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**ON THE CLASSIFICATION OF COMPACT QUANTUM
GROUPS OF $SU(N)$ -TYPE**

Master Thesis
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Introduction

In this master's thesis, we study the partial solution to a classification problem formulated by Woronowicz in [Wor88]. In that seminal work, Woronowicz established Tannaka-Krein duality for compact quantum groups and stated the following problem:

Determine all compact quantum groups \mathbb{G} of $SU(N)$ -type for $N \geq 2$, i.e. all compact quantum groups for which there exists a dimension-preserving isomorphism between the fusion semiring of \mathbb{G} and the fusion semiring of $SU(N)$.

This problem was resolved in the non-Kac case by Neshveyev and Yamashita, see [NY17b; NY16] for an overview and detailed treatment. The objective of this thesis is to present a self-contained exposition of the known solutions to Woronowicz's problem, largely following, the approach developed by Neshveyev and Yamashita in the 2010s.

Quantum groups emerged from two perspectives, an algebraic one and an analytic one. In 1985, Drinfeld and Jimbo [Dri85; Jim85] used Hopf algebraic methods in the context of the Yang-Baxter equation and R -matrices, which originate in quantum physics, more precisely in quantum many-body problems. They introduced deformations of the universal enveloping algebras of semisimple Lie algebras. It is the algebra, not the group, that is deformed, in this sense the term quantum group can be misleading. A few years later, Woronowicz's analytical approach was inspired by the Gelfand philosophy for C^* -algebras: if G is a compact group, then the pair $(C(G), \Delta)$, consisting of the C^* -algebra of continuous functions on G together with the comultiplication, contains enough information to reconstruct the group. Replacing the commutative C^* -algebra $C(G)$ by a non-commutative C^* -algebra equipped with a $*$ -homomorphism, the comultiplication, leads to the notion of compact quantum groups. It turns out that we can generalise different notions of duality for compact quantum groups, which might fail for non-abelian compact groups, as the Pontryagin duality. Woronowicz proved, in [Wor88], that for compact quantum groups we have a reconstruction theorem, namely Tannaka-Krein duality. Given a rigid C^* -tensor category \mathcal{C} , i.e. a category with a tensor structure and C^* -like properties, e.g. the category Hilb_f of finite-dimensional Hilbert spaces, with a fiber functor, one can reconstruct a compact quantum group from these data. More precisely we obtain a compact quantum group \mathbb{G} , such that its representation category $\text{Rep}(\mathbb{G})$ is monoidally equivalent to \mathcal{C} . Hence the representation theory of compact quantum

groups is central for understanding their structure and properties. This motivates the classification problem for compact quantum groups, whose representation theory resembles that of a semisimple compact Lie group, such a \mathbb{G} is then said to be of Lie-type. More precisely, our aim is to classify non-Kac compact quantum groups of $SU(N)$ -type. In the spirit of Tannaka-Krein, this classification proceeds in three steps as stated in [NY16]:

- classification of rigid C^* -tensor categories with the fusion rules of $SU(N)$, [KW93; Jor14],
- classification of monoidal autoequivalences, [NT11b; NT12],
- classification of unitary dimension-preserving fiber functors, [NY16].

Before giving a brief overview of the contents of this thesis, let us note that the classification of $SU(N)$ -type quantum groups is not merely a mathematical problem. Lie groups such as $SU(N)$ are of considerable physical relevance and constitute the backbone of many theories in mathematical physics, including the Standard Model of particle physics and approaches to quantum gravity via non-commutative geometry. For instance, quantum gravity suggests that spacetime may become non-commutative at very small scales. Understanding quantum deformations of such fundamental Lie groups is therefore essential.

As an overview of the chapters, the following will serve:

Chapter I introduces compact quantum groups in the sense of Woronowicz, and provides concrete examples, such as the class of compact matrix quantum groups. We also recall the basics of the representation theory of compact quantum groups in order to access the representation category and its structure. The chapter concludes with cohomology theory for the discrete dual of a compact quantum group. Cohomology theory is, in a sense, a tool to control deformations of compact quantum groups, for example, the unitary 2-cohomology $H^2(\widehat{\mathbb{G}}, \mathbb{T})$ yields a full parametrisation of unitary dimension-preserving fiber functors up to natural unitary monoidal isomorphisms, and the invariant unitary 3-cohomology $H^3_{\mathbb{G}}(\widehat{\mathbb{G}}, \mathbb{T})$ can be used to construct new associativity morphisms in the representation category $\text{Rep}(\mathbb{G})$, hence providing another deformation. The main sources for this chapter are [Tim08] and [NT13].

In **Chapter II**, our goal is to understand the categorical structure of the representation category and therefore define rigid C^* -tensor categories. This chapter is rather technical and abstract, but it provides the language needed throughout the thesis. We also sketch the basic idea of the proof of Woronowicz's Tannaka-Krein duality and mention the connection between cohomology theory and the classification of fiber functors. Finally, we introduce twisted q -deformations of compact Lie groups. For this, we recall the necessary notions for compact Lie groups G and their corresponding Lie algebras \mathfrak{g} . We then define the Drinfeld-Jimbo algebra $U_q(\mathfrak{g})$, which is a deformation of the universal enveloping algebra $U(\mathfrak{g})$, and construct the quantum

group G_q via Tannaka-Krein duality. By studying its C^* -tensor category, we can then apply a cochain twist.

Chapter III begins by introducing the categorical Poisson boundary. This is a suitable analogue of the non-commutative Poisson boundary for compact quantum groups introduced by Izumi, now formulated in the language of C^* -tensor categories. It consists of a pair (\mathcal{P}, Π) , where \mathcal{P} is a C^* -tensor category and Π is a tensor functor, satisfying a universal property with respect to the intrinsic dimension. This tool yields a bijection between the isomorphism classes of unitary fiber functors on the maximal Kac quantum subgroup and dimension-preserving unitary fiber functors. We then determine the maximal Kac quantum subgroup of twisted q -deformations G_q^c of a simply connected semisimple compact Lie group G . As in the easier case of G_q , [Tom07], it turns out that the maximal torus $T \subseteq G$ is the maximal Kac quantum subgroup. This implies that compact quantum groups of the form $SU_q^c(N)$ are closed under cocycle twisting.

In **Chapter IV** we present the classification result for all non-Kac compact quantum groups of $SU(N)$ -type:

Theorem. *Let $N \geq 2$ and \mathbb{G} be a non-Kac compact quantum group of $SU(N)$ -type. Then there exist $q \in (0, 1)$ and a \mathbb{T} -valued 2-cochain c such that ∂c descends to a 3-cocycle on $\widehat{Z}(\widehat{G}) \cong \mathbb{Z}/N\mathbb{Z}$ and $\mathbb{G} \cong SU_q^c(N)$.*

To prove this, we rely on the results established in earlier chapters and, in particular, on the Kazhdan-Wenzl-Jordans theorem [KW93; Jor14], which we prove in the first section of this chapter. Kazhdan and Wenzl showed in the algebraic case that every rigid monoidal category with the same fusion rules as $SL(N)$ must be monoidally equivalent to a category of the form $(\text{Rep}(SL_q(N)), \Phi)$. The C^* -analogue was then proved twenty years later by Jordans [Jor14]. This establishes the first step of the classification mentioned above, which is also the most important step. We then study monoidal autoequivalences, which turn out to be quite concrete.

The answer to the non-Kac case follows directly from this and the preceding results. We also briefly explain why we restrict our answer to Woronowicz's question to this specific case, rather than treating the problem in full generality.

Comment on the use of AI tools. AI tools, such as ChatGPT, Gemini and DeepL, were used exclusively for literature research, for generating summaries of articles for personal orientation, and for language support (spell and grammar check). All referenced sources were independently verified. The content, explanations and arguments presented in this thesis are entirely human-generated. No text passages were adopted verbatim from AI-generated output.

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Chapter I.

Compact Quantum Groups

In this first chapter, we recall the basics of compact quantum groups, introduced by Woronowicz in [Wor87; Wor98], and look at their representation theory as well as the cohomology theory on their duals. We also discuss further structural properties and the algebraic description via Hopf algebras. This chapter is mostly based on [Tim08] and [NT13].

Remark on prior knowledge and notation. We assume the reader has a solid knowledge of functional analysis, C^* -algebras and some basics of category theory. The precise meaning of the tensor product \otimes should be clear from the context, for C^* -algebras we always use the minimal tensor product. Moreover, for compositions of maps we will most of the time omit the symbol \circ .

1. Basics of compact quantum groups

As already mentioned, the notion of compact quantum groups has two origins. One is mostly algebraic, and goes back to the independent work of Drinfeld [Dri85] and Jimbo [Jim85] in connection with the Yang-Baxter equation, we elaborate on this at a later point. The other origin is analytical as done by Woronowicz [Wor87; Wor98], where compact quantum groups are understood as non-commutative function spaces, in the spirit of the Gelfand-Naimark philosophy for C^* -algebras. This approach was motivated by generalising the Pontryagin duality.

Definition 1.1 (Compact quantum group): Let A be a unital C^* -algebra and let $\Delta: A \rightarrow A \otimes A$ be a unital $*$ -homomorphism such that

- $(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta$ holds and
- the spans of $(A \otimes 1)(\Delta(A))$ and $(1 \otimes A)(\Delta(A))$ are dense in $A \otimes A$.

We call Δ the comultiplication of A . In the sense of Gelfand-Naimark we write $C(G)$ for the unital C^* -algebra and denote $\mathbb{G} = (C(G), \Delta)$ for the *compact quantum group*.

If \mathbb{G} and \mathbb{H} are compact quantum groups, a unital $*$ -homomorphism $\varphi: C(G) \rightarrow C(H)$ that intertwines the comultiplications, i.e. $(\varphi \otimes \varphi)\Delta_{\mathbb{G}} = \Delta_{\mathbb{H}}\varphi$, is called a *homomorphism of compact quantum groups*. We also write $\varphi: \mathbb{G} \rightarrow \mathbb{H}$. We call \mathbb{H} a *quantum subgroup* of a compact quantum group \mathbb{G} if there exists a surjective homomorphism $\mathbb{G} \rightarrow \mathbb{H}$.

Before discussing the existence of a *Haar state* and further algebraic properties, we present two basic examples of compact quantum groups and introduce a large class of compact quantum groups, namely *compact matrix quantum groups*.

Example 1.2: (i) Let G be a compact group. Define

$$\Delta: C(G) \rightarrow C(G) \otimes C(G) \cong C(G \times G)$$

via

$$\Delta(f)(s, t) = f(st), \quad s, t \in G.$$

Then $(C(G), \Delta)$ is a compact quantum group. Conversely, if (A, Δ) is a commutative compact quantum group, then by the Gelfand-Naimark theorem we find a compact group G such that there is an isomorphism $A \cong C(G)$.

(ii) Historically, the following is the first example of a compact quantum group. Define the q -deformation of the special unitary group $SU(2)$ by

$$C(SU_q(2)) = C^* \left(1, \alpha, \gamma \left| \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is unitary} \right. \right)$$

for $q \in [-1, 1] \setminus \{0\}$, and define a comultiplication on the generators by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Then $SU_q(2) = (C(SU_q(2)), \Delta)$ is a compact quantum group, called the *q -deformation of $SU(2)$* .

In the first example, the pair $(C(G), \Delta)$ contains complete information about the compact group G , since the group structure can be recovered from the comultiplication. The last example, in the more general case $SU_q(N)$ for $N \in \mathbb{N}$, will be our most important example in this thesis, thus we will return to it later again.

Definition 1.3 (Compact Matrix Quantum Groups): Let $N \in \mathbb{N}$ and let $C(G)$ be a unital C^* -algebra, which is generated by the entries of an invertible matrix $u = (u_{ij}) \in M_N(C(G))$, such that u^* is also invertible. Assume that the map

$$\Delta: C(G) \rightarrow C(G) \otimes C(G), \quad u_{ij} \mapsto \sum_{k=1}^N u_{ik} \otimes u_{kj}$$

is a $*$ -homomorphism. The pair $(C(G), u)$ is called a *compact matrix quantum group* and the matrix u is called the *fundamental representation*. One can show that every compact matrix quantum group is indeed a compact quantum group, hence we will also write \mathbb{G} for it.

This class of compact matrix quantum groups contains quite interesting quantum analogues of classical matrix groups such as S_N , O_N and U_N as we introduce in the following example.

Example 1.4: Let $N \in \mathbb{N}$. We define the following compact matrix quantum groups.

CMQG	Additional relations on u_{ij}
U_N^+	u and u^T unitary
O_N^+	$u_{ij}^* = u_{ij}$, u is orthogonal
S_N^+	$u_{ij}^2 = u_{ij}^* = u_{ij}$, $\sum_i u_{ik} = \sum_i u_{ki} = 1$

These examples were introduced by Wang in [Wan93; Wan98]. One can see that these constructions are indeed generalisations. For instance, in the case of the quantum permutation group S_N^+ , one has

$$C(S_N) \cong C(S_N^+) / \langle u_{ij}u_{kl} - u_{kl}u_{ij} \rangle,$$

i.e. the continuous functions on S_N are isomorphic to the “abelianization” of $C(S_N^+)$.

By the Riesz representation theorem, any compact group G has a unique (left- and right-invariant) Haar measure μ , which means that

$$\int_G f(gx) d\mu(x) = \int_G f(x) d\mu(x) = \int_G f(xg) d\mu(x)$$

for $g \in G$. Using the defined comultiplication from Example 1.2, and defining $h(f) = \int_G f(x) d\mu$, we can rewrite this as

$$(\text{id} \otimes h)\Delta(f) = (h \otimes \text{id})\Delta(f) = h(f)1.$$

The same holds for compact quantum groups.

Theorem 1.5 (Haar state): *Let \mathbb{G} be a compact quantum group. Then there exists a unique state $h: C(\mathbb{G}) \rightarrow \mathbb{C}$ such that*

$$(\text{id} \otimes h)\Delta(x) = h(x)1 = (h \otimes \text{id})\Delta(x).$$

The existence of the Haar state was first proved by Woronowicz under additional assumptions on the C^* -algebra in [Wor87], and later in full generality for any compact quantum group in [Wor98].

Definition 1.6 (Kac-type): A compact quantum group \mathbb{G} is said to be of *Kac-type*, if its Haar state h is a trace, i.e. $h(xy) = h(yx)$.

In Proposition 1.24 we see other equivalent characterisations of being Kac for a compact quantum group.

Already at this point, it should be mentioned that our goal, the classification of compact quantum groups of $SU(N)$ -type, will only hold for the non-Kac case. This may appear counterintuitive, since compact quantum groups of Kac-type are usually “easier” to describe. We will return later to why in this setting, the Kac-case is harder (and probably not solvable).

Until now, we have only considered the C^* -algebraic theory of compact quantum groups due to Woronowicz. There is also an algebraic approach via Hopf $*$ -algebras, see [Tim08].

Definition 1.7 (Hopf $*$ -algebra): A *Hopf $*$ -algebra* is a unital $*$ -algebra A , together with

- a *comultiplication* $\Delta: A \rightarrow A \otimes A$, which is a $*$ -homomorphism,
- a *counit* $\varepsilon: A \rightarrow \mathbb{C}$, which is a $*$ -homomorphism, such that $(\varepsilon \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \varepsilon)\Delta$,
- an *antipode* $S: A \rightarrow A$, which is a linear map such that $\mu(S \otimes \text{id})\Delta = \mu(\text{id} \otimes S)\Delta = \eta \circ \varepsilon$.

Here $\mu: A \otimes A \rightarrow A$ denotes the multiplication $a \otimes b \mapsto ab$, $\eta: \mathbb{C} \rightarrow A$ the natural embedding $\lambda \mapsto \lambda 1$.

Recall that an algebra with a comultiplication and a counit is a bialgebra, i.e. it carries both an algebra structure and a coalgebra structure.

One can prove that any compact quantum group contains a dense Hopf $*$ -algebra, which allows us to work in both worlds and use for instance additional algebraic structure and properties.

Proposition 1.8: *Let \mathbb{G} be a compact quantum group. Then there exists a dense Hopf $*$ -algebra $(\mathbb{C}[G], \Delta|_{\mathbb{C}[G]})$ in $C(G)$.*

We will also denote the comultiplication of this dense Hopf $*$ -algebra by Δ . The meaning should be clear from the context.

In the next section we describe this dense Hopf $*$ -algebra. Before doing so, let us emphasise that historically Hopf algebras are one of the original starting points of quantum groups. They arise quite naturally in the context of physics, more precisely in the study of the Yang-Baxter equation, quantum integrable systems and the quantum inverse scattering method. The first appearance of the notion of quantum groups in the 1980s came from Drinfeld and Jimbo, [Jim85; Jim86] and [Dri85], who

introduced Hopf algebras arising from *deformations of universal enveloping algebras of semisimple Lie groups*. For more historical comments, see the preface of [NT13] and later in Remark 2.29.

At the same time, the analytic C^* -algebraic object $(C(G), \Delta)$ is tightly linked to finite-dimensional corepresentation theory: matrix coefficients of representations span the Hopf $*$ -algebra $\mathbb{C}[G]$ and encode much of the structure of \mathbb{G} . For this reason we now turn to representation theory, which will later provide the input for the categorical approach and reconstruction results discussed in Chapter II.

2. Representation Theory

In this section we introduce the representation theory of compact quantum groups, which will be quite essential throughout this thesis. In particular, Chapter II will be on C^* -tensor categories and Tannaka-Krein duality and we will show how representation-theoretic data can be used both to analyse the structure of a compact quantum group and, in a precise meaning, how to reconstruct it.

Throughout we will only consider finite-dimensional representations.

Remark on leg-numbering notation. The leg-numbering notation is very useful when defining and working with representations, see [NT13]. Let A be a unital algebra and let $n \leq m$ then we can define embeddings $A^{\otimes n} \rightarrow A^{\otimes m}$ via injective maps $\sigma: [n] \rightarrow [m]$. For $X \in A^{\otimes n}$ we denote its image by $X_{j_1 \dots j_n}$ where $j_i = \sigma(i)$. For example, for $n = 2$ and $m = 3$ we have

$$(x \otimes y)_{13} = x \otimes 1 \otimes y.$$

One may also consider different algebras in the tensor factors, one only has to ensure that all expressions are well-defined.

Definition 1.9 (Representation, Intertwiner): (i) Let \mathbb{G} be a compact quantum group. A *representation* of \mathbb{G} on a N -dimensional Hilbert space \mathcal{H}_U ($N \in \mathbb{N}$) is an invertible element $U \in B(\mathcal{H}_U) \otimes C(G) \cong M_N(C(G))$ such that

$$(\text{id} \otimes \Delta)(U) = U_{12}U_{13} \in B(\mathcal{H}_U) \otimes C(G) \otimes C(G) \cong M_N(C(G) \otimes C(G)).$$

The representation is called *unitary*, if the matrix U is unitary. We write $\dim(U) := \dim(\mathcal{H}_U) = N$.

(ii) Given representations U and V on finite-dimensional Hilbert spaces \mathcal{H}_U and \mathcal{H}_V , a linear map $T: \mathcal{H}_U \rightarrow \mathcal{H}_V$ is called an *intertwiner* of U and V if

$$(T \otimes \text{id})U = V(T \otimes \text{id}).$$

We denote the *space of intertwiners* by $\text{Mor}(U, V)$. We call U and V *equivalent* if there exists an invertible element in $\text{Mor}(U, V)$. If U and V are unitary

representations and there exists a unitary element in $\text{Mor}(U, V)$, then U and V are called *unitarily equivalent*. Finally a representation U is called *irreducible*, if $\text{End}(U) := \text{Mor}(U, U) = \mathbb{C}$.

We will later see that for a compact quantum group \mathbb{G} one can form the representation category $\text{Rep}(\mathbb{G})$ by taking finite-dimensional unitary representations as objects and intertwiners as morphisms. This category and its categorical properties will be our main object of interest in Chapter II, in particular because of its rich tensorial structure, see Example 2.2.

The representation theory of compact quantum groups shares many familiar features with the classical theory (e.g. Schur's lemma) and in addition the following fundamental fact holds, see [NT13].

Proposition 1.10: *Let \mathbb{G} be a compact quantum group. If U is a finite-dimensional representation of \mathbb{G} , then it is equivalent to a unitary representation.*

Moreover every irreducible representation is known to be finite-dimensional, see [Wor87]. Thus, when studying irreducible representations, we may restrict ourselves to finite-dimensional unitary ones. For readability, from now on we will simply speak of (unitary) representations, always meaning finite-dimensional unitary representations, unless stated otherwise.

As in the classical representation theory of groups, starting with two representations we can form new representations, by taking direct sums, by taking tensor products and by constructing the contragredient representation.

Definition 1.11: Let U and V be two representations, on Hilbert spaces \mathcal{H}_U and \mathcal{H}_V of dimensions n and m .

- (i) The *direct sum* $U \oplus V$ is defined as the matrix direct sum $U \oplus V \in M_{n+m}(C(G))$, which is a representation on $\mathcal{H}_U \oplus \mathcal{H}_V$.
- (ii) The *tensor product* $U \otimes V$ is the representation on $\mathcal{H}_U \otimes \mathcal{H}_V$ defined by $U \otimes V = U_{13}V_{23} \in M_{nm}(C(G))$.

Other symbols for the tensor product are sometimes used in literature, but we always use the classical tensor product symbol \otimes . It will be clear from the context, what is meant.

Proposition 1.12: *Every finite-dimensional representation decomposes as a direct sum of irreducible representations.*

Let $j: B(\mathcal{H}) \rightarrow B(\mathcal{H}^*)$ be the map that sends an operator to its dual operator. Here, \mathcal{H}^* denotes the dual space of \mathcal{H} .

Definition 1.13: Let U be a representation of a compact quantum group \mathbb{G} . Define

$$U^c = (j \otimes \text{id})(U^{-1}) \in B(\mathcal{H}^*) \otimes C(G).$$

We call U^c the *contragredient representation*.

If $U = (u_{ij})$ is a unitary representation, then $U^c = (u_{ij}^*)$. The contragredient has the following properties, see again [NT13].

Proposition 1.14: *Let U be a representation. Then*

- (i) U^c is invertible,
- (ii) U^{cc} is equivalent to U ,
- (iii) U^c is irreducible if and only if U is irreducible.

Moreover, if U is irreducible then $\text{Mor}(U, U^{cc})$ is one-dimensional.

Now let U be an irreducible representation. By the previous proposition, the space $\text{Mor}(U, U^{cc})$ is one-dimensional and hence is generated by a positive invertible operator, write ρ_U , unique up to a scalar factor. Normalizing this operator ρ_U , such that $\text{Tr}(\rho_U) = \text{Tr}(\rho_U^{-1})$, we define the *quantum dimension* of U by

$$\dim_q(U) := \text{Tr}(\rho_U).$$

By the Cauchy-Schwarz inequality $\dim_q(U) \geq \dim(U)$, with equality if and only if $\rho_U = 1$. Moreover $\rho_U = 1$ is equivalent to U^c being unitary, thus the ratio $\dim_q(U)/\dim(U)$ measures how far the contragredient representation is from being unitary.

Using the contragredient representation we can define the conjugate representation.

Definition 1.15: Let U be a representation of a compact quantum group \mathbb{G} . The *conjugate representation* \bar{U} is defined by

$$\bar{U} = (j(\rho_U)^{1/2} \otimes 1)U^c(j(\rho_U)^{-1/2} \otimes 1) \in B(\mathcal{H}_U^*) \otimes C(G).$$

Proposition 1.16: *Let U, V be representations of a compact quantum group \mathbb{G} . Then*

- (i) $\bar{\bar{U}} = U$,
- (ii) $\overline{U \oplus V} = \bar{U} \oplus \bar{V}$,
- (iii) $\dim_q(\bar{U}) = \dim_q(U)$.

The next two propositions provide the basic bridge between the analytic C^* -algebraic object $(C(G), \Delta)$ and the algebraic data coming from finite-dimensional representations. In analogy with the classical Peter-Weyl theorem, one isolates the *polynomial* part of $C(G)$, which is the linear span of matrix coefficients of representations. This span forms a Hopf $*$ -algebra that is dense in $C(G)$. This Hopf $*$ -algebra is the starting point for the reduced and universal completions, and conversely it allows one to reconstruct a compact quantum group from purely algebraic corepresentation data. Hence these propositions can be seen as reconstruction results.

Proposition 1.17: *Let \mathbb{G} be a compact quantum group. The linear span of all matrix coefficients of representations is a dense Hopf $*$ -algebra in $C(G)$. We denote it by $\mathbb{C}[G]$.*

We may also consider reduced and universal versions of the compact quantum group $(C(G), \Delta)$. Let $C_r(G)$ be the image of $C(G)$ under the induced GNS-representation $\pi_h: C(G) \rightarrow B(L^2(G))$ with respect to the Haar state h on $C(G)$. The universal version $C_u(G)$ is the C^* -enveloping algebra of the Hopf $*$ -algebra $\mathbb{C}[G]$, which exists since $\mathbb{C}[G]$ is spanned by matrix coefficients of representations. Both $C_r(G)$ and $C_u(G)$ can be equipped with comultiplications Δ_r and Δ_u , yielding reduced and universal forms of $(C(G), \Delta)$.

Proposition 1.18: *Let A be a Hopf $*$ -algebra generated by matrix coefficients of finite-dimensional unitary corepresentations of a Hopf $*$ -algebra. Then there exists a compact quantum group \mathbb{G} such that $(A, \Delta) = (\mathbb{C}[G], \Delta)$.*

Having isolated the Hopf $*$ -algebra $\mathbb{C}[G]$ of matrix coefficients, it is natural to pass to its linear dual. In the classical case, the dual of the algebra of representative functions can be identified with finitely supported functions on the discrete dual group, equipped with convolution. For compact quantum groups, this viewpoint leads to the *discrete dual quantum group* $\widehat{\mathbb{G}}$, whose algebra of functions is defined as the convolution $*$ -algebra dual to $\mathbb{C}[G]$. This algebra will serve as our main algebraic model for working on the dual side.

Definition 1.19: Let \mathbb{G} be a compact quantum group. We define the *algebra of functions on the discrete dual quantum group* $\widehat{\mathbb{G}}$ of a compact quantum group as the $*$ -algebra $\mathcal{U}(\mathbb{G}) = \mathbb{C}[G]^*$, with convolution $\omega * \nu = (\omega \otimes \nu)\Delta$ as multiplication and involution defined by $\omega^* = \overline{\omega}S$.

We have the following isomorphism

$$\mathcal{U}(\mathbb{G}) \cong \prod_{[U] \in \text{Irr}(\mathbb{G})} B(\mathcal{H}_U), \quad \omega \mapsto ((\text{id} \otimes \omega)(U))_{[U] \in \text{Irr}(\mathbb{G})}$$

where $\text{Irr}(\mathbb{G})$ denotes the set of equivalence classes of irreducible representations. We can equip the $*$ -algebra $\mathcal{U}(\mathbb{G})$ with a Hopf $*$ -algebra structure as follows:

- $\widehat{\Delta}: \mathcal{U}(\mathbb{G}) \rightarrow \mathcal{U}(\mathbb{G}^2)$ with $\widehat{\Delta}(\omega)(a \otimes b) = \omega(ab)$ for $a, b \in \mathbb{C}[G]$,
- $\widehat{\varepsilon}: \mathcal{U}(\mathbb{G}) \rightarrow \mathbb{C}$ with $\widehat{\varepsilon}(\omega) = \omega(1)$,
- $\widehat{S}: \mathcal{U}(\mathbb{G}) \rightarrow \mathcal{U}(\mathbb{G})$ with $\widehat{S}(\omega) = \omega S$.

One can now prove that with these maps, $(\mathcal{U}(\mathbb{G}), \widehat{\Delta})$ becomes a Hopf $*$ -algebra. This Hopf $*$ -algebra will play an important role in this thesis. Moreover, we will also consider the space of *n-point functions on the discrete dual* $\widehat{\mathbb{G}}$

$$\mathcal{U}(\mathbb{G}^n) = (\mathbb{C}[G]^{\otimes n})^* \cong \prod_{[U_1], \dots, [U_n] \in \text{Irr}(\mathbb{G})} B(\mathcal{H}_{U_1}) \otimes \dots \otimes B(\mathcal{H}_{U_n}).$$

We will not go into further details on the discrete dual $\widehat{\mathbb{G}}$, since for our purposes we will mostly work with $(\mathcal{U}(\mathbb{G}), \widehat{\Delta})$.

