_y, e_y \rangle e_be_b^* \\ =& \sum_{y\in Y} \sum_{b\in B} \left(\sum_{x\in X} \sum_{a\in A} p(a,b|x,y) \langle p_X e_x, e_x \rangle \right) \langle p_Y e_y, e_y \rangle e_be_b^* \\ =& \sum_{y\in Y} \sum_{b\in B} 0 \langle p_Y e_y, e_y \rangle e_be_b^* = 0. \end{aligned}$$

Analogous follows the second condition of a QNS correlation.

Conversely, let Γ_p be a QNS correlation. Define $\rho = e_xe_x^* \otimes e_ye_y^* - e_{x'}e_{x'}^* \otimes e_ye_y^*$. We then have

$$\begin{aligned} 0 =& \Gamma_p(\rho) \\ =& \sum_{\tilde{x}\in X, \tilde{y}\in Y} \sum_{a\in A, b\in B} p(a,b|\tilde{x},\tilde{y}) \langle \rho(e_{\tilde{x}} \otimes e_{\tilde{y}}), e_{\tilde{x}} \otimes e_{\tilde{y}} \rangle e_ae_a^* \otimes e_be_b^* \\ =& \sum_{\tilde{x}\in X, \tilde{y}\in Y} \sum_{a\in A, b\in B} p(a,b|\tilde{x},\tilde{y}) \langle \rho(e_{\tilde{x}} \otimes e_{\tilde{y}}), e_{\tilde{x}} \otimes e_{\tilde{y}} \rangle e_ae_a^* \otimes e_be_b^* \\ =& \sum_{a\in A, b\in B} (p(a,b|x,y) - p(a,b|x',y)) e_ae_a^* \otimes e_be_b^* \\ \Rightarrow& \sum_{a\in A} p(a,b|x,y) = \sum_{a\in A} p(a,b|x',y) \ \forall x, x' \in X, y \in Y, b \in B. \end{aligned}$$

Analogous the other condition for no-signalling correlations follows by using $\rho = e_xe_x^* \otimes e_ye_y^* - e_xe_x^* \otimes e_{y'}e_{y'}^*$. □

Let X, Y, A, B be finite sets. This Proposition shows that the no-signalling correlations over (A, B, X, Y) are exactly the QNS correlations $\Gamma\colon M_{XY} \to M_{AB}$ such that Γ is $(X \times Y, A \times B)$-classical. With these insights into how classical correlations are embedded into the quantum correlations, we can define classical-to-quantum correlations:

Definition 7.16. Let X, Y, A, B be finite sets and let $\mathcal{E}\colon D_{XY} \to M_{AB}$ be a classical-to-quantum channel.

(i) $\mathcal{E}\colon D_{XY} \to M_{AB}$ is a classical-to-quantum no-signalling correlation (CQNS) iff for all $\rho_x \in D_X$

$$\mathrm{Tr}_A\, \mathcal{E}(\rho_X \otimes \rho_Y) = 0 \ \forall \rho_Y \in D_Y, \mathrm{Tr}(\rho_Y) = 0$$

and for all $\rho_Y \in D_Y$

$$\mathrm{Tr}_B\, \mathcal{E}(\rho_X \otimes \rho_Y) = 0 \ \forall \rho_X \in D_X, \mathrm{Tr}(\rho_X) = 0.$$

(ii) Let $\varphi\colon P^{\mathrm{cl}}_{XY} \to P_{AB}$ be a classical-to-quantum nonlocal game. $\mathcal{E}$ is called a perfect strategy for φ iff for all $P \in P^{\mathrm{cl}}_{XY}$ holds $\langle \mathcal{E}(P), \varphi(P)_\perp \rangle = 0$.

Now we can also show that CQNS correlations can be embedded into the QNS correlations. This is Theorem 7.3 in [25].

Proposition 7.17. *Let X, Y, A, B be finite sets and $\mathcal{E}\colon D_{XY} \to M_{AB}$ be CQNS correlation. We define*

$$\Gamma_{\mathcal{E}}\colon M_{XY} \to M_{AB}, \Gamma_{\mathcal{E}} = \mathcal{E} \circ \Delta_{XY}.$$

Then $\Gamma_{\mathcal{E}}$ is a QNS correlation. And conversely if $\Gamma : M_{XY} \to M_{AB}$ is a QNS correlation, we get that $\Gamma_{|D_{XY}}$ is a CQNS correlation.

Proof. That $\Gamma_{|D_{XY}}$ is a CQNS correlation if Γ is a QNS correlation follows directly from the definition of CQNS correlations.

Now let $\rho_X \in M_X, \rho_X \in M_Y$ be quantum states with $\mathrm{Tr}(\rho_X) = 0$. Then we get

$$\begin{aligned}
\mathrm{Tr}_A(\Gamma_{\mathcal{E}}(\rho_X \otimes \rho_Y)) &= \mathrm{Tr}_A(\mathcal{E} \circ \Delta_{XY}(\rho_X \otimes \rho_Y))) \\
&= \mathrm{Tr}_A(\mathcal{E}(\sum_{x \in X, y \in Y} \langle \rho_X e_x, e_x \rangle \langle \rho_Y e_y, e_y \rangle e_x e_x^* \otimes e_y e_y^*)) \\
&= \mathrm{Tr}_A(\mathcal{E}(\sum_{y \in Y} \langle \rho_Y e_y, e_y \rangle (\sum_{x \in X} \langle \rho_X e_x, e_x \rangle e_x e_x^* \otimes e_y e_y^*)) \\
&= \sum_{y \in Y} \mathrm{Tr}_A(\mathcal{E}(\langle \rho_Y e_y, e_y \rangle (\sum_{x \in X} \langle \rho_X e_x, e_x \rangle e_x e_x^* \otimes e_y e_y^*)) \\
&= 0
\end{aligned}$$

Analogous we can show the second condition for a QNS correlation. Thus $\Gamma_{\mathcal{E}}$ is a QNS correlation. $\square$

7.3 Quantum commuting QNS correlations

The aim of this section will be to introduce a second class of correlations for quantum nonlocal games. This class is called quantum commuting quantum no-signalling correlations. These strategies are the “quantum” analogue of quantum commuting strategies for nonlocal games. These correlations were introduced in [25] and most of this section is taken from that article. To define quantum commuting QNS strategies, we first need to introduce stochastic operator matrices. Recall that slice maps were defined in 7.7. Using slice maps, we can define the partial trace Tr_A in a more general setting as the slice map L_{I_A}.

Definition 7.18. Let X, A be finite sets and let H be a Hilbert space. A positive operator $E \in M_X \otimes M_A \otimes B(H)$ such that $\mathrm{Tr}_A E = I_X \otimes I_H$ is called stochastic operator matrix. For $x, x' \in X$ and $a, a' \in A$, we define $E_{x,x',a,a'} = L_{e_x e_{x'}^* \otimes e_a e_{a'}}(E)$, $E_{a,a'} = L_{e_a e_{a'}^*}(E)$ and $E_{x,x'} = L_{e_x e_{x'}^*}(E)$.

Remark 7.19. Consider a stochastic operator matrix $E \in D_X \otimes D_A \otimes B(\mathbb{C})$. From the proof of Proposition 7.2, it follows that $E = \sum_{x \in X, a \in A} \lambda_{x,a} e_x e_x^* \otimes e_a e_a^*$ such that $\lambda_{x,a} \geq 0$. Identifying $D_X \otimes D_A$ with $M_{|X|,|A|}$ the stochastic operator Matrices become row stochastic (scalar-valued) matrices because

$$I_X = \mathrm{Tr}_A E \Leftrightarrow 1 = \sum_{a \in A} \lambda_{x,a} \forall x \in X$$

and the fact that positive elements in $D_X \otimes D_A$ only have positive entries.

First, we prove some useful facts for stochastic operator matrices. As stochastic operator matrices are positive, the following property holds:

Lemma 7.20. *Let X, A be finite sets, H be a Hilbert space and $E \in M_X \otimes M_A \otimes B(H)$ be a positive operator. Then we have that*

(i) *$E_{x,x} \in M_A \otimes B(H)$ is positive for all $x \in X$,*
(ii) *$E_{a,a} \in M_X \otimes B(H)$ is positive for all $a \in A$,*
(iii) *$E_{x,x,a,a} \in B(H)$ is positive for all $x \in X, a \in A$.*

Proof. Define the maps

$$\begin{aligned}
&\phi_{E,X} : M_X \to M_A \otimes B(H), e_a e_{a'}^* \mapsto L_{e_a e_{a'}^*}(E),\\
&\phi_{E,A} : M_A \to M_X \otimes B(H), e_x e_{x'}^* \mapsto L_{e_x e_{x'}^*}(E),\\
&\phi_{E,X,A} : M_A \otimes M_X \to B(H), e_x e_{x'}^* \otimes e_a e_{a'}^* \mapsto L_{e_x e_{x'}^* \otimes e_a e_{a'}^*}(E).
\end{aligned}$$

For all three of these maps, we have that their Choi matrix is E. Therefore by Proposition 4.17, we have that all maps are completely positive and therefore positive. Also for all $x \in X, a \in A$, we have that $e_x e_x^*, e_a e_a^*$ and $e_x e_x^* \otimes e_a e_a^*$ are positive therefore $\phi_{E,X}(e_x e_x^*) = E_{x,x}, \phi_{E,A}(e_a e_a^*) = E_{a,a}$ and $\phi_{E,X,A}(e_x e_x^* \otimes e_a e_a^*) = E_{x,x,a,a}$ are positive. ☐

Lemma 7.21. *Let X, A be finite sets and H be a Hilbert space. For a positive element $E \in (M_X \otimes M_A \otimes B(H))^+$, the following are equivalent:*

(i) *E is a stochastic operator matrix,*

(ii) *$E_{a,a}$ is a POVM in $M_X \otimes B(H)$.*

Proof. By Lemma 7.20, we get that the operators $E_{a,a}$ are already positive. Since

$$\mathrm{Tr}_A(E) = \sum_{a \in A} e_a e_a^* \otimes E_{a,a}$$

we get the statement. □

Recall Definition 6.9, where we defined quantum commuting correlation for nonlocal games. This proposition shows a connection between POVMs in $M_X \otimes B(H)$ and stochastic operator matrices.

Let X, A be finite sets and H be a Hilbert space. Note that every operator $E \in M_X \otimes M_A \otimes B(H)$ can be written as:

$$\sum_{x,x' \in X, a, a' \in A} e_x e_{x'}^* \otimes e_a e_{a'}^* \otimes E_{x,x',a,a'}$$

This motivates the following definition of classical and semi-classical stochastic operator matrices:

Definition 7.22. Let X, A be finite sets and H be a Hilbert space. Also, let $E \in M_X \otimes M_A \otimes B(H)$ be a stochastic operator matrix.

(i) We call E classical if there exist sets of POVMs $(A_{x,a})_{a \in A} \subseteq B(H)$ such that

$$E = \sum_{x \in X} \sum_{a \in A} E_{x,x} \otimes E_{a,a} \otimes A_{x,a}.$$

(ii) We call E semi-classical if there exist $(A_x)_{x \in X} \subseteq (M_X \otimes B(H))^+$ such that

$$E = \sum_{x \in X} E_{x,x} \otimes A_x.$$

Let H be a Hilbert space. Recall that in Proposition 3.13, it was shown the dual of the traceclass operators $T(H)$ is $B(H)$. Let $B \in B(H)$ the isomorphism was given by $B \mapsto (T \mapsto \phi_B)$ where $\phi_B : T(H) \to \mathbb{C}, T \mapsto \mathrm{Tr}(BT)$. Thus we can identify $T(H)^d$ by $B(H)$ and write for $T \in T(H), B \in B(H)$:

$$\langle B, T \rangle = \mathrm{Tr}(BT) \text{ or } \langle T, B \rangle = \mathrm{Tr}(BT).$$

Thus we can use this duality for slice maps for elements from $B(H)$ and $T(H)$. Since $T(H)$ can be embedded into its bidual space, we also use this identification the other way around for slice maps.

