Nuclearity of Hypergraph C^*-Algebras

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Contents

1 Introduction

Directed graphs give a lucid description of many C^*-algebras, including all finite-dimensional C^*-algebras, the Toeplitz algebra and the algebra of compact operators on a separable Hilbert space. These are called graph C^*-algebras and there is a useful connection between properties of the C^*-algebra and easily accessible properties of the graph. A good reference for this topic is [Rae05].

The theory of graph C^*-algebras traces back to algebras generated by isometries that were introduced by Cuntz in 1977 [Cun77]. Later called Cuntz algebras, they feature many interesting properties that inspired the further development of C^*-algebra theory. A first generalization of Cuntz algebras by Cuntz and Krieger in 1980 led to so-called Cuntz-Krieger algebras which are associated to matrices with values from $\{0, 1\}$ [CK80]. In 1982, Watatani interpreted these matrices as adjacency matrices of a (finite) directed graph, and that gave birth to the notion of a graph C^*-algebra [Wat82]. However, it was not until 1997 that graph C^*-algebras were rediscovered by Kumjian, Pask, Raeburn, and Renault [Kum+97]. From that point on, graph C^*-algebras became an active field of research. A more detailed account of the history of graph C^*-algebras can be found in [DT05].

Hypergraph C^*-algebras naturally generalize the concept of graph C^*-algebras by passing from directed graphs to directed hypergraphs where an edge can have multiple vertices in its range or source. These algebras have been introduced in the group of Moritz Weber at Saarland University in two successive theses by Dean Zenner [Zen21] and Mirjam Trieb [Tri22]. Special cases of hypergraph C^*-algebras are algebras associated to ultragraphs that were introduced earlier in [Tom03]. It emerged that hypergraph C^*-algebras truly extend the class of graph C^*-algebras. In particular, unlike the latter, hypergraph C^*-algebras can be non-nuclear with an example given by the unital free product $C(S^1) *_{\mathbb{C}} \mathbb{C}^2$ [Zen21, Proposition 3.12].

The present thesis continues the study of hypergraph C^*-algebras and aims at a characterization of finite hypergraphs with nuclear C^*-algebra in terms of forbidden hypergraph minors. We introduce a tailor-made definition of (directed) hypergraph minors and obtain a result of the following form: For any hypergraph $\mathrm{H}\Gamma$ one can construct a hypergraph minor $\mathrm{H}\Delta$ of $\mathrm{H}\Gamma$ such that $C^*(\mathrm{H}\Gamma)$ is nuclear if, and only if, the same holds for $C^*(\mathrm{H}\Delta)$. Further, if the minors of $\mathrm{H}\Delta$ include one of four forbidden minors, then $C^*(\mathrm{H}\Delta)$ is not nuclear. Otherwise, the hypergraph $\mathrm{H}\Delta$ is – in a suitable sense – a “simple” hypergraph.

Graph minors are an important concept for the characterization of particular classes of undirected graphs. A famous theorem in the field is *Wagner’s Theorem* which states that an undirected graph is planar if, and only if, it has neither the complete graph with five vertices K_5 nor the complete bipartite graph with six vertices $K_{3,3}$ as graph minor. What’s more, the *Graph Minor Theorem* of Robertson and Seymour asserts that any class of undirected graphs which is closed under the minor operations can be described by a finite set of forbidden minors. A good reference for this topic is the survey [Lov05].

Generalizations of graph minors have been discussed for directed graphs and undirected hypergraphs, see e.g. [DM18] and [AGK12], respectively. However, in these contexts there is no agreed-upon definition of the minor relation and general results are scarce. For directed hypergraphs there seems to be no research so far.

This thesis is organized as follows. In **Chapter 2** we introduce basic notions and facts needed in the later work while the next **Chapter 3** presents the main results. In **Chapter 4** the seven hypergraph minor operations are introduced and investigated on the C^*-algebra side. Next, **Chapter 5** shows how nuclearity of a hypergraph C^*-algebra can be reduced to nuclearity of a simpler hypergraph C^*-algebra. Finally, in **Chapter 6** we discuss the C^*-algebras associated to the four forbidden hypergraph minors.

2 Preliminaries

Throughout this thesis, the variables A, B, C, D denote C^*-algebras. A $*$-homomorphism $\varphi : A \to B$ is called an embedding if it is injective. Whenever there is an embedding of B into A we write $B \subset A$, and if A and B are $*$-isomorphic we write $A = B$. An ideal $I \subset A$ is generally two-sided and closed. If $S \subset A$ is a subset of a C^*-algebra, then (S) denotes the ideal generated by S. M_k is the matrix algebra of dimension k, and we denote its standard matrix units consistently E_{ij} for $i, j \leq k$. We denote by $M_n(A)$ the C^*-algebra of $n \times n$-matrices with entries from A. We write S^1 for the unit circle in $\mathbb{C}$. The spectrum of an element $a \in A$ is denoted $\sigma(a)$.

We assume that the reader knows about (universal) C^*-algebras as well as projections and (partial) isometries in a C^*-algebra. A good reference for these topics is [Bla06].

2.1 Graph C^*-Algebras

This section presents the basic facts about graph C^*-algebras along the lines of [Rae05].

Definition 2.1 (directed graph). *A* directed graph Γ *is a tuple* (E^0, E^1, r, s) *where*

- $E^0 = E^0(\Gamma)$ *is the (countable) set of* vertices *of* Γ*,*
- $E^1 = E^1(\Gamma)$ *is the (countable) set of* edges *of* Γ*,*
- $r = r_\Gamma : E^1 \to E^0$ *maps every edge to its* range *(vertex),*
- $s = s_\Gamma : E^1 \to E^0$ *maps every edge to its* source *(vertex).*

We call Γ finite *if both* E^0 *and* E^1 *are finite. Further,* Γ *is* row-finite *if for every vertex* v *the set* $\{e \in E^1 : r(e) = v\}$ *is finite.*

If $s(e) = v$ we say that the edge e starts from v or that e is emitted from v; if $r(e) = v$ we say that e ends at v or that e is received by v. If for a vertex v there is no edge e with $s(e) = v$, then v is called a *sink*.

Definition 2.2 (graph C^*-algebra). *Let* Γ *be a row-finite directed graph. Then the* graph C^*-algebra $C^*(\Gamma)$ *is the universal* C^**-algebra generated by pairwise orthogonal projections* p_v *and partial isometries* s_e *for* $v \in E^0$*,* $e \in E^1$*, respectively, such that the following* Cuntz-Krieger relations *hold:*

(CK1) $$s_e^* s_f = \delta_{ef} p_{r(e)} \text{ for all } e, f \in E^1,$$

(CK2) $$p_v = \sum_{e:\, s(e)=v} s_e s_e^* \text{ for all } v \in E^0 \text{ with } \{e : s(e) = v\} \neq \emptyset.$$

Some authors formulate the relations (CK1)-(CK2) with the roles of ranges and sources swapped. Note that the convention used here is in accordance with [Zen21] and [Tri22] but diverges from [Rae05].

Recall that in general the universal C^*-algebra generated by certain elements satisfying certain relations need not exist as the norm might not be well-defined. However, it can be shown that for every row-finite directed graph Γ the C^*-algebra $C^*(\Gamma)$ exists and if Γ is finite, the C^*-algebra $C^*(\Gamma)$ is unital where its unit is given by $\sum_{v \in E^0} p_v$.

A (Cuntz-Krieger) Γ-family in a C^*-algebra A is a family $\{P_v, S_e\}_{v \in E^0(\Gamma), e \in E^1(\Gamma)} \subset A$ where the P_v are pairwise orthogonal projections, the S_e are partial isometries and (CK1)-(CK2) are satisfied with P_v (S_e) in the place of p_v (s_e). The following proposition spells out the universal property of $C^*(\Gamma)$.