In combination with $\mathcal{U}(\mathbb{G})$, the positive invertible operator ρ_U introduced above can also be used to define a family of linear functionals.

Definition 1.20: Let \mathbb{G} be a compact quantum group. The family $(f_z)_{z \in \mathbb{C}}$ of linear functionals $f_z: \mathbb{C}[G] \rightarrow \mathbb{C}$, defined by

$$(\text{id} \otimes f_z)(U) = \rho_U^z$$

for all unitary representations U of \mathbb{G} , is called the *Woronowicz characters*.

The Woronowicz characters are well-defined and admit an explicit description. Let (e_{ij}) denote the standard matrix units, and write

$$U = \sum_{i,j} e_{ij} \otimes u_{ij}.$$

Then define

$$f_z(u_{ij}) = (\rho_U^z)_{ji}.$$

In particular

$$(\text{id} \otimes f_z)(U) = \sum_{i,j} e_{ij} \otimes f_z(u_{ij}) = \sum_{i,j} e_{ij} (\rho_U^z)_{ji} = \rho_U^z.$$

Since $\mathbb{C}[G]$ is spanned by the matrix coefficients of representations, this defines f_z for every $z \in \mathbb{C}$. Note that $f_0 = \varepsilon$.

Moreover, one has $f_z = f_1^z$. Indeed, $f_z \in \mathcal{U}(\mathbb{G}) = \mathbb{C}[G]^*$ and $\mathcal{U}(\mathbb{G})$ is as mentioned above a product of matrix algebras. Hence we may identify f_z with the family $((\text{id} \otimes f_z)(U))_{[U] \in \text{Irr}(\mathbb{G})} = (\rho_U^z)_{[U] \in \text{Irr}(\mathbb{G})}$, and functional calculus yields $f_z = f_1^z$.

Proposition 1.21: Let \mathbb{G} be a compact quantum group, let $(f_z)_{z \in \mathbb{C}}$ be its Woronowicz characters and let h be its Haar state. Then

$$h(ab) = h(b\sigma(a)),$$

for all $a, b \in \mathbb{C}[G]$, where $\sigma(a) = f_1 * a * f_1$ with $a * f_z = (f_z * \text{id})\Delta(a)$ and $f_z * a = (\text{id} \otimes f_z)\Delta(a)$.

This proposition follows from the orthogonality relations for the Haar state on matrix coefficients of irreducible representations, see [Wor87].

Consequently, the Woronowicz character f_1 can be used to decide whether the compact quantum group is of Kac-type. If \mathbb{G} is Kac, then $\sigma(a) = a$ for all $a \in \mathbb{C}[G]$, hence $f_1 = 1$. Conversely if $f_1 = 1$ then obviously $h(ab) = h(ba)$ for all $a, b \in \mathbb{C}[G]$, so \mathbb{G} is Kac. We also introduce the *scaling group*.

Definition 1.22: Let \mathbb{G} be a compact quantum group, define $\tau_z: \mathbb{C}[\mathbb{G}] \rightarrow \mathbb{C}[\mathbb{G}]$ by

$$\tau_z(a) = (f_{iz} * a * f_{-iz}).$$

The one-parameter group $(\tau_t)_{t \in \mathbb{R}}$ of $*$ -automorphisms is called the *scaling group* of \mathbb{G} .

The next lemma records two basic facts.

Lemma 1.23: *Let \mathbb{G} be a compact quantum group, then*

(i) $S^2 = \tau_{-i}$,

(ii) *if \mathbb{H} is a quantum subgroup of \mathbb{G} with surjection $\iota_{\mathbb{H}}: C(\mathbb{G}) \rightarrow C(\mathbb{H})$, then*

$$\tau_t^{\mathbb{H}} \circ \iota_{\mathbb{H}} = \iota_{\mathbb{H}} \circ \tau_t^{\mathbb{G}}$$

for all $t \in \mathbb{R}$, where $(\tau_t^{\mathbb{H}})$, $(\tau_t^{\mathbb{G}})$ are the scaling groups of \mathbb{H} and \mathbb{G} , respectively.

For a proof of the second part of the previous lemma, we refer to [Tom07, Lemma 2.9]. We conclude with the following characterisation of Kac compact quantum groups.

Proposition 1.24: *Let \mathbb{G} be a compact quantum group with Haar state h . The following are equivalent:*

(i) \mathbb{G} is Kac, i.e. h is tracial,

(ii) $f_z = f_1 = \varepsilon$ for all $z \in \mathbb{C}$,

(iii) $S^2 = \text{id}$ (hence S is also $*$ -preserving),

(iv) $\tau_t = \text{id}$ for all $t \in \mathbb{R}$,

where ε is the counit and S the antipode of $\mathbb{C}[\mathbb{G}]$.

Example 1.25: We consider $SU_q(2)$ from Example 1.2. Its antipode is given on generators by

$$S(\alpha) = \alpha^*, \quad S(\gamma) = -q\gamma.$$

Hence $S^2(\alpha) = \alpha$ and $S^2(\gamma) = q^2\gamma$ and $SU_q(2)$ is Kac if and only if $q = \pm 1$. More generally $SU_q(N)$ for $N \geq 2$ (see Example 2.31) is Kac if and only if $q = \pm 1$.

Having introduced the basic structure theory on the C^* -algebraic and Hopf-algebraic levels, we next discuss a cohomology theory on the discrete dual side. In low degrees, these cohomology groups provide a convenient language for deformation problems (in particular twists and associators), which will reappear from a categorical perspective in the next chapter.

3. Cohomology Theory

In this section we look at basic cohomology theory for compact quantum groups. The construction is strongly motivated by classical cohomology theory for groups, so we begin by recalling some standard definitions and basic constructions for discrete groups.

Let G be a discrete group and let A be an abelian group equipped with an action of G . Define $C^n(G, A)$ to be the set of all functions from G^n to A . For $n \in \mathbb{N}_0$ we define

$$\begin{aligned} \partial^n : C^n(G, A) &\rightarrow C^{n+1}(G, A), \\ (\partial f)(g_1, \dots, g_{n+1}) &= \sum_{i=0}^{n+1} (\partial_i^n f)(g_1, \dots, g_{n+1}). \end{aligned}$$

Here

$$\begin{aligned} \partial_0^n f(g_1, \dots, g_{n+1}) &= g_1 f(g_2, \dots, g_{n+1}), \\ \partial_k^n f(g_1, \dots, g_{n+1}) &= f(g_1, \dots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \dots, g_{n+1}), \quad \text{for } 1 \leq k \leq n, \\ \partial_{n+1}^n f(g_1, \dots, g_{n+1}) &= f(g_1, \dots, g_n). \end{aligned}$$

Note that $\partial^{n+1} \circ \partial^n = 0$. We call $\partial = (\partial^n)_n$ the *coboundary operator* and we obtain the *cobar complex*

$$0 \rightarrow C^0(G, A) \xrightarrow{\partial^0} C^1(G, A) \xrightarrow{\partial^1} C^2(G, A) \xrightarrow{\partial^2} C^3(G, A) \xrightarrow{\partial^3} \dots$$

Set $Z^n(G, A) = \ker(\partial^{n+1})$ and $B^n(G, A) = \text{im}(\partial^n)$ for $n \in \mathbb{N}$ (with $B^0(G, A) = 0$). The *cohomology group* $H^n(G, A)$ is defined as the quotient

$$H^n(G, A) = Z^n(G, A) / B^n(G, A).$$

Elements of $Z^n(G, A)$ are called *n-cocycles* and elements in $B^n(G, A)$ are called *n-coboundaries*.

The first three cohomology groups admit concrete interpretations:

- $H^0(G, A)$ is the fixed point group A^G with respect to the action, in particular $H^0(G, A) = A$ if the action is trivial.
- If the action is again trivial, then $H^1(G, A) = \text{Hom}(G, A)$.
- If the action is again trivial, then $H^2(G, A)$ corresponds to a set of extensions of G and A .

For $H^3(G, A)$ there is no equally direct interpretation. One can say, however, that it classifies *strict 2-groups*, which is a groupoid, “which looks like a group”, and that 3-cocycles play the role of *associators*. A 3-cocycle ω satisfies

$$g\omega(h, k, l) - \omega(gh, k, l) + \omega(g, hk, l) - \omega(g, h, kl) + \omega(g, h, k) = 0,$$

which is the pentagon equation, see Definition 2.1. In this sense 3-cocycles can be viewed as *deformations of associativity morphisms*, a point of view that will also be relevant for compact quantum groups.

As we have seen many interesting group invariants already appear in low-degree cohomology groups. Our goal is therefore to develop an analogous cohomology theory as a tool for compact quantum groups.

Recall that $\mathcal{U}(\mathbb{G}) = \mathbb{C}[G]^* \cong \prod_{[U] \in \text{Irr}(\mathbb{G})} B(\mathcal{H}_U)$. More generally, we can define the *space of k -point functions on the discrete dual $\widehat{\mathbb{G}}$* by

$$\mathcal{U}(\mathbb{G}^k) := (\mathbb{C}[G]^{\otimes k})^* \cong \prod_{[U_1], \dots, [U_k] \in \text{Irr}(\mathbb{G})} B(\mathcal{H}_{U_1}) \otimes \dots \otimes B(\mathcal{H}_{U_k}).$$

We now need a suitable coboundary map. For a n -cochain, which is an invertible element $\omega \in \mathcal{U}(\mathbb{G}^n)$, define

$$\begin{aligned} \partial_0^n \omega &= 1 \otimes \omega, \\ \partial_k^n \omega &= (\text{id} \otimes \dots \otimes \text{id} \otimes \hat{\Delta} \otimes \text{id} \otimes \dots \otimes \text{id})(\omega), \quad \text{for } 1 \leq k \leq n, \\ \partial_{n+1}^n \omega &= \omega \otimes 1, \end{aligned}$$

where $\hat{\Delta}$ is placed in the k -th position. Then we set

$$\partial^n \omega = \partial_0^n(\omega) \partial_2^n(\omega) \dots \partial_1^n(\omega^{-1}) \partial_3^n(\omega^{-1}) \dots,$$

continuing in this alternating fashion up to $k = n + 1$.

Definition 1.26: A \mathbb{C}^* - (\mathbb{T}) -valued n -cocycle on $\widehat{\mathbb{G}}$ is a (unitary) n -cochain $\omega \in \mathcal{U}(\mathbb{G}^n)$, such that $\partial^n \omega = 1$.

Two (unitary) n -cochains $\omega_1, \omega_2 \in \mathcal{U}(\mathbb{G}^n)$ are called *cohomologous*, if there exists a $(n - 1)$ -cochain $\rho \in \mathcal{U}(\mathbb{G}^{n-1})$ such that

$$\omega_1 = \omega_2 \rho := (\partial_0^n(\omega) \partial_2^n(\omega) \dots) \rho (\partial_1^n(\omega^{-1}) \partial_3^n(\omega^{-1}) \dots).$$

If for some $\omega \in \mathcal{U}(\mathbb{G}^n)$ there exists $\rho \in \mathcal{U}(\mathbb{G}^{n-1})$ such that $\omega = \partial \rho$, we call ω *n -coboundary*.

Finally, we call a n -cochain *invariant* if it commutes with the images of $\hat{\Delta}^{(n-1)}$.

With this construction we do not obtain a complex as in the group case and several familiar properties fail. Nevertheless, the resulting cohomologies in degrees $n = 1, 2, 3$ are useful for controlling deformation problems for compact quantum groups. We discuss their interpretation in the context of C^* -tensor categories in the next chapter, Section 3.

To conclude this section and this chapter, let us look at 1-, 2- and 3-cocycles.

Remark 1.27: (i) Let $n = 1$, a 1-cochain ω is a 1-cocycle on $\widehat{\mathbb{G}}$ if and only if $\widehat{\Delta}(\omega) = \omega \otimes \omega$. This follows directly from the definition. Such elements are called *group-like*. They form the group $H^1(\widehat{\mathbb{G}}, \mathbb{C}^*)$. Restricting to unitaries we obtain the group $H^1(\widehat{\mathbb{G}}, \mathbb{T})$. Moreover invariant 1-cocycles are precisely the central group-like elements, i.e. those commute with everything. Hence we obtain two further groups, denoted $H_{\mathbb{G}}^1(\widehat{\mathbb{G}}, \mathbb{C}^*)$ and $H_{\mathbb{G}}^1(\widehat{\mathbb{G}}, \mathbb{T})$.

(ii) For $n = 2$, the 2-cocycle condition is

$$(\omega \otimes 1)(\widehat{\Delta} \otimes \text{id})(\omega) = (1 \otimes \omega)(\text{id} \otimes \widehat{\Delta})(\omega).$$

This is the already mentioned Yang-Baxter equation, which is satisfied by R -matrices. We need to define $H^2(\widehat{\mathbb{G}}, \mathbb{C}^*)$ in a different manner, namely as the set of equivalence classes with being cohomologous as equivalence relation. As above one similarly constructs $H^2(\widehat{\mathbb{G}}, \mathbb{T})$, $H_{\mathbb{G}}^2(\widehat{\mathbb{G}}, \mathbb{C}^*)$ and $H_{\mathbb{G}}^2(\widehat{\mathbb{G}}, \mathbb{T})$. However, only the last two are indeed groups, since the product of two 2-cocycles is in general not a 2-cocycle. Counital 2-cocycles are also called (*Drinfeld*) *twists*. Note that every 2-cocycle is cohomologous to a counital one.

(iii) Finally for $n = 3$ we obtain

$$(1 \otimes \omega)(\text{id} \otimes \widehat{\Delta} \otimes \text{id})(\omega)(\omega \otimes 1) = (\text{id} \otimes \text{id} \otimes \widehat{\Delta})(\omega)(\widehat{\Delta} \otimes \text{id} \otimes \text{id})(\omega).$$

Observe that we cannot construct $H^3(\widehat{\mathbb{G}}, \mathbb{C}^*)$ and $H^3(\widehat{\mathbb{G}}, \mathbb{T})$ in the same way as for $n = 2$, since being cohomologous is not symmetric as a relation in general. Fortunately, for invariant 3-cocycles it is again an equivalence relation, and we can construct $H_{\mathbb{G}}^3(\widehat{\mathbb{G}}, \mathbb{C}^*)$ and $H_{\mathbb{G}}^3(\widehat{\mathbb{G}}, \mathbb{T})$. A counital element $\Phi \in H_{\mathbb{G}}^3(\widehat{\mathbb{G}}, \mathbb{T})$ is called an *associator* for \mathbb{G} . This terminology reflects the fact, that such elements induce a new associativity morphism $(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ in $\text{Rep}(\mathbb{G})$. We will return to this in the following chapter.

These three low-degree cases are the only ones of interest for compact quantum groups.

Chapter II.

C^* -Tensor Categories and Woronowicz's Tannaka-Krein Duality

This chapter aims to explain Woronowicz's Tannaka-Krein duality for compact quantum groups. We begin by introducing the category-theoretic framework for C^* -tensor categories. We assume the reader is familiar with basic notions from category theory. We then discuss the categorical meaning of low-degree cohomology and its interpretation in terms of C^* -tensor categories and deformations. Finally we recall the necessary background on compact Lie groups and Drinfeld-Jimbo quantum groups, and introduce twisted q -deformations via cochain twisting. This chapter is mainly based on [NT13] with additional background and perspectives from [Eti+15; LR97; KS97] and [NY15].

1. C^* -Tensor Categories

Given a category \mathcal{C} , we denote its class of objects by $\text{Ob}(\mathcal{C})$. For objects $A, B \in \text{Ob}(\mathcal{C})$, the set of morphisms from A to B by $\text{Mor}(A, B)$. In particular, we write $\text{Mor}(A, A) =: \text{End}(A)$.

The main idea of this section is to study structural properties of the representation category of a compact quantum group \mathbb{G} . This category carries substantial additional structure, and our goal is to capture the properties shared with the category of Hilbert spaces with bounded linear operators as morphisms, see Example 2.2.

Definition 2.1 (C^* -tensor category): A \mathbb{C} -linear category \mathcal{C} is called a C^* -category if

- each $\text{Mor}(U, V)$ is a Banach space and if

$$\text{Mor}(V, W) \otimes \text{Mor}(U, V) \rightarrow \text{Mor}(U, W), (S, T) \mapsto ST$$

is bilinear and satisfies $\|ST\| \leq \|S\| \|T\|$ for all $U, V, W \in \text{Ob}(\mathcal{C})$,

- it is equipped with an antilinear contravariant functor $*$: $\mathcal{C} \rightarrow \mathcal{C}$, which is the identity on objects, and for $T \in \text{Mor}(U, V)$, then $T^* \in \text{Mor}(V, U)$ and that

satisfies $T^{**} = T$ and the C^* -identity $\|T^*T\| = \|T\|^2$ for every $T \in \text{Mor}(U, V)$ and $U, V \in \text{Ob}(\mathcal{C})$.

Moreover we call \mathcal{C} a C^* -tensor category if it is equipped with

- a bilinear bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, $(U, V) \mapsto U \otimes V$,
- a unit object $I \in \text{Ob}(\mathcal{C})$, such that $U \otimes I \cong U$ for all $U \in \text{Ob}(\mathcal{C})$,
- natural unitary isomorphisms $\alpha_{U,V,W}: (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ for $U, V, W \in \text{Ob}(\mathcal{C})$, called *associativity morphisms*, and
- natural unitary isomorphisms $\lambda_U: I \otimes U \rightarrow U$ and $\rho_U: U \otimes I \rightarrow U$ for $U \in \text{Ob}(\mathcal{C})$,

such that the pentagon diagram

$$\begin{array}{ccc}
 & ((U \otimes V) \otimes W) \otimes X & \\
 \alpha \otimes \text{id} \swarrow & & \searrow \alpha \\
 (U \otimes (V \otimes W)) \otimes X & & (U \otimes V) \otimes (W \otimes X) \\
 \downarrow \alpha & & \downarrow \alpha \\
 U \otimes ((V \otimes W) \otimes X) & \xrightarrow{\alpha} & U \otimes (V \otimes (W \otimes X))
 \end{array}$$

and the triangle diagram

$$\begin{array}{ccc}
 (U \otimes I) \otimes V & \xrightarrow{\alpha} & U \otimes (I \otimes V) \\
 \downarrow \text{id} \otimes \rho & & \swarrow \text{id} \otimes \lambda \\
 & U \otimes V &
 \end{array}$$

commute for $U, V, W, X \in \text{Ob}(\mathcal{C})$, where we suppress the indices of α, λ and ρ . Moreover we assume the following properties

- $(S \otimes T)^* = S^* \otimes T^*$ for all morphisms S and T ,
- the category \mathcal{C} has *finite direct sums*, i.e. for any objects U, V there exist an object W and isometries $u \in \text{Mor}(U, W)$ and $v \in \text{Mor}(V, W)$ such that $uu^* + vv^* = 1$,
- the unit object I is *simple*, i.e. $\text{End}(I) = \mathbb{C}1$,
- the category has *subobjects*, meaning that for any projection $p \in \text{End}(U)$ there exist an object V and an isometry $v \in \text{Mor}(V, U)$ such that $p = vv^*$,
- the category is small, i.e. the class of objects is a set.

We call \mathcal{C} *strict* if $(U \otimes V) \otimes W = U \otimes (V \otimes W)$, $I \otimes U = U = U \otimes I$ and α, λ, ρ are the identity morphisms.

At first glance, this definition may seem rather overwhelming. However it is essentially a formalisation of the basic properties and constructions familiar from the category of finite-dimensional Hilbert spaces. The assumption that \mathcal{C} needs to be small, is a technical assumption, and we will not discuss it in detail.

For technical convenience we will always assume our categories to be strict. By *MacLane's coherence theorem*, see [Eti+15], every monoidal category is *monoidally equivalent*, see Definition 2.5, to a strict one. While the notion of a monoidal category is slightly more general than that of a C^* -tensor category, every C^* -tensor category is, in particular, a monoidal category.

Example 2.2: (i) The basic example of a C^* -tensor category is the *category of finite-dimensional Hilbert spaces* Hilb_f , whose objects are finite-dimensional Hilbert spaces, and whose morphisms are bounded linear maps. This category becomes a C^* -tensor category with the usual tensor product. The associativity morphisms are given by $(\xi \otimes \eta) \otimes \mu \mapsto \xi \otimes (\eta \otimes \mu)$. To obtain a strict C^* -tensor category, one may fix a realization of the tensor product using the Cuntz algebra \mathcal{O}_∞ . Namely consider a linear map $\beta_{\mathcal{H}}: \mathcal{H} \rightarrow \mathcal{O}_\infty$ such that $\beta_{\mathcal{H}}(\xi)^* \beta_{\mathcal{H}}(\eta) = \langle \eta, \xi \rangle 1$ and define $\mathcal{H}_1 \otimes \mathcal{H}_2$ the closed span of elements $\xi \otimes \eta = \beta_{\mathcal{H}_1}(\xi) \beta_{\mathcal{H}_2}(\eta)$. In this setting strictness holds because

$$(\xi \otimes \eta) \otimes \mu = (\beta_{\mathcal{H}_1}(\xi) \beta_{\mathcal{H}_2}(\eta)) \beta_{\mathcal{H}_3}(\mu) = \beta_{\mathcal{H}_1}(\xi) (\beta_{\mathcal{H}_2}(\eta) \beta_{\mathcal{H}_3}(\mu)) = \xi \otimes (\eta \otimes \mu).$$

Clearly $I = \mathbb{C}$ and by construction we also have $\mathbb{C} \otimes \mathcal{H} = \mathcal{H} \otimes \mathbb{C} = \mathcal{H}$.

Note that in order to make Hilb_f small, one should fix a set of finite-dimensional Hilbert spaces, that is sufficiently large for our purposes.

(ii) The main example of a C^* -tensor category in this thesis is the *representation category* $\text{Rep}(\mathbb{G})$ of a compact quantum group \mathbb{G} . Its objects are unitary finite-dimensional representations of \mathbb{G} and its morphisms are the intertwiners. To ensure compatibility and smallness, we should fix the same underlying Hilbert spaces as in the first part of the example. The tensor product is given by the tensor product of representations as defined in Chapter I. In general strictness is not automatic, nevertheless, by the coherence theorem, we will assume $\text{Rep}(\mathbb{G})$ to be strict.

As usual we now need to introduce a mapping, which respects our C^* -tensor structure on a category.

Definition 2.3: Let \mathcal{C} and \mathcal{C}' be C^* -tensor categories. A functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ is called a *tensor functor* if it is linear on morphisms, together with an isomorphism $F_0: I' \rightarrow F(I)$ and natural isomorphisms $F_2: F(U) \otimes F(V) \rightarrow F(U \otimes V)$ such that the diagrams

$$\begin{array}{ccccc} (F(U) \otimes F(V)) \otimes F(W) & \xrightarrow{F_2 \otimes \text{id}} & F(U \otimes V) \otimes F(W) & \xrightarrow{F_2} & F((U \otimes V) \otimes W) \\ \downarrow \alpha & & & & \downarrow F \circ \alpha \\ F(U) \otimes (F(V) \otimes F(W)) & \xrightarrow{\text{id} \otimes F_2} & F(U) \otimes F(V \otimes W) & \xrightarrow{F_2} & F(U \otimes (V \otimes W)) \end{array}$$

and

$$\begin{array}{ccc}
F(I) \otimes F(U) & \xrightarrow{F_2} & F(I \otimes U) \\
F_0 \otimes \text{id} \uparrow & & \downarrow F \circ \lambda \\
I' \otimes F(U) & \xrightarrow{\lambda'} & F(U)
\end{array}
\qquad
\begin{array}{ccc}
F(U) \otimes F(I) & \xrightarrow{F_2} & F(U \otimes I) \\
\text{id} \otimes F_0 \uparrow & & \downarrow F \circ \rho \\
F(U) \otimes I' & \xrightarrow{\rho'} & F(U)
\end{array}$$

commute for all $U, V, W \in \text{Ob}(\mathcal{C})$.

A tensor functor is called *unitary* if $F(T)^* = F(T^*)$ for every morphism T and if both $F_2: F(U) \otimes F(V) \rightarrow F(U \otimes V)$ and F_0 are unitary.

As mentioned in [NT13] the morphism F_0 is uniquely determined by F and F_2 . Moreover there is usually a canonical choice for F_0 , hence we will omit F_0 . Thus we will sometimes write the tensor functor F as a pair $(F, F_2) := (F_0, F, F_2)$.

Example 2.4: (i) Let \mathbb{H} be a quantum subgroup of a compact quantum group \mathbb{G} . Then the restriction functor $F: \text{Rep}(\mathbb{G}) \rightarrow \text{Rep}(\mathbb{H})$ sends a representation of \mathbb{G} to its restriction to \mathbb{H} and intertwiners to their restrictions. Let F_2 be the identity. Then F is a tensor functor.

(ii) The *forgetful functor* $F: \text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$ given by $U \mapsto \mathcal{H}_U$ and acting as the identity on morphisms, is a tensor functor when equipped with $F_2 = \text{id}$. This is an example of a *fiber functor*, which we will discuss in the next section.

Definition 2.5: Let $\mathcal{C}, \mathcal{C}'$ be C^* -tensor categories.

(i) Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two tensor functors. A natural isomorphism $\eta: F \rightarrow G$ is called *monoidal* if the two diagrams

$$\begin{array}{ccc}
F(U) \otimes F(V) & \xrightarrow{F_2} & F(U \otimes V) \\
\eta \otimes \eta \downarrow & & \downarrow \eta \\
G(U) \otimes G(V) & \xrightarrow{G_2} & G(U \otimes V)
\end{array}
\qquad
\begin{array}{ccc}
& I' & \\
F_0 \swarrow & & \searrow G_0 \\
F(I) & \xrightarrow{\eta} & G(I)
\end{array}$$

commute for all $U, V \in \text{Ob}(\mathcal{C})$.

(ii) If there exist tensor functors $F: \mathcal{C} \rightarrow \mathcal{C}'$ and $G: \mathcal{C}' \rightarrow \mathcal{C}$ such that the compositions FG and GF are naturally monoidally isomorphic to the identity functors on \mathcal{C}' and on \mathcal{C} , respectively, then we call \mathcal{C} and \mathcal{C}' *monoidally equivalent*.

In this thesis we will work with *semisimple categories*, i.e. categories in which every object is semisimple (equivalently, a direct sum of simple objects). The following lemma is a useful criterion.

Lemma 2.6: *Let \mathcal{C} and \mathcal{C}' be semisimple C^* -tensor categories and (U_α) a family of representatives of the isomorphism classes of simple objects of \mathcal{C} . Then a tensor functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ is a monoidal equivalence if and only if the objects $F(U_\alpha)$ are simple, $F(U_\alpha) \not\cong F(U_\beta)$ for $\alpha \neq \beta$ and every simple object V in \mathcal{C}' is isomorphic to $F(U_\gamma)$ for some γ .*

This lemma follows from a standard criterion, that is F is a monoidal equivalence if and only if F is *fully faithful* (that means $\text{Mor}(U, V) \cong \text{Mor}(F(U), F(V))$) and essentially surjective, i.e. for every object V in \mathcal{C}' , we have $V \cong F(U)$ for an object U in \mathcal{C} .

Besides semisimple C^* -tensor categories, we will also mainly work with *rigid* C^* -tensor categories.

Definition 2.7: Let \mathcal{C} be a C^* -tensor category. It is called *rigid*, if for every object U , there exist an object \bar{U} in \mathcal{C} and morphisms $R_U \in \text{Mor}(I, \bar{U} \otimes U)$ and $R_{\bar{U}} \in \text{Mor}(I, U \otimes \bar{U})$ such that the *conjugate equations*

$$(R_{\bar{U}}^* \otimes \text{id})(\text{id} \otimes R_U) = \text{id} \quad \text{and} \quad (R_U^* \otimes \text{id})(\text{id} \otimes R_{\bar{U}}) = \text{id},$$

hold. We call \bar{U} *conjugate* to U .

Recall that R_U^* and $R_{\bar{U}}^*$ are morphisms $\bar{U} \otimes U \rightarrow I$ and $U \otimes \bar{U} \rightarrow I$ respectively. The morphisms R_U and $R_{\bar{U}}$ are sometimes called *coevaluation maps*, and $R_U^*, R_{\bar{U}}^*$ the *evaluation maps*. This terminology is standard in classical literature about C^* -tensor categories, see e.g. [LR97] and [Eti+15]. A quite natural approach is to use string diagrams, as in [HM23]. The evaluation maps are depicted as *cups* and the coevaluation maps as *caps*, and our conjugate equations become the *snake equations*.