Lemma 7.23. *Let X, A be finite sets, let H be a Hilbert space, let $\sigma \in T(H)$ be a state and let $E \in M_X \otimes M_A \otimes B(H)$ be a stochastic operator matrix. The linear map $\Gamma_{E,\sigma} \colon M_X \to M_A, M \mapsto L_{M \otimes \sigma}(E)$ is a quantum channel.*

Proof. Define $\phi_E \colon M_A \to M_X \otimes B(H), e_a e_{a'}^* \mapsto L_{e_a e_{a'}^*}(E) = E_{a,a'}$. Since E is positive it follows directly from Proposition 4.17 that ϕ_E is completely positive. Let ϕ_* be the predual of ϕ_E. Then $\phi_* : M_X \otimes T(H) \to M_A$ since $T(H)$ is the predual of $B(H)$ and M_X, M_A are selfdual. The predual ϕ_* of ϕ_E satisfies by definition

$$\langle \phi_*(\rho), \omega \rangle = \langle \rho, \phi_E(\omega) \rangle \; \forall \rho \in M_X \otimes T(H), w \in M_A.$$

In [25] it is stated (without proof) that the predual of a completely positive map is completely positive. Let $\sigma \in T(H)$ be a state and $x, x' \in X, a, a\prime \in A$, then we have

$$\begin{aligned}\langle \phi_*(e_x e_{x'}^* \otimes \sigma), e_a e_a^* \rangle &= \langle e_x e_{x'}^* \otimes \sigma, \phi_E(e_a e_a^*) \rangle \\ &= \langle e_x e_{x'}^* \otimes \sigma, E_{a,a'} \rangle \\ &= \left\langle e_x e_{x'}^* \otimes \sigma, \sum_{\tilde{x},\tilde{x}' \in X} e_{\tilde{x},\tilde{x}'} \otimes E_{\tilde{x},\tilde{x}',a,a'} \right\rangle \\ &= \langle \sigma, E_{x,x',a,a'} \rangle.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle L_{e_x e_{x'}^* \otimes \sigma}(E), e_a e_{a'}^* \rangle &= \left\langle L_{e_x e_{x'}^* \otimes \sigma} \left(\sum_{\tilde{x},\tilde{x}' \in X} \sum_{\tilde{a},\tilde{a}' \in A} e_{\tilde{x},\tilde{x}'} \otimes e_{\tilde{a},\tilde{a}'} \otimes E_{\tilde{x},\tilde{x}',\tilde{a},\tilde{a}'} \right), e_a e_{a'}^* \right\rangle \\ &= \left\langle \sum_{\tilde{a},\tilde{a}' \in A} L_\sigma(e_{\tilde{a}} e_{\tilde{a}'}^* \otimes E_{x,x',\tilde{a},\tilde{a}'}), e_a e_{a'}^* \right\rangle \\ &= \sum_{\tilde{a},\tilde{a}' \in A} \langle \sigma, E_{x,x',\tilde{a},\tilde{a}'} \rangle \langle e_{\tilde{a}} e_{\tilde{a}'}^*, , e_a e_{a'}^* \rangle \\ &= \langle \sigma, E_{x,x',a,a'} \rangle.\end{aligned}$$

Since $\{e_x e_{x'}^* x, x' \in X\}$ and $\{e_a e_{a'}^*; a, a' \in A\}$ form a basis of M_X and M_A respectively, we get that $\Gamma_{E,\sigma} = L_{e_x e_{x'}^* \otimes \sigma}(E) = \phi_*(e_x e_{x'}^* \otimes \sigma)$. Since ϕ_* is completely positive and σ is positive, we get that $\Gamma_{E,\sigma}$ is completely positive. $\Gamma_{E,\sigma}$ is trace preserving as $\mathrm{Tr}(\sigma) = 1$ and by definition $\mathrm{Tr}_A(E) = I_X \otimes I_A$. Thus $\Gamma_{E,\sigma}$ is a quantum channel. □

Lemma 7.24. *Let X, A be finite sets, let H be a Hilbert space, let $\sigma \in T(H)$ be a state and let $E \in M_X \otimes M_A \otimes B(H)$ be a stochastic operator matrix. For all $x, x' \in X$ and $a, a' \in A$, we have*

$$\langle \Gamma_{E,\sigma}(e_x e_{x'}^*), e_a e_{a'}^* \rangle = \langle \sigma, E_{x,x',a,a'} \rangle.$$

Proof. This follows directly from the proof of Lemma 7.23. □

Using Lemma 7.23, we can define quantum channels with stochastic operator matrices and vector states. These quantum channels are also classical iff the stochastic operator matrix is classical. This is the following proposition which is Lemma 7.2 in [25]. Following this article we write

$$\begin{aligned}\tilde{\Delta}_X &= \Delta_X \otimes \mathrm{id}_A \otimes \mathrm{id}_{B(H)} \colon M_X \otimes M_A \otimes B(H) \to D_X \otimes M_A \otimes B(H), \\ \tilde{\Delta}_X &= \mathrm{id}_X \otimes \Delta_A \otimes \mathrm{id}_{B(H)} \colon M_X \otimes M_A \otimes B(H) \to M_X \otimes D_A \otimes B(H), \\ \tilde{\Delta}_{XA} &= \Delta_X \otimes \Delta_A \otimes \mathrm{id}_{B(H)} \colon M_X \otimes M_A \otimes B(H) \to D_X \otimes D_A \otimes B(H).\end{aligned}$$

Proposition 7.25. *Let X, A be finite sets and H be a Hilbert space. Also, let $E \in M_X \otimes M_A \otimes B(H)$ be a stochastic operator matrix and $\sigma \in T(H)$ be a state. Set $E' = \tilde{\Delta}_X(E)$ and $E'' = \tilde{\Delta}_{XA}(E)$. Then E' is a semi-classical stochastic operator matrix and E'' is a classical operator matrix. Moreover, we have*

$$\Gamma_{E,\sigma} \circ \Delta_X = \Gamma_{E',\sigma} \text{ and } \Delta_A \circ \Gamma_{E,\sigma} \circ \Delta_X = \Gamma_{E'',\sigma}. \tag{12}$$

Proof. We can write

$$E = \sum_{x,x' \in X} \sum_{a,a' \in A} e_x e_{x'}^* \otimes e_a e_{a'}^* \otimes E_{x,x',a,a'}.$$

Therefore we get

$$\begin{aligned} E' = \tilde{\Delta}_X(E) &= \sum_{x \in X} \sum_{a,a' \in A} e_x e_x^* \otimes e_a e_{a'}^* \otimes E_{x,x,a,a'} = \sum_{x \in X} e_x e_x^* \otimes E_{x,x}, \\ E'' = \tilde{\Delta}_{XA}(E) &= \sum_{x \in X} \sum_{a \in A} e_x e_x^* \otimes e_a e_a^* \otimes E_{x,x,a,a}. \end{aligned}$$

By Lemma 7.20, we have that E' is a semi-classical operator stochastic operator matrix. By

$$I_X \otimes I_H = \mathrm{Tr}_A(E) = \sum_{x,x' \in X} \sum_{a \in A} e_x e_{x'}^* \otimes E_{x,x',a,a} \Rightarrow \sum_{a \in A} E_{x,x',a,a} = \delta_{x,x'} I_H$$

and Lemma 7.20, we get that E'' is a classical stochastic operator matrix. By Lemma 7.24, we get that

$$\begin{aligned} &\langle \Gamma_{E,\sigma}(\Delta_X(e_x e_{x'}^*)), e_a e_a^* \rangle = \delta_{x,x'} \langle \sigma, E_{x,x',a,a'} \rangle = \langle \Gamma_{E',\sigma}(e_x e_{x'}^*), e_a e_a^* \rangle, \\ &\langle \Delta_A(\Gamma_{E,\sigma}(\Delta_X(e_x e_{x'}^*))), e_a e_a^* \rangle = \delta_{x,x'} \delta_{a,a'} \langle \sigma, E_{x,x',a,a'} \rangle = \langle \Gamma_{E'',\sigma}(e_x e_{x'}^*), e_a e_a^* \rangle. \end{aligned}$$

Since $\{e_x e_{x'}^* x, x' \in X\}$ and $\{e_a e_{a'}^*; a, a' \in A\}$ form a basis of M_X and M_A respectively, we get (12). □

This Proposition shows that the channel corresponding to a stochastic operator matrix is classical iff the stochastic operator matrix is classical and similarly, it has classical inputs iff the stochastic operator matrix is semi-classical.

From Lemma 7.23, we can define quantum channels with stochastic operator matrices and vector states. But these are not necessarily QNS correlations. So if we want to define a "quantum" analogue for quantum commuting strategies these should be QNS correlations as quantum commuting strategies for classical nonlocal games were no-signalling strategies as well. Thus we will introduce a special class of stochastic operator matrices.

Definition 7.26. Let X, Y, A, B be finite set and let $E \in M_X \otimes M_A \otimes B(H)$, $F \in M_Y \otimes M_B \otimes B(H)$ be stochastic operator matrices. The pair (E, F) is called commuting if for all $x, x' \in X, y, y' \in Y, a, a' \in A, b, b' \in B$:

$$E_{x,x',a,a'} F_{y,y',b,b'} = F_{y,y',b,b'} E_{x,x',a,a'}.$$

Now we can show that a pair of quantum commuting stochastic operator matrices defines a QNS correlation. This was shown in Proposition 4.1 in [25].

Proposition 7.27. *Let X, Y, A, B be finite sets and H be a Hilbert space. Let $E \in M_X \otimes M_A \otimes B(H)$, $F \in M_Y \otimes M_B \otimes B(H)$ form a commuting pair of stochastic operator matrices. Then there exists a unique operator $E \cdot F$ such that*

$$\begin{aligned} &\forall \rho_X \in M_X, \rho_Y \in M_Y, \rho_A \in M_A, \rho_B \in M_B, \sigma \in T(H): \\ &\langle E \cdot F, \rho_X \otimes \rho_Y \otimes \rho_A \otimes \rho_B \otimes \sigma \rangle = \langle L_{\rho_X \otimes \rho_A}(E) L_{\rho_Y \otimes \rho_B}(F), \sigma \rangle. \end{aligned}$$

Moreover, $E \cdot F$ is a stochastic operator matrix and for $\sigma \in T(H)$, $\Gamma_{E \cdot F, \sigma}$ is a QNS correlation.

Proof. Define $E \cdot F = (E_{x,x',a,a'}F_{y,y',b,b'}) \in M_{XY} \otimes M_{AB} \otimes B(H)$. Denote by $\mathcal{A}$ the C^*-algebra generated by $\{E_{x,x',a,a'}; x, x' \in X, a, a' \in A\}$ and denote by $\mathcal{B}$ the C^*-algebra generated by $\{F_{y,y',b,b'}; y, y' \in Y, b, b' \in B\}$. Since $\mathcal{A} \subseteq B(H)$ and $\mathcal{B} \subseteq B(H)$, we get representations

$$\pi_{\mathcal{A}}\colon M_{XA}(\mathcal{A}) \to M_{XYAB}(B(H)), S \mapsto S \otimes I_{YB},$$
$$\pi_{\mathcal{B}}\colon M_{YB}(\mathcal{B}) \to M_{XYAB}(B(H)), T \mapsto I_{XA} \otimes T.$$

Since the ranges of $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ commute, we get a representation

$$\pi\colon M_{XA}(\mathcal{A}) \otimes_{C^*\max} M_{YB}(\mathcal{B}) \to M_{XYAB}(B(H)), S \otimes T \mapsto \pi_{\mathcal{A}}(S)\pi_{\mathcal{B}}(T).$$

Set $E \cdot F = \pi(E \otimes F)$. Since E and F are positive, there exist $A \in M_X \otimes M_A \otimes B(H)$ such that $A^*A = E$ and $B \in M_Y \otimes M_B \otimes B(H)$ such that $B^*B = F$, therefore

$$E \cdot F = \pi(E \otimes F) = \pi(A^*A \otimes B^*B) = \pi(A \otimes B)^*\pi(A \otimes B) \geq 0.$$