Proposition 2.3 (universal property). *Let Γ be a row-finite directed graph and let $\{P_v, S_e\}_{v,e} \subset A$ be a Γ-family. Then there exists a unique $*$-homomorphism $\pi : C^*(\Gamma) \to A$ with $\pi(p_v) = P_v$ and $\pi(s_e) = S_e$ for all $v \in E^0(\Gamma)$, $e \in E^1(\Gamma)$, respectively.*

A *path* of length n in Γ is a finite sequence $e_1 \dots e_n$ of edges such that $r(e_i) = s(e_{i+1})$ holds for all $i < n$. A vertex is considered as a path of length zero and the set of all paths in Γ is denoted $E^* = E^*(\Gamma)$. The range and source functions are easily extended to E^* via $r(e_1 \dots e_n) = r(e_n)$ and $s(e_1 \dots e_n) = s(e_1)$. For a path v of length zero one sets $r(v) = s(v) = v$. We call a path $e_1 \dots e_n$ *closed* if $n \geq 1$ and $r(e_1) = r(e_n)$. A *cycle* starting at $v \in E^0$ is a path $\mu = e_1 \dots e_n$ of non-zero length such that $s(\mu) = r(\mu) = v$ and $r(e_i) = s(e_j)$ implies $j = i + 1$ for all $i < n$. To each path $\mu = e_1 \dots e_n$ there is an associated element $s_\mu := s_{e_1} \cdots s_{e_n}$ in $C^*(\Gamma)$. In particular, $s_v = p_v$ for all $v \in E^0(\Gamma)$. An inspection of $C^*(\Gamma)$ yields the following useful proposition.

Proposition 2.4. *[Rae05, Corollary 1.16] Let $\Gamma = (E^0, E^1, r, s)$ be a row-finite directed graph. Then*

$$C^*(\Gamma) = \overline{\operatorname{span}}(s_\mu s_\nu^* \,|\, \mu, \nu \in E^*, r(\mu) = r(\nu))$$

It follows that whenever Γ is finite and contains no cycle, then $C^*(\Gamma)$ is finite dimensional. In this case there is a concrete formula for $C^*(\Gamma)$.

Proposition 2.5. *[Rae05, Proposition 1.18] Let $\Gamma = (E^0, E^1, r, s)$ be a finite directed graph with no cycles and let $v_1, \dots, v_n$ be the sinks in Γ. Then*

$$C^*(\Gamma) = \bigoplus_{i=1}^{n} M_{|r^{-1}(v_i)|}, \qquad \textit{where } r^{-1}(v_i) = \{\mu \in E^* \,|\, r(\mu) = v_i\}.$$

The isomorphism maps a projection $p_v \in C^*(\Gamma)$ to a tuple $(x_i)_{i=1}^n \in \bigoplus_{i=1}^n M_{|r^{-1}(v_i)|}$ where for every x_i we have that x_i is a one-dimensional projection in $M_{|r^{-1}(v_i)|}$ if there is a path from v to v_i and otherwise $x_i = 0$.

Note, that in the previous proposition the set $r^{-1}(v_i)$ contains as one element the zero-length path v_i.

If a Γ-family $\{P_v, S_e\}_{v,e} \subset A$ is given, it is usually not trivial to show that the subalgebra $C^*(\{P_v, S_e\}_{v,e}) \subset A$ generated by this family is isomorphic to the graph C^*-algebra $C^*(\Gamma)$. However, there are two famous uniqueness theorems for graph C^*-algebras which ensure the latter. The *Cuntz-Krieger uniqueness theorem* applies as soon as Γ has a special property, while the *gauge invariant uniqueness theorem* requires the existence of a special action on A.

Theorem 2.6 (Cuntz-Krieger uniqueness theorem). *Let Γ be a row-finite directed graph such that every cycle in Γ has an entry, i.e. if $\mu = e_1 \dots e_n$ is a cycle, then there exists an edge f and $i \leq n$ with $s(f) = s(e_i)$ and $f \notin \{e_i : i \leq n\}$. Further, assume that $\{P_v, S_e\}_{v,e} \subset A$ is a Γ-family with $P_v \neq 0$ for all $v \in E^0(\Gamma)$ and let $\pi : C^*(\Gamma) \to A$ be given by the universal property. Then π is injective.*

Definition 2.7 (gauge action). *An* action *of a locally compact group G on A is a map $\alpha : G \to \mathrm{Aut}(A)$ such that*

- *$\alpha(st) = \alpha(s) \circ \alpha(t)$ for all $s, t \in G$,*
- *for every fixed $a \in A$ the map $G \ni t \mapsto \alpha(t)(a) =: \alpha_t(a) \in A$ is continuous.*

On $C^*(\Gamma)$ a canonical action α of S^1 is given by

$$\alpha_\lambda(p_v) = p_v \quad \text{and} \quad \alpha_\lambda(s_e) = \lambda s_e \quad \text{for all } \lambda \in S^1$$

via the universal property of $C^*(\Gamma)$. Then α is called the gauge action *of S^1 on $C^*(\Gamma)$.*

Theorem 2.8 (gauge uniqueness theorem). *Let Γ be a row-finite directed graph and let $\{P_v, S_e\} \subset A$ be a Γ-family in A. Further, let $\pi : C^*(\Gamma) \to A$ be given by the universal property of $C^*(\Gamma)$. If $P_v \neq 0$ holds for all $v \in E^0$ and there is an action β of S^1 on A such that*

1. *$\beta_\lambda(S_e) = \lambda S_e$ for all $\lambda \in S^1, e \in E^1$,*
2. *$\beta_\lambda(P_v) = P_v$ for all $\lambda \in S^1, v \in E^0$,*

then the map π is injective. The conditions (1)-(2) are equivalent to $\pi \circ \alpha_\lambda = \beta_\lambda \circ \pi$ for all $\lambda \in S^1$ and the gauge action α on $C^*(\Gamma)$.

Example 2.9. *Let A be generated by a unitary s and assume that A admits an action β of S^1 with $\beta_\lambda(s) = \lambda s$ for all $\lambda \in S^1$. Then $A = C(S^1)$.*

Proof. We know that $C(S^1)$ is the universal C^*-algebra generated by a unitary element u. A moment's thought shows that the latter is the C^*-algebra generated by the graph Γ with one vertex v and one edge e such that $s(e) = r(e) = v$. Now, s and 1_A form a Γ-family since s is a partial isometry with $ss^* = s^*s = 1_A$. Hence, the universal property of $C^*(\Gamma)$ provides a surjective map $\pi : C^*(\Gamma) \to A$. It is $\pi(v) = 1_A \neq 0$ and the action β on A satisfies the requirements from the previous theorem; thus π is injective. □

2.2 Hypergraph C^*-Algebras

Below we summarize the basic facts about hypergraph C^*-algebras according to Dean Zenner's Bachelor's Thesis [Zen21] and the subsequent Master's Thesis by Mirjam Trieb [Tri22]. At the same time we slightly generalize the concept of a hypergraph C^*-algebra so that hypergraph edges are allowed to have empty range in a meaningful way.

Definition 2.10. *A hypergraph $\mathrm{H}\Gamma$ is a tuple (E^0, E^1, r, s), where*

- *$E^0 = E^0(\mathrm{H}\Gamma)$ is the (countable) set of vertices of $\mathrm{H}\Gamma$,*
- *$E^1 = E^1(\mathrm{H}\Gamma)$ is the (countable) set of edges of $\mathrm{H}\Gamma$,*

- $r = r_{\mathrm{H}\Gamma} : E^1 \to \mathcal{P}(E^0)$ *maps every edge to its* range *(set),*
- $s = s_{\mathrm{H}\Gamma} : E^1 \to \mathcal{P}(E^1) \setminus \{\emptyset\}$ *maps every edge to its* source *(set).*

We call $\mathrm{H}\Gamma$ finite *if both* E^0 *and* E^1 *are finite sets, and we call* $\mathrm{H}\Gamma$ undirected *if every edge* $e \in E^1(\mathrm{H}\Gamma)$ *has empty range.*

From now on, every hypergraph is understood to be finite. A specific hypergraph is commonly given by a sketch as below.

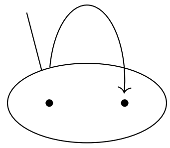


Figure 2.1: Example Sketch of a Hypergraph

This describes the hypergraph $\mathrm{H}\Gamma$ with vertices $E^0(\mathrm{H}\Gamma) = \{v, w\}$ and edges $E^1(\mathrm{H}\Gamma) = \{e, f\}$ such that $s(e) = s(f) = \{v, w\}$, $r(e) = \emptyset$ and $r(f) = \{w\}$. Note that the line without an arrow corresponds to the edge with empty range.