By the Joyal-Street-Verity theorem, [JSV96], any two topologically equivalent string diagrams represent the same morphisms in a symmetric monoidal category, hence we can freely apply topological deformations to string diagrams.

Proposition 2.8: *Let \mathcal{C} be a rigid C^* -tensor category.*

- (i) *The conjugate object \bar{U} to an object U is unique up to isomorphism. Moreover given two solutions $(R_U, R_{\bar{U}})$ and $(R'_U, R'_{\bar{U}})$ of the conjugate equations for U, \bar{U} there exists an invertible morphism $T \in \text{End}(U)$ such that*

$$R'_U = (\text{id} \otimes T)R_U, \quad \text{and} \quad R'_{\bar{U}} = (T^{-1} \otimes \text{id})R_{\bar{U}}.$$

- (ii) *For every object U the C^* -algebra $\text{End}(U)$ is finite-dimensional. In particular \mathcal{C} is semisimple.*

Example 2.9: (i) The C^* -tensor category Hilb_f is rigid. Fix an ONB $(e_i)_{i=1, \dots, n}$ of \mathcal{H} and define $R_{\mathcal{H}}: \mathbb{C} \rightarrow \mathcal{H}^* \otimes \mathcal{H}$ be the usual embedding $1 \mapsto \sum_{i=1}^n e_i^* \otimes e_i$. Then $R_{\mathcal{H}}^*: \mathcal{H}^* \otimes \mathcal{H} \rightarrow \mathbb{C}$ is the usual evaluation map. Likewise we define the

corresponding morphisms $R_{\overline{\mathcal{H}}}$ and $R_{\overline{\mathcal{H}}}^*$, where $\overline{\mathcal{H}} = \mathcal{H}^*$ is the dual space. These maps clearly satisfy the conjugate equations. If we instead consider the category of (possibly infinite-dimensional) Hilbert spaces Hilb , then the category is not rigid, since such a coevaluation map does not exist.

- (ii) The representation category $\text{Rep}(\mathbb{G})$ of a compact quantum group \mathbb{G} is rigid. Recalling the construction of the conjugate representation \overline{U} , one may consider the operators

$$R = (1 \otimes \rho_{\overline{U}}^{-1/2})R_{\mathcal{H}_U}$$

and

$$\overline{R} = (\rho_U^{1/2} \otimes 1)R_{\mathcal{H}_{\overline{U}}}.$$

Then (R, \overline{R}) fulfils the conjugate equations.

We can now define the *intrinsic dimension* of an object U .

Definition 2.10 (Intrinsic dimension): Let \mathcal{C} be a rigid C^* -tensor category. The *intrinsic dimension* of U is defined by

$$d^{\mathcal{C}}(U) = \inf \|R_U\| \|R_{\overline{U}}\|,$$

where $(R_U, R_{\overline{U}})$ ranges over all solutions of the conjugate equations.

For a simple object U , we have

$$d^{\mathcal{C}}(U) = \|R_U\| \|R_{\overline{U}}\|$$

for any solution $(R_U, R_{\overline{U}})$. Since rigid C^* -tensor categories are semisimple, every object $U = \bigoplus_k U_k$ decomposes as a direct sum of simple objects, and the intrinsic dimension decomposes as

$$d^{\mathcal{C}}(U) = \sum_k d^{\mathcal{C}}(U_k).$$

In particular, this dimension is independent of concrete realisations. A solution $(R_U, R_{\overline{U}})$ such that $d^{\mathcal{C}}(U)^{1/2} = \|R_U\| = \|R_{\overline{U}}\|$ is called a *standard solution*.

Example 2.11: (i) Consider again $\mathcal{C} = \text{Hilb}_f$. Since every object is a finite number of copies of the simple object \mathbb{C} , we have $d^{\mathcal{C}}(\mathcal{H}) = \dim(\mathcal{H})$.

- (ii) Now consider $\mathcal{C} = \text{Rep}(\mathbb{G})$ and let U be a simple object, hence an irreducible representation, and fix an ONB $(e_i)_{i=1, \dots, n}$ for \mathcal{H}_U . Then

$$\begin{aligned} \|R\| &= \left\| (1 \otimes \rho_U^{-1/2})R_{\mathcal{H}_U}(1) \right\| = \left\| \sum_{i=1}^n e_i^* \otimes \rho_U^{-1/2} e_i \right\| \\ &= \text{Tr}(\rho_U^{-1})^{1/2} = \dim_q(U)^{1/2}. \end{aligned}$$

Likewise $\|\overline{R}\| = \dim_q(U)^{1/2}$, and hence $d^{\mathcal{C}}(U) = \dim_q(U)$.

For two rigid C^* -tensor categories unitary tensor functors are contractive as shown in the following proposition.

Proposition 2.12: *Let $\mathcal{C}, \mathcal{C}'$ be two rigid C^* -tensor categories and let $F: \mathcal{C} \rightarrow \mathcal{C}'$ be a unitary tensor functor. Then*

$$d^{\mathcal{C}'}(F(U)) \leq d^{\mathcal{C}}(U)$$

for all objects U in \mathcal{C} .

Proof. Let U be an object of \mathcal{C} and let $(R_U, R_{\bar{U}})$ be a standard solution. We construct a solution for $F(U)$, such that the product of the norms equals $d^{\mathcal{C}}(U)$. Define morphisms $R: I \rightarrow F(\bar{U}) \otimes F(U)$ and $\bar{R}: I \rightarrow F(U) \otimes F(\bar{U})$ by

$$R = F_2^* F(R_U) F_0, \quad \text{and} \quad \bar{R} = F_2^* F(R_{\bar{U}}) F_0.$$

Then (R, \bar{R}) is a solution for the conjugate equations of $F(U)$. Clearly,

$$R^* R = F_0^* F(R_U^*) F_2^* F_2 F(R) F_0 = F_0^* F(R_U^* R_U) F_0 = \|R_U\|^2 1,$$

and $\bar{R}^* \bar{R} = \|R_{\bar{U}}\|^2 1$. Hence $\|R\| \|\bar{R}\| = d^{\mathcal{C}}(U)$ and by definition of the intrinsic dimension, we obtain $d^{\mathcal{C}'}(F(U)) \leq d^{\mathcal{C}}(U)$. \square

We call a compact quantum group \mathbb{G} *coamenable* if the canonical map $C_u(\mathbb{G}) \rightarrow C_r(\mathbb{G})$ is an isomorphism. In this case the intrinsic dimension is the smallest dimension function, see [NT13]. As assumption for equality in the previous proposition, we need the notion of *amenability* and therefore we need to introduce the *fusion semiring* of a C^* -tensor category \mathcal{C} .

Definition 2.13: Let \mathcal{C} be a semisimple C^* -tensor category. Two objects U, V in \mathcal{C} are called *isomorphic* if $\text{Mor}(U, V)$ contains an isomorphism. The equivalence class of U under isomorphism is denoted by $[U]$. The *fusion (or Grothendieck) semiring* $R^+(\mathcal{C})$ is the semiring generated by the equivalence classes of simple objects $[U]$ with addition and multiplication defined as follows

$$[U] + [V] = [U \oplus V], \quad [U] \cdot [V] = [U \otimes V].$$

Right multiplication by a simple object $[U] \in \text{Irr}(\mathcal{C})$ on $R^+(\mathcal{C})$ defines an operator $\Gamma_U \in \ell^2(\text{Irr}(\mathcal{C}))$. A dimension function $d: R^+(\mathcal{C}) \rightarrow \mathbb{R}$, i.e. $d([U]) > 0$ and $d([U]) = d([\bar{U}])$ for all $[U] \in \text{Irr}(\mathcal{C})$, is called *amenable* if

$$\|\Gamma_U\| = d([U])$$

for all $[U] \in \text{Irr}(\mathcal{C})$. If the intrinsic dimension on \mathcal{C} is amenable, then \mathcal{C} is called *amenable*.

In general one has

$$\|\Gamma_U\| \leq d([U]).$$

Additionally, we may define the *fusion semiring* $K^+(\mathbb{G})$ of a compact quantum group \mathbb{G} as

$$K^+(\mathbb{G}) := R^+(\text{Rep}(\mathbb{G})).$$

We are especially interested in rigid C^* -tensor categories, respectively compact quantum groups, with the same fusion semiring.

Definition 2.14 (Lie-type): Let \mathcal{C} be a rigid C^* -tensor category. We say that \mathcal{C} is of *Lie-type*, if there exists a connected compact Lie group G such that there is a dimension-preserving isomorphism $R^+(\mathcal{C}) \cong R^+(G)$. A compact quantum group \mathbb{G} is of *Lie-type*, if $\text{Rep}(\mathbb{G})$ is of Lie-type.

Now, with the notion of amenability, we can state the following.

Proposition 2.15: *Let $\mathcal{C}, \mathcal{C}'$ be two rigid C^* -tensor categories, and let $F: \mathcal{C} \rightarrow \mathcal{C}'$ be a unitary tensor functor. If \mathcal{C} is amenable, then*

$$d^{\mathcal{C}'}(F(U)) = d^{\mathcal{C}}(U)$$

for all objects U in \mathcal{C} .

Since any compact group G is coamenable (indeed $C_r(G) = C_u(G)$), we obtain the following corollary by using the fact that the coamenability of the compact quantum group coincides with the amenability of the representation category if and only if G is of Kac-type.

Corollary 2.16: *Let G be a compact group, and $F: \text{Rep}(G) \rightarrow \text{Hilb}_f$ be a unitary fiber functor, see Definition 2.17, then $\dim(F(U)) = \dim(U)$.*

This corollary is one reason why our classification result will not hold in the Kac-case, since every fiber functor on a representation category $\text{Rep}(SU(N))$ gives rise to a compact quantum group of Kac-type. We will comment on why a classification in this setting is not feasible in Remark 4.15.

2. Tannaka-Krein Duality

After our preparatory work on rigid C^* -tensor categories, we are now in a position to explain the connection between these categories and compact quantum groups. This provides in some sense an additional “philosophical” viewpoint on compact quantum groups. Rather than describing them in C^* -algebraic or Hopf-algebraic terms, one can reconstruct them from their representation theory. The resulting duality goes back to Woronowicz’s first articles on compact quantum groups, [Wor88].

Definition 2.17 (Fiber functor): Let \mathcal{C} be a rigid C^* -tensor category. A tensor functor $F: \mathcal{C} \rightarrow \text{Hilb}_f$ is called a *fiber functor*.

More generally, for an arbitrary C^* -tensor category, one requires F to be faithful, i.e. F is injective on morphisms, and exact. In the rigid case these properties are automatic.

The simplest example is the *canonical fiber functor* $F: \text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$ defined by $F(U) = \mathcal{H}_U$ and with F_2 equal to the identity.

Theorem 2.18 (Woronowicz's Tannaka-Krein duality): *Let \mathcal{C} be a C^* -tensor category with conjugates and let $F: \mathcal{C} \rightarrow \text{Hilb}_f$ be a unitary fiber functor. Then there exist a compact quantum group \mathbb{G} and a unitary monoidal equivalence $E: \mathcal{C} \rightarrow \text{Rep}(\mathbb{G})$ such that F is naturally (unitarily) monoidally isomorphic to the composition of E with the canonical fiber functor.*

Sketch of the proof.

By MacLane's coherence theorem, we may assume that \mathcal{C} is strict, and we may also assume that $F(I) = \mathbb{C}$ and F_0 to be the identity. The first step is to construct a suitable Hopf $*$ -algebra that will turn out to be isomorphic to $(\mathcal{U}(\mathbb{G}), \widehat{\Delta})$ for a suitable compact quantum group \mathbb{G} . Denote by $\text{End}(F)$ the $*$ -algebra of natural transformations from F to itself. A natural transformation $\eta \in \text{End}(F)$ is determined by maps $\eta_{U_\alpha}: F(U_\alpha) \rightarrow F(U_\alpha)$, where $\{U_\alpha\}_\alpha$ is a family of representatives of the isomorphism classes of simple objects. Hence

$$\text{End}(F) \cong \prod_{\alpha} B(F(U_\alpha)).$$

On this $*$ -algebra we can define a comultiplication

$$\tilde{\Delta}: \text{End}(F) \rightarrow \text{End}(F^{\otimes 2}) \cong \prod_{\alpha, \beta} B(F(U_\alpha) \otimes F(U_\beta)).$$

Together with the remaining structure maps this yields a Hopf $*$ -algebra $(\text{End}(F), \tilde{\Delta})$. This structure is in some sense the dual of the Hopf $*$ -algebra we want to construct. Let

$$A = \bigoplus_{\alpha} B(F(U_\alpha))^*.$$

The algebra A can be equipped with the structure of a Hopf $*$ -algebra, for instance the comultiplication is given by

$$\Delta: A \rightarrow A \otimes A, \quad \Delta(a)(\eta \otimes \mu) = a(\eta\mu).$$

One shows that this Hopf $*$ -algebra is generated as an algebra by matrix coefficients of finite-dimensional unitary corepresentations. Hence by Proposition 1.18, we obtain $(A, \Delta) \cong (\mathbb{C}[G], \Delta)$ for some compact quantum group \mathbb{G} .

Uniqueness follows from duality arguments, since our construction depends functorially on $(\text{End}(F), \widehat{\Delta})$, which is isomorphic to $(\mathcal{U}(\mathbb{G}), \widehat{\Delta})$. More precisely if $F': \text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$ is the canonical fiber functor for some other compact quantum group \mathbb{G}' , then one can identify $(\text{End}(F'), \widehat{\Delta}') = (\mathcal{U}(\mathbb{G}'), \widehat{\Delta}')$, and we can construct a natural unitary monoidal isomorphism

$$(\mathcal{U}(\mathbb{G}), \widehat{\Delta}) \cong (\mathcal{U}(\mathbb{G}'), \widehat{\Delta}').$$

By duality this implies $(\mathbb{C}[G], \Delta) \cong (\mathbb{C}[G'], \Delta')$. □

A weaker variant is the following.

Theorem 2.19: *Let (A, Δ) be a Hopf $*$ -algebra, and let $\mathcal{D} \subseteq \text{Rep}(A)$ (the corepresentation category of the Hopf $*$ -algebra) be a full rigid C^* -subcategory. Then there exists a compact quantum group \mathbb{G} and a unitary monoidal equivalence $F: \mathcal{D} \rightarrow \text{Rep}(\mathbb{G})$.*

Not every C^* -tensor category admits a fiber functor. Moreover two different fiber functors may yield non-isomorphic compact quantum groups, even though the resulting representation categories are monoidally equivalent. Hence in order to classify compact quantum groups one needs to classify monoidal autoequivalences as well as fiber functors. This was already mentioned in the introduction.

Before turning to the connection between the classification of C^* -tensor categories $\text{Rep}(\mathbb{G})$ and their cohomology theory, which we introduced in Chapter I, we explain how to deform a compact quantum group.

Let \mathbb{G} be a compact quantum group and let $\omega \in H^2(\widehat{\mathbb{G}}, \mathbb{T})$ be a \mathbb{T} -valued 2-cocycle. Define a fiber functor $F: \text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$ to be the identity on objects and morphisms and with F_2 given by the action of ω^* . This is well-defined. Via Tannaka-Krein duality this produces a new compact quantum group \mathbb{G}_ω . By construction, \mathbb{G}_ω is unitarily monoidally equivalent to \mathbb{G} , which means that $\text{Rep}(\mathbb{G})$ is unitarily monoidally equivalent to $\text{Rep}(\mathbb{G}_\omega)$. However the Hopf $*$ -algebraic structure of $(\mathcal{U}(\mathbb{G}_\omega), \widehat{\Delta}_\omega)$ differs from that of $(\mathcal{U}(\mathbb{G}), \Delta)$. Indeed the comultiplication is deformed by

$$\widehat{\Delta}_\omega = \omega \widehat{\Delta}(\cdot) \omega^*.$$

Its antipode is also obtained by a deformation of \widehat{S}

$$\widehat{S}_\omega = u \widehat{S}(\cdot) u^*, \quad \text{where } u = m(\text{id} \otimes \widehat{S})(\omega).$$

In the language of Hopf $*$ -algebras we call $(\mathcal{U}(\mathbb{G}_\omega), \widehat{\Delta}_\omega)$ a (cocycle-) twisting of $(\mathcal{U}(\mathbb{G}), \widehat{\Delta})$. One can also describe the twisting on the level of the dual $(\mathbb{C}[G_\omega], \Delta)$, but it is more difficult to describe, hence in the spirit of Tannaka-Krein we therefore stay on the level of $(\mathcal{U}(\mathbb{G}_\omega), \widehat{\Delta}_\omega)$. This kind of twisting is not particularly interesting, since the representation category is not substantially deformed.

3. Categorical Meaning of Low-degree Cohomology

In this section we explain the connection between deformations of quantum groups and cohomology theory, which, as already mentioned, can be viewed as a framework for controlling deformations of compact quantum groups. See the chapter on cohomology theory in [NT13].

Categorical interpretation of $H^1(\widehat{\mathbb{G}}, \mathbb{C}^)$, $H^1(\widehat{\mathbb{G}}, \mathbb{T})$.*

As we saw in the proof sketch of Theorem 2.18, we may identify $\mathcal{U}(\mathbb{G})$ with the algebra $\text{End}(F)$ for the canonical fiber functor $F: \mathbb{G} \rightarrow \text{Hilb}_f$. Hence by construction, $H^1(\widehat{\mathbb{G}}, \mathbb{C}^*)$ is the group of monoidal automorphisms of the canonical fiber functor and $H^1(\widehat{\mathbb{G}}, \mathbb{T})$ is the group of unitary monoidal automorphisms of the canonical fiber functor.

Categorical interpretation of $H^2(\widehat{\mathbb{G}}, \mathbb{C}^)$, $H^2(\widehat{\mathbb{G}}, \mathbb{T})$, $H_{\mathbb{G}}^2(\widehat{\mathbb{G}}, \mathbb{T})$.*

Let F denote the forgetful functor, i.e. the functor $\text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$ sending $U \mapsto \mathcal{H}_U$ and acting as the identity on morphisms. For an invertible element $\omega \in \mathcal{U}(\mathbb{G}^2)$ we obtain a natural isomorphism given by the action of ω^{-1} ,

$$F_2: F(U) \otimes F(V) \xrightarrow{\omega^{-1}} F(U \otimes V).$$

Then ω is a 2-cocycle if and only if the diagram

$$\begin{array}{ccc} F(U) \otimes F(V) \otimes F(W) & \xrightarrow{\omega^{-1} \otimes \text{id}} & F(U \otimes V) \otimes F(W) \\ \downarrow 1 \otimes \omega^{-1} & & \downarrow (\widehat{\Delta} \otimes \text{id})(\omega^{-1}) \\ F(U) \otimes F(V \otimes W) & \xrightarrow{(\text{id} \otimes \widehat{\Delta})(\omega^{-1})} & F(U \otimes V \otimes W) \end{array}$$

commutes. Equivalently (F, ω^{-1}) is a tensor functor. Moreover two 2-cocycles are cohomologous if and only if the induced tensor functors are naturally monoidally equivalent.

These tensor functors, which are also fiber functors, exhaust all dimension-preserving fiber functors $E: \text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$, i.e. those satisfying $d(E(U)) = \dim(U)$. This follows from the discussion above and from the observation that every dimension-preserving functor is naturally equivalent to the forgetful functor $\text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$. If E is in addition a tensor functor, then as mentioned above, it is induced by some 2-cocycle.

Consequently $H^2(\widehat{\mathbb{G}}, \mathbb{C}^*)$ is the set of naturally monoidal isomorphism classes of dimension-preserving fiber functors. Analogously $H^2(\widehat{\mathbb{G}}, \mathbb{T})$ is the set of natural unitary monoidal isomorphism classes of dimension-preserving unitary fiber functors. Finally if ω is an invariant unitary 2-cocycle, then multiplication by ω^{-1} on $\mathcal{H}_U \otimes \mathcal{H}_V$ can be viewed as a unitary endomorphism of $U \otimes V$ in the representation category, and hence induces then a natural unitary transformation of the tensor bifunctor. The

cocycle condition then means that this transformation is a monoidal autoequivalence of $\text{Rep}(\mathbb{G})$. In this way, any autoequivalence of $\text{Rep}(\mathbb{G})$, that fixes the isomorphism classes of irreducible objects is obtained up to equivalence. Thus $H_{\mathbb{G}}^2(\widehat{\mathbb{G}}, \mathbb{T})$ can be seen as the subgroup of (tensor-) autoequivalences $\text{Aut}^{\otimes}(\text{Rep}(\mathbb{G}))$, consisting of those that preserve the isomorphism classes of objects.

Categorical interpretation of $H_{\mathbb{G}}^3(\widehat{\mathbb{G}}, \mathbb{T})$.

As stated in Remark 1.27, every unitary counital 3-cocycle Φ is also called an associator, since such a cocycle defines new associativity morphisms, and hence gives rise to a different C^* -tensor category $(\text{Rep}(\mathbb{G}), \Phi)$. Two associators are cohomologous if and only if the corresponding C^* -tensor categories are unitarily monoidally equivalent, in such a way that this equivalence preserves the isomorphism classes of objects. Note that $(\text{Rep}(\mathbb{G}), \Phi)$ need not be rigid, however, if Φ acts as a scalar on $\mathcal{H}_U \otimes \mathcal{H}_V \otimes \mathcal{H}_W$, then $(\text{Rep}(\mathbb{G}), \Phi)$ is rigid, see later.

Moreover, every unitary invariant 3-cocycle is cohomologous to a unitary counital invariant 3-cocycle. In particular, Φ and Ψ are cohomologous, if there exists $\omega \in \mathcal{U}(\mathbb{G}^2)$ such that

$$\Psi = \Phi_{\omega} = (1 \otimes \omega)(\text{id} \otimes \widehat{\Delta})(\omega)\Phi(\widehat{\Delta} \otimes \text{id})(\omega^{-1})(\omega^{-1} \otimes 1).$$

This is again an invariant 3-cocycle.

In summary, $H_{\mathbb{G}}^3(\widehat{\mathbb{G}}, \mathbb{T})$ parametrises the C^* -tensor categories of the form $(\text{Rep}(\mathbb{G}), \Phi)$ up to equivalence, preserving the isomorphism classes of objects. In the case of $H_{\mathbb{G}}^3(\widehat{\mathbb{G}}, \mathbb{C}^*)$, there is no similarly convenient categorical description.

4. Twisted q -deformations of Compact Lie Groups

In this section we recall basic notions from compact Lie groups and introduce twisted q -deformations.

Recall that a Lie group G is a smooth manifold G that is also a group such that the multiplication and inversion maps are smooth. We call a Lie group G *compact* if the underlying manifold is compact. Associated to a Lie group G is its *Lie algebra* \mathfrak{g} . This is a vector space \mathfrak{g} equipped with a bilinear Lie bracket $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, satisfying the *Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

and $[x, x] = 0$ for all $x, y, z \in \mathfrak{g}$. For instance, for an algebra A one may take the commutator $[x, y] = xy - yx$ as a Lie bracket. Note that a Lie algebra is in general not an associative unital algebra.

We will only work with finite-dimensional semisimple Lie algebras, so we briefly recall the necessary notions in this setting. An element $x \in \mathfrak{g}$ is called *semisimple* if $\text{ad}(x) = [x, \cdot]$ is diagonalizable. A maximal commutative Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$

consisting of semisimple elements is called a *Cartan subalgebra*. Then \mathfrak{g} decomposes as a direct sum of a Cartan subalgebra \mathfrak{h} and the *root subspaces*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha.$$

The *roots* $\alpha \in \mathfrak{h}^*$ are those linear functionals for which there exists nonzero $x \in \mathfrak{g}$ with $[h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}$, i.e. such that \mathfrak{g}_α is a non-trivial eigenspace. These root spaces are one-dimensional, we refer to [KS97] for further details and properties. We may choose a basis h_1, \dots, h_l of \mathfrak{h} , where l is called *rank* of \mathfrak{g} , such that $\alpha(h_i) \in \mathbb{R}$ for all $1 \leq i \leq l$ and all roots α . Moreover one can choose basis vectors e_α such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ ($-\alpha$ is also a root). This yields a basis for \mathfrak{g} , which we call a *Cartan-Weyl-basis*, which fulfils the commutation relations

$$[h_i, h_j] = 0, \quad [h_i, e_\alpha] = \alpha(h_i)e_\alpha, \quad [e_\alpha, e_{-\alpha}] = h_\alpha.$$

A root α is called *positive* if the first non-zero entry among $\alpha(h_1), \alpha(h_2), \dots, \alpha(h_l)$ is positive, we write $\alpha > 0$. A positive root α is *simple* if it cannot be written as a sum of two other positive roots. The number of simple roots equals the rank of \mathfrak{g} , and every positive root is a linear combination of simple roots. We also have the following decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-,$$

where $\mathfrak{g}_\pm = \sum_{\alpha > 0} \mathfrak{g}_{\pm\alpha}$. For a fixed choice of simple roots $\alpha_1, \dots, \alpha_l$ (not uniquely determined) the Lie subalgebra \mathfrak{g}_+ is generated by the generators of the root subspaces $E_{\alpha_1}, \dots, E_{\alpha_l}$ and likewise \mathfrak{g}_- is generated by elements $E_{-\alpha_1}, \dots, E_{-\alpha_l}$. Moreover $H_i = [E_{\alpha_i}, E_{-\alpha_i}]$ forms a basis of the Cartan subalgebra \mathfrak{h} . Hence for a fixed set of simple roots $\alpha_1, \dots, \alpha_l$ the elements $E_{\alpha_1}, \dots, E_{\alpha_l}, E_{-\alpha_1}, \dots, E_{-\alpha_l}$ generate \mathfrak{g} .

An important tool for the classification of semisimple Lie algebras \mathfrak{g} is the *Cartan matrix* $A = (a_{ij}) \in M_l(\mathbb{C})$, defined by

$$a_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

Here $\langle \cdot, \cdot \rangle$ on \mathfrak{h}^* is induced by the *Killing form*

$$\langle \cdot, \cdot \rangle: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}, \quad (x, y) \mapsto \text{Tr}(\text{ad } x \circ \text{ad } y).$$

Using the vector space isomorphism $\mathfrak{h} \rightarrow \mathfrak{h}^*$, we obtain the induced pairing on roots in \mathfrak{h}^* . If $i \neq j$, then $a_{ij} \leq 0$ and $a_{ii} = 2$, note $\det(A) \neq 0$. A finite-dimensional semisimple Lie algebra is determined up to isomorphism by its Cartan matrix.

Finally, we define the *root lattice* Q and the *weight lattice* P of a semisimple Lie algebra \mathfrak{g} . We set

$$Q = \mathbb{Z}\langle \alpha \mid \alpha \text{ is a root} \rangle \quad \text{and,}$$

$$P = \left\{ \lambda \in \mathfrak{h}^* \mid \lambda \left(\frac{2\alpha}{\langle \alpha, \alpha \rangle} \right) \in \mathbb{Z} \quad \forall \alpha \right\}.$$

The weight lattice P is generated by the *fundamental weights* ω_i , characterized by

$$\omega_i \left(\frac{2\alpha_j}{\langle \alpha_j, \alpha_j \rangle} \right) = \delta_{ij},$$

for the fixed set of simple roots. We call a weight *integral* if

$$\lambda \left(\frac{2\alpha}{\langle \alpha, \alpha \rangle} \right) \in \mathbb{N}_0.$$

We call an integral weight *dominant* if it is a non-negative integer combination of fundamental weights.

The root lattice Q is contained in the weight lattice P . This yields a useful connection to its underlying Lie group. The Pontryagin dual of the maximal torus T , which is the maximal connected compact abelian Lie subgroup, whose Lie algebra is a Cartan subalgebra, is naturally identified with P , i.e.

$$\widehat{T} = P.$$

If moreover G is simply connected as a Lie group, then we have an isomorphism

$$\widehat{Z(G)} \cong P/Q.$$

As a final Lie-theoretic object, let us briefly introduce the so-called *Weyl group*. For a root α define

$$s_\alpha: \mathfrak{h} \rightarrow \mathfrak{h}, \quad x \mapsto x - \frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha,$$

where we use the identification of \mathfrak{h} and \mathfrak{h}^* . The map s_α is called *Weyl reflection*. The group W generated by these reflections is called *Weyl group*. Equivalently one can also define it as the quotient of the normalizer of the maximal torus T in G by the torus T , i.e.

$$W = N_G(T)/T.$$

There exists a *longest element* w_0 in W , which reflects all roots, i.e. it sends all positive roots to negative ones.