So if $\mathrm{Tr}_{AB}(E \cdot F) = I_{XY} \otimes I_H$, we have that $E \cdot F$ is a stochastic operator matrix. First note that

$$I_X \otimes I_H = \mathrm{Tr}_A(E) = \sum_{a \in A}(E_{x,x',a,a})_{x,x'}$$

and therefore $E_{x,x',a,a} = \delta_{x,x'}I$ for all $x, x' \in X$. Analogous we get for F that $F_{y,y',b,b} = \delta_{y,y'}I$ for all $y, y' \in Y$. Thus we can conclude

$$\mathrm{Tr}_{AB}(E \cdot F) = \sum_{a \in A}\sum_{b \in B}\sum_{x,x' \in X}\sum_{y,y' \in Y} e_x e_{x'}^* \otimes e_y e_{y'}^* \otimes E_{x,x',a,a}F_{y,y',b,b}$$
$$= \sum_{a \in A}\sum_{b \in B}(E_{x,x',a,a}F_{y,y',b,b})_{x,x',y,y'} = (\delta_{x,x'}\delta_{y,y'}I)_{x,x',y,y'} = I_{XY} \otimes I_H.$$

First note that we can write

$$E \cdot F = \sum_{x,x' \in X}\sum_{y,y' \in Y}\sum_{a,a' \in A}\sum_{b,b' \in B} e_x e_{x'}^* \otimes e_a e_{a'}^* \otimes e_y e_{y'}^* \otimes e_a e_{a'}^* \otimes e_b e_{b'}^* \otimes E_{x,x',a,a'}F_{y,y',b,b'}.$$

For $x, x' \in X, y, y' \in Y, a, a' \in A, b, b' \in B$ and $\sigma \in T(H)$, we have

$$\begin{aligned} &\langle E \cdot F, e_x e_{x'}^* \otimes e_a e_{a'}^* \otimes e_y e_{y'}^* \otimes e_a e_{a'}^* \otimes e_b e_{b'}^* \otimes \sigma\rangle \\ &= \langle E_{x,x',a,a'}F_{y,y',b,b'}, \sigma\rangle = \left\langle L_{e_x e_{x'}^* \otimes e_a e_{a'}^*}(E)L_{e_y e_{y'}^* \otimes e_b e_{b'}^*}(F), \sigma\right\rangle. \end{aligned} \tag{13}$$

By Linearity we now get

$$\forall \rho_X \in M_X, \rho_Y \in M_Y, \rho_A \in M_A, \rho_B \in M_B, \sigma \in T(H)\colon$$
$$\langle E \cdot F, \rho_X \otimes \rho_Y \otimes \rho_A \otimes \rho_B \otimes \sigma\rangle = \langle L_{\rho_X \otimes \rho_A}(E)L_{\rho_Y \otimes \rho_B}(F), \sigma\rangle.$$

The last statement we need to show is that $\Gamma_{E \cdot F,\sigma}$ is a QNS correlation. Let $\rho_X \in M_X, \rho_Y \in M_Y$ with $\mathrm{Tr}(\rho_X) = 0$ and $\tau_B \in M_B$,we have

$$\begin{aligned} \langle \mathrm{Tr}_A \Gamma_{E \cdot F,\sigma}(\rho_X \otimes \rho_Y), \tau_B\rangle &= \langle \Gamma_{E \cdot F,\sigma}(\rho_X \otimes \rho_Y), I_A \otimes \tau_B\rangle \\ &= \langle L_{\rho_X \otimes \rho_Y \otimes \sigma}(E \cdot F), I_A \otimes \tau_B\rangle \\ &= \langle L_{\rho_X \otimes \rho_Y}(E \cdot F), I_A \otimes \tau_B \otimes \sigma\rangle \\ &= \langle E \cdot F, \rho_X \otimes \rho_Y I_A \otimes \tau_B \otimes \sigma\rangle \\ &= \langle \mathrm{Tr}_A L_{\rho_X}(E)L_{\rho_Y \otimes \tau_B}(F), \sigma\rangle = 0 \end{aligned}$$

The first, third, fourth and fifth identity follows similarly by decomposition like (13). Analogous follows the other condition for a QNS correlation. Thus we get that $\Gamma_{E\cdot F,\sigma}$ is a QNS correlation. $\square$

By using semi-classical stochastic operator matrices, we can also get CQNS correlations in a similar fashion:

Proposition 7.28. *Let X, Y, A, B be finite sets, H be a Hilbert space, $\sigma \in T(H)$ be a state and $E \in M_X \otimes M_A \otimes B(H), F \in M_Y \otimes M_B \otimes B(H)$ be a commuting pair of semi-classical stochastic operator matrices. Denote $E_x = L_{e_x e_x^*}(E)$ and $F_y = L_{e_y e_y^*}(F)$, then*

$$\mathcal{E} : D_{XY} \to M_{AB}, E_{x,x} \otimes E_{y,y} \mapsto L_\sigma(E_x \cdot F_y)$$

is a CQNS correlation.

Proof. First note that E_x and F_y are stochastic operator matrices in $\mathbb{C} \otimes M_A \otimes B(H)$ and $\mathbb{C} \otimes M_B \otimes B(H)$ since $\operatorname{Tr}_A E_x = I_H$ and $\operatorname{Tr}_B F_y = I_H$ and it is positive by the definition of a semi-classical stochastic operator matrix. That E_x and F_y commute, follows directly from the fact that E and F commute. Thus the map is well defined and it remains to show that it is a CQNS correlation. By Proposition 7.27, we get that $\Gamma_{E\cdot F,\sigma}$ is a QNS correlation. It is easy to check that $(\Gamma_{E\cdot F,\sigma})_{|D_{XY}} = \mathcal{E}$ and therefore by Proposition 7.17, we have that $\mathcal{E}$ is a CQNS correlation. $\square$

Lemma 7.29. *Let H be a Hilbert space and $\xi \in H$ be a unit vector. Then the map*

$$\xi\xi^* \colon H \to H, h \mapsto \langle h, \xi\rangle\xi$$

is a state in $T(H)$.

Proof. First, we show that $\xi\xi^*$ is positive. Thus let $h \in H$, then we have

$$\langle\langle h, \xi\rangle\xi, h\rangle = \langle h, \xi\rangle\langle\xi, h\rangle = |\langle h, \xi\rangle|^2.$$

Now let I be an index set and $(h_i)_{i\in I}$ be an orthonormal basis of H. Then we have

$$\|\xi\xi^*\|_{\mathrm{Tr}} = \sum_{i\in I}\langle\langle h_i, \xi\rangle\xi, h_i\rangle = \sum_{i\in I}\langle\xi, \langle h_i, \xi\rangle h_i\rangle = \langle\xi, \xi\rangle = \|\xi\|^2 = 1.$$

$\square$

The last two propositions, let us define quantum commuting QNS correlations and quantum commuting CQNS correlations. Let H be a Hilbert space and $\xi \in H$ be a unit vector, we then write $\Gamma_{E,\xi}$ for $\Gamma_{E,\xi\xi^*}$.

Definition 7.30. Let X, Y, A, B be finite sets.

(i) A QNS correlation $\Gamma\colon M_{XY} \to M_{AB}$ is called quantum commuting if there exists a Hilbert space H, a unit vector $\xi \in H$ and a commuting pair of stochastic matrices $E \in M_X \otimes M_A \otimes B(H), F \in M_Y \otimes M_B \otimes B(H)$ such that $\Gamma = \Gamma_{E\cdot F,\xi}$.

(ii) A CQNS correlation $\mathcal{E}\colon D_{XY} \to M_{AB}$ is called quantum commuting if there exists a Hilbert space H, a unit vector $\xi \in H$ and a commuting pair of semi-classical stochastic matrices $E \in M_X \otimes M_A \otimes B(H), F \in M_Y \otimes M_B \otimes B(H)$ such that $\mathcal{E}(E_{x,x} \otimes E_{y,y}) = L_\sigma(E_x \cdot F_y)$.

We defined quantum commuting QNS correlations in a way that just allows vector states in $T(H)$. This is done because there is no gain of generality for allowing any state of $T(H)$ in this definition. This is shown in the following lemma which is Proposition 4.3 in [25].

Lemma 7.31. *Let X, Y, A, B be finite sets, H be a Hilbert space, $\sigma \in T(H)$ be a state and $E \in M_X \otimes M_A \otimes B(H), F \in M_Y \otimes M_B \otimes B(H)$ be a commuting pair of stochastic matrices. Then there exists a Hilbert space $\tilde{H}$ unit vector $\xi \in \tilde{H}$ and a commuting pair of stochastic operator matrices $\tilde{E} \in M_X \otimes M_A \otimes B(\tilde{H}), \tilde{F} \in M_Y \otimes M_B \otimes B(\tilde{H})$ such that $\Gamma_{E\cdot F,\sigma} = \Gamma_{\tilde{E}\cdot\tilde{F},\xi}$*

Proof. Since $T(H) \subseteq K(H)$ by Lemma 3.12 and σ is a state, we get that there exists a sequence of unit vectors $(\xi_i)_{i=1}^{\infty} \subseteq H$ such that $\sigma = \sum_{i=1}^{\infty} \lambda_i \xi_i \xi_i^*$ and $\sum_{i=1}^{\infty} \lambda_i = 1$. Set $\tilde{H} = H \otimes \ell^2$ and $\xi = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i \otimes e_i$. Then $\|\xi\| = \sum_{i=1}^{\infty} \sqrt{\lambda_i}^2 = 1$. Also for all $T \in T(H)$, we have that $\langle \xi\xi^*, T \otimes I_{\ell^2} \rangle = \langle \sigma, T \rangle$. Let $\tilde{E} = E \otimes I_{\ell^2}$ and $\tilde{F} = F \otimes I_{\ell^2}$. Thus $\tilde{E}, \tilde{F}$ are stochastic operator matrices acting on $\tilde{H}$. Also since E, F are a commuting pair of stochastic operator matrices, we get that $\tilde{E}, \tilde{F}$ are a commuting pair of stochastic operator matrices. Moreover for $\rho_X \in M_X, \rho_Y \in M_Y$ and $\tau_A \in M_A, \tau_B \in M_B$, we have by Proposition 7.27:

$$\begin{aligned}
\left\langle \Gamma_{\tilde{E}\cdot\tilde{F},\xi}(\rho_X \otimes \rho_Y), \tau_A \otimes \tau_B \right\rangle &= \left\langle \tilde{E} \cdot \tilde{F}, \rho_X \otimes \rho_Y \otimes \tau_A \otimes \tau_B \otimes \xi\xi^* \right\rangle \\
&= \left\langle L_{\rho_X \otimes \tau_A}(\tilde{E}) L_{\rho_Y \otimes \tau_B}(\tilde{F}), \xi\xi^* \right\rangle \\
&= \langle L_{\rho_X \otimes \tau_A}(E) L_{\rho_Y \otimes \tau_B}(F) \otimes I_{\ell^2}, \xi\xi^* \rangle \\
&= \langle L_{\rho_X \otimes \tau_A}(E) L_{\rho_Y \otimes \tau_B}(F), \sigma \rangle \\
&= \langle E \cdot F, \rho_X \otimes \rho_Y \otimes \tau_A \otimes \tau_B \otimes \sigma \rangle \\
&= \langle \Gamma_{E\cdot F,\sigma}(\rho_X \otimes \rho_Y), \tau_A \otimes \tau_B \rangle .
\end{aligned}$$

This shows that $\Gamma_{\tilde{E}\cdot\tilde{F},\xi} = \Gamma_{E\cdot F,\sigma}$. □

The following proposition shows that quantum commuting QNS correlations form a proper generalization of the quantum commuting strategies for nonlocal games. Recall that for a classical correlation p, we associated a channel Γ_p in Proposition 7.15 and quantum commuting strategies and their correlations were defined in Definition 6.5 and Definition 6.9.

Proposition 7.32. *Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game and H be a Hilbert space. Consider a quantum commuting strategy for a correlation p that $\mathcal{G}$ consists of a unit vector $\xi \in H$ and commuting POVMs of operators $(A_{x,a})_{a\in A} \subseteq B(H)$ and $(B_{y,b})_{b\in B} \subseteq B(H)$. Define*

$$E = \sum_{x\in X, a\in A} e_x e_x^* \otimes e_a e_a^* \otimes A_{x,a} \text{ and } F = \sum_{y\in Y, b\in B} e_y e_y^* \otimes e_b e_b^* \otimes B_{y,b},$$

then (E, F) forms a commuting pair of classical stochastic operator matrices such that $\Gamma_{E\cdot F,\xi} = \Gamma_p$.