Definition 2.11 (hypergraph C^*-algebra)**.** *Let* $\mathrm{H}\Gamma$ *be a hypergraph. The hypergraph* C^**-algebra* $C^*(\mathrm{H}\Gamma)$ *is the universal* C^**-algebra generated by pairwise orthogonal projections* p_v *and partial isometries* s_e *for* $v \in E^0$*,* $e \in E^1$*, respectively, such that the following holds:*

$$\text{(HR1)} \qquad s_e^* s_f = \begin{cases} \delta_{ef} \sum_{v \in r(e)} p_v, & r(e) \neq \emptyset, \\ \delta_{ef} s_e, & \textit{otherwise}, \end{cases} \quad \textit{for all } e, f \in E^1,$$

$$\text{(HR2)} \qquad s_e s_e^* \leq \sum_{v \in s(e)} p_v \textit{ for all } e \in E^1,$$

$$\text{(HR3)} \qquad p_v \leq \sum_{e : v \in s(e)} s_e s_e^* \textit{ for all } v \in E^0 \textit{ with } \{e : v \in s(e)\} \neq \emptyset.$$

In (HR1) this definition diverges from [Zen21] and [Tri22]. In their definition, edges with empty range are effectively forbidden whereas our relation (HR1) implies that for edges e with $s(e) = \emptyset$ the partial isometry s_e is in fact a projection. It will become clear later that this does not alter the class of hypergraph C^*-algebras up to Morita equivalence. Moreover, none of the following facts about hypergraph C^*-algebras is affected.

It can be shown that in the above situation $C^*(\mathrm{H}\Gamma)$ exists in the sense of a universal C^*-algebra and that $C^*(\mathrm{H}\Gamma)$ has a unit given by $\sum_{v \in E^0} p_v$, see [Zen21, Theorem 3.9].

Similarly as in the previous section, $\{P_v, S_e\}_{v \in E^0, e \in E^1} \subset A$ is an $\mathrm{H}\Gamma$-family if the P_v are pairwise orthogonal projections and the S_e are partial isometries such that the relation (HR1)-(HR3) are satisfied with P_v (S_e) in the place of p_v (s_e). The following universal property holds.

Proposition 2.12 (universal property)**.** *Let* $\mathrm{H}\Gamma$ *be a hypergraph and let* $\{P_v, S_e\}_{v,e} \subset A$ *be an* $\mathrm{H}\Gamma$*-family. Then there exists a unique* **-homomorphism* $\pi : C^*(\mathrm{H}\Gamma) \to A$ *with* $\pi(p_v) = P_v$ *and* $\pi(s_e) = S_e$ *for all* $v \in E^0(\mathrm{H}\Gamma), e \in E^1(\mathrm{H}\Gamma)$*, respectively.*

In $\mathrm{H}\Gamma$ we say that an edge $e \in E^1$ starts from a set $V \subset E^0$ if $s(e) \cap V \neq \emptyset$. A vertex $v \in E^0$ is called a sink if there is no edge $e \in E^1$ with $v \in s(e)$. A path of length n is a sequence $\mu = e_1 \dots e_n$ of edges with $r(e_i) \cap s(e_{i+1}) \neq \emptyset$ for all $i < n$ and the vertices are considered as paths of length zero. E^* denotes the set of all paths. For every path μ the element $s_\mu \in C^*(\mathrm{H}\Gamma)$ and $r(\mu), s(\mu)$ are defined similarly as for graphs. Further, a path μ is called *closed* if μ has non-zero length and $r(\mu) \cap s(\mu) \neq \emptyset$. A closed path $\mu = e_1 \dots e_n$ is a cycle if $r(e_i) \cap s(e_j) \neq \emptyset$ implies $j = i+1$ for all $i < n$.

Unfortunately, for hypergraph C^*-algebras we do not have a nice dense subset as in Proposition 2.4. Instead, the following holds.

Proposition 2.13. *[Tri22, Corollary 2.24] We have*

$$C^*(\mathrm{H}\Gamma) = \overline{\operatorname{span}} \left(s_{\mu_1}^{\epsilon_1} \cdots s_{\mu_n}^{\epsilon_n} | n \in \mathbb{N}, \mu_i \in E^*, \epsilon_1, \dots, \epsilon_n \in \{1, *\}, \epsilon_i \neq \epsilon_{i+1} \text{ for } i < n \right).$$

Better results can be obtained using certain requirements on the paths in a hypergraph. For these we refer the interested reader to [Tri22] where the notion of a (quasi-)perfect path is introduced and studied.

For our purposes we reformulate the previous proposition as follows.

Lemma 2.14. *Let $\mathrm{H}\Gamma$ be a hypergraph. A dense subset of $C^*(\mathrm{H}\Gamma)$ is spanned by products of the form*

$$x = x_1 \dots x_n \qquad \text{with } n \in \mathbb{N}, x_i \in \{p_v, s_e, s_e^* : v \in E^0, e \in E^1\},$$

where for every $i < n$ neither of the following is true:

a) $x_i x_{i+1} = s_e^* s_f$ for some edges $e, f \in E^1$.

b) $x_i x_{i+1} = s_e p_v$ or $x_i x_{i+1} = p_v s_e^*$ for some $e \in E^1, v \in E^0$ with $r(e) \neq \emptyset$, and either $v \notin r(e)$ or $\{v\} = r(e)$.

c) $x_i x_{i+1} = s_e p_v$ or $x_i x_{i+1} = p_v s_e^*$ for some $e \in E^1, v \in E^0$ with $r(e) = \emptyset$, and either $v \notin s(e)$ or $\{v\} = s(e)$.

d) $x_i x_{i+1} = p_v s_e$ or $x_i x_{i+1} = s_e^* p_v$ for some $e \in E^1, v \in E^0$ with $v \notin s(e)$ or $\{v\} = s(e)$.

e) $x_i x_{i+1} = s_e s_f$ or $x_i x_{i+1} = s_f^* s_e^*$ for some $e, f \in E^1$ with $r(e) \cap s(f) = \emptyset$.

f) $x_i x_{i+1} = s_e s_f^*$ for some $e, f \in E^1$ with $r(e) \neq \emptyset$ and $r(e) \cap r(f) = \emptyset$.

g) $x_i x_{i+1} = s_e s_f^*$ for some $e, f \in E^1$ with $r(e) = \emptyset$ and $s(e) \cap r(f) = \emptyset$.

Proof. It follows from Proposition 2.13 that words of the given form span a dense subset of $C^*(\mathrm{H}\Gamma)$. It remains to show that we can do without words which satisfy one of the conditions (a) – (e). For that, we show that in each of those situations either $x = 0$ or x can be expressed as linear combination of similar products x_i which are shorter in the sense that they contain fewer factors. Then the claim follows by induction.

Ad (a): For $e \neq f$ this is obvious. If $e = f$ and $r(e) \neq \emptyset$, then we have

$$x = x_1 \dots x_{i-1} s_e^* s_e x_{i+1} \dots x_n = x_1 \dots x_{i-1} \left(\sum_{v \in r(e)} v \right) x_{i+1} \dots x_n = \sum_{v \in r(e)} x_1 \dots x_{i-1} v x_{i+2} \dots x_n.$$

Finally, if $r(e) \neq \emptyset$, then $s_e^* s_e = s_e$ and therefore

$$x = x_1 \dots x_{i-1} s_e^* s_e x_{i+2} x_n = x_1 \dots x_{i-1} s_e x_{i+2} \dots x_n.$$

Ad (b): By symmetry, it suffices to consider the case $x_i x_{i+1} = s_e p_v$. If $v \notin r(e)$, then $s_e p_v = s_e s_e^* s_e v = 0$ which implies $x = 0$. If $\{v\} = r(e)$ then [Tri22, Proposition 2.12] yields $s_e p_v = s_e$ and therefore

$$x = x_1 \ldots x_{i-1} s_e p_v s_{i+2} \ldots x_n = x_1 \ldots x_{i-1} s_e x_{i+2} \ldots x_n.$$

Ad (c): Again it suffices to consider the case $x_i x_{i+1} = s_e p_v$. If $v \notin s(e)$, then one checks $s_e p_v = 0$. If $\{v\} = s(e)$, then one has $s_e p_v = s_e$.

Ad (d): Analogous to (b).

Ad (e), (f): In these situations, the product x is zero by [Tri22, Proposition 2.15].

Ad (g): In this situation $s_e s_f^* = s_e^* s_f^*$ and one obtains $x = 0$ from (e). □

An important observation of [Tri22] is that, without changing the associated C^*-algebra, in a hypergraph the ranges of edges can be decomposed so that $|r(e)| = 1$ holds for all $e \in E^1$. More precisely we have the following theorem.

Theorem 2.15. *[Tri22, Theorem 4.1] Let* $\mathrm{H}\Gamma$ *be a given hypergraph and define* $\mathrm{H}\Delta$ *by*

- $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma)$,
- $E^1(\mathrm{H}\Delta) = \{(e, v) : e \in E^1(\mathrm{H}\Gamma), v \in r_{\mathrm{H}\Gamma}(e)\}$,
- $r_{\mathrm{H}\Delta}((e, v)) = \{v\}$,
- $s_{\mathrm{H}\Delta}((e, v)) = s_{\mathrm{H}\Gamma}(e)$.