Before defining the *Drinfeld-Jimbo q -deformation*, we introduce the *universal enveloping Lie algebra* $U(\mathfrak{g})$. Since a Lie algebra \mathfrak{g} is only equipped with a non-associative Lie bracket, it does not come with an associative multiplication. The universal enveloping algebra provides an associative algebra associated to \mathfrak{g} .

Definition 2.20 (Universal enveloping algebra): Let \mathfrak{g} be a semisimple Lie algebra with fixed simple roots $\alpha_1, \dots, \alpha_l$ and Cartan matrix A . The *universal enveloping*

algebra $U(\mathfrak{g})$ is the algebra generated by $E_i = E_{\alpha_i}$, $F_i = E_{-\alpha_i}$ and $H_i = [E_i, F_i]$ for $1 \leq i \leq l$ with the following relations

$$\begin{aligned} [H_i, H_j] &= 0, & [E_i, F_j] &= 0 \text{ for } i \neq j, \\ [H_i, E_j] &= a_{ij}E_j, & [H_i, F_j] &= -a_{ij}F_j \\ (\text{ad } E_i)^{1-a_{ij}} E_j &= 0, & (\text{ad } F_i)^{1-a_{ij}} F_j &= 0, \text{ for } i \neq j. \end{aligned}$$

It is also possible and perhaps more intuitive to define $U(\mathfrak{g})$ as the quotient of the tensor algebra $T(\mathfrak{g}) = \mathbb{C} \oplus \bigoplus_{k \geq 1} \mathfrak{g}^{\otimes k}$ by the two-sided ideal $\langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$. Serre, [Ser66], showed that $U(\mathfrak{g})$ admits the above presentation by generators and relations, which we take as our definition. Moreover the algebra $U(\mathfrak{g})$ carries a natural Hopf $*$ -algebra structure.

Before turning to the quantized version, let us make a brief ‘‘philosophical’’ remark. Compact Lie groups are known to be *rigid*, in the sense that they admit no non-trivial deformations. This is a result due to Nijenhuis and Richardson, [NR67]. Since compact groups are inverse limits of compact Lie groups, the same holds for compact groups in general. Thus, rather than deforming the group (or the Lie algebra) itself, one should instead deform an associated algebraic object such as the group algebra or the universal enveloping algebra, which both can be deformed within the category of Hopf $*$ -algebras. This point of view is similar to the Gelfand-Naimark philosophy.

Definition 2.21 (Drinfeld-Jimbo algebra): Let \mathfrak{g} be a finite-dimensional semisimple Lie algebra of rank l with a fixed set of simple roots $\alpha_1, \dots, \alpha_l$ and Cartan matrix A . Fix $q > 0$ and $q \neq 1$, set $d_i := \langle \alpha_i, \alpha_i \rangle / 2$ and $q_i := q^{d_i}$. The *Drinfeld-Jimbo algebra* or *quantized universal enveloping algebra* $U_q(\mathfrak{g})$ is the algebra generated by elements E_i, F_i, K_i and K_i^{-1} for $1 \leq i \leq l$ such that the following relations hold

$$\begin{aligned} K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1, & K_i E_j K_i^{-1} &= q_i^{a_{ij}} E_j, \\ K_i F_j K_i^{-1} &= q_i^{-a_{ij}} F_j, & E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r &= 0 = \sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r, \end{aligned}$$

where

$$\binom{n}{r}_q = \frac{[n]_q!}{[r]_q! [n-r]_q!} \quad \text{and} \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

defines the *q-binomial coefficient*.

These relations are motivated by those of the universal enveloping algebra. The last two relations are the *q-analogues* (the *q-Serre relations*) of the Serre relations in

the definition of $U(\mathfrak{g})$. Indeed one checks that

$$(\text{ad } E_i)^{1-a_{ij}} E_j = \sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0.$$

An analogous identity holds for F_i .

Proposition 2.22: *There is a unique Hopf*-algebra structure on $U_q(\mathfrak{g})$ with comultiplication $\widehat{\Delta}_q$ and antipode \widehat{S}_q determined on generators by*

$$\begin{aligned} \widehat{\Delta}_q(K_i) &= K_i \otimes K_i, & \widehat{\Delta}_q(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \widehat{\Delta}_q(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i \\ \widehat{S}_q(K_i) &= K_i^{-1}, & \widehat{S}_q(E_i) &= -K_i^{-1} E_i, & \widehat{S}_q(F_i) &= -F_i K_i. \end{aligned}$$

We now construct a compact quantum group from this data. For this we need to find a suitable rigid C^* -tensor category, namely the category of *admissible* $U_q(\mathfrak{g})$ -modules.

Definition 2.23: A $U_q(\mathfrak{g})$ -module V is called *admissible* if it admits a weight decomposition

$$V = \bigoplus_{\lambda \in P} V(\lambda)$$

with

$$V(\lambda) = \{x \in V \mid K_i x = q_i^{\lambda(i)} x \text{ for all } i\},$$

where $\lambda(i)$ is the i -th coefficient of λ in the basis of fundamental weights $\omega_1, \dots, \omega_r$ of the weight lattice P .

We denote by $\mathcal{C}_q(\mathfrak{g})$ the C^* -category of finite-dimensional admissible $U_q(\mathfrak{g})$ -modules.

Remark 2.24: In the representation theory of semisimple Lie algebras, the *highest weight theorem* says that irreducible finite-dimensional representations (equivalently, simple modules) are classified by their *highest weights*. We order weights by saying $\lambda \geq \mu$ if $\lambda - \mu$ is a non-negative integer linear combination of positive roots. The theorem then asserts that every irreducible representation has a (unique) highest weight, which is integral and dominant, two irreducible representations with the same highest weight are isomorphic, and conversely every integral dominant weight occurs as the highest weight of some irreducible representation. For details, see [Hal15]. This also works in the case of $U_q(\mathfrak{g})$.

Lemma 2.25: *The C^* -category of finite-dimensional admissible $U_q(\mathfrak{g})$ -modules is a rigid C^* -tensor subcategory of $\text{Rep } U_q(\mathfrak{g})$.*

Therefore by Theorem 2.19, we obtain a compact quantum group.

Definition 2.26 (Drinfeld-Jimbo quantum group): Let $q > 0$, $q \neq 1$. The *Drinfeld-Jimbo q -deformation* of a simply connected semisimple compact Lie group G is the compact quantum group G_q with Hopf $*$ -algebra $(\mathbb{C}[G_q], \Delta_q)$, where $\mathbb{C}[G_q]$ is the subspace of $U_q(\mathfrak{g})^*$ spanned by the matrix coefficients of all finite dimensional admissible unitary modules. The multiplication, comultiplication and involution are given by

$$(ab)(\omega) = (a \otimes b)\widehat{\Delta}_q(\omega), \quad \Delta_q(a)(\omega \otimes \nu) = a(\omega\nu), \quad a^*(\omega) = \overline{a(\widehat{S}_q(\omega)^*)}.$$

Let us note that the C^* -tensor categories $\text{Rep}(G_q)$ and $\text{Rep}(G)$ need not be monoidally equivalent. One way to see this is to compare the intrinsic dimensions of simple objects with the following lemma, see [NT13].

Lemma 2.27: *Let G be a simply connected semisimple compact Lie group and $q > 0$, then the Woronowicz character f_1 is given by $f_1 = q^{-2\rho^*}$, where ρ^* is half the sum of the simple positive roots.*

Remark 2.28: Let $q > 0$ and let G be a simply connected semisimple compact Lie group, then G_q and $G_{q^{-1}}$ are isomorphic since $U_q(\mathfrak{g})$ and $U_{q^{-1}}(\mathfrak{g})$ are isomorphic as Hopf $*$ -algebras. Concretely, an isomorphism is given by

$$\tilde{E}_i \mapsto q_i^{-1} E_i^*, \quad \tilde{F}_i \mapsto q_i F_i^*, \quad \tilde{K}_i \mapsto K_i,$$

where E, F, K are the generators of $U_q(\mathfrak{g})$ and $\tilde{E}, \tilde{F}, \tilde{K}$ are the generators of $U_{q^{-1}}(\mathfrak{g})$.

Before considering a concrete example, namely $SU_q(2)$, let us briefly place the work of Drinfeld and Jimbo in a historical context.

Remark 2.29: In the 1980s Drinfeld-Jimbo quantum groups emerged from close interactions between mathematical physics and quantum mechanics, in particular through the study of integrable systems in statistical mechanics and QFT. Drinfeld's approach was motivated by the study of quantum Yang-Baxter equation and R -matrices. Drinfeld worked with quantized Lie algebras, while Jimbo independently introduced q -deformations of universal enveloping algebras via symmetries appearing in integrable systems. Both physical starting points are closely connected to the *quantum inverse scattering method*, which is an algebraic method for solving integrable systems in $1+1$ dimensions. A standard example is the Heisenberg-spin-1/2-chain, describing interacting spins on a one-dimensional lattice chain. Given a suitable R -matrix in this physical setting solving the quantum Yang-Baxter equation, the quantum inverse scattering method produces a matrix whose entries generate a Drinfeld-Jimbo quantum group, equivalently the algebra generated by these entries is a Drinfeld-Jimbo quantum group. See [TF79; Fad98] for further details on the physical aspects.

Example 2.30 ($SU_q(2)$): Consider the Lie group $SU(2)$ and its corresponding Lie algebra $\mathfrak{sl}(2)$. We verify that $SU_q(2)$ defined above coincides with the one from Example 1.2. In this example the latter will be denoted as $\widetilde{SU_q(2)}$. We will show that the corresponding Hopf *-algebras are isomorphic.

Since $\mathfrak{sl}(2)$ has rank 1, we work with the generators K, E, F . For Lie algebras of type A_N , see also Chapter IV, which are Lie algebras of the form $\mathfrak{sl}(N+1)$, it suffices to consider the fundamental corepresentation (ρ, V) , here $V = \mathbb{C}^2$ and

$$\rho(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad \rho(E) = \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix}, \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix},$$

see [Ros90].

The contragredient representation ρ^c defined by $\rho^c(x) = \rho(\widehat{S}_q(x))^t$, is given by

$$\rho^c(K) = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}, \quad \rho^c(E) = - \begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix}, \quad \rho^c(F) = - \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix}.$$

The matrix

$$J = \begin{pmatrix} 0 & -q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix}$$

intertwines ρ and ρ^c . If now we let

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}[SU_q(2)]).$$

From the equation $JU^c = UJ$, where U^c was defined by taking the adjoints entrywise, we obtain $d = a^*$ and $b = -qc^*$. Hence by this we obtain a surjective Hopf *-algebra homomorphism

$$\mathbb{C}[\widetilde{SU_q(2)}] \rightarrow \mathbb{C}[SU_q(2)], \quad \alpha \mapsto a, \quad \gamma \mapsto c.$$

Injectivity follows by considering monomials in $\alpha, \alpha^*, \gamma, \gamma^*$. Therefore Woronowicz's $SU_q(2)$ agrees with Drinfeld's and Jimbo's $SU_q(2)$.

Similar arguments apply to $SU_q(N)$.

Example 2.31 ($SU_q(N)$): We describe the Hopf *-algebra $\mathbb{C}[SU_q(N)]$. It is generated by the matrix entries of $U = (u_{i,j})_{1 \leq i,j \leq N}$ satisfying the relations

$$\begin{aligned} u_{ik}u_{jk} &= qu_{jk}u_{ik}, & u_{ki}u_{kj} &= qu_{kj}u_{ki}, & \text{for } i < j, \\ u_{il}u_{jk} &= u_{jk}u_{il}, & & & \text{for } i < j, k < l, \\ u_{ik}u_{jl} - u_{jl}u_{ik} &= (q - q^{-1})u_{jk}u_{il}, & & & \text{for } i < j, k < l, \\ \det_q(U) &= 1, \\ u_{ij}^* &= (-q)^{j-i} \det_q(U_{i,j}), \end{aligned}$$

where $U_{i,j}$ denotes the minor matrix, obtained by deleting the i -th row and the j -th column, and

$$\det_q(U) = \sum_{\sigma \in S_N} (-q)^{l(\sigma)} u_{\sigma(1)1} \cdots u_{\sigma(N)N}.$$

Here $l(\sigma)$ is the number of inversions of σ , i.e. the number of pairs $i < j$ with $\sigma(i) > \sigma(j)$.

The matrix U is the fundamental representation of $SU_q(N)$. If $q = 1$, we recover the classical Hopf $*$ -algebra $\mathbb{C}[SU(N)]$. Indeed, all generators commute, \det_q reduces to the usual determinant, and the last relation becomes $U^{-1} = U^*$.

Having defined the Drinfeld-Jimbo q -deformation, we can now introduce *twisted q -deformations* of a simply connected semisimple compact Lie group G .

We first define twisting on $\text{Rep}(\mathbb{G})$ by an invariant 3-cocycle $\Phi \in H^3(\widehat{\mathbb{G}}, \mathbb{T})$ for some compact quantum group \mathbb{G} . Letting Φ act on $\mathcal{H}_U \otimes \mathcal{H}_V \otimes \mathcal{H}_W$ by multiplication, we obtain a new associator $(U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$ on the representation category of $\text{Rep}(\mathbb{G})$. This yields a new C^* -tensor category, which we will denote by $(\text{Rep}(\mathbb{G}), \Phi)$. Since this C^* -tensor category will not necessarily correspond to a compact quantum group, we need to do some assumptions. Assume that $(\text{Rep}(\mathbb{G}), \Phi)$ is rigid and that there exists a unitary 2-cocycle $F \in H^2(\widehat{\mathbb{G}}, \mathbb{T})$ such that $\Phi = \partial F$ that is

$$\Phi = (\text{id} \otimes \widehat{\Delta})(F^*)(1 \otimes F^*)(F \otimes 1)(\widehat{\Delta} \otimes 1)(F).$$

As in Theorem 2.19, we can construct a compact quantum group \mathbb{G}_F . Let $\psi: (\text{Rep}(\mathbb{G}), \Phi) \rightarrow \text{Rep}(\mathbb{G}_F)$ be the forgetful functor, which is the identity on objects and morphisms, and on the tensor structure $\psi(U) \otimes \psi(V) \rightarrow \psi(U \otimes V)$ it is given by the action of F^* , namely

$$\mathcal{H}_U \otimes \mathcal{H}_V \rightarrow \mathcal{H}_U \otimes \mathcal{H}_V, \quad x \otimes y \mapsto F^*(x \otimes y).$$

This tensor functor is a unitary monoidal equivalence, see [NY15]. Intuitively twisting associativity on the categorical side corresponds to twisting the comultiplication on the algebraic side via F .

We now turn to cochain twistings of q -deformations, which we call twists of q -deformations, and for this we consider and construct \mathbb{T} -valued 2-cochains on $P = \widehat{T}$.

Let G be a simply connected semisimple compact Lie group. We are interested in associators Φ that arise as 3-cocycles on $\widehat{Z}(G) = P/Q$. The reason why we only consider these associators will be seen in Chapter IV. For this we need to explain how to lift such 3-cocycles and we need to construct a suitable extension of $U_q(\mathfrak{g})$.

Definition 2.32 (Extended $U_q(\mathfrak{g})$): Let G be a simply connected semisimple compact Lie group, and let $U_q(\mathfrak{g})$ be its Drinfeld-Jimbo algebra. The extended Drinfeld-Jimbo

algebra $\tilde{U}_q(\mathfrak{g})$ is the universal $*$ -algebra generated by $U_q(\mathfrak{g})$ together with unitary central elements C_1, \dots, C_r , where r is the rank of \mathfrak{g} . It also carries a Hopf $*$ -structure, and there is a Hopf $*$ -algebra homomorphism $\tilde{U}_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ sending $C_i \mapsto 1$ and restricting to the identity on the remaining generators.

For $U_q(\mathfrak{g})$ we introduced the notion of admissibility, for the extension we need the notion of τ -admissibility, where $\tau = (\tau_1, \dots, \tau_r) \in Z(G)^r$.

Definition 2.33: Let $\tau = (\tau_1, \dots, \tau_r) \in Z(G)^r$ be a r -tuple. A representation of $\tilde{U}_q(\mathfrak{g})$ is called τ -admissible, if its restriction on $U_q(\mathfrak{g})$ is admissible and each central element C_i acts on $V(\lambda)$ by the scalar $\langle \tau, \lambda \rangle$.

As in Definition 2.26 and Theorem 2.19 we can define the compact quantum group G_q^τ .

We can identify $\mathcal{U}(G_q) \cong \mathcal{U}(G_q^\tau)$. Indeed every admissible representation of $U_q(\mathfrak{g})$ extends uniquely to a τ -admissible representation on $\tilde{U}_q(\mathfrak{g})$ and conversely every τ -admissible representation of $\tilde{U}_q(\mathfrak{g})$ arises in this way. Under this identification let $U_q^\tau(\mathfrak{g})$ denote the image of $\tilde{U}_q(\mathfrak{g})$. This $*$ -algebra is generated by $E_i, F_i, K_i, K_i^{-1}, \tau_i$, and it carries a deformed comultiplication and can be seen as a quotient of $\tilde{U}_q(\mathfrak{g})$ by the relations identifying the set of central elements C_1, \dots, C_r with τ_1, \dots, τ_r .

We now explain how to lift τ in some appropriate sense to a 3-cocycle on $\widehat{Z(G)}$. We consider the following \mathbb{T} -valued map, a 2-cochain, on $P \times P$

$$c_\tau: P \times P \rightarrow \mathbb{T}, \text{ such that } c_\tau(\lambda, \mu + Q) = c_\tau(\lambda, \mu), \quad c_\tau(\lambda + \alpha_i, \mu) = \langle \tau_i, \mu \rangle c_\tau(\lambda, \mu).$$

One then can compute that

$$\partial c_\tau(\lambda, \mu, \nu) = c_\tau(\lambda, \mu) c_\tau(\lambda + \mu, \nu)^{-1} c_\tau(\lambda, \mu + \nu) c_\tau(\lambda, \mu)^{-1},$$

which is invariant in each component under translation by Q . Hence we may identify ∂c_τ as 3-cocycle on $P/Q = \widehat{Z(G)}$. Moreover its cohomology class only depends on τ and also the twisted coproduct $\widehat{\Delta}_{c_\tau} = c_\tau \widehat{\Delta}_q c_\tau^*$ is independent on the choice of c_τ . We also have $(\partial c_\tau)^* \in \mathcal{U}(Z(G)^3)$ and can therefore be seen as an invariant 3-cocycle Φ^τ , with

$$\Phi^c = (\text{id} \otimes \widehat{\Delta}_q)(c^*)(1 \otimes c^*)(c \otimes 1)(\widehat{\Delta}_q \otimes \text{id})(c),$$

where $c := c_\tau$, as already discussed below Theorem 2.19. This defines a group homomorphism

$$\Theta: Z(G)^r \rightarrow H^3(P/Q, \mathbb{T})$$

by sending τ to the cohomology class of Φ^τ in $H^3(P/Q, \mathbb{T})$. Finally for τ, ν we have $G_q^\tau \cong G_q^\nu$ if and only if

$$\prod_{i=1}^{n-1} \tau_i^i = \prod_{i=1}^{n-1} \nu_i^i. \quad (*)$$

For more details, including the following two propositions, see [NY15].

Proposition 2.34: *The comultiplications $\widehat{\Delta}_f$ on $\mathcal{U}(G_q)$ and $\widehat{\Delta}$ on $U_q^\tau(\mathfrak{g})$ coincide. Moreover $\text{Rep}(G_q^\tau)$ is unitarily monoidally equivalent to $(\text{Rep}(G_q), \Phi^\tau)$.*

Proposition 2.35: *Let G be a simply connected semisimple compact Lie group then every 3-cocycle $\Phi \in H^3(P/Q, \mathbb{T})$ is cohomologous to Φ^τ for $\tau \in Z(G)^r$ if and only if Φ lifts to a coboundary of P .*

If $P/Q = \widehat{Z(G)}$ is generated by at most two elements, then every 3-cocycle $\Phi \in H^3(P/Q, \mathbb{T})$ is cohomologous to Φ^τ for $\tau \in Z(G)^r$. Hence the compact quantum groups G_q^τ exhaust all associators of $\text{Rep} G_q$ in this case.

Remark 2.36: Note that in the case $G = SU(N)$ one has

$$P/Q = \widehat{Z(G)} \cong \mathbb{Z}/N\mathbb{Z}$$

and therefore

$$H^3(P/Q, \mathbb{T}) = H^3(\mathbb{Z}/N\mathbb{Z}, \mathbb{T}) \cong \mathbb{Z}/N\mathbb{Z}.$$

Via this identification, we can explicitly define a generator ϕ by

$$\phi(a, b, c) = \omega_N^{\alpha_N(a,b)c}, \quad \alpha_N(a, b) = \left\lfloor \frac{a+b}{N} \right\rfloor - \left\lfloor \frac{a}{N} \right\rfloor - \left\lfloor \frac{b}{N} \right\rfloor,$$

where $\omega_N = e^{2\pi i/N}$, hence we may identify $H^3(P/Q, \mathbb{T})$ with the group of units of order N . Since $Z(G) = \mathbb{Z}/N\mathbb{Z}$ we can also identify it with the group of N -th roots of unity and thus $\tau \in Z(G)^{N-1}$ as a tuple of N -th roots of unity.

Let us also consider 2-cochains c , which we want to decompose, see [NY16, below Theorem 5.1].

For this it is useful to enlarge the maximal torus T of $SU(N)$ by the maximal torus \mathbb{T}^{N-1} of $U(N)$, and we view T as $T = \mathbb{T}^{N-1} \cap SU(N)$. Then 2-cocycles on \widehat{T} can be lifted to $\widehat{\mathbb{T}^{N-1}}$, and can be seen (up to coboundaries) as skew-symmetric bicharacters $\omega: \mathbb{Z}^N \times \mathbb{Z}^N \rightarrow \mathbb{T}$ as the dual of \mathbb{T}^{N-1} is given as \mathbb{Z}^N , which needs to satisfy $\omega(L_1 + \dots + L_N, x) = 1$ for all $x \in \mathbb{Z}^N$, where (L_i) is the base of \mathbb{Z}^N dual to the base of the Lie algebra of diagonal matrices. The matrix $(\omega_{ij}) \in M_N$ defined by $\omega_{ij} = \omega(L_i, L_j)$ satisfies $\omega_{ii} = 1$, $\omega_{ij} = \overline{\omega_{ji}}$ and $\prod_i \omega_{ij} = 1$ for every $1 \leq j \leq N$. Two matrices (ω_{ij}) and $(\tilde{\omega}_{ij})$ represent the same element of $H^2(P, \mathbb{T})$ if and only if $\omega_{ij}^2 = \tilde{\omega}_{ij}^2$ for all $1 \leq i, j \leq N$.

Now fix 2-cochains c_1, \dots, c_N , such that $\partial c_1, \dots, \partial c_N$ exhaust $H^3(P/Q, \mathbb{T})$. Let the simple roots of $SU(N)$ be given as $\alpha_i = L_i - L_{i+1}$ and define $|\cdot|: P \rightarrow \mathbb{Z}$ via $|L_1| = N - 1$ and $|L_i| = -1$ otherwise, then the cochain c_τ for a $N - 1$ -tuple τ of N -th roots of unity fulfils

$$c_\tau(\lambda, \mu + \alpha_i) = c_\tau(\lambda + \mu), \quad c_\tau(\lambda + \alpha_i, \mu) = \langle \tau_i, \mu \rangle c_\tau(\lambda, \mu) = \tau_i^{|\mu|} c_\tau(\lambda, \mu).$$

By (*) we may choose $c_k := c_{\tau_k}$ with

$$\tau_k = (e^{2\pi(k-1)i/N}, 1, \dots, 1).$$

Then we can first twist $SU_q(N)$ by c_k and then we can twist it by the skew-symmetric bicharacter ω . In some sense the bicharacter ω encodes the difference of c and a suitable c_k , to obtain $c = c_k\omega$. See also Corollary 4.19.

Chapter III.

The Poisson Boundary and Factorisation of Fiber Functors

In this chapter we aim to construct the Poisson boundary of a C^* -tensor category, generalising the Poisson boundary for compact quantum groups, see [Izu04]. The Poisson boundary will be useful for determining the dimension-preserving unitary fiber functors, since these can be factorised through the maximal Kac quantum subgroup of a compact quantum group.

Throughout this chapter let \mathcal{C} be a strict rigid C^* -tensor category, in particular closed under finite direct sums and subobjects, and such that $\text{Irr}(\mathcal{C})$ is at most countable.

This chapter is mainly based on [NY17a] and [NY16].

1. Poisson Boundary

In this section we briefly introduce the construction of the classical Poisson boundary for groups. We then generalise it to the categorical Poisson boundary of a rigid C^* -tensor category, and prove its *universality*.

Let G be a discrete group, and let μ be a probability measure on G . A function $f: G \rightarrow \mathbb{C}$ is called μ -harmonic if

$$f(g) = P^\mu f(g) := \sum_{h \in G} \mu(h) f(gh), \quad \text{for all } g \in G.$$

We call P^μ the *Markov operator* and we denote by $H^\infty(G, \mu)$ the space of bounded (with respect to the sup-norm) μ -harmonic functions. Endowed with a suitable product and involution, this space is a unital commutative C^* -algebra, hence it is isomorphic to $C(\Pi_\mu)$ for some compact space Π_μ . Moreover there exists a measure ν on Π_μ such that P^ν defines an isomorphism $C(\Pi_\mu) \rightarrow H^\infty(G, \mu)$. We call (Π_μ, ν) the Poisson boundary of (G, μ) .

One can also construct the Poisson boundary of (G, μ) via right random walks on (G, μ) , which encode the asymptotic behaviour of the random walk on G . A *(right) random walk* on (G, μ) is a time-discrete Markov chain with transition probability $p(g \rightarrow h) = \mu(g^{-1}h)$, i.e. the position is given by the previous position by right multiplication with a random element in G . The boundary Π_μ can then be realised as quotient of the trajectory space $G^{\mathbb{N}}$ by a suitable equivalence relation.

This Poisson boundary has interesting applications in group theory, such as the proved *Fürstenberg conjecture*; see [Für73]. It states that G is amenable if and only if G possesses a probability measure μ with full support for which no non-trivial μ -boundaries exist. Here, μ -boundaries are quotients of the Poisson boundary. Motivated by this, Izumi [Izu04] defined the non-commutative Poisson boundary for discrete quantum groups. We will not review this construction here, and instead study the Poisson boundary of C^* -tensor categories. Note that an analogue of the Fürstenberg conjecture holds for the Poisson boundary of C^* -tensor categories.

Our starting point is a rigid C^* -tensor category \mathcal{C} and a probability measure μ on the set of isomorphism classes of simple objects $\text{Irr}(\mathcal{C})$. We now need to find a suitable analogue of the Markov operator. For this let us fix some notation.

Notations. We mainly follow the notation in [NY17a]. For an object $U \in \text{Ob}(\mathcal{C})$ we define the functor

$$\text{id} \otimes U: \mathcal{C} \rightarrow \mathcal{C}, \quad X \mapsto X \otimes U.$$

Recall that a *natural transformation* between two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ is a family of morphisms $(\eta_X: F(X) \rightarrow G(X))_X$ such that $\eta_Y \circ F(f) = G(f) \circ \eta_X$ for every $f \in \text{Mor}(X, Y)$ and $X, Y \in \text{Ob}(\mathcal{C})$. We denote the set of natural morphisms between two functors F and G by $\text{Nat}(F, G)$. On the space $\text{Nat}(\text{id} \otimes U, \text{id} \otimes V)$ for objects $U, V \in \text{Ob}(\mathcal{C})$ we may define a linear operator

$$P_X: \text{Nat}(\text{id} \otimes U, \text{id} \otimes V) \rightarrow \text{Nat}(\text{id} \otimes U, \text{id} \otimes V), \quad (\eta_Y) \mapsto (\text{tr}_X \otimes \text{id})(\eta_{X \otimes Y}),$$

where $(\text{tr}_X \otimes \text{id})$ denotes the *partial categorical trace*: Let $U \in \text{Ob}(\mathcal{C})$ and let $(R_U, R_{\overline{U}})$ be a fixed standard solution of the conjugate equations. Then first define the categorical trace

$$\text{tr}_U: \text{End}(U) \rightarrow \mathbb{C}, \quad \text{tr}_U(T) = d^{\mathcal{C}}(U)^{-1} R_U^*(\text{id} \otimes T) R_U.$$

At first we give an example for this, before considering the partial categorical trace.