Proof. Since $e_x e_x^*, e_a e_a^*, A_{x,a}$ are all positive, we have that $e_x e_x^* \otimes e_a e_a^* \otimes A_{x,a}$ is positive for all $x \in X, a \in A$ and sums of positive elements are positive by Lemma 2.8. Therefore E, F are positive. Because $(A_{x,a})_{a\in A}$ are POVMs, we have

$$\mathrm{Tr}_A(E) = \sum_{x\in X} e_x e_x^* \otimes I_H = I_X \otimes I_H.$$

Analogous we have that $\mathrm{Tr}_B(F) = I_Y \otimes I_H$. Therefore E and F are stochastic operator matrices that commute because the $A_{x,a}$ and $F_{y,b}$ commute. And it is easy to see that $E \cdot F$ is also a classical stochastic operator matrix. So it remains to show that $\Gamma_p = \Gamma_{E\cdot F,\xi}$. By Proposition 7.15 and Proposition 7.25, we get that we only need to check $\Gamma_p = \Gamma_{E\cdot F,\xi}$ on D_{XY}. Let $x \in X, y \in Y$ then

$$\begin{aligned}
\Gamma_{E\cdot F,\xi}(e_x e_x^* \otimes e_y e_y^*) &= L_{e_x e_x^* \otimes e_y e_y^* \otimes \xi\xi^*}(E \cdot F) \\
&= \sum_{a\in A}\sum_{b\in B} e_a e_a^* \otimes e_b e_b^* \langle E_{x,a}F_{y,b}, \xi\xi^*\rangle \\
&= \sum_{a\in A}\sum_{b\in B} e_a e_a^* \otimes e_b e_b^* \,\mathrm{Tr}(E_{x,a}F_{y,b}\xi\xi^*) \\
&= \sum_{a\in A}\sum_{b\in B} e_a e_a^* \otimes e_b e_b^* \sum_{i\in I}\langle E_{x,a}F_{y,b}\xi\langle h_i,\xi\rangle, h_i\rangle \\
&= \sum_{a\in A}\sum_{b\in B} e_a e_a^* \otimes e_b e_b^* \sum_{i\in I}\langle E_{x,a}F_{y,b}, h_i\langle h_i,\xi\rangle\rangle \\
&= \sum_{a\in A}\sum_{b\in B} e_a e_a^* \otimes e_b e_b^* \langle E_{x,a}F_{y,b}\xi, \xi\rangle \\
&= \sum_{a\in A}\sum_{b\in B} e_a e_a^* \otimes e_b e_b^* p(a,b|x,y) \\
&= \Gamma_p
\end{aligned}$$

By linearity we get $\Gamma_p = \Gamma_{E\cdot F,\xi}$. □

Recall that in Proposition 7.17, we defined a QNS correlation for a corresponding CQNS correlation. This QNS correlation is also quantum commuting iff the corresponding CQNS correlation is quantum commuting. This was shown in Theorem 7.3 of [25] and is the next proposition. Recall that we associated a QNS correlation to a CQNS correlation in Proposition 7.17.

Proposition 7.33. *Let X, Y, A, B be finite sets and $\mathcal{E} : D_{XY} \to M_{AB}$ be a CQNS correlation. Then $\mathcal{E}$ is quantum commuting iff there exists a commuting pair of semi-classical stochastic operator matrices $E \in M_X \otimes M_A \otimes B(H), F \in M_Y \otimes M_B \otimes B(H)$ and a state $\sigma \in T(H)$ such that $\Gamma_{\mathcal{E}} = \Gamma_{E\cdot F,\sigma}$.*

Proof. Let $\mathcal{E}$ be a quantum commuting CQNS correlation. Therefore there exists a pair of commuting semi-classical stochastic operator matrices E, F such that $\mathcal{E}(e_x e_x^* \otimes e_y e_y^*) = L_\sigma(E_x \cdot F_y)$. First we show that $\tilde{\Delta}_{XY}(E \cdot F) = \tilde{\Delta}_X(E) \cdot \tilde{\Delta}_Y(F)$. Notice that we can write

$$E \cdot F = \sum_{x\in X}\sum_{y\in Y}\sum_{a,a'\in X}\sum_{b,b'\in B} e_x e_x^* \otimes e_y e_y^* \otimes e_a e_{a'}^* \otimes e_b e_{b'}^* \otimes E_{x,x,a,a'}F_{y,y,b,b'}$$

Therefore $E \cdot F = \tilde{\Delta}_{XY}(E \cdot F)$ and because E, F are semi-classical we have that $\tilde{\Delta}_X(E) = E$ and $\tilde{\Delta}_Y(F) = F$. Therefore it follows from Proposition 7.25 that $\Gamma_{E\cdot F,\sigma} = \Gamma_{E\cdot F,\sigma} \circ \Delta_{XY}$. Therefore it is sufficient to check $\Gamma_{\mathcal{E}} = \Gamma_{E\cdot F,\sigma}$ on D_{XY}. Now let $x \in X, y \in Y$, then

$$\Gamma_{E\cdot F,\sigma}(e_x e_x^* \otimes e_y e_y^*) = L_{e_x e_x^* \otimes e_y e_y^*}(E \cdot F) = L_\sigma(E_x \cdot F_y) = \mathcal{E}(e_x e_x^* \otimes e_y e_y^*).$$

The result follows by linearity.

Conversely let $\mathcal{E}$ be a CQNS correlation such that there exists a a pair of commuting semi-classical stochastic operator matrices E, F such that $\Gamma_{E\cdot F,\sigma} = \Gamma_{\mathcal{E}}$. Now define $\mathcal{E}_{E\cdot Y} : D_{XY} \to M_{AB}, e_x e_x^* \otimes e_y e_y^* \mapsto L_\sigma(E_x \cdot F_y)$. Analogous to the converse direction we get $\Gamma_{\mathcal{E}_{E\cdot Y}} = \Gamma_{E\cdot F,\sigma} = \Gamma_{\mathcal{E}}$. Therefore $\mathcal{E}_{E\cdot Y} = \mathcal{E}$ and this shows that $\mathcal{E}$ is a quantum commuting CQNS correlation. □

7.4 Quantum no-signalling strategies via states

In this subsection, we give a classification for QNS and quantum commuting QNS correlation by states of an operator system, similar to the classification for no-signalling and quantum commuting correlation in Theorem 6.15. These results were presented in [25].

First, we need to introduce the operator system used for the classification results. We present a brief overview of the construction of this operator system and some of its properties. The complete construction and the proof of these properties is presented in [25].

A ternary ring is a complex vector space $\mathcal{V}$, equipped with a map

$$[\cdot,\cdot,\cdot] \colon \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

that is conjugate linear in the middle variable, linear on the outer two variables such that

$$[s,t,[u,v,w]] = [s,[v,u,t],w] = [[s,t,u].v.w] \ \forall s,t,,u,v,w \in \mathcal{V}.$$

Let X, A be finite sets. Let $\{v_{a,x}; a \in A, x \in X\}$ be a set satisfying the relations

$$\sum_{a\in A} [v_{a'',x''}, v_{a,x}, v_{a,x'}] = \delta_{x,x'} v_{a'',x''} \ \forall x,x',x'' \in X, a'' \in A.$$

Let $\mathcal{V}^0_{X,A}$ be the ternary ring generated by this set. Note that this implies that

$$\sum_{a\in A} [u, v_{a,x}, v_{a,x'}] = \delta_{x,x'} u \ \forall x,x' \in X, u \in \mathcal{V}^0_{X,A}.$$

Let H, K be Hilbert spaces. Because $[V_1,V_2,V_3] = V_1 V_2^* V_3$ is a ternary map for all $V_1, V_2, V_3 \in B(H,K)$, we can form ternary rings of operators and define ternary representations, i.e. linear maps θ from a ternary ring into $B(H,K)$ that respect the ternary map, that is for all u, v, w in the ternary ring, we have

$$\theta([v,w,u]) = \theta(v)\theta(w)^*\theta(u).$$

Using the structure of $B(H,K)$ and these ternary representations, we can construct an object $\mathcal{V}_{X,A} \subseteq B(H,K)$ for some Hilbert spaces H, K that is still a ternary ring. From this we can now define the needed C^*-algebra:

$$\mathcal{C}_{X,A} = \overline{\operatorname{span}\{S^*T; S,T \in \mathcal{V}_{x,a}\}}$$

As $\mathcal{V}_{x,a}$ was constructed from $\mathcal{V}^0_{x,a}$, it still contains elements $v_{a,x}$ that satisfy

$$\sum_{a\in A} v_{a''x''} v^*_{a,x} v_{a,x'} = \delta_{x,x'} v_{a'',x''} \ \forall x,x',x'' \in X, a'' \in A. \tag{14}$$

Define $e_{x,x',a,a'} = v_{a,x}^* v_{a',x'} \in \mathcal{C}_{X,A}$ for all $x, x' \in X$ and $a, a' \in A$. Then we define the operator system

$$\mathcal{T}_{X,A} = \operatorname{span}\{e_{x,x',a,a'}; x, x' \in X, a, a' \in A\} \subseteq \mathcal{C}_{X,A}.$$

Note that (14) implies that for all $x, x' \in X$

$$\sum_{a \in A} e_{x,x',a,a} = \delta_{x,x'} 1.$$

Now we present some properties of $\mathcal{C}_{X,A}$ and $\mathcal{T}_{X,A}$, these properties were shown in Section 5 of [25].

Proposition 7.34. *Let X, A be finite sets, H be a Hilbert space and $\phi\colon \mathcal{T}_{X,A} \to B(H)$ be a linear map. The following are equivalent:*

(i) *ϕ is a unital completely positive map,*
(ii) *$(\phi(e_{x,x',a,a'})) \in M_X \otimes M_A \otimes B(H)$ is a stochastic operator matrix,*
(iii) *there exists a representation $\pi\colon \mathcal{C}_{X,A} \to B(H)$ such that $\phi = \pi_{|\mathcal{T}_{X,A}}$.*

Recall the definition of the maximal C^*-algebra was given in Definition 4.40. If we choose the inclusion map of $\mathcal{T}_{X,A}$ into $\mathcal{C}_{X,A}$, we get from Proposition 7.34 the following corollary.

Corollary 7.35. *Let X, A be finite sets. The maximal C^*-algebra of $\mathcal{T}_{X,A}$ is $C_u^*(\mathcal{T}_{X,A}) = \mathcal{C}_{X,A}$.*

Similar to Proposition 7.34, we can also characterize the completely positive maps that are not necessarily unital.

Proposition 7.36. *Let X, A be finite sets, H be a Hilbert space and $\phi\colon \mathcal{T}_{X,A} \to B(H)$ be a linear map. The following are equivalent:*

(i) *ϕ is a completely positive map,*
(ii) *$(\phi(e_{x,x',a,a'})) \in (M_X \otimes M_A \otimes B(H))^+$,*

Now that we have defined the operator system that is needed for the classification of the QNS correlations and also collected some useful properties. The next step is to associate linear maps to linear functionals from this operator system.

Definition 7.37. Let X, Y, A, B be finite sets and let

$$s_1\colon \mathcal{T}_{X,A} \otimes \mathcal{T}_{Y,B} \to \mathbb{C}, s_2\colon \mathcal{C}_{X,A} \otimes \mathcal{C}_{Y,B} \to \mathbb{C}, s_3\colon \mathcal{C}_{X,A} \otimes_{C^*\max} \mathcal{C}_{Y,B},$$

be linear maps. For $i \in \{1,2,3\}$, we define the linear map $\Gamma_{s_i}\colon M_{XY} \to M_{AB}$ by

$$\Gamma_{s_i}(e_x e_{x'}^* \otimes e_y e_{y'}^*) = \sum_{a,a' \in A} \sum_{b,b' \in B} s_i(e_{x,x',a,a'} \otimes f_{y,y',b,b'}) e_a e_{a'}^* \otimes e_b e_{b'}^*.$$

Before we can give the classification of QNS strategies we need a lemma that is needed for the proof. This is Proposition 5.5 in [25].

Lemma 7.38. *Let X, A be finite sets. Denote by*

$$\mathcal{L}_{X,A} = \left\{ (\lambda_{x,x',a,a'}) \in M_{XA}; \exists c \in \mathbb{C} \colon \forall\, x, x' \in X \colon \sum_{a \in A} \lambda_{x,x',a,a'} = \delta_{x,x'} c \right\}.$$

The linear map $\Lambda \colon \mathcal{T}^d_{X,A} \to \mathcal{L}_{X,A}, \phi \mapsto (\phi(e_{x,x',a,a'}))$ is a well defined complete order isomorphism.