Then $C^*(\mathrm{H}\Delta) = C^*(\mathrm{H}\Gamma)$.

Remark 2.16. *To simplify notation, from now on we will identify vertices and edges in* $\mathrm{H}\Gamma$ *with their respective elements in* $C^*(\mathrm{H}\Gamma)$. *Thus, we will do without the notations* p_v, s_e, s_μ *and simply consider* v, e *and* μ *as elements of* $C^*(\mathrm{H}\Gamma)$. *More precisely, we make the following identifications:*

$$\begin{cases} v = p_v & \text{for all } v \in E^0, \\ e = s_e & \text{for all } e \in E^1, \\ \mu = e_1 \cdots e_n & \text{for all } \mu = e_1 \ldots e_n \in E^*, \\ r(e) = \sum_{v \in r(e)} v, & \text{for all } e \in E^1, \\ s(e) = \sum_{v \in s(e)} v, & \text{for all } e \in E^1. \end{cases}$$

Then the hypergraph relations take the following form:

(HR1) $$e^* f = \begin{cases} \delta_{ef} r(e), & r(e) \neq \emptyset, \\ \delta_{ef} e, & \text{otherwise}, \end{cases} \quad \text{for all } e, f \in E^1,$$

(HR2) $$ee^* \leq s(e) \text{ for all } e \in E^1,$$

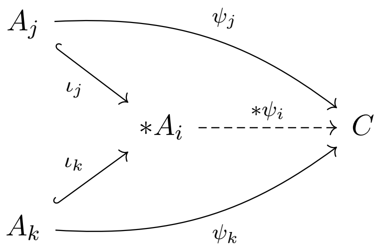
(HR3) $$v \leq \sum_{e : v \in s(e)} ee^* \text{ for all } v \in E^0 \text{ with } \{e : v \in s(e)\} \neq \emptyset.$$

2.3 (Amalgamated) Free Products

As mentioned in the introduction the first example of a non-nuclear hypergraph C^*-algebra is given by the unital free product $C(S^1) *_{\mathbb{C}} \mathbb{C}^2$. This section deals with the definition and first properties of the underlying concept of (amalgamated) free products of C^*-algebras.

Definition 2.17 (Free Product)**.** *Let $(A_i)_{i\in I}$ be a family of C^*-algebras. The* free product $*A_i$ *is the unique C^*-algebra $*A_i$ with embeddings $\iota_j : A_j \to *A_i$ such that the following universal property holds:*

For any C^*-algebra C and any family of $*$-homomorphisms $\psi_i : A_i \to C$ $(i \in I)$ there is a unique $*$-homomorphism $*\psi_i : *A_i \to C$ such that the diagram below commutes for all j, k.



To construct the free product more explicitly, one forms the universal C^*-algebra generated by copies of the A_i without any additional relations [Bla06, II.8.3.4]. As usually, we often do not mention the embeddings ι_i and understand that A_i is a subalgebra of $*A_i$. The following is an immediate consequence of the universal property.

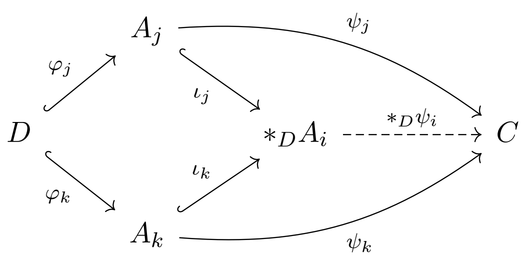
Proposition 2.18. *Let $*A_i$ be a free product. Then*

$$*A_i = \overline{\operatorname{span}}(x_1 \dots x_m \,|\, x_k \in A_{i_k} \text{ for } k \leq m \text{ and suitable } i_k \in I).$$

If the A_i have a common subalgebra D we can glue together the copies of D in $*A_i$. This leads to the more general concept of an amalgamated free product.

Definition 2.19 (Amalgamated Free Product)**.** *Let $\varphi_i : D \to A_i$ $(i \in I)$ be embeddings. The* amalgamated free product $*_D A_i$ *is the unique C^*-algebra with embeddings $\iota_j : A_j \to *_D A_i$ such that the following universal property holds:*

For any C^*-algebra C and any family of $*$-homomorphisms $\psi_i : A_i \to C$ $(i \in I)$ with $\psi_i \circ \varphi_i = \psi_j \circ \varphi_j$ there is a unique $*$-homomorphism $*_D\psi_i : *_D A_i \to C$ such that the diagram below commutes for all j, k.



Remark 2.20. *i) One can construct the amalgamated free product as the quotient of the free product $*A_i$ over the ideal generated by $\{\varphi_j(d) - \varphi_k(d) \mid d \in D, j, k\}$ [Ped99, p. 247]. In other words, the different copies of D contained in the A_i are identified. In particular, Proposition 2.18 remains valid for the amalgamated free product. However, it is not trivial to show that the maps ι_i are injective. To prove this, one needs suitable amplifications of the universal representations of A_i; for the details see [Bla78, Theorem 3.1].*

ii) The notation $*_D A_i$ can be deceptive as it hides the dependence on the maps φ_i. To make this more transparent we denote the amalgamated free product alternatively $*_{\varphi_i} A_i$ (or $A *_{\varphi_i} B$ if $|I| = 2$). In the latter case, we understand that φ_1 is an embedding of some D into A and φ_2 an embedding of D into B. If $D = \mathbb{C}$ we also write $A *_{\varphi_1(1)=\varphi_2(1)} B$ for $A *_{\varphi_i} B$.

iii) If the A_i are unital and $\varphi_i : \mathbb{C} \to A_i$ is the unique unital homomorphism, then we call $*_{\mathbb{C}} A_i$ the unital free product. Unless specified otherwise, $*_{\mathbb{C}} A_i$ is the unital free product.

The following proposition is about amalgamated free products of *universal* C^*-algebras that are defined via prescribed relations. The proof is a simple exercise using the respective universal properties.

Proposition 2.21. *Let A be the universal C^*-algebra generated by elements $(a_j)_j$ that satisfy some relations $\mathcal{R}_A$ and let $B, (b_k)_k, \mathcal{R}_B$ and $D, (d_\ell)_\ell, \mathcal{R}_D$ be analogous. Assume that φ_1, φ_2 is an embedding of D into A or B, respectively. Then*

$$A *_{\varphi_i} B = C^*(a_i, b_j \mid \mathcal{R}_A, \mathcal{R}_B, \varphi_1(d_\ell) = \varphi_2(d_\ell)).$$

Below we use the previous proposition to find an embedding of $M_2 *_{\mathbb{C}} \mathbb{C}^2$ into $M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2$. We will use this embedding later on.

Example 2.22. *There is an embedding $\varphi : M_2 *_{\mathbb{C}} \mathbb{C}^2 \to M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2$ with*

$$\varphi : \begin{cases} \hat{E}_{ij} \mapsto E_{(i+1)(j+1)}, & i, j \leq 2, \\ p_k \mapsto p_k, & k \leq 2, \end{cases} \tag{$*$}$$

where we write $\hat{E}_{ij}, E_{ij}$ for the matrix units of M_2 and M_3, respectively, and p_k for the standard units of $\mathbb{C}^2$. Let $\mathcal{R}_{M_2}$, $\mathcal{R}_{M_3}$ and $\mathcal{R}_{\mathbb{C}^2}$ be the relations required from the $\hat{E}_{ij}$, E_{ij} and p_k, respectively. Then we have by the previous proposition

$$M_2 *_{\mathbb{C}} \mathbb{C}^2 = C^*(\hat{E}_{ij}, p_k | \mathcal{R}_{M_2}, \mathcal{R}_{\mathbb{C}^2}, \hat{E}_{11} + \hat{E}_{22} = p_1 + p_2)$$

and

$$M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2 = C^*(E_{ij}, p_k | \mathcal{R}_{M_3}, \mathcal{R}_{\mathbb{C}^2}, E_{22} + E_{33} = p_1 + p_2).$$

Clearly, the universal property of the first C^*-algebra above yields the existence of a map φ with $(*)$. It remains to show that φ is injective.