Example 3.1: In the setting of Example 2.9, for an object \mathcal{H} with dimension $d(\mathcal{H}) = \dim(\mathcal{H}) = n$ and an operator $T \in B(\mathcal{H}) \cong M_n(\mathbb{C})$, we obtain

$$\text{tr}_{\mathcal{H}}(T) = \frac{1}{n} R_{\mathcal{H}}^*(\text{id} \otimes T) R_{\mathcal{H}}(1) = \frac{1}{n} R_{\mathcal{H}}^* \left(\sum_{i=1}^n e_i^* \otimes T e_i \right) = \frac{1}{n} \sum_{i=1}^n \langle T e_i, e_i \rangle = \text{tr}(T),$$

which is the normalised trace on \mathcal{H} .

The abstract categorical trace is tracial. This can be seen by considering string diagrams,

$$\mathrm{tr}_U(ST) = \begin{array}{c} \text{---} \\ \boxed{T} \\ \text{---} \\ \boxed{S} \\ \text{---} \end{array} \bigcirc = \begin{array}{c} \text{---} \\ \boxed{S} \\ \text{---} \\ \boxed{T} \\ \text{---} \end{array} \bigcirc = \mathrm{tr}_U(TS).$$

The diagrams are topologically equivalent by passing S through the circle clockwise. By the mentioned Joyal-Street-Verity theorem, tr_U is tracial. Note that this *spherical property* is in a sense a consequence of the assumed rigidity.

We now define the *partial trace*: for fixed object U , X and Y , set

$$\begin{aligned} \mathrm{tr}_U \otimes \mathrm{id}: \mathrm{Mor}(U \otimes X, U \otimes Y) &\rightarrow \mathrm{Mor}(X, Y), \\ (\mathrm{tr}_U \otimes \mathrm{id})(T) &= d^{\mathcal{C}}(U)^{-1}(R_U^* \otimes \mathrm{id})(\mathrm{id} \otimes T)(R_U \otimes \mathrm{id}). \end{aligned}$$

In the same manner as above, one checks that for $\mathcal{C} = \mathrm{Hilb}_f$, this is indeed the classical normalised partial trace. Both kinds of traces were introduced in [JSV96].

Moreover let

$$\hat{\mathcal{C}}(U, V) = \{(\eta_Y) \in \mathrm{Nat}(\mathrm{id} \otimes U, \mathrm{id} \otimes V) \mid \sup_Y \|\eta_Y\| < \infty\} \subseteq \mathrm{Nat}(\mathrm{id} \otimes U, \mathrm{id} \otimes V).$$

Then $\hat{\mathcal{C}}(U, V)$ is a Banach space and it is isomorphic to the ℓ^∞ -direct sum by choosing representatives U_s for each simple object in $\mathrm{Irr}(\mathcal{C})$

$$\hat{\mathcal{C}}(U, V) \cong \bigoplus_{s \in \mathrm{Irr}(\mathcal{C})} \mathrm{Mor}(U_s \otimes U, U_s \otimes V).$$

This follows from the fact that natural transformations are already determined by their action on simple objects.

The operator P_X defined above is a contraction on this Banach space, and depends only on the isomorphism class of X . Hence we fix representatives (U_s) in $\mathrm{Irr}(\mathcal{C})$ and only consider $P_s := P_{U_s}$. We can define the *Markov operator* by

$$P_\mu = \sum_s \mu(s) P_s,$$

which is again a contraction on $\hat{\mathcal{C}}(U, V)$. We call a natural transformation $\eta \in \hat{\mathcal{C}}(U, V)$ P_μ -*harmonic* if $P_\mu(\eta) = \eta$. Note that both the classical Markov operators and the categorical Markov operators are kind of averaging operators and share similar properties.

The composition of two P_μ -harmonic natural transformations is not necessarily P_μ -harmonic. However we can define the following associative product, which exists by arguments in [Izu12].

Proposition 3.2: *Let $\nu, \eta: \text{id} \otimes U \rightarrow \text{id} \otimes V$ be bounded P_μ -harmonic natural transformations, then*

$$(\nu \cdot \eta)_X := \lim_{n \rightarrow \infty} P_\mu^n(\nu\eta)_X$$

exists for all objects X and defines a bounded P_μ -harmonic natural transformation $\text{id} \otimes U \rightarrow \text{id} \otimes V$. Moreover \cdot is associative.

Before constructing the Poisson boundary of \mathcal{C} , we need a categorical auxiliary construction.

Remark 3.3 (Idempotent completion): Given a C^* -category \mathcal{C} with morphism sets $\text{Mor}(X, Y)$ for objects X, Y , one can construct a new category $\widetilde{\mathcal{C}}_0$ by taking the same objects and enlarging the morphism sets, i.e. we consider morphism sets $\widetilde{\text{Mor}}(X, Y) \supseteq \text{Mor}(X, Y)$. This category is in general not closed under subobjects, since there may be projections $p \in \widetilde{\text{End}}(X)$, that are not contained in $\text{End}(X)$. We apply the *idempotent completion* on it, sometimes called *Karoubi envelope*, by adding projections $p \in \widetilde{\text{End}}(X)$ as objects of $\widetilde{\mathcal{C}}$, and taking $q\text{Mor}(X, Y)p$ as the morphism set from the object $p \in \widetilde{\text{End}}(X)$ to $q \in \widetilde{\text{End}}(Y)$. We denote the resulting C^* -category as $\widetilde{\mathcal{C}}$. Furthermore the natural embedding $\text{Mor}(X, Y) \rightarrow \widetilde{\text{Mor}}(X, Y)$ can be considered as a C^* -functor.

If one works with C^* -tensor categories, one first needs to define a suitable tensor bifunctor

$$\otimes: \widetilde{\mathcal{C}}_0 \times \widetilde{\mathcal{C}}_0 \rightarrow \widetilde{\mathcal{C}}_0,$$

before performing the completion. In this case the idempotent completion preserves rigidity.

Construction of the Poisson boundary. In this spirit, let \mathcal{P}_0 be the category with the same objects as \mathcal{C} and let the morphism set $\widetilde{\text{Mor}}(U, V)$ be the space of bounded P_μ -harmonic natural transformations $\eta: \text{id} \otimes U \rightarrow \text{id} \otimes V$, and with composition given by the product in Proposition 3.2 as composition. Note that $\text{Mor}(U, V) \subseteq \widetilde{\text{Mor}}(U, V)$, indeed for any morphism $T \in \text{Mor}(U, V)$ the natural transformation $(\text{id}_Y \otimes T)_Y$ is contained in $\text{Nat}(\text{id} \otimes U, \text{id} \otimes V)$, is bounded and is P_μ -harmonic, since

$$\begin{aligned} P_X(\text{id}_Y \otimes T)_Y &= (\text{tr}_X \otimes \text{id})(\text{id}_{X \otimes Y} \otimes T) \\ &= (\text{tr}_X \otimes \text{id})(\text{id}_X \otimes (\text{id}_Y \otimes T)) = \text{id}_Y \otimes T \end{aligned}$$

for all X , and hence the same holds for P_μ .

The tensor product on objects is the same as in \mathcal{C} , but we need a different tensor product on morphisms.

Let $\eta: \text{id} \otimes U \rightarrow \text{id} \otimes V$ and $\nu: \text{id} \otimes W \rightarrow \text{id} \otimes Z$ be bounded P_μ -harmonic natural transformations and define

$$\eta \otimes \nu = (\eta \otimes \text{id}_Z) \cdot (\text{id}_U \otimes \nu),$$

where

$$\begin{aligned} \eta \otimes \text{id}_Z: \text{id} \otimes U \otimes Z &\rightarrow \text{id} \otimes V \otimes Z, & \text{with } (\eta \otimes \text{id}_Z)_X &= \eta_X \otimes \text{id}_Z, \\ \text{id}_U \otimes \nu: \text{id} \otimes U \otimes W &\rightarrow \text{id} \otimes U \otimes Z, & \text{with } (\text{id}_U \otimes \nu)_X &= \nu_{X \otimes U}. \end{aligned}$$

By construction clearly $\eta \otimes \text{id}_Z$ and $\text{id}_U \otimes \nu$ are P_μ -harmonic. Moreover by naturality of η we have $(\eta \otimes \nu) = (\text{id} \otimes \nu) \cdot (\eta \otimes \text{id})$. Thus $\otimes: \mathcal{P}_0 \times \mathcal{P}_0 \rightarrow \mathcal{P}_0$ is a bifunctor.

Applying the idempotent completion to \mathcal{P}_0 , which yields a strict rigid C^* -tensor category \mathcal{P} . Finally define

$$\Pi: \mathcal{C} \rightarrow \mathcal{P}, \quad \text{identity on objects and on morphisms } T \mapsto (\text{id}_X \otimes T)_X.$$

This is a well-defined unitary tensor functor.

Definition 3.4 (Poisson boundary): Let \mathcal{C} be a rigid C^* -tensor category and let μ be a probability measure. The pair (\mathcal{P}, Π) , constructed above, is called the *Poisson boundary* of (\mathcal{C}, μ) . The Poisson boundary (\mathcal{P}, Π) is called *trivial*, if the only bounded P_μ -harmonic natural transformations $\text{id} \otimes U \rightarrow \text{id} \otimes V$ are transformations of the form $(\text{id}_X \otimes T)_X$ for some $T \in \text{Mor}(U, V)$.

The Poisson boundary has a universal property, which we will discuss after the next example. Note that it is also possible to give concrete realisations of Poisson boundaries. Since this is not necessary for this thesis, we refer to [NY17a].

Example 3.5: Consider the tensor unit object I and in particular the algebra $\text{End}(I)$. The functor induced by I is simply

$$\text{id} \otimes I: \mathcal{C} \rightarrow \mathcal{C}, \quad X \mapsto X \otimes I = X,$$

by strictness. Thus we are looking at natural transformations from the identity functor to itself, i.e. a family of endomorphisms $\eta_X: X \rightarrow X$.

For simple objects $U_s \in \text{Irr}(\mathcal{C})$, Schur's lemma gives $\text{End}(U_s) = \mathbb{C} \text{id}$ and hence

$$\eta_{U_s} = \lambda_s \text{id}_{U_s}, \quad \text{for some } \lambda_s \in \mathbb{C}.$$

Therefore η is completely determined by a bounded function

$$\lambda: \text{Irr}(\mathcal{C}) \rightarrow \mathbb{C}, \quad [U_s] \mapsto \lambda_s,$$

and we may view $\text{End}(I)$ as a subspace of $\ell^\infty(\text{Irr}(\mathcal{C}))$. Moreover

$$P_{U_s}(\eta)_{U_t} = (\text{tr}_{U_t} \otimes \text{id})(\eta_{U_t \otimes U_s}).$$

As in Definition 2.13, we can decompose

$$U_t \otimes U_s = \bigoplus_r N_{t,s}^r U_r$$

with fusion coefficients $N_{t,s}^r$ and since η acts on U_r as $\lambda_r \text{id}$ we obtain

$$(P_{U_s} \lambda)(U_t) = \sum_r \frac{N_{t,s}^r d(U_t)}{d(U_s) d(U_r)} \lambda(U_r).$$

Consequently

$$(P_\mu \lambda)(U_t) = \sum_s \mu(U_s) \sum_r \frac{N_{t,s}^r d(U_t)}{d(U_s) d(U_r)} \lambda(U_r).$$

If we set

$$p_\mu(U_t \rightarrow U_r) := \sum_s \mu(U_s) \frac{N_{t,s}^r d(U_t)}{d(U_s) d(U_r)},$$

then the Markov operator is of the form

$$(P_\mu \lambda)(U_t) = \sum_r p_\mu(U_t \rightarrow U_r) \lambda(U_r).$$

This corresponds to a classical random walk on $\text{Irr}(\mathcal{C})$ and hence $\text{End}(I)$ is determined by this random walk.

If the tensor unit of \mathcal{P} is simple with respect to the probability measure μ , then we call μ *ergodic*.

We need to introduce (weakly) amenable C^* -tensor categories, for the universality theorem. Amenability implies also the existence of an ergodic probability measure.

Definition 3.6: We call \mathcal{C} (weakly) amenable if there exist a left invariant mean on $\ell^\infty(\text{Irr}(\mathcal{C}))$, i.e. a state m such that $m(P_s(f)) = m(f)$ for all $f \in \ell^\infty(\text{Irr}(\mathcal{C}))$ and all $U_s \in \text{Irr}(\mathcal{C})$.

Proposition 3.7: \mathcal{C} is weakly amenable if and only if there exists an ergodic probability measure μ .

We now prove the universality of the Poisson boundary.

Theorem 3.8: Let \mathcal{C} be a weakly amenable strict C^* -tensor category and μ be an ergodic probability measure and let (\mathcal{P}, Π) be the corresponding Poisson boundary. Then the functor Π realises the minimal intrinsic dimension function, i.e.

$$d_{\min}^{\mathcal{C}}(U) = d^{\mathcal{P}}(\Pi(U))$$

for all objects U in \mathcal{C} , where

$$d_{\min}^{\mathcal{C}}(U) := \inf d^{\mathcal{A}}(F(U)),$$

and the infimum is taken over all pairs (\mathcal{A}, F) consisting of a rigid C^* -tensor category \mathcal{A} and a unitary tensor functor $F: \mathcal{C} \rightarrow \mathcal{A}$.

Moreover (\mathcal{P}, Π) is universal in the following sense that if \mathcal{A} is another rigid C^* -tensor category and unitary tensor functor $F: \mathcal{C} \rightarrow \mathcal{A}$, such that $d_{\min}^{\mathcal{C}} = d^{\mathcal{A}}F$, then there exists a unique up to a natural unitary monoidal isomorphism, unitary tensor functor $\Lambda: \mathcal{P} \rightarrow \mathcal{A}$ such that $\Lambda\Pi \cong F$. Finally the resulting Poisson boundary does not depend on the choice of ergodic probability measure.

We only sketch the proof. For the following we have to work with morphism spaces in different categories. For a category \mathcal{C} we write $\mathcal{C}(U, V) := \text{Mor}(U, V)$ and $\mathcal{C}(U) := \mathcal{C}(U, U)$.

The following two lemmas are needed.

Lemma 3.9: *Let \mathcal{A} be a strict rigid C^* -tensor category and let $\mathcal{C} \subseteq \mathcal{A}$ be a C^* -tensor subcategory. For every object U in \mathcal{C} there exists a unique invertible element $a_U \in \mathcal{A}(U)$ such that*

$$(\text{id} \otimes a_U^{1/2})R_U \quad \text{and} \quad (a_U^{-1/2} \otimes \text{id})\bar{R}_U$$

form a standard solution of the conjugate equations of U in \mathcal{A} , where (R_U, \bar{R}_U) is a solution of the conjugate equations of U in \mathcal{C} .

Proof. Recall from Proposition 2.8 that there exists an invertible morphism $T \in \mathcal{A}(U)$ such that $(\text{id} \otimes T)R_U$ and $((T^*)^{-1} \otimes \text{id})\bar{R}_U$ is a standard solution. By polar decomposition we can write $T = v|T| = va_U^{1/2}$ for some unitary v . Then

$$\begin{aligned} (\text{id} \otimes T)R_U &= (\text{id} \otimes v)(\text{id} \otimes a_U^{1/2})R_U, \\ ((T^*)^{-1} \otimes \text{id})\bar{R}_U &= (v \otimes \text{id})(a_U^{-1/2} \otimes \text{id})\bar{R}_U. \end{aligned}$$

Standard solutions are invariant under multiplication by a unitary. Hence $(\text{id} \otimes a_U^{1/2})R_U$ and $(a_U^{-1/2} \otimes \text{id})\bar{R}_U$ are standard solutions.

Any other standard solution is of the form $(\text{id} \otimes ua_U^{1/2})R_U$ and $(ua_U^{-1/2} \otimes \text{id})\bar{R}_U$ for a unitary u . Since the polar decomposition is unique $ua_U^{1/2}$ is positive if and only if $u = 1$. \square

Sketch of the proof of Theorem 3.8. First we may assume that $F: \mathcal{C} \rightarrow \mathcal{A}$ is an embedding functor, hence \mathcal{C} may be regarded as a subcategory of \mathcal{A} .

Step 1. Define a ucp map

$$\Theta_{U,V}: \mathcal{A}(U, V) \rightarrow \mathcal{P}(U, V),$$

so that $\Theta_{U,V}(T)$ is P_X -harmonic for every object X in \mathcal{C} . These maps can be viewed as projections onto the Poisson boundary. In the classical setting they would correspond to

$$\Theta(f) = \lim_{n \rightarrow \infty} n^{-1} \sum_{i=0}^{n-1} P_\mu^i(f),$$

where $\Theta(f)$ is μ -harmonic.

Step 2. Show that $d^{\mathcal{P}}(U) \leq d^{\mathcal{A}}(U)$ for all objects U in \mathcal{C} . In other words, the intrinsic dimension function on \mathcal{P} is the smallest dimension function on \mathcal{C} . Let (R_U, \bar{R}_U) be solutions of the conjugate equations of U . Then consider the solutions

$$r_U = (\text{id} \otimes \Theta_U(a_U)^{1/2})R_U \quad \text{and} \quad \bar{r}_U = (\Theta_U(a_U)^{-1/2} \otimes \text{id})\bar{R}_U.$$

Then

$$\begin{aligned} r_U^* r_U &= R_U^* (\text{id} \otimes \Theta_U(a_U)) R_U = \Theta_U(R_U^* (\text{id} \otimes a_U) R_U) \\ &= \Theta_U\left(\left((\text{id} \otimes a_U^{1/2})R_U\right)^* \left((\text{id} \otimes a_U^{1/2})R_U\right)\right). \end{aligned}$$

The last step follows from the properties of Θ_U .

Moreover if one has a standard solution (R, \bar{R}) , then by construction R^*R is of the form λid . Since $\|R^*R\| = \|R\|^2 = d^{\mathcal{A}}(U)$, it follows that $\lambda = d^{\mathcal{A}}(U)$. Hence using the C^* -identity and the linearity of Θ_U we obtain

$$\|r_U\| = d^{\mathcal{A}}(U).$$

Using a Jensen's inequality argument however we get

$$\|\bar{r}_U\| \leq d^{\mathcal{A}}(U).$$

This proves $d^{\mathcal{P}}(U) \leq d^{\mathcal{A}}(U)$. Thus by construction

$$d_{\min}^{\mathcal{C}}(U) = d^{\mathcal{P}}(U).$$

Step 3. Construct the minimal C^* -subcategory $\tilde{\mathcal{B}} \subseteq \mathcal{A}$ that contains \mathcal{C} and morphisms of the form $(\text{id}_V \otimes a_U \otimes \text{id}_W)$ for objects U, V, W in \mathcal{C} . Denote by \mathcal{B} the idempotent completion of $\tilde{\mathcal{B}}$. This can be considered again as a subcategory of \mathcal{A} (possibly after adjoining some objects).

Via the maps $\Theta_{U,V}$ defining a unitary tensor functor $\tilde{\mathcal{B}} \rightarrow \mathcal{P}$, we may consider its image, which yields a C^* -subcategory $\tilde{\mathcal{P}} \subseteq \mathcal{P}$. Let \mathcal{P}' be the idempotent completion of $\tilde{\mathcal{P}}$. By construction, \mathcal{P}' is unitarily monoidally equivalent to \mathcal{B} .

The embedding functor $\mathcal{P}' \rightarrow \mathcal{P}$ is a unitary monoidal equivalence, with $d_{\min}^{\mathcal{C}}(U) = d^{\mathcal{P}'}(U)$ for all objects U in \mathcal{C} , which follows by construction. Moreover by this equivalence, also \mathcal{P} and \mathcal{B} are unitarily monoidally equivalent, say through $\Lambda: \mathcal{P} \rightarrow \mathcal{B}$,

which we may assume to be the identity on \mathcal{C} . The functor Λ can also be seen as a functor $\mathcal{P} \rightarrow \mathcal{A}$, this yields the wanted factorisation of $F: \mathcal{C} \rightarrow \mathcal{A}$, and the theorem is proven. \square

The construction in the third step can also be viewed as an explicit description of the Poisson boundary.

Proposition 3.10: *Let \mathcal{C} be a rigid amenable C^* -tensor category and (\mathcal{P}, Π) its Poisson boundary. Then*

$$d_{\min}^{\mathcal{C}}(U) = \|\Gamma_U\|.$$

Note that a compact quantum group \mathbb{G} is coamenable if and only if $\|\Gamma_U\| = \dim(U)$ for every representation U .

Moreover, if \mathbb{G} is coamenable, then $\text{Rep}(\mathbb{G})$ is amenable, i.e. in this case

$$d_{\min}^{\text{Rep}(\mathbb{G})}(U) = \dim(U).$$

In the next section we will see that the Poisson boundary of the representation category of a coamenable compact quantum group is given by the forgetful functor $\text{Rep}(\mathbb{G}) \rightarrow \text{Rep}(\mathbb{K})$, where $\mathbb{K} \subseteq \mathbb{G}$ is the maximal Kac quantum subgroup.

2. Factorisation of Fiber Functors

As mentioned above we consider amenable compact quantum groups \mathbb{G} and find their Poisson boundary. The maximal Kac quantum subgroup $\mathbb{K} \subseteq \mathbb{G}$ will be important and we will see how it is established at the categorical level. Recall that a compact quantum group is Kac if the antipode is selfinvolutive on the Hopf $*$ -algebra, i.e. $S^2(a) = a$ for all $a \in \mathbb{C}[G]$. For any compact quantum group \mathbb{G} we can describe the maximal Kac quantum subgroup as follows. Let

$$I := \langle S^2(a) - a \rangle.$$

Then I is a Hopf $*$ -ideal and the quotient Hopf $*$ -algebra $\mathbb{C}[G]/I$ is Kac. It corresponds to the maximal Kac quantum subgroup $\mathbb{K} \subseteq \mathbb{G}$. This construction is due to van Daele and Sołtan. For the following proof, see [NY16].

Theorem 3.11 (Neshveyev-Yamashita): *Let \mathbb{G} be a coamenable compact quantum group and \mathbb{K} its maximal Kac compact quantum subgroup. Then there is a bijection of the isomorphism classes of unitary fiber functors $\tilde{F}: \text{Rep}(\mathbb{K}) \rightarrow \text{Hilb}_f$ and the isomorphism classes of unitary fiber functors $F: \text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$ such that $d(F(U)) = \dim U$. Equivalently, the map*

$$H^2(\widehat{\mathbb{K}}, \mathbb{T}) \rightarrow H^2(\widehat{\mathbb{G}}, \mathbb{T})$$

is a bijection.

Proof. Since \mathbb{G} is coamenable, $\text{Rep}(\mathbb{G})$ is amenable, and hence we may consider its Poisson boundary (\mathcal{P}, Π) with respect to an ergodic measure. The forgetful functor $\text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$ factorises through the universal functor $\Pi: \text{Rep}(\mathbb{G}) \rightarrow \mathcal{P}$ and hence, by Tannaka-Krein duality, we can identify Π with the forgetful functor $\text{Rep}(\mathbb{G}) \rightarrow \text{Rep}(\mathbb{H})$, where $\mathbb{H} \subseteq \mathbb{G}$ is a closed quantum subgroup of \mathbb{G} .

By universality of the Poisson boundary the intrinsic dimension function of $\text{Rep}(\mathbb{H})$ is the classical dimension, which is if and only if $\rho_U = 1$ for all representations U , and hence \mathbb{H} is Kac. Since $\mathbb{H} \subseteq \mathbb{K}$, every intertwiner for \mathbb{K} is also an intertwiner in \mathbb{H} , so we have $\text{End}_{\mathbb{K}}(U) \subseteq \text{End}_{\mathbb{H}}(U)$ for all representations U . By the factorisation property we can embed $\text{End}_{\mathbb{H}}(U)$ in $\text{End}_{\mathbb{K}}(U)$, thus $\text{End}_{\mathbb{K}}(U) = \text{End}_{\mathbb{H}}(U)$ for all representations U and consequently $\mathbb{H} = \mathbb{K}$.

The statement follows from this and the factorisation property. Indeed, if

$$F_0: \text{Rep}(\mathbb{K}) \rightarrow \text{Hilb}_f$$

is a unitary fiber functor, then $F := F_0\Pi$ defines a unitary fiber functor $\text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$ and since \mathbb{K} is Kac, we have $\dim(U) = d(F(U))$. Conversely, if $F: \text{Rep}(\mathbb{G}) \rightarrow \text{Hilb}_f$ is a unitary fiber functor such that $\dim(U) = d(F(U))$, then we can factorise to obtain a unitary fiber functor $F_0: \text{Rep}(\mathbb{K}) \rightarrow \text{Hilb}_f$. \square

We conclude for coamenable compact quantum groups \mathbb{G} the classification of dimension-preserving unitary fiber functors reduces to the classification of unitary fiber functors on the maximal Kac quantum subgroup \mathbb{K} . It suffices to determine $H^2(\widehat{\mathbb{K}}, \mathbb{T})$.

It will turn out in the following section that the maximal Kac quantum subgroup of G_q^c with $0 < q < 1$ and c a 2-cochain with some properties is given by the maximal torus T .

Proposition 3.12: *Let \mathbb{G} be a compact quantum group, \mathbb{K} its maximal Kac quantum subgroup, and let ω be a unitary 2-cocycle on $\widehat{\mathbb{K}}$, then \mathbb{K}_ω is the maximal Kac quantum subgroup of \mathbb{G}_ω .*

Proof. Recall the construction of \mathbb{G}_ω . We have $\text{Rep}(\mathbb{G}) \cong \text{Rep}(\mathbb{G}_\omega)$, the same holds for $\text{Rep}(\mathbb{K}_\omega) = \text{Rep}(\mathbb{K})$, and hence the intrinsic dimension functions coincide. Since \mathbb{K} is Kac, and the intrinsic dimension is the classical dimension, also \mathbb{K}_ω is Kac.

Assume that $\mathbb{K}_\omega \subseteq \mathbb{H}$ is a Kac quantum subgroup of \mathbb{G}_ω . Note that ω^* is a unitary 2-cocycle on $\widehat{\mathbb{G}_\omega}$ and we have $(\mathbb{G}_\omega)_{\omega^*} \cong \mathbb{G}$ and $(\mathbb{K}_\omega)_{\omega^*} \cong \mathbb{K}$ by construction. One has $\mathbb{K} \subseteq \mathbb{H}_{\omega^*}$, and since \mathbb{K} is the maximal Kac quantum subgroup of \mathbb{G} , we obtain that $\mathbb{K}_\omega = \mathbb{H}$. Therefore \mathbb{K}_ω is the maximal Kac quantum subgroup of \mathbb{G}_ω . \square

3. Maximal Kac Quantum Subgroup of G_q^c

We now determine the maximal Kac quantum subgroup of the twisted q -deformation of a connected semisimple compact Lie group in order to apply our result above. We

work in the same setting as in Chapter II: G is a connected semisimple compact Lie group, denote by \mathfrak{g} its (complexified) Lie algebra, T the maximal torus of G , P the weight lattice and Q its root lattice, such that $\widehat{Z(G)} \cong P/Q$. Moreover let c be a \mathbb{T} -valued 2-cochain on the dual group $\widehat{T} = P$, such that ∂c defines a 3-cocycle on $\widehat{Z(G)}$.

Proposition 3.13: *Let $q > 0$, then the maximal Kac quantum subgroup of G_q is the maximal torus $T \subseteq G$.*

This is a result by Tomatsu, [Tom07], and we want to give the idea of the proof. For this we recall the C^* -algebraic structure of $C(G_q)$, to do so let us briefly introduce its generators.

Let $(\alpha_i)_{i=1}^n$ be the set of simple roots of G . Let λ be an integral dominant weight, we denote V^λ for the irreducible (hence also admissible, recall Remark 2.24) $U_q(\mathfrak{g})$ -module with highest weight λ and denote by $V^\lambda(\mu)$ the weight space of μ in V^λ , defined as in Definition 2.23, and fix an ONB $(\xi_r^\lambda)_r$ for each weight space. Define the generators

$$C_{\xi_s^\mu, \xi_r^\nu}^\lambda \in \mathbb{C}[\widehat{G}_q] \subseteq U_q(\mathfrak{g})^*,$$

by

$$C_{\xi_r^\mu, \xi_s^\nu}^\lambda(x) = \langle x \xi_s^\nu, \xi_r^\mu \rangle,$$

where $x \xi_s^\nu$ denotes the action of $x \in U_q(\mathfrak{g})$ on V^λ . Then

$$\mathbb{C}[G_q] = \text{span} \left\{ C_{\xi_r^\mu, \xi_s^\nu}^\lambda \right\}_{\lambda, \mu, \nu, r, s}.$$

The C^* -completion of $\mathbb{C}[G_q]$ is denoted by $C(G_q)$.