Proof. Let $\phi \colon \mathcal{T}^d_{X,A} \to \mathbb{C}$ be a positive functional. By Proposition 4.19, we have that ϕ is completely positive. Therefore we have $(\phi(e_{x,x',a,a'})) \in M^+_{XA}$ by Proposition 7.36. Thus we can define the map $\Lambda_+ \colon (\mathcal{T}^d_{X,A})^+ \to \mathcal{L}_{X,A}, \phi \mapsto (\phi(e_{x,x',a,a'}))$. Since all maps in $(\mathcal{T}^d_{X,A})^+$ are linear we get that the map Λ_+ satisfies

$$\Lambda_+(\lambda\phi_1 + \phi_2) = \lambda\Lambda_+(\phi_1) + \Lambda_+(\phi_2), \forall\lambda \geq 0, \phi_1, \phi_2 \in (\mathcal{T}^d_{X,A})^+. \tag{15}$$

Since we can decompose every selfadjoint functional $\phi \in \mathcal{T}^d_{X,A}$ into two functionals $\phi_1, \phi_2 \in (\mathcal{T}^d_{X,A})^+$ such that $\phi = \phi_1 - \phi_2$, we define

$$\Lambda_{\text{sa}} \colon (\mathcal{T}^d_{X,A})_{\text{sa}} \to \mathcal{L}_{X,A}, \phi \mapsto \Lambda_+(\phi_1) - \Lambda_+(\phi_2). \tag{16}$$

Let $\psi_1, \psi_2 \in (\mathcal{T}^d_{X,A})^+$ such that $\psi_1 - \psi_2 = \phi_1 - \phi_2$, to show that Λ_{sa} is well defined we need $\Lambda_+(\psi_1) - \Lambda_+(\psi_2) = \Lambda_+(\phi_1) - \Lambda_+(\phi_2)$. This follows from the additivity as $\Lambda_+(\phi_1) + \Lambda_+(\psi_2) = \Lambda_+(\psi_1) + \Lambda_+(\phi_2)$. The map Λ_{sa} is $\mathbb{R}$-linear by (15) and (16). Every element in $\phi \in \mathcal{T}^d_{X,A}$ can be decomposed into

$$\phi = \frac{\phi + \phi^*}{2} + i\frac{\phi - \phi^*}{2i}, \ \frac{\phi - \phi^*}{2i}, \frac{\phi + \phi^*}{2} \in (\mathcal{T}^d_{X,A})_{\text{sa}}.$$

Thus we can extend Λ_{sa} into a $\mathbb{C}$-linear map $\Lambda \colon \mathcal{T}^d_{X,A} \to \mathcal{L}_{X,A}$. Let $(\phi_{i,j}) \in M_m(\mathcal{T}^d_{X,A})^+$. Thus the map $\Phi \colon \mathcal{T}_{X,A} \to M_m, u \mapsto (\phi_{i,j}(u))$ is completely positive. By Proposition 7.36, we get that $(\phi_{i,j}(e_{x,x',a,a'}))_{i,j} \in (M_{XA} \otimes M_m)^+$. Therefore Λ is completely positive. Because $\{e_{x,x',a,a'}; x, x' \in X, a, a' \in A\}$ generates $\mathcal{T}_{X,A}$, we have that $\Lambda(\phi) = 0$ implies $\phi = 0$. Thus Λ is injective. Let $M \in \mathcal{L}_{X,A}$, then

$$\operatorname{Tr}_A(M) = \sum_{x,x' \in X} \sum_{a \in A} e_x e^*_{x'} \otimes \lambda_{x,x',a,a} = cI_x.$$

Therefore we get by Proposition 7.34 that $\mathcal{L}^+_{X,A} \subseteq \operatorname{Im}(\Lambda)$. But since these elements span $\mathcal{L}_{X,A}$, we get that Λ is surjective. So it only remains to show that the inverse is completely positive. Let $(\phi_{i,j}) \in M_m(\mathcal{T}^d_{X,A})$ such that $(\Lambda(\phi_{i,j})) \in M_m(\mathcal{L}_{X,A})^+$ and let $\Phi \colon \mathcal{T}_{X,A} \to M_m, u \mapsto (\phi_{i,j}(u))$. Then $(\phi(e_{x,x',a,a'})) \in M_m(\mathcal{L}_{X,A})^+$ and therefore by Proposition 7.36, we have that ϕ is completely positive. Thus $(\phi_{i,j}) \in M_m(\mathcal{T}^d_{X,A})^+$ and this shows that Λ^{-1} is completely positive. □

The following is Theorem 6.2 in [25] and gives the classification of QNS correlations by the state space of $\mathcal{T}_{X,A} \otimes_{\max} \mathcal{T}_{Y,B}$. Therefore it is the analogue result for QNS correlation to Theorem 6.15. Note that if we have $\mathcal{T}_{X,A}$ and $\mathcal{T}_{Y,B}$, we denote the generators of $\mathcal{T}_{X,A}$ by $e_{x,x',a,a'}$ but the generators of $\mathcal{T}_{Y,B}$ by $f_{y,y',b,b'}$.

Theorem 7.39. *Let X, Y, A, B be finite sets and $\Gamma \colon M_{XY} \to M_{AB}$ be a linear map. The following are equivalent:*

(i) Γ *is a QNS correlation,*
(ii) *there exists a state* $s\colon \mathcal{T}_{X,A} \otimes_{max} \mathcal{T}_{Y,B} \to \mathbb{C}$ *such that* $\Gamma = \Gamma_s$.

Proof. *(i)* $\Rightarrow$ *(ii)*:
As $\Gamma\colon M_{XY} \to M_{AB}$ is linear we can write it as

$$\Gamma(e_x e_{x'}^* \otimes e_y e_{y'}^*) = \sum_{a,a' \in A} \sum_{b,b' \in B} C_{a,a',b,b'}^{x,x',y,y'} e_a e_{a'}^* \otimes e_b e_{b'}^*,$$

for $x, x' \in X, y, y' \in Y, a, a' \in A, b, b' \in B, C_{a,a',b,b'}^{x,x',y,y'} \in \mathbb{C}$. By Proposition 4.17, we get that the Choi matrix $C = (C_{a,a',b,b'}^{x,x',y,y'}) \in M_{XY} \otimes M_{AB}$ is positive. Let $x, x' \in X$ and $y, y' \in Y$, since $\mathrm{Tr}(e_x e_x^* - e_{x'} e_{x'}^*) = 0$ and Γ is a QNS correlation, we have

$$\begin{aligned}
0 &= \mathrm{Tr}_A(\Gamma((e_x e_x^* - e_{x'} e_{x'}^*) \otimes e_{y,y'})) \\
&= \mathrm{Tr}_A(\sum_{a,a' \in A} \sum_{b,b' \in B} C_{a,a',b,b'}^{x,x,y,y'} e_a e_{a'}^* \otimes e_b e_{b'}^*) - \mathrm{Tr}_A(\sum_{a,a' \in A} \sum_{b,b' \in B} C_{a,a',b,b'}^{x',x',y,y'} e_a e_{a'}^* \otimes e_b e_{b'}^*) \\
&= \sum_{a \in A} \sum_{b,b' \in B} C_{a,a,b,b'}^{x,x,y,y'} e_b e_{b'}^* - \sum_{a \in A} \sum_{b,b' \in B} C_{a,a,b,b'}^{x',x',y,y'} e_b e_{b'}^* \Rightarrow \sum_{a \in A} C_{a,a,b,b'}^{x,x,y,y'} = \sum_{a \in A} C_{a,a,b,b'}^{x',x',y,y'}.
\end{aligned}$$

We also have if $x \neq x'$

$$0 = \mathrm{Tr}_A((\Gamma(e_x e_{x'}^* \otimes e_y e_{y'}^*)) = \sum_{a \in A} \sum_{b,b' \in B} C_{a,a,b,b'}^{x,x',y,y'} e_b e_{b'}^* \Rightarrow \sum_{a \in A} C_{a,a,b,b'}^{x,x',y,y'} = 0.$$

Thus we get that there exist $c_{b,b'}^{y,y'} \in \mathbb{C}$ such that

$$\sum_{a \in A} C_{a,a',b,b'}^{x,x',y,y'} = \delta_{x,x'} c_{b,b'}^{y,y'} \ \forall y, y' \in Y, b, b' \in B.$$

Analogous we can show with the other condition in the Definition of QNS correlations that there exist $d_{a,a'}^{x,x'} \in \mathbb{C}$ such that

$$\sum_{b \in B} C_{a,a',b,b'}^{x,x',y,y'} = \delta_{y,y'} d_{a,a'}^{x,x'} \ \forall x, x' \in X, a, a' \in A.$$

From this we get that $L_{y_x e_{y'} \otimes e_b e_{b'}^*}(C) \in \mathcal{L}_{X,A}$ and $L_{e_x e_{x'} \otimes e_a e_{a'}^*}(C) \in \mathcal{L}_{Y,B}$. Therefore we have:

$$C = \sum_{y,y' \in Y} \sum_{b,b' \in B} \sum_{a,a' \in A} \sum_{x,x' \in X} L_{y_x e_{y'} \otimes e_b e_{b'}^*}(C) \otimes L_{e_x e_{x'} \otimes e_a e_{a'}^*}(C) \in \mathcal{L}_{X,A} \otimes \mathcal{L}_{Y,B}.$$

Since C is positive in M_{XYAB}, we have $C \in (\mathcal{L}_{X,A} \otimes \mathcal{L}_{Y,B})^+$. Since $\mathcal{L}_{X,A}$ and $\mathcal{L}_{Y,B}$ are finite dimensional, we get by Proposition 4.34

$$\mathcal{L}_{X,A} \otimes \mathcal{L}_{Y,B} = \mathcal{L}_{X,A} \otimes_{C^*\max} \mathcal{L}_{Y,B} = \mathcal{L}_{X,A} \otimes_{\max} \mathcal{L}_{Y,B}.$$

By Proposition 4.30, we get that $C \in (\mathcal{L}_{X,A} \otimes_{\min} \mathcal{L}_{Y,B})^+$. By Proposition 7.38, we have that $\mathcal{T}_{X,A}^d \cong \mathcal{L}_{X,A}$. By Proposition 4.46, we get

$$\mathcal{L}_{X,A} \otimes_{\min} \mathcal{L}_{Y,B} \cong \mathcal{T}_{X,A}^d \otimes_{\min} \mathcal{T}_{X,A}^d \cong (\mathcal{T}_{X,A} \otimes_{\max} \mathcal{T}_{Y,B})^d.$$

The state $s\colon \mathcal{T}_{X,A} \otimes_{\max} \mathcal{T}_{Y,B} \to \mathbb{C}$ is given by $C^{x,x',y,y'}_{a,a',b,b'} = s(e_{x,x',a,a'} \otimes f_{y,y',b,b'})$. And therefore we get that $\Gamma = \Gamma_s$.
$(ii) \Rightarrow (i)$:

Let $C = (C^{x,x',y,y'}_{a,a',b,b'})$ be Choi Matrix of Γ_s. We have $C^{x,x',y,y'}_{a,a',b,b'} = s(e_{x,x',a,a'} \otimes f_{y,y',b,b'})$ for all $x, x' \in X, y, y' \in Y, a, a' \in A, b, b' \in B$. By representing $\mathcal{C}_{X,A}$ faithfully on a Hilbert space and by Proposition 7.34, we get that

$$E = (e_{x,x',a,a'})_{x,x' \in X, a,a' \in A} \in (M_{XA} \otimes \mathcal{C}_{X,A})^+,$$
$$F = (f_{y,y',b,b'})_{y,y' \in Y, b,b' \in B} \in (M_{YB} \otimes \mathcal{C}_{Y,B})^+.$$