For that, let $\rho : M_2 *_{\mathbb{C}} \mathbb{C}^2 \to B(\mathcal{H})$ be the universal representation given by the GNS construction. Further, let $\mathcal{K}$ be a Hilbert space of the same dimension as $\rho(\hat{E}_{11})\mathcal{H}$. Then, in $\mathcal{B}(\mathcal{K} \oplus \mathcal{H})$ there are partial isometries V_1 and V_2

from $\mathcal{K}$ to $\rho(\hat{E}_{11})\mathcal{H}$ and $\rho(\hat{E}_{22})\mathcal{H}$, respectively. Now, use the universal property of $M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2$ to construct a representation π of $M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2$ on $\mathcal{K} \oplus \mathcal{H}$ with

$$\pi : \begin{cases} \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Id}_{\mathcal{K}} \oplus 0 & V_1^* & V_2^* \\ V_1 & 0 \oplus \rho\left(\hat{E}_{11}\right) & 0 \oplus \rho\left(\hat{E}_{12}\right) \\ V_2 & 0 \oplus \rho\left(\hat{E}_{21}\right) & 0 \oplus \rho\left(\hat{E}_{22}\right) \end{pmatrix}, \\ p_k \mapsto 0 \oplus \rho(p_k), \qquad\qquad k = 1, 2. \end{cases}$$

Indeed, the necessary relations are easily checked. Observe that ρ is equal to $\pi \circ \varphi$ restricted to $\mathcal{H}$. Thus, for any $x \in M_2 *_{\mathbb{C}} \mathbb{C}^2$ we have $x \in \ker(\varphi) \implies x \in \ker(\rho)$. The latter entails $x = 0$ and this concludes the proof.

2.4 Group C^*-Algebras

To every discrete group G one can associate a C^*-algebra $C^*(G)$. In particular, this yields a useful description of such well-known C^*-algebras like $\mathbb{C}^n$ or $C(S^1)$. In this section, we scratch the surface of this topic and collect only a few facts which we will refer to later.

Definition 2.23. *Let G be a (discrete) group. The (full or maximal)* group C^*-algebra *$C^*(G)$ is the universal C^*-algebra generated by unitaries $(u_t)_{t\in G}$ such that $u_{st} = u_s u_t$ holds for all $s, t \in G$.*

Example 2.24.

1. *For all $n \in \mathbb{N}$ it is $\mathbb{C}^n = C^*(\mathbb{Z}_n)$, where $\mathbb{Z}_n = \mathbb{Z}/(n)$ is the cyclic group generated by one element of order n. Indeed, $C^*(\mathbb{Z}_n)$ is generated by one single unitary u with $u^n = 1$. It follows that $C^*(\mathbb{Z}_n)$ is a commutative, n-dimensional C^*-algebra, i.e. it is isomorphic to $\mathbb{C}^n$.*
2. *It is $C(S^1) = C^*(\mathbb{Z})$. Indeed, let u be the unitary generator of the algebra on the right-hand side. Then u is normal, and we obtain $C^*(\mathbb{Z}) = C(\sigma(u))$ from Gelfand theory. It is not difficult to show that $\sigma(u) = S^1$ and this yields the claim.*

The following facts are easily checked.

Proposition 2.25. *Let G_1, G_2 be groups. Then*

1. *$G_1 \subset G_2$ implies $C^*(G_1) \subset C^*(G_2)$,*
2. *$C^*(G_1) *_{\mathbb{C}} C^*(G_2) = C^*(G_1 * G_2)$,*
3. *$C^*(G_1) \oplus C^*(G_2) = C^*(G_1 \oplus G_2)$.*

The next property will be important later when we are interested in nuclearity of a group C^*-algebra.

Definition 2.26. *A (discrete) group G is* amenable *if there exists a state μ on $\ell^\infty(G)$ that is left-invariant, i.e.*

$$\mu((x_t)_t)) = \mu((x_{st})_t)) \quad \text{for all } s \in G, (x_t)_t \in \ell^\infty(G).$$

In this case, μ is called an invariant mean.

An example of a non-amenable group is given by $\mathbb{F}_2$, the free group generated by two elements.

2.5 Nuclearity

The tensor product $A \otimes B$ of two C^*-algebras is generally not well-defined as the algebraic tensor product $A \odot B$ might admit different C^*-norms. If the latter is not the case A is called *nuclear*. In this section, we present the definition and important closure properties of the class of nuclear C^*-algebras. The primary references for this and the following section are [BO08] and [Bla06].

We trust that the reader knows the definition of the algebraic tensor product $A \odot B$ of two C^*-algebras. Generally, it can be shown that there are two C^*-norms $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ on $A \odot B$ such that

$$\|x\|_{\min} \leq \|x\| \leq \|x\|_{\max}$$

holds for all $x \in A \odot B$, see e.g. [Bla06, Section II.9.1]. The closure of $A \odot B$ with respect to the norm $\|\cdot\|_{\min}$ is called the minimal tensor product of A and B, written $A \otimes_{\min} B$. Similarly, one obtains the maximal tensor product $A \otimes_{\max} B$ by taking the closure of $A \odot B$ with respect to $\|\cdot\|_{\max}$.

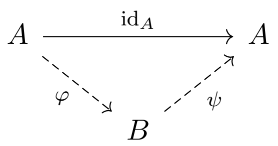
A linear map $\varphi : A \to B$ is called *completely positive* if for all $n \in \mathbb{N}$ the map $\varphi_n : M_n(A) \to M_n(B)$ given by $\varphi_n((a_{ij})_{i,j\leq n}) = (\varphi(a_{ij}))_{i,j\leq n}$ is positive.

Definition 2.27 (Nuclearity)**.** *A C^*-algebra A is* nuclear *if the following equivalent conditions hold.*

1. *For all C^*-algebras B the algebraic tensor product $A \odot B$ admits only one C^* norm.*
2. *For every $\epsilon > 0$ there is a finite-dimensional C^*-algebra B together with contractive completely positive maps $\varphi : A \to B$ and $\psi : B \to A$ such that for all $a \in A$*

$$|\psi \circ \varphi(a) - a| < \epsilon.$$

In this case, we say that the diagram below commutes up to ϵ.



Proposition 2.28. *The following C^*-algebras are nuclear.*

- *finite-dimensional C^*-algebras*
- *commutative C^*-algebras*
- *graph C^*-algebras [Rae05, Remark 4.3]*
- *group C^*-algebras associated to amenable groups*

Moreover, the class of nuclear C^*-algebras is closed under taking

- *tensor products with other nuclear C^*-algebras,*
- *quotients,*

- *extensions, in the sense that whenever we have a short exact sequence*

$$0 \to I \to A \to A/I \to 0$$

  of C^*-algebras where both I and A/I are nuclear, then A is nuclear as well,
- *crossed products with amenable groups,*
- *ideals.*

What's more, a group C^*-algebra associated to a discrete group G is nuclear if, and only if, the group G is amenable. In particular, it follows that $C^*(\mathbb{F}_2)$ is not nuclear.

2.6 Exactness

Exactness is a slightly weaker property for C^*-algebras than nuclearity. Its definition parallels that of nuclearity, although it does not relate as directly to the C^*-tensor product. The major advantage of exactness over nuclearity is that the former is preserved by taking subalgebras. We will make use of this fact later on. At this point, we exhibit the essential facts about exact C^*-algebras and find examples of non-exact C^*-algebras.

Definition 2.29 (Exactness)**.** *A is* exact *if the following equivalent conditions hold:*

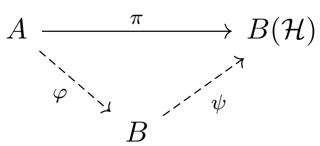
1. *For any exact sequence*

$$0 \to J \to B \to B/J \to 0$$

   of C^*-algebras, the sequence

$$0 \to A \otimes_{\min} J \to A \otimes_{\min} B \to A \otimes_{\min} (B/J) \to 0$$

   is exact as well.
2. *There is a faithful representation $\pi : A \to B(\mathcal{H})$ such that for every $\epsilon > 0$ there are contractive completely positive maps φ, ψ and a finite-dimensional C^*-algebra B such that the diagram below commutes up to ϵ.*



Proposition 2.30. *Every nuclear C^*-algebra is exact. Moreover, the class of exact C^*-algebras is closed under taking subalgebras and quotients.*

Note that the class of exact C^*-algebras is not closed under extensions.

The next proposition collects examples of non-exact C^*-algebras.

Proposition 2.31. *The following C^*-algebras are not exact.*

1. $C^*(\mathbb{F}_2) = C(S^1) *_{\mathbb{C}} C(S^1)$,

2. *the universal unital C^*-algebra $\mathcal{P}_1$ generated by one partial isometry,*
3. *$C^*(G_1) *_{\mathbb{C}} C^*(G_2)$ for non-trivial groups G_1, G_2 where G_2 is of order strictly greater than two.*

Proof. Ad (1): Note that $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ and $C(S^1) = C^*(\mathbb{Z})$. The proof of non-exactness is not trivial, and we refer to [Was76].