We will also need the scaling group, which is given on these generators by

$$\tau_t(C_{\xi_r^\mu, \xi_s^\nu}^\lambda) = q^{i(\mu-\nu, 2\rho)t} C_{\xi_r^\mu, \xi_s^\nu}^\lambda,$$

where $\rho = \frac{1}{2} \sum_{i=1}^n \alpha_i$. Denote by

$$r_i: \mathbb{C}[G_q] \rightarrow \mathbb{C}[SU_{q_i}(2)]$$

the canonical surjection, which is induced by the embedding $\mathfrak{sl}(2) \rightarrow \mathfrak{g}$, respectively

$$U_q(\mathfrak{sl}(2)) \rightarrow U_q(\mathfrak{g}),$$

since any semisimple Lie algebra has a subalgebra isomorphic to $\mathfrak{sl}(2)$. Finally, recall that any one-dimensional $*$ -homomorphism $\pi: \mathbb{C}[G_q] \rightarrow \mathbb{C}$ is given by $\pi_t = \chi_t \circ \iota_T$, where $\chi_t: C(T) \rightarrow \mathbb{C}$ denotes the evaluation map at $t \in T$.

Proof of Proposition 3.13. Assume that \mathbb{H} is a Kac quantum subgroup of \mathbb{G}_q . We show that $C(\mathbb{H})$ needs to be commutative. Suppose, for contradiction, that \mathbb{H} is not commutative. Then there exists a $*$ -representation $\rho: C(\mathbb{H}) \rightarrow B(H_\rho)$ with $\dim(H_\rho) \geq 2$. Define $\pi = \rho \circ \iota_{\mathbb{H}}$. Since the scaling group of \mathbb{H} is trivial, we have by Lemma 1.23 $\iota_{\mathbb{H}} = \iota_{\mathbb{H}} \circ \tau_t$ for all $t \in \mathbb{R}$. Hence

$$\iota_{\mathbb{H}}(C_{\xi_r^\mu, \xi_s^\nu}^\lambda) = 0,$$

since $q^{i(\mu-\nu, 2\rho)t} \neq 0$, it follows that $\pi(C_{\xi_r^\mu, \xi_s^\nu}^\lambda) = 0$ for such matrix coefficients.

However, this is not true: one can show that it suffices to consider $*$ -representations of the form $\pi_{s_{i_1}, t}$. Indeed, by a *theorem of Soibelman and Korogodsky*, [KS98], every representation can be written as

$$\pi = \pi_{w,t} = (\pi_{i_1} \circ r_{i_1} \otimes \dots \otimes \pi_{i_n} \circ r_{i_n} \otimes \pi_t) \circ \Delta^{(k)},$$

where $w = s_{i_1} \dots s_{i_n}$ in the Weyl group and by construction these representations have the property that

$$\pi_{s_{i_1}, t}(C_{\xi_{\omega_j}, \xi_{\omega_j - \alpha_j}}^{\omega_j}) \neq 0,$$

where ω_j is the fundamental weight to α_j . See [Tom07] for more details on the definition and the properties.

This yields a contradiction. Hence $C(\mathbb{H})$ is commutative, and \mathbb{H} is a classical compact group.

Let $\chi_x^{\mathbb{H}}: C(\mathbb{H}) \rightarrow \mathbb{C}$ be the character $*$ -homomorphism, i.e. evaluation at $x \in \mathbb{H}$. We use the same notation for $C(T)$. Since every one-dimensional representation π of $C(G_q)$ can be factorised as $\pi = \chi_t \circ \iota_T$, for each $x \in \mathbb{H}$ there exists $\varphi(x) \in T$ such that

$$\chi_x^{\mathbb{H}} \circ \iota_{\mathbb{H}} = \chi_{\varphi(x)}^T \circ \iota_T.$$

The map $\varphi: \mathbb{H} \rightarrow T$ is continuous and one can prove that it is also a group homomorphism. Indeed, let $x, y \in \mathbb{H}$. Then

$$\begin{aligned} \chi_{\varphi(xy)}^T \circ \iota_T &= \chi_{xy}^{\mathbb{H}} \circ \iota_{\mathbb{H}} = (\chi_x^{\mathbb{H}} \otimes \chi_y^{\mathbb{H}}) \circ \Delta_{\mathbb{H}} \circ \iota_{\mathbb{H}} \\ &= (\chi_x^{\mathbb{H}} \otimes \chi_y^{\mathbb{H}}) \circ (\iota_{\mathbb{H}} \otimes \iota_{\mathbb{H}}) \circ \Delta_{G_q} \\ &= (\chi_{\varphi(x)}^T \otimes \chi_{\varphi(y)}^T) \circ (\iota_T \otimes \iota_T) \circ \Delta_{G_q} \\ &= (\chi_{\varphi(x)}^T \otimes \chi_{\varphi(y)}^T) \circ \Delta_T \circ \iota_T \\ &= \chi_{\varphi(x)\varphi(y)}^T \circ \iota_T. \end{aligned}$$

Injectivity follows by the simple fact that $\chi_e^T \circ \iota_T = \varepsilon$, hence the kernel needs to be trivial. Thus every Kac quantum subgroup is contained in T , and hence T is the maximal Kac quantum subgroup, as stated. \square

In the following theorem we see that for G_q^c the maximal Kac quantum subgroup is actually also given by the maximal torus T of G . We first need to establish the following lemma, which states that the Woronowicz character of G_q^c agrees with that of G_q .

Lemma 3.14: *Let $1 \neq q > 0$ and c be a \mathbb{T} -valued 2-cochain on P , such that ∂c descends to a 3-cocycle on P/Q . The Woronowicz character f_1 of G_q^c is given by $q^{-2\rho^*}$.*

Proof. We may assume that Φ^c is normalized, i.e. if one of the arguments is the unit, then it acts trivially, since G_q^c does not change by multiplication by a 2-cochain on $\widehat{Z(G)} = P/Q$. Note that the associator $\Phi^c(U, V, W)$ for irreducible representations U, V, W acts as a scalar $\lambda_{U,V,W}$.

For an irreducible representation U , let the scalar λ_U denote the scalar that acts on $\mathcal{H}_{\bar{U}} \otimes \mathcal{H}_U \otimes \mathcal{H}_{\bar{U}}$. Now let (R_U, \bar{R}_U) be a standard solution of the conjugate equations of U . Note that by construction $(\lambda_U R_U, \bar{R}_U)$ is a solution for the conjugate equations of U in the twisted category $(\text{Rep}(G_q), \Phi^c)$. Applying the induced fiber functor $F: \text{Rep}(G_q) \rightarrow \text{Hilb}_f$, defining G_q^c , to the solution, we obtain $(\lambda_U c R_U, c \bar{R}_U)$. Since $c \in \mathcal{U}(T \times T)$, c acts on weight vectors $(\xi_i)_i$ by multiplication with scalars in \mathbb{T} . Choose an ONB of weight vectors $(\xi_i)_i$ in \mathcal{H}_U . By Example 2.9 and since torus elements act on weight vectors as a scalar, we can write

$$c R_U(I) = c(1 \otimes \rho^{-1/2}) \sum \xi_i^* \otimes \xi_i = c \sum \xi_i^* \otimes \rho^{-1/2} \xi_i = \sum \xi_i^* \otimes \beta_i \rho^{-1/2} \xi_i,$$

where $\beta_i \in \mathbb{T}$ are such scalars, and ρ is the Woronowicz character of G_q . Comparing with the definition of the Woronowicz character, this forces the new Woronowicz character $f_{-1/2}$ to take the form $\rho^{-1/2} v$ for some unitary operator v . Because the Woronowicz character must be a positive operator and $\rho^{-1/2}$ is already positive, this unitary v must be the identity, concluding that $f_1 = \rho = q^{-2\rho^*}$ by Lemma 2.27. \square

We only give a sketch of the proof of the next theorem, [NY16], and highlight the main steps.

Theorem 3.15: *Let $1 \neq q > 0$ and c be a \mathbb{T} -valued 2-cochain on P , such that ∂c descends to a 3-cochain on P/Q . Then the maximal Kac quantum subgroup of G_q^c is the maximal torus T .*

Sketch of the proof.

We consider 2-cochains of the form c_τ , where $\tau \in Z(G)^r$. Let T be the maximal torus of G , and let $T_\tau \subseteq T$ be the subgroup generated by the entries of τ . There is a group homomorphism $\psi: \widehat{T}_\tau \rightarrow T/Z(G)$, defined by $\langle \psi(\chi), \alpha_i \rangle = \tau_i$ for $\chi \in \widehat{T}_\tau$ and all simple roots α_i .

We work at the C^* -algebraic level of $G_q^{c_\tau} =: G_q^\tau$.

Step 1. The introduced group homomorphism ψ induces an action on the level of Hopf $*$ -algebras $\text{ad}(\psi): \widehat{T}_\tau \curvearrowright \mathbb{C}[G_q]$ on $C(G_q^\tau)$ by composition with conjugation action $T/Z(G) \curvearrowright C(G_q)$. The crossed product $C(G_q) \rtimes_{\text{ad}(\psi)} \widehat{T}_\tau$ carries an additional action $T_\tau \curvearrowright C(G_q) \rtimes_{\text{ad}(\psi)} \widehat{T}_\tau$. One then has an isomorphism

$$C(G_q^\tau) \cong \left(C(G_q) \rtimes_{\text{ad}(\psi)} \widehat{T}_\tau \right)^{T_\tau},$$

where $\left(C(G_q) \rtimes_{\text{ad}(\psi)} \widehat{T}_\tau \right)^{T_\tau}$ denotes the fixed-point algebra under the action of T_τ . The set of primitive ideals, i.e. the primitive spectrum, decomposes

$$\text{Prim}(C(G_q^\tau)) = \bigcup_{w \in W} (\theta_w(\widehat{T}_\tau) \backslash T/T_\tau) \times \widehat{\theta_w^{-1}(T_\tau)},$$

where W is the Weyl group, and θ_w are some maps. Note that $\text{Prim}(C(G_q))$ is given as disjoint union of copies of the maximal torus T indexed over W . Using this and some Morita equivalence relation of the fixed point algebra representation of $C(G_q^\tau)$, one obtains this fact.

Step 2. By Lemma 3.14 the scaling group is the same as for G_q , i.e. given by the conjugation action by $q^{-2it\rho^*} \in T$. The scaling group can be extended on $C(G_q) \rtimes_{\text{ad}(\psi)} \widehat{T}_\tau$. As mentioned above, we may use a Morita equivalent C^* -algebra of $C(G_q^\tau)$, which is here given by

$$C(G_q) \rtimes T_\tau \rtimes \widehat{T}_\tau,$$

where we omit the explicit actions in the crossed products. Since this is a more concrete representation, we now let (τ_t) act on $C(G_q)$ as an action $\mathbb{R} \curvearrowright C(G_q) \rtimes T_\tau \rtimes \widehat{T}_\tau$, such that it only acts on $C(G_q)$ and trivially on the copies of T_τ and \widehat{T}_τ .

Step 3. One then deduces that on the primitive spectrum the scaling group τ_t acts only on the component that is indexed over W , where it acts as $q^{-2it(w\rho^* - \rho^*)}$. The only element in W , such that $w\rho^* = \rho^*$ is given by the neutral element e , and the part of the spectrum indexed with e consists precisely of irreducible one-dimensional representations, i.e. point evaluations on T . We may summarise that the only irreducible representations of $C(G_q^\tau)$ fixed by the scaling group are point evaluations on the maximal torus T .

Step 4. Now we can use again the same arguments as in the proof of Proposition 3.13.

For more details see [NY15] for the results on $C(G_q^c)$ and [NY16] for the proof in this setting. □

Since we are particularly interested in quantum groups of the form $SU_q^c(N)$, let us briefly recall its maximal torus.

Example 3.16: Let $N \geq 2$ and consider $SU_q^c(N)$ for $q \neq 1$ and c be a \mathbb{T} -valued 2-cochain on P , such that ∂c descends to a 3-cochain on P/Q . Then the maximal Kac quantum subgroup is given by the maximal torus of $SU(N)$, namely

$$T = \left\{ \left(\begin{array}{cccc} e^{i\theta_1} & & & \\ & \ddots & & \\ & & e^{i\theta_{n-1}} & \\ & & & e^{-i(\theta_1+\dots+\theta_{n-1})} \end{array} \right) \middle| \theta_j \in \mathbb{R} \right\}.$$

Finally by Theorem 3.8 and Theorem 3.15 we obtain the following important corollary.

Corollary 3.17: *Let $1 \neq q > 0$ and let c be a \mathbb{T} -valued 2-cochain on P , such that ∂c descends to a 3-cochain on P/Q . Then any dimension-preserving unitary fiber functor $\text{Rep}(G_q^c) \rightarrow \text{Hilb}_f$ factors uniquely (up to isomorphism) through $\text{Rep}(T)$. Equivalently, we have a bijection $H^2(\widehat{T}, \mathbb{T}) \rightarrow H^2(\widehat{G}_q^c, \mathbb{T})$.*

In particular the class of quantum groups of the form $SU_q^c(N)$ is closed under cocycle twisting.

Recall that cocycle twisting as defined below Theorem 2.19, corresponds to equipping the representation category with a new induced dimension-preserving unitary fiber functor F .

Chapter IV.

Classification of Non-Kac Compact Quantum Groups of $SU(N)$ -type

The goal of this chapter is to prove the classification theorem for non-Kac compact quantum groups of $SU(N)$ -type. We begin by recalling the Kazhdan-Wenzl-Jordans theorem, which describes the structure of rigid C^* -tensor categories of $SU(N)$ -type and thus provides the categorical starting point of the classification. Since our interest lies in compact quantum groups, we then explain why every such category is realised as the representation category of a quantum group of $SU(N)$ -type. The key input is the corollary of the factorisation result on the maximal torus, showing that this class of quantum groups is closed under cocycle twisting. Finally, we address the uniqueness problem, we determine when two compact quantum groups of the form $SU_q^c(N)$ are isomorphic, which leads us to study monoidal autoequivalences of $SU_q^c(N)$, in certain cases these are controlled by the automorphism group of the based root datum.

These observations indicate that the classification of non-Kac quantum groups of $SU(N)$ -type follows a largely “classical pattern”.

1. Theorem of Kazhdan-Wenzl-Jordans

Before proving the classification result in the next section, we first establish the Kazhdan-Wenzl-Jordans theorem.

The main tool will be *Hecke algebras*, since every category of $SU(N)$ -type carries natural representations of Hecke algebras. These representations allow one to recover the fusion semiring, and hence much of the category, in a way that is largely independent of the specific realisation.

Definition 4.1 (Hecke algebra): Let $n \in \mathbb{N}$ and $q \in \mathbb{C}$. The *Hecke algebra* $H_n(q)$ is the unital algebra generated by elements g_1, \dots, g_{n-1} subject to the relations

$$\begin{aligned} g_i g_j &= g_j g_i, & \text{if } |i - j| \geq 2, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, & i = 1, \dots, n-2, \\ g_i^2 &= (q-1)g_i + q, & i = 1, \dots, n-1. \end{aligned}$$

If $q \neq 0$, then each generator is invertible, with inverse $q^{-1}(g_i + 1 - q)$. For $q = 1$ the quadratic relation becomes $g_i^2 = 1$, and together with the first two relations this yields the Coxeter relations for the transpositions $(i, i+1)$ in S_n , hence $H_n(1) = \mathbb{C}[S_n]$. More generally, if q is not a root of unity, then $H_n(q)$ is isomorphic (as an abstract \mathbb{C} -algebra) to $\mathbb{C}[S_n]$. In this sense, Hecke algebras can be viewed as a “ q -deformation” of $\mathbb{C}[S_n]$.

If $q > 0$ or $|q| = 1$, we can equip $H_n(q)$ with a $*$ -structure by declaring the idempotents $e_i := (q - g_i)(q + 1)^{-1}$ to be self-adjoint, i.e. $e_i^* := e_i$, and extending anti-linearly and multiplicatively.

Remark 4.2: Consider the category Hilb_f introduced in Chapter 2, which is a C^* -tensor category. Let $\mathcal{H} := \mathbb{C}^d$ be a fixed object of Hilb_f for the whole section, fix an orthonormal basis $\{\psi_i\}_{i=1}^d$ of \mathcal{H} , and write $q = \mu^2$. Define an operator $T \in \text{End}(\mathcal{H} \otimes \mathcal{H})$ by

$$T(\psi_i \otimes \psi_j) = \begin{cases} \mu \psi_j \otimes \psi_i, & i > j, \\ q \psi_i \otimes \psi_i, & i = j, \\ (q-1)\psi_i \otimes \psi_j + \mu \psi_j \otimes \psi_i, & i < j. \end{cases}$$

By construction, T satisfies the quadratic Hecke relation. For instance, if $i = j$, then

$$T^2(\psi_i \otimes \psi_i) = q^2 \psi_i \otimes \psi_i = (q-1)T(\psi_i \otimes \psi_i) + q \text{id}(\psi_i \otimes \psi_i),$$

and if $i < j$, then

$$\begin{aligned} T^2(\psi_i \otimes \psi_j) &= T((q-1)\psi_i \otimes \psi_j + \mu \psi_j \otimes \psi_i) = (q-1)T(\psi_i \otimes \psi_j) + \mu T(\psi_j \otimes \psi_i) \\ &= (q-1)T(\psi_i \otimes \psi_j) + \mu^2 \psi_i \otimes \psi_j = (q-1)T(\psi_i \otimes \psi_j) + q \psi_i \otimes \psi_j. \end{aligned}$$

The case $i > j$ is analogous. We thus obtain a representation of $H_n(q)$ on $\mathcal{H}^{\otimes n}$ by

$$\eta: H_n(q) \rightarrow \text{End}(\mathcal{H}^{\otimes n}), \quad g_i \mapsto T_i := \text{id}^{\otimes(i-1)} \otimes T \otimes \text{id}^{\otimes(n-i-1)}.$$

Using the computations above one checks that η satisfies the defining relations of $H_n(q)$. See [Jim86] for further details.

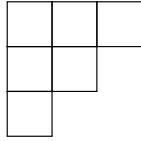
Finally, define $\tilde{S}: \mathbb{C} \rightarrow \mathcal{H}^{\otimes d}$ by $c \mapsto cS$, where

$$S = \sum_{\sigma \in S_d} (-\mu)^{l(\sigma)} \psi_{\sigma(1)} \otimes \dots \otimes \psi_{\sigma(d)},$$

and $l(\sigma)$ denotes the length of $\sigma \in S_d$. As stated in [PR11], the representation category $\text{Rep}(SU_q)$ is, up to equivalence, the smallest C^* -tensor category (with subobjects and direct sums) in Hilb_f containing the (fundamental) object \mathcal{H} and the morphisms $T \in \text{End}(\mathcal{H}^{\otimes 2})$ and $\tilde{S} \in \text{Mor}(\mathbb{C}, \mathcal{H}^{\otimes d})$, see also [Pin07].

Let \mathcal{C} be a rigid C^* -tensor category of $SU(N)$ -type, and let X denote its fundamental object, i.e. the object corresponding to the fundamental representation of $SU(N)$. Our goal is to extract an invariant $\mu_{\mathcal{C}}$ associated with \mathcal{C} .

We briefly recall the parametrization of the irreducible representations of $SU(N)$ and fix some notation. One can proceed via the highest-weight classification, or equivalently via *Young diagrams*. A Young diagram is of the form



It encodes a partition of an integer $n \in \mathbb{N}$ as a sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ with

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r, \quad \sum_{i=1}^r \lambda_i = n.$$

The i -th row of the diagram contains λ_i boxes. Irreducible representations of $SU(N)$ are in bijection with Young diagrams with at most N rows, equivalently, one may restrict to at most $N - 1$ rows, since adding a column of height N does not change the corresponding $SU(N)$ -representation. Thus we parameterize them by

$$\Lambda_N = \{(\lambda_1, \dots, \lambda_N) \in \mathbb{N}_0^N \mid \lambda_1 \geq \dots \geq \lambda_N\}.$$

The *fusion rules*, i.e. the decomposition of tensor products of irreducibles, can be written as

$$V^\mu \otimes V^\nu = \bigoplus_{\lambda \in \Lambda_N} m_\lambda^{\mu, \nu} V^\lambda,$$

where $m_\lambda^{\mu, \nu} = \dim \text{Mor}(V^\lambda, V^\mu \otimes V^\nu)$ is the corresponding multiplicity.

Moreover, when the second Young diagram consists of a single row, this decomposition can be computed explicitly (*Pieri rule*) by adding the boxes of that row in all possible ways, with at most one new box in each column. For example,

$$(2, 1) \otimes (2) = (4, 1) \oplus (3, 2) \oplus (3, 1, 1) \oplus (2, 2, 1).$$

Now fix, for each $\lambda \in \Lambda_N$, a simple object X_λ .

Lemma 4.3: For every $n \in \mathbb{N}$ we have

$$X^{\otimes n} = \bigoplus_{\lambda \in \Lambda_N} m_{\lambda,n} X_\lambda.$$

Moreover, $m_{\lambda,n} = 0$ unless $|\lambda| \equiv n \pmod{N}$. In particular, if $m \not\equiv n \pmod{N}$, then $\text{Mor}(X^{\otimes m}, X^{\otimes n}) = 0$.

Example 4.4: We can compute $X^{\otimes 3}$ using Young diagrams:

$$(1) \otimes (1) \otimes (1) = ((2) \oplus (1, 1)) \otimes (1) = (3) \oplus (2, 1) \oplus (2, 1) \oplus (1, 1, 1).$$

Therefore,

$$X^{\otimes 3} = \begin{cases} X_{(3)} \oplus X_{(2,1)} \oplus X_{(2,1)}, & N = 2, \\ X_{(3)} \oplus X_{(2,1)} \oplus X_{(2,1)} \oplus X_{(1,1,1)}, & N > 2. \end{cases}$$

Note that for $N = 2$ the Young diagram $(1, 1, 1)$ is not defined.

By Schur's lemma, we obtain

$$\text{End}(X^{\otimes 3}) \cong \begin{cases} \mathbb{C} \oplus M_2(\mathbb{C}), & N = 2, \\ \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C}, & N > 2. \end{cases}$$

Proposition 4.5: Let \mathcal{C} be a C^* -tensor category of $SU(N)$ -type. Denote by X its fundamental object, and let $a \in \text{End}(X^{\otimes 2})$ be the projection onto $X_{(1,1)} \subseteq X^{\otimes 2}$. Set $a_1 = a \otimes \text{id}$ and $a_2 = \text{id} \otimes a$. Then there exists a constant $\gamma_{\mathcal{C}} \in (0, 1]$ such that

$$a_1 a_2 a_1 - \gamma_{\mathcal{C}} a_1 = a_2 a_1 a_2 - \gamma_{\mathcal{C}} a_2.$$

As in [Jor14], we only construct $\gamma_{\mathcal{C}}$, we do not prove that $\gamma_{\mathcal{C}} \neq 0$ (see [KW93] for details).

Proof. Since $(1, 1) \otimes (1) = (1, 1, 1) \oplus (2, 1)$ in terms of Young diagrams, we have

$$X_{(1,1)} \otimes X = X_{(1,1,1)} \oplus X_{(2,1)}.$$

Hence $X_{(1,1,1)}$ is a subobject of $X_{(1,1)} \otimes X$, so there exist a projection $p \in \text{End}(X_{(1,1)} \otimes X)$ and a morphism $v \in \text{Mor}(X_{(1,1,1)}, X_{(1,1)} \otimes X)$ such that $v^* v = \text{id}_{X_{(1,1,1)}}$ and $v v^* = p$. Similarly, we can choose $w \in \text{Mor}(X_{(1,1)}, X^{\otimes 2})$ such that $w^* w = \text{id}_{X_{(1,1)}}$ and $w w^* = a$.

The restriction of a_1 to $X_{(1,1,1)}$ equals the identity on $X_{(1,1,1)}$, since

$$\begin{aligned} a_1 |_{X_{(1,1,1)}} &= v^*(w^* \otimes \text{id}) a_1 (w \otimes \text{id}) v \\ &= v^*(w^* \otimes \text{id})(w w^* \otimes \text{id})(w \otimes \text{id}) v \\ &= v^* \text{id}_{X_{(1,1)} \otimes X} v = \text{id}_{X_{(1,1,1)}}. \end{aligned}$$

The same calculation yields $a_2|_{X_{(1,1,1)}} = \text{id}_{X_{(1,1,1)}}$.

Since $X_{(3)}$ is neither a subobject of $X_{(1,1)} \otimes X$ nor of $X \otimes X_{(1,1)}$, the restrictions of both a_1 and a_2 to $X_{(3)}$ are zero. Moreover, a_1 and a_2 act as rank-one projections in $\text{End}(X_{(2,1)} \oplus X_{(2,1)})$, since

$$\dim(\text{Mor}(X_{(2,1)}, X_{(1,1)} \otimes X)) = \dim(\text{Mor}(X_{(2,1)}, X \otimes X_{(1,1)})) = 1.$$

Using Example 4.4, we may identify a_i with $(1, f_i, 0) \in \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$, where f_1, f_2 are rank-one projections in $M_2(\mathbb{C})$.

Since $\text{ran}(f_1 f_2 f_1) \subseteq \text{ran}(f_1)$ and $\text{ran}(f_2 f_1 f_2) \subseteq \text{ran}(f_2)$, there exist $\gamma_1, \gamma_2 \in \mathbb{C}$ such that $f_1 f_2 f_1 = \gamma_1 f_1$ and $f_2 f_1 f_2 = \gamma_2 f_2$. Then

$$\gamma_1 f_1 f_2 = f_1 f_2 f_1 f_2 = \gamma_2 f_1 f_2.$$

Hence either $\gamma_1 = \gamma_2$ or $f_1 f_2 = 0$, in the latter case we set $\gamma_1 = \gamma_2 = 0$. Define $\gamma_{\mathcal{C}} := \gamma_1 = \gamma_2$.

Since f_1 and f_2 are projections (hence positive), we have $\gamma_{\mathcal{C}} \in [0, 1]$. As noted above, we omit the technical argument showing $\gamma_{\mathcal{C}} \neq 0$. \square

Choose $\mu_{\mathcal{C}}$ such that

$$\mu_{\mathcal{C}} + \mu_{\mathcal{C}}^{-1} = \gamma_{\mathcal{C}}^{-1} - 2 \quad \Leftrightarrow \quad \gamma_{\mathcal{C}} = \frac{\mu_{\mathcal{C}}}{(1 + \mu_{\mathcal{C}})^2}.$$

Note $\mu_{\mathcal{C}} \in (0, 1] \cup \{e^{i\alpha} \mid 0 < \alpha < \frac{2\pi}{3}\}$. However in the case of a strict C^* -tensor category of $SU(N)$ -type, $\mu_{\mathcal{C}} \in (0, 1]$.

Proposition 4.6: *The maps*

$$\theta_n: H_n(\mu_{\mathcal{C}}) \rightarrow \text{End}(X^{\otimes n}), \quad e_i \mapsto a_i$$

extend to $$ -representations. Moreover, the kernel depends only on $\mu_{\mathcal{C}}$, and*

$$\theta_n(H_n(\mu_{\mathcal{C}})) = \text{End}(X^{\otimes n}).$$

So far we have only extracted an invariant $\mu_{\mathcal{C}}$, but we still need a twist. Recall from Remark 2.36 that all associators on $SU_q(N)$ correspond to an N -th root of unity, which we denote by ω .

The maps $\theta_n: H_n(\mu_{\mathcal{C}}) \rightarrow \text{End}(X^{\otimes n})$ can be extended to an inductive-limit map $\theta: H_{\infty}(\mu_{\mathcal{C}}) \rightarrow M_{\mathcal{C}}$, where $H_{\infty}(\mu_{\mathcal{C}})$ is the inductive limit of $(H_n(\mu_{\mathcal{C}}), \iota_{m,n})$, with canonical embeddings $\iota_{m,n}: H_m(\mu_{\mathcal{C}}) \rightarrow H_n(\mu_{\mathcal{C}})$ for $m \leq n$, and $M_{\mathcal{C}}$ is the inductive limit $(\text{End}(X^{\otimes n}), \tilde{\iota}_{m,n})$, with $\tilde{\iota}_{m,n}: \text{End}(X^{\otimes m}) \rightarrow \text{End}(X^{\otimes n})$, $T \mapsto T \otimes \text{id}^{\otimes(n-m)}$.