By the definition of the maximal tensor product and $\mathcal{T}_{X,A} \subseteq \mathcal{C}_{X,A}$, we have that $I(E \otimes F)I \in M_{XYAB}(\mathcal{T}_{X,A} \otimes \mathcal{T}_{Y,B})^+$. By Proposition 4.19, we have that s is completely positive and thus C is positive. And therefore we have by Proposition 4.17 that Γ_s is completely positive. Since $\sum_{a \in A} e_{x,x',a,a'} = \delta_{x,x'}1$ and $\sum_{b \in B} f_{y,y',b,b'} = \delta_{y,y'}1$ for all $x, x' \in X, y, y' \in Y$, we get that Γ_s is trace-preserving and we also get for $\rho_X \in M_X$ with $\mathrm{Tr}(\rho_X) = 0$ and $\rho_Y \in M_Y$:

$$\begin{aligned}
\mathrm{Tr}_A(\Gamma_s(\rho_X \otimes \rho_Y)) &= \mathrm{Tr}_A\left(\sum_{x,x' \in X} \sum_{y,y' \in Y} \sum_{a,a' \in A} \sum_{b,b' \in B} (\rho_X)_{x,x'} (\rho_Y)_{y,y'} C^{x,x',y,y'}_{a,a',b,b'} e_a e_{a'}^* \otimes e_b e_{b'}^*\right) \\
&= \sum_{x,x' \in X} \sum_{y,y' \in Y} \sum_{b,b' \in B} \sum_{a \in A} (\rho_X)_{x,x'} (\rho_Y)_{y,y'} C^{x,x',y,y'}_{a,a,b,b'} e_b e_{b'}^* \\
&= \sum_{x \in X} \sum_{y,y' \in Y} \sum_{b,b' \in B} \sum_{a \in A} (\rho_X)_{x,x} (\rho_Y)_{y,y'} C^{x,x,y,y'}_{a,a,b,b'} e_b e_{b'}^* \\
&= \left(\sum_{x \in X} (\rho_X)_{x,x}\right) \left(\sum_{y,y' \in Y} \sum_{b,b' \in B} \sum_{a \in A} (\rho_Y)_{y,y'} C^{x,x,y,y'}_{a,a,b,b'} e_b e_{b'}^*\right) \\
&= \mathrm{Tr}(\rho_X) \left(\sum_{y,y' \in Y} \sum_{b,b' \in B} \sum_{a \in A} (\rho_Y)_{y,y'} C^{x,x,y,y'}_{a,a,b,b'} e_b e_{b'}^*\right) = 0.
\end{aligned}$$

Analogous we get the other condition for a QNS correlation and therefore Γ_s is a QNS correlation. □

The next result, we give is Theorem 6.3 in [25] and gives the classification of the quantum commuting CQNS correlations via the state spaces of $\mathcal{C}_{X,A} \otimes_{C^*\max} \mathcal{C}_{Y,B}$ and $\mathcal{T}_{X,A} \otimes_c \mathcal{T}_{Y,B}$. Therefore it is the analogue result for quantum commuting QNS correlation to Theorem 6.15 and Corollary 6.17.

Theorem 7.40. *Let X, Y, A, B be finite sets and $\Gamma\colon M_{XY} \to M_{AB}$ be a linear map. The following are equivalent:*

(i) *Γ is a quantum commuting QNS correlation*
(ii) *there exists a state $s\colon \mathcal{T}_{X,A} \otimes_c \mathcal{T}_{Y,B} \to \mathbb{C}$ such that $\Gamma = \Gamma_s$*
(iii) *there exists a state $s\colon \mathcal{C}_{X,A} \otimes_{C^*max} \mathcal{C}_{Y,B} \to \mathbb{C}$ such that $\Gamma = \Gamma_s$*

Proof. $(i) \Rightarrow (iii)$:

Since Γ is quantum commuting QNS correlation, there exists a Hilbert space and a pair of commuting stochastic operator matrices $E \in M_X \otimes M_Y \otimes B(H)$, $F \in M_A \otimes M_B \otimes B(H)$ and a unit vector $\xi \in H$ such that we have $\Gamma_{E \cdot F,\xi} = \Gamma$. By

Proposition 7.34, there exist representations $\pi_X\colon \mathcal{C}_{X,A} \to B(H)$ such that for all $x, x' \in X, y, y' \in Y$, $E_{x,x',a,a'} = \pi_X(e_{x,x',a,a'})$ and $\pi_Y\colon \mathcal{C}_{Y,B} \to B(H)$ such that for all $y, y' \in Y, b, b' \in B$, we have $F_{y,y',b,b'} = \pi_X(f_{y,y',b,b'})$. As $\{e_{x,x',a,a'}; x, x' \in X, a, a' \in A\}$ and $\{f_{y,y',b,b'}; y, y' \in Y, b, b' \in B\}$ generate $\mathcal{C}_{X,A}$ and respectively $\mathcal{C}_{Y,B}$, we get, because E, F are a commuting pair, that π_X and π_Y have commuting ranges. By the universal property, there exists a map

$$\pi_X \otimes \pi_Y\colon \mathcal{C}_{X,A} \otimes_{C^{\max}} \mathcal{C}_{Y,B} \to B(H), u \otimes v \mapsto \pi_X(u)\pi_Y(v).$$

Let $(h_i)_{i\in I} \in H$ be an orthonormal basis of H, we then get

$$\begin{aligned}
\langle \Gamma_{E\cdot F,\xi}, e_a e_{a'}^* \otimes e_b e_{b'}^* \rangle &= \langle E_{x,x',a,a'} F_{y,y',b,b'}, \xi\xi^* \rangle \\
&= \mathrm{Tr}(E_{x,x',a,a'} F_{y,y',b,b'} \xi\xi^*) \\
&= \sum_{i\in I} \langle E_{x,x',a,a'} F_{y,y',b,b'} \xi \langle h_i, \xi\rangle, h_i \rangle \\
&= \sum_{i\in I} \langle E_{x,x',a,a'} F_{y,y',b,b'} \xi, h_i \langle h_i, \xi\rangle \rangle \\
&= \langle E_{x,x',a,a'} F_{y,y',b,b'} \xi, \xi \rangle \\
&= \langle \pi_X \otimes \pi_Y(e_{x,x',a,a'} \otimes f_{y,y',b,b\prime})\xi, \xi \rangle\,.
\end{aligned}$$

By linearity, we get that $s\colon \mathcal{C}_{X,A} \otimes_{C\max} \mathcal{C}_{Y,B} \to \mathbb{C}, u \mapsto \langle \pi_X \otimes \pi_Y(u)\xi, \xi\rangle$ is a state with $\Gamma_s = \Gamma$.

(iii) $\Rightarrow$ *(i)*: Let s: $s\colon \mathcal{C}_{X,A} \otimes_{C^*\max} \mathcal{C}_{Y,B} \to \mathbb{C}$ and let (H_s, π_s, ξ_s) be the GNS representation of s from Proposition 2.11. Since $e_{x,x',a,a'}$ and $f_{y,y',b,b'}$ commute for all $x, x' \in X, y, y' \in Y, a, a' \in A, b, b' \in B$, we have that $E = (\pi_s(e_{x,x',a,a'}))_{x,x'\in X,a,a'\in A}$ and $F = (\pi_s(f_{y,y',b,b'}))_{y,y'\in Y,b,b'\in B}$. We also have that

$$\begin{aligned}
\left\langle \Gamma_s(e_x e_{x'}^* \otimes e_y e_{y'}^*), e_a e_{a'}^* \otimes e_b e_{b'}^* \right\rangle &= s(e_{x,x',a,a'} \otimes f_{y,y',b,b'}) \\
&= \langle \pi_s(e_{x,x',a,a'} \otimes f_{y,y',b,b'})\xi_s, \xi_s \rangle \\
&= \langle E_{x,x',a,a'} F_{y,y',b,b'} \xi_s, \xi_s \rangle \\
&= \langle \Gamma_{E\cdot F,\xi_s}(e_x e_{x'}^* \otimes e_y e_y^*), e_a e_{a'}^* \otimes e_b e_{b'} \rangle
\end{aligned}$$

By linearity follows that $\Gamma_s = \Gamma_{E\cdot,\xi_s}$ and therefore Γ_s is a quantum commuting QNS correlation.

(iii) $\Leftrightarrow$ *(ii)*: From Corollary 7.35, we get $C_u^*(\mathcal{T}_{X,A}) = \mathcal{C}_{X,A}$ and $C_u^*(\mathcal{T}_{Y,B}) = \mathcal{C}_{Y,B}$ and therefore we have by Proposition 4.41 that $\mathcal{T}_{X,A} \otimes_c \mathcal{T}_{Y,B} \subseteq_{\text{c.o.i.}} \mathcal{C}_{X,A} \otimes_{C^*\max} \mathcal{C}_{Y,B}$. Therefore we have by Krein's Theorem(Proposition 4.18) that a state on $\mathcal{T}_{X,A} \otimes_c \mathcal{T}_{Y,B}$ can be extended to a state on $\mathcal{C}_{X,A} \otimes_{C^*\max} \mathcal{C}_{Y,B}$. On the other hand if we have a state $s\colon \mathcal{C}_{X,A} \otimes_{C^*\max} \mathcal{C}_{Y,B} \to \mathbb{C}$ that $s_{|\mathcal{T}_{X,A}\otimes_c \mathcal{T}_{Y,B}}$ is still a state.

□

Corollary 7.41. *The set of quantum commuting QNS correlations is closed and convex.*

Proof. The proof follows the same arguments as the proof of Proposition 6.18 except that the functions over $X \times Y \times A \times B$ in Proposition 6.18 become linear maps on $M_X \otimes M_Y \to M_A \otimes M_B$. □

The last results are similar results for QNS and quantum commuting QNS strategies to some of the results, we had in Section 6.4 for no-signalling and quantum commuting no-signalling strategies. Therefore the question arises, whether we can also get classification results for classical-to-quantum games. These results are also taken from [25].

Let X, Y, A, B be finite sets. Recall that we defined $\mathcal{B}_{X,A}$ in Example 2.16 and we defined $\mathcal{R}_{X,A}$ in Example 4.23. We denoted the set of canonical generators of $\mathcal{B}_{X,A}$ by $e_{x,a,a'}$. If we now have both $\mathcal{B}_{X,A}$ and $\mathcal{B}_{Y,B}$, we still denote the generators of $\mathcal{B}_{X,A}$ by $e_{x,a,a'}$ but we denote the generators of $\mathcal{B}_{Y,B}$ by $f_{y,b,b'}$.

Similar to Definition 7.37, we can associate linear maps to states of an operator system.

Definition 7.42. Let X, Y, A, B be finite sets and let

$$t_1\colon \mathcal{R}_{X,A} \otimes \mathcal{R}_{Y,B} \to \mathbb{C}, t_2\colon \mathcal{B}_{X,A} \otimes \mathcal{B}_{Y,B} \to \mathbb{C}, t_3\colon \mathcal{B}_{X,A} \otimes_{C^*\max} \mathcal{B}_{Y,B},$$

be linear maps. For $i \in \{1,2,3\}$, we define the linear map $\mathcal{E}_{s_i}\colon D_{XY} \to M_{AB}$ by

$$\mathcal{E}_{t_i}(e_x e_x^* \otimes e_y e_y^*) = \sum_{a,a' \in A} \sum_{b,b' \in B} t_i(e_{x,a,a'} \otimes f_{y,b,b'}) e_a e_{a'}^* \otimes e_b e_{b'}^*.$$

In Section 7.2, we already had some results that established connections between CQNS and QNS strategies. Similarly, we had results in Section 7.3 that gave connections between quantum commuting CQNS strategies and quantum commuting QNS strategies. We want to use these connections to get the classification results for CQNS strategies from Theorem 7.39 and similarly for quantum commuting CQNS strategies from Theorem 7.40. Therefore we first need some preliminary lemmas.

Lemma 7.43. *Let H be a Hilbert space, X, A be finite sets and $\phi\colon \mathcal{R}_{X,A} \to B(H)$ be a linear map. The following are equivalent:*

(i) *ϕ is a unital completely positive map,*
(ii) *$((\phi(e_{x,a,a'}))_{a,a' \in A})_{x \in X} \in D_X \otimes M_A \otimes B(H)$ is a semi-classical stochastic operator matrix.*

Proof. *(i) $\Rightarrow$ (ii)*: Let ϕ_x be the restriction of ϕ to the x-th copy of M_A. Then ϕ_x is still unital and completely positive. Therefore by Proposition 4.17, we have that $(\phi(e_{x,a,a'}))_{x \in X}$ is positive. Thus we also have that $E = ((\phi(e_{x,a,a'}))_{a,a' \in A})_{x \in X}$ is positive. It remains to show that $\mathrm{Tr}_A(E) = I_X \otimes I_H$.

$$\mathrm{Tr}_A(E) = \sum_{x \in X} \sum_{a \in A} e_{x,x} \otimes \phi(e_{x,a,a}) = \sum_{x \in X} e_{x,x} \otimes \phi(I) = I_X \otimes I_H.$$

Thus E is a semi-classical stochastic operator matrix.