Ad (2): In [BN12] it is proven that the universal C^*-algebra $\mathcal{P}$ generated by one partial isometry is not exact. Ultimately this result traces back to the non-exactness of $C^*(\mathbb{F}_2)$. Clearly, $\mathcal{P} \subset \mathcal{P}_1$ and this yields the claim.

Ad (3): In this situation one knows from group theory that $\mathbb{F}_2 \subset G_1 * G_2$, see for instance [Tri22, Lemma 3.17]. Using Proposition 2.25 we have

$$C^*(\mathbb{F}_2) \subset C^*(G_1 * G_2) = C^*(G_1) *_{\mathbb{C}} C^*(G_2).$$

As the class of exact C^*-algebras is closed under taking subalgebras (see 2.30), the claim follows from (1). □

2.7 Morita Equivalence

Morita equivalence is an important equivalence relation for C^*-algebras. Two algebras that are Morita equivalent share the same K-theory and have the same behavior with respect to nuclearity and exactness. We do not discuss the general definition of Morita equivalence. Instead, we focus on one particular situation where two C^*-algebras are Morita equivalent.

If A and B are *Morita equivalent*, we write $A =_M B$.

Definition 2.32. *Let A, B be C^*-algebras. Then B is a* full corner *in A if there is a projection $p \in A$ such that $B = pAp$ and there is no non-trivial ideal $B \subset I \subset A$.*

The following lemma is well-known, see for instance [Tri22, Proposition 1.27].

Lemma 2.33. *Assume that A is a full corner in B. Then A and B are Morita equivalent.*

In the rest of this thesis we will often consider the situation described in the next example.

Example 2.34. *Let A be unital and assume that $p \in A$ is a projection. Then pAp and (pAp) are Morita equivalent. Indeed, it suffices to note that there is no non-trivial ideal $pAp \subset I \subset (pAp)$.*

Finally, nuclearity is preserved by Morita equivalence. More precisely, we have the following theorem.

Theorem 2.35. *[HRW05, Theorem 15] Let A and B be Morita equivalent. Then A is nuclear if, and only if, the same holds for B.*

2.8 Unital Free Products of Finite Dimensional C^*-Algebras

This section discusses nuclearity and exactness for some unital free products of C^*-algebras. Moreover, we prove Morita equivalence of two particular amalgamated free product algebras which we will use later on.

Proposition 2.36. *The free product $\mathbb{C}^m *_{\mathbb{C}} \mathbb{C}^n$ is not exact if $2 \le m$ and $3 \le n$ or if $3 \le m$ and $2 \le n$. Otherwise, $\mathbb{C}^m *_{\mathbb{C}} \mathbb{C}^n$ is nuclear.*

Proof. If $2 \le m$ and $3 \le n$ holds, observe $C^m *_{\mathbb{C}} \mathbb{C}^n = C^*(\mathbb{Z}_m * \mathbb{Z}_n)$ and apply Proposition 2.31 to obtain non-exactness. The case $3 \le m$ and $2 \le n$ is completely analogous.

If $n = 1$ we have $\mathbb{C}^n *_{\mathbb{C}} \mathbb{C}^m = \mathbb{C} *_{\mathbb{C}} \mathbb{C}^m = \mathbb{C}^m$ and this algebra is obviously nuclear. Again, the case $m = 1$ is completely analogous. It remains to show that $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2$ is nuclear.

For that, one shows that $A := \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2 = C^*(a, b \,|\, a, b \text{ self-adjoint unitaries})$ is isomorphic to $B := C^*(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}_2$, where $\alpha(1)$ swaps the generating unitary u of $C^*(\mathbb{Z})$ with its adjoint. Then

$$B = C^*(u, v \,|\, u, v \text{ unitaries}, v = v^*, u^* = vuv)$$

and one constructs $\varphi : A \to B$ and $\psi : B \to A$ with

$$\begin{aligned} \varphi : \; a &\mapsto uv, \quad b \mapsto v \\ \psi : \; u &\mapsto ab, \quad v \mapsto b \end{aligned}$$

via the respective universal property. After checking the required relations, observe that φ and ψ are inverse to each other. Thus $A = B$. However, $C^*(\mathbb{Z}) = C(S^1)$ is nuclear as a commutative C^*-algebra and nuclearity is preserved under crossed products with amenable groups. Since $\mathbb{Z}_2$ is amenable, this proves that A is nuclear. □

Proposition 2.37. *The algebra $M_2 *_{\mathbb{C}} \mathbb{C}^2$ is not nuclear.*

The proof of this fact is beyond the scope of this thesis. It uses the notion of $*$-wildness and the fact that every factor-representation of a nuclear C^*-algebra is hyperfinite. For the details, see [Alb+06, Proposition 3+6].

Note that for $k \ge 3$ the algebra $M_k *_{\mathbb{C}} \mathbb{C}^2$ is not exact since $\mathbb{C}^k *_{\mathbb{C}} \mathbb{C}^2$ is a non-exact subalgebra. Moreover, one can show $M_2 *_{\mathbb{C}} \mathbb{C}^2 = (\mathbb{C}^2 \otimes \mathbb{C}^2) *_{\mathbb{C}} \mathbb{C}^2 \subset (\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2) \otimes (\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2)$. As $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2$ is exact the closure properties of the class of exact C^*-algebras can be used to infer exactness of $M_2 *_{\mathbb{C}} \mathbb{C}^2$.

The proposition below will be used later in this thesis.

Proposition 2.38. *It is $M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2 =_M M_2 *_{\mathbb{C}} \mathbb{C}^2$.*

Proof. Let us write $\hat{E}_{ij}, i, j \le 2$, for the matrix units of M_2 and p_1, p_2 for the standard units of $\mathbb{C}^2$. First, we show $M_2 *_{\mathbb{C}} \mathbb{C}^2 = (E_{22} + E_{33})(M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2)(E_{22} + E_{33}) =: A$.

Recall the embedding $\varphi : M_2 *_{\mathbb{C}} \mathbb{C}^2 \to M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2$ from Example 2.22 with

$$\varphi : \begin{cases} \hat{E}_{ij} \mapsto E_{(i+1)(j+1)}, & i, j \le 2, \\ p_k \mapsto p_k. & k \le 2, \end{cases} \tag{$*$}$$

It suffices to show that the image of φ is equal to A.

From Proposition 2.18 we know that a dense subset of $M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2$ is spanned by elements of the form

$$x = x_1 y_1 x_2 y_2 \dots y_n x_{n+1} \quad \text{with } x_i \in M_3, y_j \in \mathbb{C}^2.$$

By the possibility of setting x_1 or x_{n+1} to $E_{22} + E_{33}$ this incorporates those products that start or end with an element from $\mathbb{C}^2$. It follows that a dense subset of A is spanned by elements of the form

$$\begin{aligned} x &= (E_{22} + E_{33}) x_1 y_1 x_2 \dots y_n x_{n+1} (E_{22} + E_{33}) \\ &= (E_{22} + E_{33}) x_1 (E_{22} + E_{33}) y_1 (E_{22} + E_{33}) x_2 \dots \\ &\quad (E_{22} + E_{33}) y_n (E_{22} + E_{33}) x_{n+1} (E_{22} + E_{33}), \end{aligned}$$

where the second equality is obtained using the amalgamation $E_{22} + E_{33} = 1_{\mathbb{C}^2}$. The latter, however, is also a dense subset of $M_2 *_{\mathbb{C}} \mathbb{C}^2$ under the embedding φ and this yields the claim.

Finally, observe that A is a full corner in $M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2$. Indeed, $E_{ij} \in A$ for $i, j > 1$ and $\mathbb{C}^2 \subset A$ are clear. The ideal generated by A contains in particular the ideal generated by $M_2 \subset M_3$ in M_3. However, matrix algebras have only trivial ideals and therefore $M_3 \cup \mathbb{C}^2 \subset (A)$. Thus, A contains all generators of the amalgamated free product and this implies $A = M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2$. □

3 Main Results

In the present thesis we give a partial answer to the following question:

For which hypergraphs $\mathrm{H}\Gamma$ *is the* C^**-algebra* $C^*(\mathrm{H}\Gamma)$ *nuclear?*

We aim at a characterization of hypergraphs $\mathrm{H}\Gamma$ with nuclear C^*-algebra by means of forbidden hypergraph minors. Leaving the details to chapter 4, we call $\mathrm{H}\Delta$ a hypergraph minor of $\mathrm{H}\Gamma$, written $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$, if the former is obtained from the latter by a combination of the following minor operations:

- edge/vertex deletion
- forward/backward edge contraction
- edge cutting
- source separation
- range decomposition

The table below lists four hypergraphs $\mathrm{H}\Gamma_1, \mathrm{H}\Gamma_2, \mathrm{H}\Gamma_3, \mathrm{H}\Gamma_4$ which turn out to account for non-nuclearity of a large portion of hypergraph C^*-algebras.