Remark 4.7: Let \mathcal{C} be a rigid C^* -tensor category of $SU(N)$ -type and let $q_{\mathcal{C}} = \sqrt{\mu_{\mathcal{C}}} \in (0, 1]$. In $\text{Rep}(SU(N))$, the trivial representation \mathbb{C} occurs as a subrepresentation of $V^{\otimes N}$ (this follows from standard Lie-theoretic arguments). Hence, in \mathcal{C} there

exists a morphism $\nu: I \rightarrow X^{\otimes N}$ such that $\nu^*\nu = \text{id}_I$, and therefore a projection $\nu\nu^* \in \text{End}(X^{\otimes N})$. Consider the composition

$$X = X \otimes I \xrightarrow{\text{id} \otimes \nu} X \otimes X^{\otimes N} \xrightarrow{\theta(g_d \dots g_1)} X^{\otimes N} \otimes X \xrightarrow{\nu^* \otimes \text{id}} I \otimes X = X.$$

Since X is simple, Schur's lemma implies that this endomorphism is scalar. We denote the corresponding scalar by $\tau_{\mathcal{C}} \in \mathbb{C}$. Equivalently, $\tau_{\mathcal{C}}$ is characterised by

$$\theta(g_d \dots g_1)(\text{id} \otimes \nu) = \tau_{\mathcal{C}}(\nu \otimes \text{id}).$$

The scalar $\tau_{\mathcal{C}}$ measures the interaction between the Hecke-algebra action and the distinguished morphism ν .

Using standard properties of the Hecke algebra, we obtain the following.

Lemma 4.8: *Let \mathcal{C} be a rigid C^* -tensor category of $SU(N)$ -type, and let $\tau_{\mathcal{C}}$ be as above. Then there exists an N -th root of unity $\omega_{\mathcal{C}}$ such that*

$$\tau_{\mathcal{C}} = \omega_{\mathcal{C}} q_{\mathcal{C}}^{N+1}.$$

An N -th root of unity $\omega_{\mathcal{C}}$ corresponds to an associator Φ , that is, to an invariant unitary 3-cocycle. Thus we can formulate the Kazhdan-Wenzl-Jordans theorem as follows. The invariants $\mu_{\mathcal{C}}$ and ω do not depend on the particular choice of the fundamental object X . This ultimately reflects the fact that the Dynkin diagram of $SU(N)$ has exactly two symmetries, namely the identity and the reflection. On $U_q(\mathfrak{g})$, the reflection acts by $E_i \mapsto E_{N-i}$, $F_i \mapsto F_{N-i}$, and $K_i \mapsto K_{N-i}$, and it sends each object to its conjugate object \bar{X} . Proceeding as in the proof of Proposition 4.13, one concludes that $\mu_{\mathcal{C}}$ and $\omega_{\mathcal{C}}$ are independent of the choice of fundamental object.

Theorem 4.9 (Kazhdan-Wenzl-Jordans): *Let \mathcal{C} be a (strict) rigid C^* -tensor category of $SU(N)$ -type. Then there exist $q \in (0, 1]$ and a 3-cocycle Φ on $P/Q = \widehat{Z(G)} = \mathbb{Z}/N\mathbb{Z}$ such that \mathcal{C} is monoidally equivalent to*

$$(\text{Rep}(SU_q(N)), \Phi).$$

The proof in [Jor14] is split into two parts. In the first part, Jordans assumes a list of properties for a rigid C^* -tensor category \mathcal{C} , formulated in terms of $\mu_{\mathcal{C}}$, $\omega_{\mathcal{C}}$, and the morphisms η and ν introduced above, see [Jor14, Assumption 6.1]. Under these assumptions, he proves that \mathcal{C} is unitarily monoidally equivalent to $\text{Rep}(SU_q(N))^{\omega}$. In the second part, he shows that every rigid C^* -tensor category of $SU(N)$ -type satisfies these assumptions.

Sketch of the proof.

Step 1. We have canonical isomorphisms

$$\text{End}(X^{\otimes n}) \rightarrow \text{End}(\mathcal{H}^{\otimes n}),$$

which we want to extend to

$$\text{Mor}(X^{\otimes m}, X^{\otimes n}) \rightarrow \text{Mor}(\mathcal{H}^{\otimes m}, \mathcal{H}^{\otimes n}),$$

using a suitable embedding

$$H_p^{m,n}: \text{Mor}(X^{\otimes m}, X^{\otimes n}) \rightarrow \text{End}(X^{\otimes p})$$

for some sufficiently large p . Roughly speaking, this embedding “inflates” morphisms by tensoring with ν and ν^* .

Step 2. Denote the isomorphism $\text{End}(X^{\otimes p}) \rightarrow \text{End}(\mathcal{H}^{\otimes p})$ by ψ_p . Then $\psi_p \circ \theta_p = \eta_p$. Moreover, the maps $\theta_p: H_p(q) \rightarrow \text{End}(X^{\otimes p})$ and $\eta_p: H_p(q) \rightarrow \text{End}(\mathcal{H}^{\otimes p})$ are surjective and satisfy $\ker(\theta_p) = \ker(\eta_p)$.

We then define

$$\psi_{m,n}: \text{Mor}(X^{\otimes m}, X^{\otimes n}) \rightarrow \text{Mor}(\mathcal{H}^{\otimes m}, \mathcal{H}^{\otimes n}),$$

by

$$\psi_{m,n} = (H_{p, \text{Rep}(SU_q(N))}^{m,n})^{-1} \circ \psi_p \circ H_{p, \mathcal{C}}^{m,n}.$$

These maps are isomorphisms whenever $m \equiv n \pmod{N}$ (see Lemma 4.3), where $H_{p, \mathcal{C}}^{m,n}$ denotes the map $H_p^{m,n}$ in the category \mathcal{C} , and similarly for $\text{Rep}(SU_q(N))$.

Step 3. Let $\tilde{\mathcal{C}}$ be the full subcategory of \mathcal{C} with objects $\text{Ob}(\tilde{\mathcal{C}}) = \{X^{\otimes n} \mid n \in \mathbb{N}\}$, and let \mathcal{D} be the full subcategory of $\text{Rep}(SU_q(N))$ with objects $\text{Ob}(\mathcal{D}) = \{\mathcal{H}^{\otimes n} \mid n \in \mathbb{N}\}$. The idempotent completions of $\tilde{\mathcal{C}}$ and \mathcal{D} are \mathcal{C} and $\text{Rep}(SU_q(N))$, respectively. Define a tensor functor $\tilde{F}: \tilde{\mathcal{C}} \rightarrow \mathcal{D}$ by $\tilde{F}(X^{\otimes n}) = \mathcal{H}^{\otimes n}$ on objects, $\tilde{F}(T) = \psi_{m,n}(T)$ on morphisms $T \in \text{Mor}(X^{\otimes m}, X^{\otimes n})$, and $\tilde{F}_2 = \text{id}$. This yields a well-defined tensor functor. Extending it to the idempotent completions gives a unitary tensor functor $F: \mathcal{C} \rightarrow \text{Rep}(SU_q(N))$. Finally, F is essentially surjective and faithful (see Lemma 2.6), hence a unitary monoidal equivalence. Therefore \mathcal{C} and $\text{Rep}(SU_q(N))$ are unitarily monoidally equivalent. \square

At this point, our classification is purely categorical. While we have classified all C^* -tensor categories of $SU(N)$ -type, reconstructing compact quantum groups requires an additional ingredient, namely the existence of a unitary fiber functor. Therefore, it remains to determine which of the twisted categories $(\text{Rep}(SU_q(N)), \Phi)$ actually admit such a functor. Nevertheless, the theorem above settles the first step of the classification problem mentioned in the introduction, and prepares the ground for the explicit classification, together with the preceding and subsequent results.

2. Monoidal Autoequivalences and Classification

For the remainder of this section, let G be a simply connected simple compact Lie group, and denote by P its weight lattice and by Q its root lattice. In this section we answer the second step of the classification, mentioned in the introduction.

Proposition 4.10: *Let $q > 0$ and let c be a \mathbb{T} -valued 2-cochain on P , such that ∂c descends to a 3-cochain on $P/Q \cong \widehat{Z(G)}$. Then*

$$H^2(P/Q, \mathbb{T}) \cong H_{G_q^c}^2(\widehat{G}_q^c, \mathbb{T}) \cong H_{G_q^c}^2(\widehat{G}_q^c, \mathbb{C}^\times)$$

induced by the inclusion $\mathcal{U}(Z(G)) \hookrightarrow \mathcal{U}(G_q^c)$.

Proof. The idea is to reduce the problem to the case where c is trivial.

Denote the comultiplication on $\mathcal{U}(G_q)$ by $\widehat{\Delta}_q$. Since the comultiplication on $\mathcal{U}(G_q^c)$ is given by $c\widehat{\Delta}_q c^*$, an element $\omega \in \mathcal{U}(G_q^c \times G_q^c)$ is an invariant 2-cochain on $\mathcal{U}(G_q^c)$ if and only if $c^*\omega c$ is an invariant 2-cochain in $\mathcal{U}(G_q)$. We claim that the same equivalence holds with cochain replaced by cocycle.

Let ω be an invariant 2-cocycle on \widehat{G}_q^c . By Remark 1.27, we have

$$\begin{aligned} (\omega \otimes 1)(\widehat{\Delta}_q^c \otimes \text{id})(\omega) &= (1 \otimes \omega)(\text{id} \otimes \widehat{\Delta}_q^c)(\omega) \\ \Leftrightarrow (\omega c \otimes 1)(\widehat{\Delta}_q \otimes \text{id})(\omega)(c^* \otimes 1) &= (1 \otimes \omega c)(\text{id} \otimes \widehat{\Delta}_q)(\omega)(1 \otimes c^*). \end{aligned}$$

Thus it suffices to prove that $c^*\omega c$ is an invariant 2-cocycle on \widehat{G}_q , i.e.

$$(c^*\omega c \otimes 1)(\widehat{\Delta}_q \otimes \text{id})(c^*\omega c) = (1 \otimes c^*\omega c)(\text{id} \otimes \widehat{\Delta}_q)(c^*\omega c).$$

Using the invariance of $c^*\omega c$, the left-hand side expands to

$$\begin{aligned} y &= (c^*\omega c \otimes 1)(\widehat{\Delta}_q \otimes \text{id})(c^*\omega c) \\ &= (\widehat{\Delta}_q \otimes \text{id})(c^*\omega c)(c^*\omega c \otimes 1) \\ &= (\widehat{\Delta} \otimes \text{id})(c^*)(\widehat{\Delta}_q \otimes \text{id})(\omega)(\widehat{\Delta} \otimes \text{id})(c)(c^*\omega c \otimes 1) \\ &= (\widehat{\Delta} \otimes \text{id})(c^*)(c^*\omega c \otimes 1)(\widehat{\Delta}_q \otimes \text{id})(\omega)(\widehat{\Delta} \otimes \text{id})(c) \\ &= (\widehat{\Delta} \otimes \text{id})(c^*)(c^* \otimes 1)(\omega c \otimes 1)(\widehat{\Delta}_q \otimes \text{id})(\omega)(\widehat{\Delta} \otimes \text{id})(c). \end{aligned}$$

Here we used, that since $c \in \mathcal{U}(T)$ we have $(\widehat{\Delta}_q \otimes \text{id})(c) = (\widehat{\Delta} \otimes \text{id})(c)$. By analogous transformations, we define the right-hand side as

$$z = (\text{id} \otimes \widehat{\Delta})(c^*)(1 \otimes c^*)(1 \otimes \omega c)(\text{id} \otimes \widehat{\Delta}_q)(\omega)(\text{id} \otimes \widehat{\Delta})(c).$$

Recall that $\partial c = (1 \otimes c)(\text{id} \otimes \widehat{\Delta})(c)(\widehat{\Delta} \otimes \text{id})(c^*)$. Then

$$z^{-1}y = (\text{id} \otimes \widehat{\Delta})(c^*)(\text{id} \otimes \widehat{\Delta}_q)(\omega)^{-1}(1 \otimes \omega c)^{-1}(\partial c)(\omega c \otimes 1)(\widehat{\Delta}_q \otimes \text{id})(\omega)(\widehat{\Delta} \otimes \text{id})(c).$$

Since $\partial c \in \mathcal{U}(Z(G)^3)$ by assumption, we may move ∂c at the beginning of the expression, obtaining

$$\begin{aligned} z^{-1}y &= (\partial c)(\text{id} \otimes \widehat{\Delta})(c^*)(\text{id} \otimes \widehat{\Delta}_q)(\omega)^{-1}(1 \otimes \omega c)^{-1}(\omega c \otimes 1)(\widehat{\Delta}_q \otimes \text{id})(\omega)(\widehat{\Delta} \otimes \text{id})(c) \\ &= (\partial c)(\text{id} \otimes \widehat{\Delta})(c^*)(\text{id} \otimes \widehat{\Delta}_q)(\omega)^{-1}(1 \otimes \omega c)^{-1}(1 \otimes \omega c)(\text{id} \otimes \widehat{\Delta}_q)(\omega)(1 \otimes c^*) \\ &= (\partial c)(\text{id} \otimes \widehat{\Delta})(c^*)(1 \otimes c^*)(c \otimes 1)(\widehat{\Delta} \otimes \text{id})(c) = 1, \end{aligned}$$

where in the last step we used, that ω is an 2-cocycle (see above). Hence $y = z$ and therefore $c^*\omega c$ is an invariant 2-cocycle on \widehat{G}_q . Consequently, it remains to prove that

$$H^2(P/Q, \mathbb{T}) \cong H_{G_q}^2(\widehat{G}_q, \mathbb{T}) \cong H_{G_q}^2(\widehat{G}_q, \mathbb{C}^*).$$

We only sketch the main ideas, see [NT12] for more details.

Recall that the map $H^2(P/Q, \mathbb{T}) \rightarrow H_{G_q}^2(G_q, \mathbb{T})$, $c \mapsto \omega_c$ is well-defined as explained at the end of Chapter II. Let P_+ be the set of dominant integral weights. One can show that there exist operators

$$T_{\mu, \nu}: V^{\mu+\nu} \rightarrow V^\mu \otimes V^\nu, \quad \xi_{\mu+\nu} \mapsto \xi_\mu \otimes \xi_\nu,$$

where ξ denote the highest weight vectors. An invariant 2-cocycle acts on $T_{\mu, \nu}$ by multiplication with a scalar $c_\omega(\mu, \nu)$. This map c_ω can be seen as a 2-cocycle on P_+ . Moreover its isomorphism class in $H^2(P_+, \mathbb{C}^*)$ depends only on the class of ω . This yields a map

$$H_{G_q}^2(\widehat{G}_q, \mathbb{C}^*) \rightarrow H^2(P_+, \mathbb{C}^*).$$

One then shows that for every invariant 2-cocycle on \widehat{G}_q , the class of c_ω in $H^2(P_+, \mathbb{C}^*)$ lies in the image of the canonical map $H^2(P/Q, \mathbb{T}) \rightarrow H^2(P_+, \mathbb{C}^*)$. Hence the map $H_{G_q}^2(\widehat{G}_q, \mathbb{C}^*) \rightarrow H^2(P/Q, \mathbb{T})$, $\omega \mapsto c_\omega$ is well-defined, and it is clearly a left inverse of $c \mapsto \omega_c$. Finally one proves injectivity, which follows since the quotient map $P_+ \rightarrow P/Q$ is surjective and a cocycle on P/Q is a coboundary if it is symmetric. This establishes the isomorphism

$$H^2(P/Q, \mathbb{T}) \cong H_{G_q}^2(\widehat{G}_q, \mathbb{C}^*) \cong H_{G_q}^2(\widehat{G}_q, \mathbb{T}).$$

The last isomorphism follows from polar decomposition, see [NT11b, Lemma 1.1].

Together with the first part of the proof this concludes the statement. \square

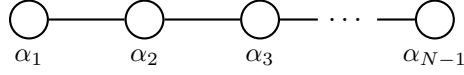
Let us briefly recall the so-called *based root datum* of a compact semisimple Lie group G . It is a tuple

$$\Psi = (X, R, \tilde{X}, \tilde{R}),$$

where X is the lattice of characters (of the maximal torus T of G), R is a fixed set of simple roots, $\tilde{X} = \text{Hom}(X, \mathbb{Z})$ and \tilde{R} is the set of simple coroots. For example for $SU(2)$ the based root datum is

$$\Psi = (\mathbb{Z}, \{2\}, \mathbb{Z}, \{1\}).$$

We also can give the based root datum a geometrical meaning, in the following way. For classification of semisimple Lie algebras we can consider *Dynkin diagrams*, which are graphs, with $\#R$ vertices, one then calculates the ‘‘angle’’ between neighboring simple roots. For instance the simple roots of $SU(N)$ can be represented as $\alpha_1 = e_1 - e_2, \dots, \alpha_{N-1} = e_{N-1} - e_N$, where $(e_i)_{i=1, \dots, N}$ denotes the standard normal basis. The associated Dynkin diagram is



A Lie algebra or group of this type is said to be of *type* A_{n-1} .

Intuitively this based root datum and the Dynkin diagram encode the structure of the Lie algebra, and is therefore a useful tool. We are particularly interested in automorphisms $\text{Aut}(\Psi)$ of a based root diagram Ψ , which correspond to automorphisms of the Dynkin diagram. For example in the case of $SU(N)$ with $N \geq 3$ one has

$$\text{Aut}(\Psi) = \{\text{id}, \theta: \alpha_i \mapsto \alpha_{N-i}\},$$

where θ is the reflection map.

Let Ψ be the based root datum of a Lie group G . There is a natural action of $\text{Aut}(\Psi)$ on $U_q(\mathfrak{g})$ by Hopf *-automorphisms, determined on the generators by $E_j \mapsto E_{\sigma j}$ and $K_j \mapsto K_{\sigma j}$ for $\sigma \in \text{Aut}(\Psi)$. This yields an action $\text{Aut}(\Psi) \curvearrowright (\mathcal{U}(G_q), \widehat{\Delta}_q)$. Its restriction to the maximal torus T , and hence to $Z(G)$, is independent of q . For a given c , we denote by $\text{Aut}(\Psi)_{c,q} \subseteq \text{Aut}(\Psi)$ the stabilizer of the image of ∂c in $H_{G_q}^3(\widehat{G}_q, \mathbb{T})$.

We will use the following remark repeatedly.

Remark 4.11: By Tannaka-Krein duality, an isomorphism of compact quantum groups $\mathbb{G}_1 \cong \mathbb{G}_2$ is equivalent to a unitary monoidal equivalence $F: \text{Rep}(\mathbb{G}_1) \rightarrow \text{Rep}(\mathbb{G}_2)$ together with unitary fiber functors $F_i: \text{Rep}(\mathbb{G}_i) \rightarrow \text{Hilb}_f$ and a natural unitary monoidal isomorphism $\eta: F_2 \circ F \rightarrow F_1$. Moreover, such an equivalence induces an isomorphism of fusion semirings $R^+(\mathbb{G}_1) \rightarrow R^+(\mathbb{G}_2)$.

Proposition 4.12: *Let $q > 0$ and let c be a \mathbb{T} -valued 2-cochain on P . Assume that ∂c descends to a 3-cochain on P/Q . Then there is a short exact sequence*

$$1 \rightarrow H^2(P/Q, \mathbb{T}) \rightarrow \text{Aut}^\otimes(\text{Rep}(G_q^c)) \rightarrow \text{Aut}(\Psi)_{c,q} \rightarrow 1.$$

Proof. By McMullen, see [McM84], there is an isomorphism $\text{Aut}(\Psi) \rightarrow \text{Aut}(R^+(G))$. Since $R^+(G_q^c) = R^+(G)$, any monoidal autoequivalence $F: \text{Rep}(G_q^c) \rightarrow \text{Rep}(G_q^c)$ induces an automorphism of the level of the fusion semiring, which then can be considered as an element in $\text{Aut}(\Psi)$. Thus we obtain a group homomorphism

$$\text{Aut}^\otimes(\text{Rep}(G_q^c)) \rightarrow \text{Aut}(\Psi).$$

Its kernel consists of those monoidal autoequivalences, that preserve the isomorphism classes of objects, which means $H_{G_q^c}^2(\widehat{G}_q^c, \mathbb{T})$. By Proposition 4.10, we have

$$H_{G_q^c}^2(\widehat{G}_q^c, \mathbb{T}) = H^2(P/Q, \mathbb{T})$$

hence the kernel is $H^2(P/Q, \mathbb{T})$.

Next we show that the image of the homomorphism $\text{Aut}^\otimes(\text{Rep}(G_q^c)) \rightarrow \text{Aut}(\Psi)$ is contained in the stabilizer subgroup $\text{Aut}(\Psi)_{c,q}$. Let $\sigma \in \text{Aut}(\Psi)$ be the element induced by a monoidal autoequivalence of $\text{Rep}(G_q^c)$. The action $\text{Aut}(\Psi) \curvearrowright \mathcal{U}(G_q^c)$, defines an isomorphism $G_q^c \cong G_q^{\sigma(c)}$, which means by definition a unitary monoidal equivalence

$$(\text{Rep}(G_q), \Phi^c) \cong (\text{Rep}(G_q), \Phi^{\sigma(c)}).$$

Since σ comes from an autoequivalence of $(\text{Rep}(G_q), \Phi^c)$, there exists a unitary monoidal equivalence $F: (\text{Rep}(G_q), \Phi^c) \rightarrow (\text{Rep}(G_q), \Phi^{\sigma(c)})$ that preserves isomorphism classes of objects. Such an equivalence F exists, since first we let σ act as an autoequivalence E on $\text{Rep}(G_q^c)$, i.e. on the categorical level. On an algebraic level, we let σ act on $\mathcal{U}(G_q^c) = \mathcal{U}(G_q)$, which induces an isomorphism $G_q^c \cong G_q^{\sigma(c)}$ and another monoidal equivalence $\tilde{E}: (\text{Rep}(G_q), \Phi^c) \rightarrow (\text{Rep}(G_q), \Phi^{\sigma(c)})$. We now can define the composition $F = \tilde{E}E^{-1}$, which preserves the isomorphism classes of objects, as stated above. By this we may assume that F acts as the identity on $\text{Rep}(G_q)$, which implies that F_2 can be seen as the action ω^{-1} of an invariant unitary 2-cochain ω on \widehat{G}_q . Then, by the associativity diagram of tensor functors, we see that $\Phi^{\sigma(c)}$ is the twisted 3-cocycle Φ_ω^c . Since Φ_ω^c and Φ^c are by definition cohomologous, we conclude that σ lies in the stabilizer subgroup $\text{Aut}(\Psi)_{c,q}$. We can use the same arguments in reverse to obtain surjectivity of $\text{Aut}^\otimes(\text{Rep}(G_q^c)) \rightarrow \text{Aut}(\Psi)_{c,q}$. This establishes the short exact sequence. \square

The following result also holds for general Lie type, except for D_{2M} . However, since we did not really introduce Dynkin diagrams for classification, we will not consider these cases.

Proposition 4.13: *Let $q > 0$ and c be a \mathbb{T} -valued 2-cochain on P , then*

$$\text{Aut}^\otimes(\text{Rep}(G_q^c)) \cong \text{Aut}(\Psi),$$

if G is of type A_N , which means that $G \cong SU(N+1)$.

Proof. By assumption P/Q is cyclic, namely $P/Q = \mathbb{Z}/N\mathbb{Z}$, hence by fact $H^2(P/Q, \mathbb{T})$ is trivial.

We only consider G of type A_{N-1} , i.e. we may assume that $G = SU(N)$. By the comments preceding Proposition 4.12, we only need to consider the reflection map $\theta: \alpha_i \mapsto \alpha_{n-i}$. The induced Hopf *-algebra homomorphism on $\mathcal{U}(\mathfrak{sl}(N))$ is again given by reflection. On $Z(G)$ which can be identified as $\mathbb{Z}/N\mathbb{Z}$ it is given as $a \mapsto -a$. The so induced map θ on the level of cocycles is therefore

$$\theta(\phi)(a, b, c) = \phi(-a, -b, -c).$$

Note that $\theta(\phi)$ and ϕ are cohomologous, since if ϕ , see Remark 2.36, is given as

$$\phi(a, b, c) = \omega(\lfloor \frac{a+b}{N} \rfloor - \lfloor \frac{a}{N} \rfloor - \lfloor \frac{b}{N} \rfloor)^c,$$

where ω is a N -th root of unity, then $\phi\theta(\phi)^{-1}$ is the coboundary of the 2-cochain

$$(a, b) \mapsto \omega^{-(\lfloor \frac{a}{N} \rfloor + \lfloor \frac{-a}{N} \rfloor)b}$$

on $\mathbb{Z}/N\mathbb{Z}$. □

A proof of the following result can again be found in [NY16]. The case $G = SU(N)$ was indirectly shown in our discussion of the Kazhdan-Wenzl-Jordans theorem.

Proposition 4.14: *Let G be a simply connected simple compact Lie group and let $q > 0$, then*

$$H^3(P/Q, \mathbb{T}) \rightarrow H_{G_q}^3(\widehat{G}_q, \mathbb{T})$$

is injective.

3. Classification Result

We can now prove the classification result, [NY16]. It provides only a partial answer to the classification problem posed by Woronowicz. The following remark explains why the Kac case is currently not feasible.

Remark 4.15: Let \mathbb{G} be a Kac compact quantum group of $SU(N)$ -type. By Tannaka-Krein duality, the classification of such quantum groups is equivalent to the classification of unitary fiber functors $\text{Rep}(SU(N)) \rightarrow \text{Hilb}_f$. By Corollary 2.16, these functors are dimension-preserving. Any such unitary fiber functor can be realized as

$$U \mapsto \text{Hom}_{SU(N)}(U, A_F).$$

More precisely, one can construct a C^* -algebra A_F (we omit the details) together with an action $\alpha: SU(N) \curvearrowright A_F$ such that $\text{Hom}_{SU(N)}(U, A_F)$ consists of the intertwiners between U_g and α_g for all $g \in SU(N)$. The resulting $SU(N)$ - C^* -algebra A_F induced by F is *ergodic*, meaning that the fixed-point algebra $A_F^\alpha = \mathbb{C}1$, and it is of *full multiplicity*, meaning that for every irreducible U we have

$$\dim(\text{Hom}_{SU(N)}(U, A_F)) = \dim(U).$$

In particular, $\dim F(U) = \dim(U)$.

Conversely, any $SU(N)$ - C^* -algebra A with an ergodic action of full multiplicity yields a dimension-preserving unitary fiber functor. Hence, the two classification problems are equivalent.

Moreover, by results of Wassermann, [Was89; Was88], classifying such actions is related to understanding *certain* subgroups of $SU(N)$, which is an open group-theoretic problem. The number of these subgroups grows quickly with N , and a complete description currently seems out of reach. Equivalently, an explicit

description of $H^2(\widehat{SU(N)}, \mathbb{T})$ is, in practice too hard. Intuitively, the difficulty in the Kac case is that one cannot reduce the problem to the maximal Kac quantum subgroup, i.e., the maximal torus. More details can be found in [NT11a].

Finally, note that for $N = 2, 3$ complete classification results are known (also in the Kac-case), see, e.g. [Ohn05]. If one restricts to the category of compact groups, then any compact group with the fusion rules of $SU(N)$ is isomorphic to $SU(N)$.

Lemma 4.16: *Assume that $\text{Rep}(G_{q_1}^{c_1})$ and $\text{Rep}(G_{q_2}^{c_2})$ are unitarily monoidally equivalent for $q_1, q_2 > 0$, where c_1, c_2 are \mathbb{T} -valued 2-cochains on P such that ∂c_1 and ∂c_2 descend to a 3-cochain on P/Q . Then either $q_1 = q_2^{-1}$ or $q_1 = q_2$.*

Proof. We may assume $q_1, q_2 \leq 1$, since G_q and $G_{q^{-1}}$ are isomorphic, and this isomorphism sends the maximal torus of G_q onto the maximal torus of $G_{q^{-1}}$. Thus it suffices to show $q_1 = q_2$.