(ii) $\Rightarrow$ (i): For each $x \in X$, define $\phi_x\colon M_A \to B(H)$ by $\phi(e_a e_{a'}^*) = \phi(e_{x,a,a'})$. Since $(\phi(e_{x,a,a'}))_{a,a' \in A}$ is positive, we get by Proposition 4.17 that ϕ_x is completely positive. Because ϕ_x is also unital, we get by Lemma 4.24 that the map ϕ is unital and completely positive. □

Lemma 7.44. *Let X, A be finite sets. There exists a unital completely positive map $\beta_{X,A}\colon \mathcal{R}_{X,A} \to \mathcal{T}_{X,A}$ with $\beta_{X,A}(e_{x,a,a'}) = e_{x,x,a,a'}$ for all $x \in X, a, a' \in A$ and a unital completely positive map $\beta'_{X,A}\colon \mathcal{R}_{X,A} \to \mathcal{T}_{X,A}$ such that $\beta'_{X,A}(e_{x,x',a,a'}) = \delta_{x,x'} e_{x,a,a'}$. We also have $\beta'_{X,A} \circ \beta_{X,A} = id_{\mathcal{R}_{X,A}}$.*

Proof. By Proposition 2.12, there exists a Hilbert space H and a unital injective representation $\pi\colon \mathcal{C}_{X,A} \to B(H)$. Note that π is completely positive as *-homomorphisms are positive and π_n is also a *-homomorphism. Therefore we get from Proposition 7.34 that $E = (\pi(e_{x,x',a,a'}))_{x,x'\in,a,a'\in A}$ is a stochastic operator matrix. By Proposition 7.25, we get that E' is also a stochastic operator matrix. Since $E' = (\pi \circ \beta_{X,A}(e_{x,a,a}))$, we get by Lemma 7.44 that $\pi \circ \beta_{X,A}$ is a completely positive unital map. Because π is injective, we can form an inverse π^{-1} on its image that is also a *-homomorphism. Therefore we get $\pi^{-1} \circ \pi \circ \beta_{X,A} = \beta_{X,A}$ is a unital completely positive map.

Similarly we get by Proposition 2.12 that there exists a Hilbert space $\tilde{H}$ and a unital injective representation $\tilde{\pi}\colon \mathcal{B}_{X,A} \to B(\tilde{H})$ and that $\tilde{E} = (\pi(e_{x,a,a'}))_{x\in X,a,a'\in A}$ is a stochastic operator matrix. But then $\tilde{E} = \tilde{\pi} \circ \beta'_{X,A}$ is also a stochastic operator matrix and therefore we get that $\beta'_{X,A}$ is unital and completely positive.

The identity $\beta'_{X,A} \circ \beta_{X,A} = \mathrm{id}_{\mathcal{R}_{X,A}}$ follows directly from the definition of these maps and the fact that $(e_{x,a,a'})_{x\in X,a,a'\in A}$ generates $\mathcal{R}_{X,A}$. □

Theorem 7.45. *Let X, Y, A, B be finite sets and let*

$$M\colon \mathcal{L}(\mathcal{R}_{X,A} \otimes \mathcal{R}_{Y,B}, \mathbb{C}) \to \mathcal{L}(D_{XY}, M_{AB}), t \mapsto \mathcal{E}_t$$

be the map from Definition 7.42. Denote by

$$CQ_{ns} = \{\mathcal{E}\colon D_{XY} \to M_{AB}; \mathcal{E} \text{ is a QNS correlation}\},$$
$$CQ_{qc} = \{\mathcal{E}\colon D_{XY} \to M_{AB}; \mathcal{E} \text{ is a quantum commuting QNS correlation}\}.$$

Then M is a bijection between the sets

(i) CQ_{ns} *and* $\{t\colon \mathcal{R}_{X,A} \otimes_{max} \mathcal{R}_{Y,B}; t \text{ is a state}\}$,
(ii) CQ_{qc} *and* $\{t\colon \mathcal{R}_{X,A} \otimes_{c} \mathcal{R}_{Y,B}; t \text{ is a state}\}$.

Proof. (i) It is easy to see that M is bijective. Since the maximal tensor product is functorial from Proposition 4.33, we get that $\beta'_{X,A} \otimes \beta'_{Y,B}$ is completely positive. Let $t\colon \mathcal{R}_{X,A} \otimes_{\max} \mathcal{R}_{Y,B} \to \mathbb{C}$ be a state. Then $s = t \circ \beta'_{X,A} \otimes \beta'_{Y,B}$ is a state on $\mathcal{T}_{X,A} \otimes_{\max} \mathcal{T}_{Y,B}$. It is straightforward to see that $\Gamma = \Gamma_s \circ \Delta_{XY}$ and therefore $(\Gamma_s)_{|D_{XY}}$ is a CQNS correlation by Proposition 7.17. We also have that $\mathcal{E}_t = (\Gamma_s)_{|D_{XY}}$. Conversely, let $\mathcal{E}$ be a CQNS correlation. Then $\Gamma_{\mathcal{E}}$ is a QNS correlation and by Theorem 7.39, we get that there is a state $s\colon \mathcal{T}_{X,A} \otimes_{\max} \mathcal{T}_{Y,B} \to \mathbb{C}$ such that $\Gamma_{\mathcal{E}} = \Gamma_s$. Then $t = s \circ \beta_{X,A} \otimes \beta_{Y,B}$ is a state since the maximal tensor product is functorial. And again it is straightforward to check that $\mathcal{E} = \mathcal{E}_t$.

(ii) Since the commuting tensor product is also functorial by Proposition 4.39, the proof follows analogous to (i) by using Theorem 7.40 and Proposition 7.33.

□

Corollary 7.46. *Let X, Y, A, B be finite sets and $\mathcal{E}\colon D_{XY} \to M_{AB}$ be a linear map. The following are equivalent:*

(i) $\mathcal{E}$ *is a quantum commuting QNS correlation*
(ii) *there exists a state* $t\colon \mathcal{R}_{X,A} \otimes_c \mathcal{R}_{Y,B} \to \mathbb{C}$ *such that* $\mathcal{E} = \mathcal{E}_t$
(iii) *there exists a state* $t\colon \mathcal{B}_{X,A} \otimes_{C^*max} \mathcal{B}_{Y,B} \to \mathbb{C}$ *such that* $\mathcal{E} = \mathcal{E}_t$

Proof. $(i) \Leftrightarrow (ii)$: follows directly from Theorem 7.45.

$(ii) \Leftrightarrow (iii)$: We have that $\mathcal{R}_{X,A} \otimes_c \mathcal{R}_{Y,B} \subseteq_{\text{c.o.i.}} \mathcal{B}_{X,A} \otimes_{C^*\max} \mathcal{B}_{Y,B}$ by Proposition 4.42. Therefore we get from Krein's Theorem(Proposition 4.18) that a state on $\mathcal{T}_{X,A} \otimes_c \mathcal{T}_{Y,B}$ can be extended to a state on $\mathcal{B}_{X,A} \otimes_{C^*\max} \mathcal{B}_{Y,B}$. On the other hand if we have a state $s\colon \mathcal{B}_{X,A} \otimes_{C^*\max} \mathcal{B}_{Y,B} \to \mathbb{C}$ that $s_{|\mathcal{R}_{X,A}\otimes_c\mathcal{R}_{Y,B}}$ is still a state. □

7.5 Perfect quantum commuting CQNS strategies for quantum output mirror games

In this subsection, we want to classify perfect quantum commuting CQNS strategies for quantum output mirror games similarly to the classification given in Proposition 6.22 for mirror games. This result was presented in [4]. First, we need a preliminary lemma to get that classification result.

Recall that we defined quantum output mirror games in 7.10. This is Lemma 3.1 in [4] but we omit the proof of this lemma in this thesis as it is long and technical.

Lemma 7.47. *Let X, Y, A, B be finite sets such that $A = B$, $\varphi \colon P^{cl}_{XY} \to P_{AB}$ be a quantum output mirror game and $s \colon \mathcal{B}_{X,A} \otimes_{C^*max} \mathcal{B}_{Y,B} \to \mathbb{C}$ be a state such that $\mathcal{E}_s \colon D_{XY} \to M_{AB}$ is a perfect quantum commuting CQNS strategy for φ. Let H be a Hilbert space, $\pi_1 \colon \mathcal{B}_{X,A} \to B(H)$ and $\pi_2 : \mathcal{B}_{Y,B} \to B(H)$ be representations with commuting ranges and $\xi \in H$ be a unit vector such that*

$$s(u_1 \otimes u_2) = \langle \pi_1(u_1)\pi_2(u_2)\xi, \xi\rangle, \; u_1 \in \mathcal{B}_{X,A}, u_2 \in \mathcal{B}_{Y,B}.$$

Define $E_x = (\pi_1(e_{x,a,a'}))_{a,a' \in A}$ and $F_y = (\pi_1(f_{y,b,b'}))_{b,b' \in B}$ and let $f \colon X \to Y$ be a function such that $\varphi(e_x e_x \otimes e_{f(x)} e_{f(x)})$ is bijective for all $x \in X$. Let $(U_i^x)_{i=1}^{r(x)}$ be partial isometries satisfying

$$\varphi(e_x e_x \otimes e_{f(x)} e_{f(x)}) = \sum_{i=1}^{r(x)} \zeta_{U_i^x} \zeta^*_{U_i^x} \; \forall x \in X, \; \sum_{i=1}^{r(x)} (U_i^x)^* U_i^x = I = \sum_{i=1}^{r(x)} U_i^x (U_i^x)^*$$

Then we have that

$$(U_i^* \otimes I)^* E_x(e_a \otimes \xi) = F^t_{f(x)} (U_i^x \otimes I)^* (e_a \otimes \xi), \; \forall i \in \{1, ..., r(x)\}, a \in A.$$

Lemma 7.48. *Let X, Y, A, B be finite sets and $\mathcal{E}_s \colon D_{XY} \to M_{AB}$ be a quantum commuting CQNS correlation. Then there exists a Hilbert space H and representations $\pi_1 \colon \mathcal{B}_{X,A} \to B(H)$ and $\pi_2 \colon \mathcal{B}_{Y,B} \to B(H)$ with commuting ranges and a unit vector $\xi \in H$ such that*

$$s(u_1 \otimes u_2) = \langle \pi_1(u_1)\pi_2(u_2)\xi, \xi\rangle, \; u_1 \in \mathcal{B}_{X,A}, u_2 \in \mathcal{B}_{Y,B}.$$

Proof. Let (H_s, π_s, ξ_s) be the GNS-representation of s which was given in Lemma 2.11. Let $a, a' \in A, b, b' \in B, y \in Y, b \in B$ be any element of the corresponding set. Now we define

$$\pi_1 \colon \mathcal{B}_{X,A} \to B(H), e_{x,a,a'} \mapsto \pi_s(e_{x,a,a'} \otimes 1)$$

and

$$\pi_2 \colon \mathcal{B}_{Y,B} \to B(H), f_{y,b,b'} \mapsto \pi_s(1 \otimes f_{y,b,b'}).$$

Since $e_{x,a,a'} \otimes 1$ and $1 \otimes f_{y,b,b'}$ commute, we get that $\pi_s(e_{x,a,a'} \otimes 1)$ and $\pi_s(1 \otimes f_{y,b,b'})$ commute. Since $\{e_{x,a,a'}; x \in X, a, a' \in A\}$ and $\{f_{y,b,b'}; y \in Y, b, b' \in B\}$ generate $\mathcal{B}_{X,A}$ and $\mathcal{B}_{Y,B}$ respectively, we get because π_s is a representation (which are continuous as well) that the entire image of π_1 and π_2 has to commute. Thus we have

$$s(u_1 \otimes u_2) = \langle \pi_s(u_1 \otimes u_2)\xi_s, \xi_s\rangle = \langle \pi_1(u_1)\pi_2(u_2)\xi_s, \xi_s\rangle \; \forall u_1 \in \mathcal{B}_{X,A}, u_2 \in \mathcal{B}_{Y,B}.$$

□

Now we show the main result of this subsection which is Theorem 3.2 in [4].