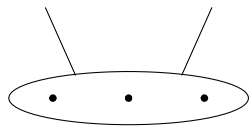
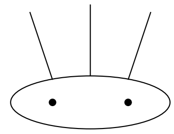
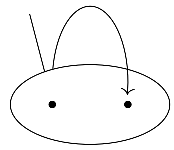
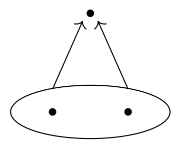
$\mathrm{H}\Gamma_1$	$E^0 = \{v_1, v_2, v_3\}$, $E^1 = \{e, f\}$,	$s(e) = s(f) = E^0$,	$r(e) = r(f) = \emptyset$	
$\mathrm{H}\Gamma_2$	$E^0 = \{v_1, v_2\}$, $E^1 = \{e, f, g\}$,	$s(e) = s(f) = s(g) = E^0$,	$r(e) = r(f) = r(g) = \emptyset$	
$\mathrm{H}\Gamma_3$	$E^0 = \{v, w\}$, $E^1 = \{e, f\}$,	$s(e) = s(f) = E^0$,	$r(e) = \emptyset$, $r(f) = \{w\}$	
$\mathrm{H}\Gamma_4$	$E^0 = \{v_1, v_2, w\}$, $E^1 = \{e, f\}$,	$s(e) = s(f) = \{v_1, v_2\}$,	$r(e) = r(f) = \{w\}$	

Table 3.1: The Forbidden Minors $\mathrm{H}\Gamma_1, \mathrm{H}\Gamma_2, \mathrm{H}\Gamma_3, \mathrm{H}\Gamma_4$

We call the $\mathrm{H}\Gamma_i$ the *forbidden minors*. Their C^*-algebras are easily determined.

Proposition 3.1. *We have*

- $C^*(\mathrm{H}\Gamma_1) = C^*(\mathrm{H}\Gamma_2) = \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3$,
- $C^*(\mathrm{H}\Gamma_3)$ *is the universal unital C^*-algebra generated by one partial isometry,*
- $C^*(\mathrm{H}\Gamma_4) =_M M_2 *_{\mathbb{C}} \mathbb{C}^2$.

In particular, the C^*-algebras $C^*(\mathrm{H}\Gamma_1), C^*(\mathrm{H}\Gamma_2), C^*(\mathrm{H}\Gamma_3)$ are not exact while $C^*(\mathrm{H}\Gamma_4)$ is not nuclear.

Proof. in Chapter 6. Note that non-exactness of $C^*(\mathrm{H}\Gamma_1), C^*(\mathrm{H}\Gamma_2)$ and $C^*(\mathrm{H}\Gamma_3)$ is essentially derived from the same property of $C^*(\mathbb{F}_2)$. □

Assume that we have given a hypergraph $\mathrm{H}\Gamma$ and its associated C^*-algebra $C^*(\mathrm{H}\Gamma)$. We will see later that certain minor operations do not change the C^*-algebra up to Morita equivalence, see Theorem 4.3. This can be used to put any given hypergraph in a certain "normalized" form. Let us first define the notion of a normal hypergraph.

Definition 3.2 (normal hypergraph)**.** *A hypergraph* $\mathrm{H}\Gamma$ *is called* normal *if it has the following properties.*

1. $|r(e)| \leq 1$ *for all edges* $e \in E^1(\mathrm{H}\Gamma)$.
2. *For every edge e there exists another edge f with $s(e) \cap s(f) \neq \emptyset$ or $\emptyset \neq r(e) \subset s(e)$.*
3. *Whenever (e, f) is a pair of distinct edges with $|s(e) \cap s(f)| = 1$, then one of the following holds:*
   a) $|s(e)| = |s(f)| = 1$.
   b) *There is an edge $g \neq e$ with $s(e) \cap s(f) \subsetneq s(e) \cap s(g)$.*

The next lemma asserts that, without changing the associated C^*-algebra up to Morita equivalence, any hypergraph $\mathrm{H}\Gamma$ can be normalized by passing to a suitable hypergraph minor.

Lemma 3.3. *Let $\mathrm{H}\Gamma$ by a hypergraph. Then there is a normal hypergraph $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$ such that $C^*(\mathrm{H}\Delta)$ is Morita equivalent to $C^*(\mathrm{H}\Gamma)$. We call $\mathrm{H}\Delta$ a* normalized version *of* $\mathrm{H}\Gamma$.

Proof. in Chapter 5. □

Moreover, Algorithm 1 in Chapter 5 gives an explicit procedure for constructing a normalized version of a given hypergraph $\mathrm{H}\Gamma$.

Finally, let us state the main result of this thesis.

Theorem 3.4. *Let* $\mathrm{H}\Gamma$ *be a hypergraph. One can construct a normal hypergraph* $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$ *such that* $C^*(\mathrm{H}\Gamma)$ *is nuclear if, and only if, the same holds for* $C^*(\mathrm{H}\Delta)$. *Further, the following is true:*

1. *If* $\mathrm{H}\Gamma_i \leq \mathrm{H}\Delta$ *holds for some* $i \leq 3$, *then* $C^*(\mathrm{H}\Gamma)$ *is not exact.*
2. *If* $\mathrm{H}\Gamma_4 \leq \mathrm{H}\Delta$ *holds, then* $C^*(\mathrm{H}\Gamma)$ *is not nuclear.*
3. *If none of the above holds, then* $\mathrm{H}\Delta$ *is an undirected hypergraph, i.e. all edges of* $\mathrm{H}\Delta$ *have empty range.*

Here $\mathrm{H}\Gamma_1, \mathrm{H}\Gamma_2, \mathrm{H}\Gamma_3$ *and* $\mathrm{H}\Gamma_4$ *are the forbidden minors from Table 3.1.*

Crucially, there is an explicit procedure for constructing $\mathrm{H}\Delta$ starting from $\mathrm{H}\Gamma$ which we describe in Chapter 5 as Algorithm 2. The proof of the previous theorem is done at the end of Chapter 6. At this point, let us sketch the main ideas.

To obtain $\mathrm{H}\Delta$ from $\mathrm{H}\Gamma$ we may first pass to a normalized version of $\mathrm{H}\Gamma$ due to Lemma 3.3 and the fact that nuclearity is preserved under Morita equivalence, see Theorem 2.35. Further, Lemmas 5.8, 5.13 and 5.16 will allow us to cut certain edges from a hypergraph without changing nuclearity of the associated C^*-algebra. Putting these operations together, Algorithm 2 in Chapter 5 will give a procedure which reduces $\mathrm{H}\Gamma$ to a normal hypergraph minor $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$ such that $C^*(\mathrm{H}\Gamma)$ is nuclear if, and only if, the same holds for $C^*(\mathrm{H}\Delta)$.

Ad (1): Assume $\mathrm{H}\Gamma_i \leq \mathrm{H}\Delta$ for some $i \leq 3$. In Theorem 4.3 we will see that exactness of a hypergraph C^*-algebra is preserved under passing to the associated algebra of a hypergraph minor. As the C^*-algebras $C^*(\mathrm{H}\Gamma_1), C^*(\mathrm{H}\Gamma_2)$ and $C^*(\mathrm{H}\Gamma_3)$ are not exact by Proposition 3.1, the claim follows immediately.

Ad (2): Assume $\mathrm{H}\Gamma_4 \leq \mathrm{H}\Gamma$. In Proposition 5.20 it will turn out that, unless (1) applies, $\mathrm{H}\Gamma_4$ can be obtained from $\mathrm{H}\Delta$ using only certain operations which preserve nuclearity of the associated C^*-algebra. As $C^*(\mathrm{H}\Gamma_4)$ is not nuclear this implies that $C^*(\mathrm{H}\Delta)$ is not nuclear. Then $C^*(\mathrm{H}\Gamma)$ is not nuclear as well.

Ad (3): Assume $\mathrm{H}\Gamma_i \not\leq \mathrm{H}\Delta$ for $i \leq 4$. In Theorem 5.19 we will see that as soon as $\mathrm{H}\Delta$ contains any edge with nonempty range, then it has one of the forbidden minors. Therefore, in the absence of forbidden minors $\mathrm{H}\Delta$ contains only edges with empty range, i.e. it is an undirected hypergraph.