As in Proposition 4.12, the monoidal equivalence induces an automorphism of the fusion semiring. Hence there exists $\sigma \in \text{Aut}(\Psi)$ and a unitary monoidal equivalence $F: \text{Rep}(G_{q_1}^{\sigma(c_1)}) \rightarrow \text{Rep}(G_{q_2}^{c_2})$ that induces the identity on the fusion semiring $R^+(G)$. Assume now that $q_1 < q_2 \leq 1$. By Lemma 3.14, the quantum dimension is independent of c , and the Woronowicz character of $\text{Rep}(G_q)$ is given by $f_1 = q^{-2\rho^*}$, where ρ^* denotes half the sum of the positive roots. Let w_0 be the longest element of the Weyl group. It sends positive roots to negative roots and satisfies $w_0(\rho^*) = -\rho^*$. Consequently, ρ^* acts on representations of G with a spectrum symmetric around 0. The quantum dimension is the trace of $q^{-2\rho^*}$, equivalently,

$$\dim_q(V) = \sum_{\mu} q^{-2\langle \mu, \rho^* \rangle},$$

where μ ranges over the weights of V . By symmetry, the terms occur in pairs $q^{-2\lambda} + q^{2\lambda}$. Note that $q \mapsto q + q^{-1}$ is strictly decreasing on $(0, 1]$, which induces that $\dim_{q_1} \neq \dim_{q_2}$. This is a contradiction to the unitary monoidal equivalence of $\text{Rep}(G_{q_1}^{c_1})$ and $\text{Rep}(G_{q_2}^{c_2})$, hence $q_1 = q_2$. \square

We can now describe precisely for which parameters q and 2-cochains c_1, c_2 we have the compact quantum groups $G_{q_1}^{c_1}$ and $G_{q_2}^{c_2}$ are isomorphic.

Proposition 4.17: *Let $q_1, q_2 \in (0, 1)$ and let c_1, c_2 be \mathbb{T} -valued 2-cochains on P such that ∂c_1 , and ∂c_2 descend to a 3-cochain on P/Q . Then $G_{q_1}^{c_1} \cong G_{q_2}^{c_2}$ if and only if $q_1 = q_2$ and there exist $\sigma \in \text{Aut}(\Psi)$ and a \mathbb{T} -valued 2-cochain b on P/Q such that $c_1 \sigma(c_2)^{-1} b^{-1}$ is a coboundary on P .*

Proof. Assume first that $q = q_1 = q_2$ and that there exist an element $\sigma \in \text{Aut}(\Psi)$ and a \mathbb{T} -valued 2-cochain b on P/Q such that $c_1 \sigma(c_2)^{-1} b^{-1}$ is a 2-coboundary on P . Let e be a 1-cochain on P such that $c_1 \sigma(c_2)^{-1} b^{-1} = \partial e$, i.e. $c_1 = \sigma(c_2) b \partial e$. Then we obtain the chain of isomorphisms

$$G_q^{c_2} \cong G_q^{\sigma(c_2)} \cong G_q^{\sigma(c_2)b} \cong G_q^{\sigma(c_2)b\partial e} = G_q^{c_1}.$$

Here the first isomorphism uses that any $\sigma \in \text{Aut}(\Psi)$ induces an automorphism of $\mathcal{U}(G_q)$. The second follows by definition, since any 2-cochain is defined on $P/Q \cong \widehat{Z(G)}$, i.e. it is central and commutes with $\widehat{\Delta}_q$. Finally, twisting by the coboundary ∂e is implemented on $\mathcal{U}(G_q)$ by conjugation with e , which can be viewed as a unitary in $\mathcal{U}(T)$.

Conversely assume $G_{q_1}^{c_1} \cong G_{q_2}^{c_2}$. By Lemma 4.16 we have $q = q_1 = q_2$.

As explained above, the isomorphism yields us an element in $\sigma^{-1} \in \text{Aut}(\Psi)$. Since $G_q^{\sigma(c_2)} \cong G_q^{c_2}$ we may assume that σ is the identity. From this again we obtain a unitary monoidal equivalence between $(\text{Rep}(G_q), \Phi^{c_1})$ and $(\text{Rep}(G_q), \Phi^{c_2})$, preserving the isomorphism classes of objects on $\text{Rep}(G_q)$. In particular Φ^{c_1} and Φ^{c_2} define the same class in $H_{G_q}^3(\widehat{G}_q, \mathbb{T})$. By the injectivity of the map $H^3(P/Q, \mathbb{T}) \rightarrow H_{G_q}^3(\widehat{G}_q, \mathbb{T})$, stated in Proposition 4.14, shows that the 3-cocycles ∂c_1 and ∂c_2 are cohomologous, so there exists a 2-cochain on P/Q such that

$$\partial c_1 = \partial c_2 \partial b.$$

Since as discussed above $G_q^{c_2 b} \cong G_q^{c_2}$, we may assume b is trivial, so $\partial c_1 = \partial c_2$. Therefore it remains to show that $c_1 c_2^{-1}$ is a coboundary on P .

Now let $F: \text{Rep}(G_q^{c_1}) \rightarrow \text{Hilb}_f$ be the unitary fiber functor corresponding to the 2-cocycle $c_2 c_1^{-1}$ on \widehat{T} . In the sense of Tannaka-Krein this fiber functor defines $G_q^{c_2}$. The isomorphism $G_q^{c_1} \cong G_q^{c_2}$ induces an autoequivalence E of $\text{Rep}(G_q^{c_1})$, such that an object is sent to an isomorphic object, such that the composition FE is the same as the canonical fiber functor $\text{Rep}(G_q^{c_1}) \rightarrow \text{Hilb}_f$. By definition E can be seen as an element in $H_{G_q}^2(\widehat{G}_q^{c_1}, \mathbb{T})$, and hence by Proposition 4.10, corresponds to a 2-cocycle b' on P/Q . We may assume again, that b' is trivial, instead of working with $c_2 b'$. Then F is naturally unitarily monoidally isomorphic to the canonical fiber functor $\text{Rep}(G_q^{c_1}) \rightarrow \text{Hilb}_f$ and thus Corollary 3.17 implies that $c_2 c_1^{-1}$ is a coboundary on $P = \widehat{T}$, the same holds for its inverse $c_1 c_2^{-1}$. \square

Note that the preceding proposition does not extend to the Kac-case $q = 1$ as Corollary 3.17 fails for $q = 1$. The map $H^2(\widehat{T}, \mathbb{T}) \rightarrow H^2(\widehat{G}_q, \mathbb{T})$ fails to be injective for $q = 1$.

Theorem 4.18 (Neshveyev-Yamashita): *Let $N \geq 2$ and let \mathbb{G} be a non-Kac compact quantum group of $SU(N)$ -type. Then there exist $q \in (0, 1)$ and a \mathbb{T} -valued 2-cochain c , such that ∂c descends to a 3-cocycle on $\widehat{Z(G)} \cong \mathbb{Z}/N\mathbb{Z}$ and $\mathbb{G} \cong SU_q^c(N)$. Moreover*

$$SU_{q_1}^{c_1}(N) \cong SU_{q_2}^{c_2}(N)$$

if and only if $q_1 = q_2$ and there exists a \mathbb{T} -valued 2-cochain b on $\widehat{Z(G)}$ such that either $c_1 c_2^{-1} b^{-1}$ or $c_1 \theta(c_2)^{-1} b^{-1}$ is a coboundary on \widehat{T} .

Proof. Now we can put all the previous work together.

Let \mathbb{G} be a non-Kac compact quantum group of $SU(N)$ -type. By Theorem 4.9 and Proposition 2.35 any rigid C^* -tensor category with the fusion rules of $SU(N)$ is unitarily monoidally equivalent to $(\text{Rep}(SU_q(N)), \Phi^c)$ for some $q \in (0, 1)$ and a \mathbb{T} -valued 2-cochain c such that ∂c descends to a 3-cocycle on the dual of the center of $SU(N)$, namely $Z(\widehat{SU(N)}) = \mathbb{Z}/N\mathbb{Z}$. Since by Corollary 3.17 shows that, for $q \neq 1$ the class of quantum groups of the form $SU_q^c(N)$ is closed under cocycle twisting for $q \neq 1$, i.e. we remain within the class of these quantum groups. This was a direct consequence from the result, that every dimension-preserving unitary fiber functor factors through the maximal Kac quantum subgroup, i.e. the maximal torus T . Thus every quantum group, which can be constructed by changing the fiber functor, is just another 2-cocycle deformation on \widehat{T} .

The isomorphism statement now follows from Proposition 4.17 and that the based root datum $\text{Aut}(\Psi)$ is of order 2 and generated by the reflection θ , see the discussion on Dynkin diagram automorphisms. \square

Finally we rewrite the classification in an explicit parameter form $SU_q^{c_k\omega}(N)$. Recall Remark 2.36. As a reminder, first we choose 2-cochains c_1, \dots, c_n , such that $\partial c_1, \dots, \partial c_n$ exhaust the group $H^3(\widehat{Z(G)}, \mathbb{T}) \cong \mathbb{Z}/N\mathbb{Z}$. Then any 2-cocycle c such that ∂c descends to a 3-cocycle on $\widehat{Z(G)}$ as $c = c_k\omega$ for some $1 \leq k \leq n$ and some skew-symmetric bicharacter ω on \widehat{T} . The result above describes how to construct ω .

Corollary 4.19: *Let $N \geq 2$ and let \mathbb{G} be a non-Kac compact quantum group of $SU(N)$ -type. Then there exist $q \in (0, 1)$, $\tau = (\tau_1, \dots, \tau_{N-1})$ be a $(N-1)$ -tuple of roots of unity of order N and ω be a skew-symmetric bicharacter ω on \widehat{T} , such that*

$$\mathbb{G} \cong SU_q^{c_\tau\omega}(N) =: SU_q^{\tau,\omega}(N).$$

Moreover $SU_q^{\tau,\omega}(N) \cong SU_{\tilde{q}}^{\tilde{\tau},\tilde{\omega}}(N)$ if and only if $q = \tilde{q}$, $\prod_{i=1}^{N-1} \tau_i^i = \prod_{i=1}^{N-1} \tilde{\tau}_i^i$ and, writing $\omega_{ij} = \omega(L_i, L_j)$ and $\tilde{\omega}_{ij} = \tilde{\omega}(L_i, L_j)$ and one of the following holds

(i)

$$\omega_{ij}^2 \prod_{k=i}^{j-1} \tau_k = \tilde{\omega}_{ij}^2 \prod_{k=i}^{j-1} \tilde{\tau}_k, \quad \text{for all } 1 \leq i < j \leq N-1. \quad (*)$$

(ii)

$$\omega_{ij}^2 \prod_{k=i}^{j-1} \tau_k = \tilde{\omega}_{(N-j)(N-i)}^2 \prod_{k=i}^{j-1} \tilde{\tau}_{N-k} \quad \text{for all } 1 \leq i < j \leq N-1.$$

Proof. The construction of $SU_q^{\tau,\omega}(N)$ was carried out in Remark 2.36. Hence it remains to prove the criterion for when two quantum groups of this form are

isomorphic. Concretely, we have to show that there exists a 2-cochain b such that $c_\tau \omega \theta (c_{\tilde{\tau}} \tilde{\omega})^{-1} b^{-1}$ is a coboundary if and only if $\prod_{i=1}^{n-1} \tau_i^i = \prod_{i=1}^{n-1} \tilde{\tau}_i^i$ and condition (i) holds. Note that $q = \tilde{q}$ follows again from Lemma 4.16. As in the proof of Proposition 4.17, we may also assume that θ is trivial.

Assume first that such a b exists. This means, cf. the steps in Lemma 4.16, that the 3-cocycles $\partial(c_\tau \omega)$ and $\partial(c_{\tilde{\tau}} \tilde{\omega})$ are cohomologous. Moreover if there exists a 2-cochain b on P/Q such that $\partial(c_\tau \omega) \partial(c_{\tilde{\tau}} \tilde{\omega})^{-1} = \partial b$, then $c_\tau \omega (c_{\tilde{\tau}} \tilde{\omega})^{-1} b^{-1}$ is a coboundary on P . The cochain b is unique up to a coboundary on P/Q .

By Remark 2.36 and since ω is a 2-cochain, i.e. $\partial \omega = 1$, we have $\partial(c_\tau \omega) = \partial(c_\tau)$ and $\partial(c_{\tilde{\tau}} \tilde{\omega}) = \partial(c_{\tilde{\tau}})$. Thus $\partial(c_\tau \omega)$ and $\partial(c_{\tilde{\tau}} \tilde{\omega})$ are cohomologous if and only if $\prod_{i=1}^{n-1} \tau_i^i = \prod_{i=1}^{n-1} \tilde{\tau}_i^i$. Define a bicharacter

$$f: P \times P \rightarrow \mathbb{T}, \quad f(L_i, L_j) = \prod_{k=i}^{N-1} \tau_k \tilde{\tau}_k^*.$$

It satisfies

$$f(\lambda, \mu + Q) = f(\lambda, \mu), \quad f(\lambda + \alpha_i, \mu) = \tau_i^{-|\mu|} \tilde{\tau}_i^{|\mu|} f(\lambda, \mu).$$

Recall that c_τ , respectively $c_{\tilde{\tau}}$, satisfies

$$c_\tau(\lambda + \alpha_i, \mu) = \langle \tau_i, \mu \rangle c_\tau(\lambda, \mu) = \tau_i^{-|\mu|} c_\tau(\lambda, \mu),$$

similar for $c_{\tilde{\tau}}$. Now define a 2-cochain b by

$$b(\lambda, \mu) = (c_\tau c_{\tilde{\tau}}^{-1})(\lambda, \mu) f(\lambda, \mu)^{-1}.$$

Then by construction

$$c_\tau c_{\tilde{\tau}}^{-1} = fb.$$

Moreover, again by construction, b is indeed a 2-cochain on P/Q , i.e. it is invariant under translation by Q in both arguments. Now note that f is also a 2-cocycle, hence the cochain b satisfies

$$\partial(c_\tau \omega) \partial(c_{\tilde{\tau}} \tilde{\omega})^{-1} = \partial(c_\tau) \partial(c_{\tilde{\tau}}) = \partial b.$$

Therefore the assumption that $c_\tau \omega (c_{\tilde{\tau}} \tilde{\omega})^{-1} b^{-1}$ is a 2-coboundary implies that the 2-cochain $f \omega \tilde{\omega}^{-1}$ is a coboundary. This is equivalent to $f \omega \tilde{\omega}^{-1}$ being symmetric. Since by assumption ω and $\tilde{\omega}$ are both skew-symmetric bicharacters, one has

$$f(L_i, L_j) \omega(L_i, L_j)^2 = f(L_j, L_i) \tilde{\omega}(L_i, L_j)^2, \quad 1 \leq i, j \leq N-1.$$

Equivalently

$$\omega(L_i, L_j)^2 \prod_{k=i}^{N-1} \tau_k \tilde{\tau}_k^* = \tilde{\omega}(L_i, L_j)^2 \prod_{k=j}^{N-1} \tau_k \tilde{\tau}_k^* \Leftrightarrow \omega_{i,j}^2 \prod_{k=i}^{j-1} \tau_k = \tilde{\omega}_{i,j}^2 \prod_{k=i}^{j-1} \tilde{\tau}_k,$$

which is exactly (i). Note that (ii) comes from the reflection map in the above theorem, and follows similarly.

The reverse implication follows by reading the above equivalences in the opposite direction. \square

Finally, note that it is also possible to find explicit generators of the corresponding Hopf $*$ -algebra and hence of the C^* -algebra. However we only refer to the end of [NY16] for this.

Example 4.20: Let us determine the possible ω 's for the case $N = 3$. By the properties mentioned in Remark 2.36, we know that the matrix ω needs to be of the form

$$\omega = \begin{pmatrix} 1 & z & \bar{z} \\ \bar{z} & 1 & z \\ z & \bar{z} & 1 \end{pmatrix}$$

for some $z \in \mathbb{T}$, since $\omega_{ii} = 1$, $\omega_{ij} = \overline{\omega_{ji}}$, and $\prod_{i=1}^3 \omega_{ij} = 1$ for all j .

Writing $x, y \in \mathbb{Z}^3$ in the basis L_1, L_2, L_3 and using the fact that the bicharacter is multiplicative, we can explicitly compute its value and obtain

$$\begin{aligned} \omega(x, y) &= \prod_{i,j=1}^3 \omega(L_i, L_j)^{x_i y_j} = \prod_{i,j=1}^3 \omega_{i,j}^{x_i y_j} \\ &= z^{x_1 y_2} z^{-x_2 y_1} z^{x_2 y_3} z^{-x_3 y_2} z^{-x_1 y_3} z^{x_3 y_1} \\ &= z^{\langle (x \times y), (1, 1, 1)^T \rangle}. \end{aligned}$$

Hence, non-Kac compact quantum groups of the form $SU_q^{\tau, \omega}(3)$ have in some sense exactly three degrees of freedom, parameterized by $(q, \tau, z) \in (0, 1) \times \mu_3 \times \mathbb{T}$ (where $\mu_3 \cong \mathbb{Z}/3\mathbb{Z}$ denotes the group of third roots of unity).

4. Outlook and Open Questions

In this section we collect a few related open problems.

The results above provide a complete classification of *non-Kac* compact quantum groups of $SU(N)$ -type, i.e. compact quantum groups whose fusion rules and classical dimension function agree with those of $SU(N)$, under the standing non-Kac assumption (namely, $q \neq 1$). It is natural to ask to what extent the methods and conclusions extend beyond this setting. We highlight several directions that are conceptually close to the present work and are still open (or at least not yet fully understood).

- (i) **The Kac case ($q = 1$) and the classification of fiber functors on $\text{Rep}(SU(N))$.** A key simplification in the non-Kac case is that every dimension-preserving unitary fiber functor factors through the maximal Kac quantum subgroup, which for the relevant q -deformations is the maximal torus T . For $q = 1$ this strategy breaks down: the map induced by T on the relevant H^2 -invariants is, in general, neither injective nor surjective, and the resulting classification of unitary fiber functors on $\text{Rep}(SU(N))$ is considerably richer. More concretely, specifying a dimension-preserving unitary fiber functor on $\text{Rep}(SU(N))$ is equivalent (via Tannaka-Krein reconstruction) to producing a compact quantum group with the same representation theory as $SU(N)$ in the Kac setting. As explained in Remark 4.15, in operator-algebraic terms this amounts to classifying certain full-multiplicity ergodic actions of $SU(N)$ on C^* -algebras. While low-rank cases can be handled, the general case appears extremely complicated, and understanding the space of such fiber functors remains one of the main open problems suggested by Woronowicz's original question.
- (ii) **Extending the $SU(N)$ -type classification to other compact simple Lie groups.** As mentioned in the introduction, from a structural point of view the classification splits into three parts:
- a) classification of rigid C^* -tensor categories with prescribed fusion rules,
 - b) classification of monoidal autoequivalences of these categories,
 - c) classification of dimension-preserving unitary fiber functors.

For $SU(n)$, step (a) is available via the Kazhdan-Wenzl-Jordan classification of $SU(n)$ -type categories (see Theorem 4.9), and steps (b)-(c) can then be completed using the techniques developed above.

For a general simply connected compact (semi-)simple Lie group G , a full analogue of the Kazhdan-Wenzl-Jordan theorem is currently not known, only partial results are available. A natural and important direction is therefore to develop a classification of rigid C^* -tensor categories (or suitable subclasses) that would allow one to run an analogous argument and obtain a corresponding classification of non-Kac compact quantum groups of G -type.

- (iii) **Poisson boundaries and maximal Kac quantum subgroups beyond the present setting.** A key tool in the classification was the categorical Poisson boundary, introduced in Chapter III, which yields a universal factorisation statement: for a coamenable compact quantum group \mathbb{G} with maximal Kac quantum subgroup \mathbb{K} , any dimension-preserving unitary fiber functor on $\text{Rep}(\mathbb{G})$ factors through $\text{Rep}(\mathbb{K})$. Since this result is structural, it is natural to ask for
- more explicit descriptions of the relevant Poisson boundaries in concrete examples,
 - extensions to broader classes of quantum groups or C^* -tensor categories (e.g. relaxing coamenability hypotheses, or isolating precisely which assumptions are indispensable).

We conclude that these directions highlight a remarkable rigidity and a kind of “torus control” in the non-Kac case, whereas the Kac case and extensions beyond $SU(N)$ ultimately depend on substantially deeper classification problems in the theory of quantum groups and C^* -tensor categories.

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Bibliography

- [Dri85] V. G. Drinfeld. “Hopf algebras and the quantum Yang-Baxter equation”. In: *Dokl. Akad. Nauk SSSR* 283.5 (1985), pp. 1060–1064.
- [Eti+15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*. Vol. 205. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015, pp. xvi+343.
- [Fad98] L. D. Faddeev. “How the algebraic Bethe ansatz works for integrable models”. In: *Symétries quantiques (Les Houches, 1995)*. North-Holland, Amsterdam, 1998, pp. 149–219. ISBN: 0-444-82867-2.
- [Für73] Harry Fürstenberg. “Boundary theory and stochastic processes on homogeneous spaces”. In: *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*. Vol. XXVI. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1973, pp. 193–229.
- [Hal15] Brian Hall. *Lie groups, Lie algebras, and representations*. Second. Vol. 222. Graduate Texts in Mathematics. An elementary introduction. Springer, Cham, 2015, pp. xiv+449.
- [HM23] Ralf Hinze and Dan Marsden. *Introducing string diagrams—the art of category theory*. Cambridge University Press, New York, 2023, pp. xiii+181.
- [Izu04] Masaki Izumi. “Non-commutative Poisson boundaries”. In: *Discrete geometric analysis*. Vol. 347. Contemp. Math. Amer. Math. Soc., Providence, RI, 2004, pp. 69–81. ISBN: 0-8218-3351-0.
- [Izu12] Masaki Izumi. “ E_0 -semigroups: around and beyond Arveson’s work”. In: *J. Operator Theory* 68.2 (2012), pp. 335–363.
- [Jim85] Michio Jimbo. “A q -difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation”. In: *Lett. Math. Phys.* 10.1 (1985), pp. 63–69.
- [Jim86] Michio Jimbo. “A q -analogue of $U(\mathfrak{gl}(N + 1))$, Hecke algebra, and the Yang-Baxter equation”. In: *Lett. Math. Phys.* 11.3 (1986), pp. 247–252.
- [Jor14] Bas P. A. Jordans. “A classification of $SU(d)$ -type C^* -tensor categories”. In: *Internat. J. Math.* 25.9 (2014), pp. 1450081, 40.

- [JSV96] André Joyal, Ross Street, and Dominic Verity. “Traced monoidal categories”. In: *Math. Proc. Cambridge Philos. Soc.* 119.3 (1996), pp. 447–468.
- [KS97] Anatoli Klimyk and Konrad Schmüdgen. *Quantum groups and their representations*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997, pp. xx+552.
- [KS98] Leonid I. Korogodski and Yan S. Soibelman. *Algebras of functions on quantum groups. Part I*. Vol. 56. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1998, pp. x+150. ISBN: 0-8218-0336-0.
- [KW93] David Kazhdan and Hans Wenzl. “Reconstructing monoidal categories”. In: *I. M. Gel’fand Seminar*. Vol. 16, Part 2. Adv. Soviet Math. Amer. Math. Soc., Providence, RI, 1993, pp. 111–136.
- [LR97] R. Longo and J. E. Roberts. “A theory of dimension”. In: *K-Theory* 11.2 (1997), pp. 103–159.
- [McM84] John R. McMullen. “On the dual object of a compact connected group”. In: *Math. Z.* 185.4 (1984), pp. 539–552.
- [NR67] Albert Nijenhuis and R. W. Richardson Jr. “Deformations of Lie algebra structures”. In: *J. Math. Mech.* 17 (1967), pp. 89–105.
- [NT11a] Sergey Neshveyev and Lars Tuset. “On second cohomology of duals of compact groups”. In: *Internat. J. Math.* 22.9 (2011), pp. 1231–1260.
- [NT11b] Sergey Neshveyev and Lars Tuset. “Symmetric invariant cocycles on the duals of q -deformations”. In: *Adv. Math.* 227.1 (2011), pp. 146–169.
- [NT12] Sergey Neshveyev and Lars Tuset. “Autoequivalences of the tensor category of $U_q\mathfrak{g}$ -modules”. In: *Int. Math. Res. Not. IMRN* 15 (2012), pp. 3498–3508.
- [NT13] Sergey Neshveyev and Lars Tuset. *Compact quantum groups and their representation categories*. Vol. 20. Cours Spécialisés [Specialized Courses]. Société Mathématique de France, Paris, 2013.
- [NY15] Sergey Neshveyev and Makoto Yamashita. “Twisting the q -deformations of compact semisimple Lie groups”. In: *J. Math. Soc. Japan* 67.2 (2015), pp. 637–662.
- [NY16] Sergey Neshveyev and Makoto Yamashita. “Classification of non-Kac compact quantum groups of $SU(n)$ type”. In: *Int. Math. Res. Not. IMRN* 11 (2016), pp. 3356–3391.
- [NY17a] Sergey Neshveyev and Makoto Yamashita. “Poisson boundaries of monoidal categories”. In: *Ann. Sci. Éc. Norm. Supér. (4)* 50.4 (2017), pp. 927–972.

- [NY17b] Sergey Neshveyev and Makoto Yamashita. “Towards a classification of compact quantum groups of Lie type”. In: *Operator algebras and applications—the Abel Symposium 2015*. Vol. 12. Abel Symp. Springer, 2017, pp. 231–263.
- [Ohn05] Christian Ohn. “Quantum $SL(3, \mathbb{C})$ ’s: the missing case”. In: *Hopf algebras in noncommutative geometry and physics*. Vol. 239. Lecture Notes in Pure and Appl. Math. Dekker, New York, 2005, pp. 245–255. ISBN: 0-8247-5759-9.
- [Pin07] Claudia Pinzari. “The representation category of the Woronowicz quantum group $S_\mu U(d)$ as a braided tensor C^* -category”. In: *Internat. J. Math.* 18.2 (2007), pp. 113–136.
- [PR11] Claudia Pinzari and John E. Roberts. “A rigidity result for extensions of braided tensor C^* -categories derived from compact matrix quantum groups”. In: *Comm. Math. Phys.* 306.3 (2011), pp. 647–662.
- [Ros90] Marc Rosso. “Algèbres enveloppantes quantifiées, groupes quantiques compacts de matrices et calcul différentiel non commutatif”. In: *Duke Math. J.* 61.1 (1990), pp. 11–40.
- [Ser66] Jean-Pierre Serre. *Algèbres de Lie semi-simples complexes*. W. A. Benjamin, Inc., New York-Amsterdam, 1966, viii+130 pp. (not consecutively paged).
- [TF79] L. A. Tahtadžjan and L. D. Faddeev. “The quantum method for the inverse problem and the XYZ Heisenberg model”. In: *Uspekhi Mat. Nauk* 34.5(209) (1979), pp. 13–63, 256.
- [Tim08] Thomas Timmermann. *An invitation to quantum groups and duality*. EMS Textbooks in Mathematics. From Hopf algebras to multiplicative unitaries and beyond. European Mathematical Society (EMS), Zürich, 2008. ISBN: 978-3-03719-043-2.
- [Tom07] Reiji Tomatsu. “A characterization of right coideals of quotient type and its application to classification of Poisson boundaries”. In: *Comm. Math. Phys.* 275.1 (2007), pp. 271–296.
- [Wan93] Shuzhou Wang. *General constructions of compact quantum groups*. Thesis (Ph.D.)—University of California, Berkeley. ProQuest LLC, Ann Arbor, MI, 1993.
- [Wan98] Shuzhou Wang. “Quantum symmetry groups of finite spaces”. In: *Comm. Math. Phys.* 195.1 (1998), pp. 195–211.
- [Was88] Antony Wassermann. “Ergodic actions of compact groups on operator algebras. II. Classification of full multiplicity ergodic actions”. In: *Canad. J. Math.* 40.6 (1988), pp. 1482–1527.

- [Was89] Antony Wassermann. “Ergodic actions of compact groups on operator algebras. I. General theory”. In: *Ann. of Math. (2)* 130.2 (1989), pp. 273–319.
- [Wor87] S. L. Woronowicz. “Compact matrix pseudogroups”. In: *Comm. Math. Phys.* 111.4 (1987), pp. 613–665.
- [Wor88] S. L. Woronowicz. “Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups”. In: *Invent. Math.* 93.1 (1988), pp. 35–76.
- [Wor98] S. L. Woronowicz. “Compact quantum groups”. In: *Symétries quantiques (Les Houches, 1995)*. North-Holland, Amsterdam, 1998, pp. 845–884. ISBN: 0-444-82867-2.