Theorem 7.49. *Let X, Y, A, B be finite sets such that $A = B$, $\varphi\colon P^{cl}_{XY} \to P_{AB}$ be a quantum output mirror game and $\mathcal{E}\colon D_{XY} \to M_{AB}$ be a perfect quantum commuting CQNS strategy for φ. Then there exists a trace $\tau : \mathcal{B}_{X,A} \to \mathbb{C}$ and a *-homomorphism $\rho\colon \mathcal{B}_{Y,B} \to \mathcal{B}_{X,A}$ such that for all $x \in X, y \in Y$*

$$\mathcal{E}(E_{x,x} \otimes E_{f(x),f(x)}) = (\tau(e_{x,a,a'}\rho(f_{y,b,b'})))_{a,a',b,b'}.$$

Proof. We first prove some helpful identities and define some notation for the proof as this theorem:

Let $f\colon X \to Y, g\colon Y \to B$ be functions such that $\varphi(e_x e_x \otimes e_{f(x)}e_{f(x)})$ and $\varphi(e_{g(y)}e_{g(y)} \otimes e_y e_y)$ are bijective for all $x \in X$. Let $(U_i^x)_{i=1}^{r(x)}$ be partial isometries satisfying

$$\varphi(e_x e_x \otimes e_{f(x)}e_{f(x)}) = \sum_{i=1}^{r(x)} \zeta_{U_i^x}\zeta_{U_i^x}^* \text{ and } \sum_{i=1}^{r(x)} (U_i^x)^* U_i^x = I = \sum_{i=1}^{r(x)} U_i^x (U_i^x)^* \;\forall x \in X.$$

By Corollary 7.46, we have that there exists a state $s\colon \mathcal{B}_{X,A} \otimes_{C^*\max} \mathcal{B}_{Y,B} \to \mathbb{C}$ such that $\mathcal{E} = \mathcal{E}_s$. By Lemma 7.48, we get that there exist representations $\pi_1 : \mathcal{B}_{X,A} \to B(H)$ and $\pi_2 : \mathcal{B}_{Y,B} \to B(H)$ with commuting ranges and a unit vector $\xi \in H$ such that

$$s(u_1 \otimes u_2) = \langle \pi_1(u_1)\pi_2(u_2)\xi, \xi\rangle,\; u_1 \in \mathcal{B}_{X,A}, u_2 \in \mathcal{B}_{Y,B}.$$

Denote $E_{x,a,a'} = \pi_1(e_{x,a,a'})$ and $E_{x,a,a'} = \pi_2(f_{y,b,b'})$. Then we have that

$$\mathcal{E}(e_x e_x^* \otimes e_y e_y^*) = \sum_{a,a' \in A}\sum_{b,b' \in B} \langle E_{x,a,a'}F_{y,b,b'}\xi, \xi\rangle e_a e_{a'}^* \otimes e_b e_{b'}^*.$$

Set $E_x = (E_{x,a,a'})_{a,a' \in A}$ and $F_y = (F_{y,b,b'})_{b,b' \in B}$. By Lemma 7.47, we have

$$(U_i^x \otimes I)^* E_x(e_a \otimes \xi) = F^t_{f(x)}(U_i^x \otimes I)^*(e_a \otimes \xi),\; \forall i \in \{1, ..., r(x)\}, a \in A. \tag{17}$$

Define $D_x = \sum_{i=1}^r (x) U_i^x$ and let $Q = ((D_x \otimes I)(f_{f(x),a,b})^t_{a,b}(D_x^* \otimes I)) \in M_A \otimes M_B$ and therefore if we write $Q = (q_{x,a,b})_{b,a}$. We have $q_{x,a,b} \in M_B$. Set

$$h_{x,a,b} = e_{x,a,b} \otimes 1 - 1 \otimes q_{x,b,a},\; x \in X, a \in A.$$

Note that $e^*_{x,a,b} = e_{x,b,a}$ and $q^*_{x,a,b} = q_{x,b,a}$ since $f^*_{f(x),a,b} = f_{f(x),b,a}$. Therefore we have

$$\begin{aligned} h^*_{x,a,b}h_{x,a,b} &= (e_{x,b,a} \otimes 1 - 1 \otimes q_{x,a,b})(e_{x,a,b} \otimes 1 - 1 \otimes q_{x,b,a}) \\ &= e_{x,b,b} \otimes 1 - e_{x,b,a} \otimes q_{x,b,a} - e_{x,a,b} \otimes q_{x,a,b} + 1 \otimes q_{x,a,a}. \end{aligned}$$

Note that from (17), we also get by summing over i and multiplying the unitary $(D_x \otimes I)$ on both sides

$$\begin{aligned} &(D_x \otimes I)^* E_x(e_a \otimes \xi) = F^t_{f(x)}(D_x \otimes I)^*(e_a \otimes \xi) \qquad (18) \\ \Leftrightarrow\; &E_x(e_a \otimes \xi) = (D_x \otimes I)F^t_{f(x)}(D_x \otimes I)^*(e_a \otimes \xi). \end{aligned}$$

From this follows that

$$\begin{aligned} s(e_{x,b,a} \otimes q_{x,b,a}) &= \left\langle E_{x,b,a}((D_x \otimes I)F^t_{f(x)}(D_x \otimes I))_{a,b}\xi, \xi \right\rangle \\ &= \left\langle ((D_x \otimes I)F^t_{f(x)}(D_x \otimes I))_{a,b}\xi, E_{x,a,b}\xi \right\rangle \\ &= \langle E_{x,a,b}\xi, E_{x,a,b}\xi\rangle = \langle E_{x,b,b}\xi, \xi\rangle . \end{aligned}$$

Using this we get

$$\begin{aligned} s(h^*_{x,a,b}h_{x,a,,b}) &= s(e_{x,b,b} \otimes 1) - s(e_{x,b,a} \otimes q_{x,b,a}) - s(e_{x,a,b} \otimes q_{x,a,b}) + s(1 \otimes q_{x,a,a}) \\ &= \langle E_{x,b,b}\xi, \xi\rangle - \langle E_{x,b,b}\xi, \xi\rangle - s(e_{x,a,b} \otimes q_{x,a,b}) + s(1 \otimes q_{x,a,a}) \\ &= \left\langle ((D_x \otimes I)F^t_{f(x)}(D_x \otimes I))_{a,a}\xi, \xi \right\rangle - \langle E_{x,a,a}\xi, \xi\rangle \\ &= \langle E_{x,a,a}\xi, \xi\rangle - \langle E_{x,a,a}\xi, \xi\rangle = 0. \end{aligned}$$

For the rest of this proof, we write $u \sim v$ if $s(u-v) = 0$ for for $u, v \in \mathcal{B}_{X,A} \otimes_{C^*\max} \mathcal{B}_{Y,B}$.

$$\langle u, v\rangle_s = s(v^*u)$$

is an inner product on $B(X, A) \otimes_{C^*\max} B(Y, B)$, because s is a state. The Cauchy-Schwarz inequality now implies for $u \in B(X, A) \otimes_{C^*\max} B(Y, B)$ that

$$\begin{aligned} |s(h^*_{x,a,b}u)|^2 &\leq s(u^*u)s(h^*_{x,a,b}h_{x,a,b}) = 0, \\ |s(uh_{x,a,b})|^2 &\leq s(h^*_{x,a,b}h_{x,a,b})s(uu^*) = 0. \end{aligned}$$

Since $h^*_{x,a,b} = h_{x,b,a}$, we have

$$uh_{x,a,b} \sim 0 \text{ and } h_{x,a,b}u \sim 0 \ \forall x \in X, a, b \in A, u \in \mathcal{B}_{X,A} \otimes_{C^*-\max} \mathcal{B}_{Y,B}. \tag{19}$$

For $z \in \mathcal{B}_{X,A}$, set $u = z \otimes 1$. Then we get from (19) that

$$ze_{x,a,b} \otimes 1 \sim z \otimes q_{x,b,a} \sim e_{x,a,b}z \otimes 1, \ \forall x \in X, a, b \in A. \tag{20}$$

For $i \in \{1, ..., d(y)\}$, let V^y_i be partial isometries such that

$$\varphi(e_{g(y)}e^*_{g(y)} \otimes e_ye^*_y) = \sum_{i=1}^{d(y)} \zeta_{V^y_i}\zeta^*_{(V^y_i)} \text{ and } \sum_{i=1}^{d(y)} V^y_i(V^y_i)^* = I = \sum_{i=1}^{d(x)} (V^y_i)^*V^y_i \ \forall y \in Y.$$

By symmetry of switching the sets X and Y, we can get a quantum output mirror game, $\varphi' \colon P^{\text{cl}}_{YX} \to P_{AB}$ such that $\varphi(e_xe^*_x \otimes e_ye^*_y) = \varphi'(e_ye^*_y \otimes e_xe^*_x)$. From Lemma 7.48, we now get that

$$(V^y_i \otimes I)^*F_y(e_a \otimes \xi) = E^t_{g(y)}(V^y_i \otimes I)^*(e_a \otimes \xi), \ \forall i \in \{1, ..., d(x)\}, a \in A.$$

Let $G_y = \sum_{i=1}^{d(y)} V^y_i$, then we can get, analogous to (18) :

$$F_y(e_a \otimes \xi) = (G_y \otimes I)E^t_{g(y),a,b}(G_y \otimes I)^*(e_a \otimes \xi).$$

Now define $P = ((G_y \otimes I)(e^t_{g(y),a,b})_{a,b}(G_y \otimes I)^*)_{a,b}$ and $P = (p_{y,a,b})_{b,a}$. If we define $g_{y,b,b'} = p_{y,b,b'} \otimes 1 - 1 \otimes f_{y,b,b'}$, we can show similarly to how we have shown for $q_{x,a,b}$ and $h_{x,a,b}$ that

$$zp_{y,b,b'} \otimes 1 \sim z \otimes f_{y,b,b'} \sim p_{y,b,b'}z \otimes 1, \ y \in Y, b, b' \in B. \tag{21}$$

Now we define the map $\tau\colon \mathcal{B}_{X,A} \to \mathbb{C}, z \mapsto s(z \otimes 1)$. Let $z \in \mathcal{B}_{X,A}$ be arbitrary, then from $\tau(z^*z \otimes 1) = s((z \otimes 1)^*(z \otimes 1))$ and $\tau(1) = s(1 \otimes 1) = 1$, we get that τ is a state. To conclude that τ is a trace, it remains to show that $\tau(zw) = \tau(wz)$ for all $z, w \in \mathcal{B}_{X,A}$. We first show by induction over the length of the word w for words $z, w \in \{e_{x,a,b}; x \in X, a, b \in A\}$. Let w be a word of length one then this follows immediately from (20). Now let w be a word of length n and we write $w = w'e$, where $|e| = 1$. Then by (20)

$$zw \otimes 1 = zw'e \otimes 1 \sim ezw' \otimes 1 \sim w'ez \otimes 1 = wz \otimes 1.$$

From this, the fact that $\mathrm{span}\{e_{x,a,b}; x \in X, a, b \in A\}$ is dense in $\mathcal{B}_{X,A}$ and τ is continuous, we get that $\tau(zw) = \tau(wz)$ for all $z, w \in \mathcal{B}_{X,A}$.

To construct the *-homomorphism, note that $\{p_{y,a,b}; a, b \in A\}$ form a basis of M_A and $p_{y,a,b}p_{y,a',b'} = \delta_{b,a'}p_{y,a,b'}$ and $p^*_{y,a,b} = p_{y,b,a}$. By the universal property and Example 2.16, we a *-homomorphism $\rho\colon \mathcal{B}_{Y,B} \to \mathcal{B}_{X,A}, f_{y,b,b'} \mapsto p_{y,b,b'}$ and from (21), we get that

$$s(e_{x,a,a'} \otimes f_{y,b,b'}) = s(e_{x,a,a'}p_{y,b,b'} \otimes 1) = \tau(e_{x,a,a'}\rho(f_{y,b,b'})).$$

This concludes the proof. □

Remark 7.50. Theorem 7.49 is the analogous result for quantum output mirror games and quantum commuting CQNS correlations of Proposition 6.22 for nonlocal games and quantum commuting correlations. But comparing the proofs of these two statements, we can also see that they are very similar with the only real difference being the “construction” of h.

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