If $\mathrm{H}\Gamma$ has none of the forbidden minors, then one checks that $\mathrm{H}\Delta$ satisfies the following:

- Every edge has empty range.
- For any distinct vertices $v, w \in E^0(\mathrm{H}\Delta)$ there are at most two distinct edges e, f with $\{v, w\} \subset s(e) \cap s(f)$.
- For any distinct edges $e, f \in E^1(\mathrm{H}\Delta)$ there are at most two distinct vertices v, w with $\{v, w\} \subset s(e) \cap s(f)$.

We believe that for any hypergraph $\mathrm{H}\Delta$ with these properties the associated C^*-algebra $C^*(\mathrm{H}\Delta)$ is nuclear. Then it would follow that any hypergraph C^*-algebra $C^*(\mathrm{H}\Gamma)$ is nuclear if, and only if, $\mathrm{H}\Delta$ has none of the forbidden minors.

4 Hypergraph Minors

This chapter introduces the concept of a hypergraph minor. We investigate the minor operations on the C^*-algebra side and find that exactness of the hypergraph C^*-algebra is preserved under taking minors.

4.1 Definition and Main Result

Let us first define the notion of a hypergraph minor.

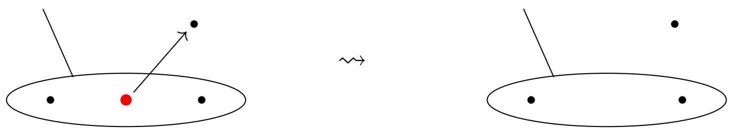
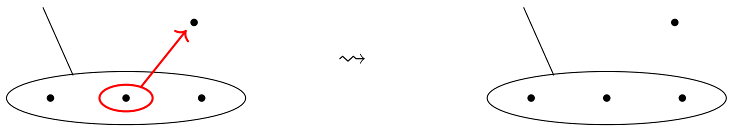
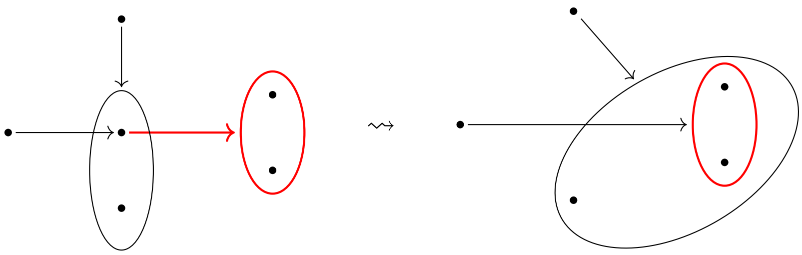
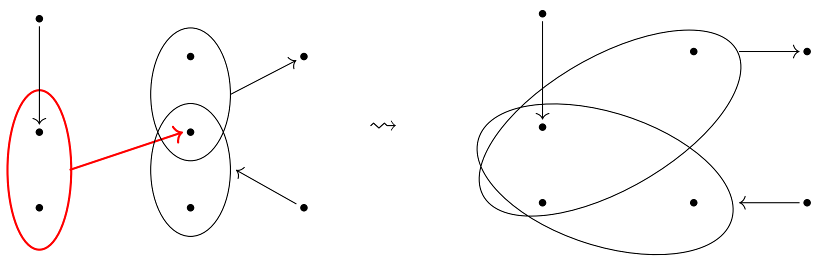
Definition 4.1 (hypergraph minor). *We say that* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by*

- vertex deletion *if there is a vertex* v *in* $\mathrm{H}\Gamma$ *such that*
  - $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma) \setminus \{v\}$,
  - $E^1(\mathrm{H}\Delta) = E^1(\mathrm{H}\Gamma) \setminus \{e \in E^1(\mathrm{H}\Gamma) : s(e) = \{v\}\}$,
  - $r_{\mathrm{H}\Delta}(e) = r_{\mathrm{H}\Gamma}(e) \setminus \{v\}$ *for all* $e \in E^1(\mathrm{H}\Delta)$,
  - $s_{\mathrm{H}\Delta}(e) = s_{\mathrm{H}\Gamma}(e) \setminus \{v\}$ *for all* $e \in E^1(\mathrm{H}\Delta)$,
- edge deletion *if there is an edge* f *in* $\mathrm{H}\Gamma$ *such that*
  - $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma)$,
  - $E^1(\mathrm{H}\Delta) = E^1(\mathrm{H}\Gamma) \setminus \{f\}$,
  - $r_{\mathrm{H}\Delta}(e) = r_{\mathrm{H}\Gamma}(e)$ *for all* $e \in E^1(\mathrm{H}\Delta)$,
  - $s_{\mathrm{H}\Delta}(e) = s_{\mathrm{H}\Gamma}(e)$ *for all* $e \in E^1(\mathrm{H}\Delta)$,
- forward edge contraction *if there is an edge* f *and a vertex* w *in* $\mathrm{H}\Gamma$ *with* $s_{\mathrm{H}\Gamma}(f) = \{w\}$ *and*
  - f *is the only edge starting from* w *in* $\mathrm{H}\Gamma$, *i.e.* $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) \neq \emptyset \implies e = f$ *for all* $e \in E^1(\mathrm{H}\Gamma)$,
  - *there is no edge* $e \in E^1(\mathrm{H}\Gamma)$ *with both* $w \in r_{\mathrm{H}\Gamma}(e)$ *and* $r_{\mathrm{H}\Gamma}(e) \cap r_{\mathrm{H}\Gamma}(f) \neq \emptyset$,
  - $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma) \setminus \{w\}$,
  - $E^1(\mathrm{H}\Delta) = E^1(\mathrm{H}\Gamma) \setminus \{f\}$,
  - $r_{\mathrm{H}\Delta}(e) = \begin{cases} r_{\mathrm{H}\Gamma}(e), & w \notin r_{\mathrm{H}\Gamma}(e), \\ (r_{\mathrm{H}\Gamma}(e) \setminus \{w\}) \cup r_{\mathrm{H}\Gamma}(f), & \textit{otherwise}, \end{cases}$ *for all* $e \in E^1(\mathrm{H}\Delta)$,
  - $s_{\mathrm{H}\Delta}(e) = s_{\mathrm{H}\Gamma}(e)$ *for all* $e \in E^1(\mathrm{H}\Delta)$,

- backward edge contraction *if there is an edge* f *and a vertex* w *in* $\mathrm{H}\Gamma$ *with* $r_{\mathrm{H}\Gamma}(f) = \{w\}$ *and*
  - f *is the only edge starting from* $s_{\mathrm{H}\Gamma}(f)$ *in* $\mathrm{H}\Gamma$*, i.e.* $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) \neq \emptyset \implies e = f$ *for all* $e \in E^1(\mathrm{H}\Gamma)$*,*
  - *there is no edge* $e \in E^1(\mathrm{H}\Gamma)$ *with both* $r_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) \neq \emptyset$ *and* $w \in r_{\mathrm{H}\Gamma}(e)$*,*
  - $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma) \setminus \{w\}$*,*
  - $E^1(\mathrm{H}\Delta) = E^1(\mathrm{H}\Gamma) \setminus \{f\}$*,*
  - $r_{\mathrm{H}\Delta}(e) = \begin{cases} r_{\mathrm{H}\Gamma}(e), & w \notin r_{\mathrm{H}\Gamma}(e), \\ (r_{\mathrm{H}\Gamma}(e) \setminus \{w\}) \cup s_{\mathrm{H}\Gamma}(f), & \text{otherwise}, \end{cases}$ *for all* $e \in E^1(\mathrm{H}\Delta)$*,*
  - $s_{\mathrm{H}\Delta}(e) = \begin{cases} s_{\mathrm{H}\Gamma}(e), & w \notin s_{\mathrm{H}\Gamma}(e), \\ (s_{\mathrm{H}\Gamma}(e) \setminus \{w\}) \cup s_{\mathrm{H}\Gamma}(f), & \text{otherwise}, \end{cases}$ *for all* $e \in E^1(\mathrm{H}\Delta)$*,*
- edge cutting *if there is an edge* f *in* $\mathrm{H}\Gamma$ *such that*
  - $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma)$*,*
  - $E^1(\mathrm{H}\Delta) = E^1(\mathrm{H}\Gamma)$*,*
  - $r_{\mathrm{H}\Delta}(e) = \begin{cases} r_{\mathrm{H}\Gamma}(e), & e \neq f, \\ \emptyset, & e = f, \end{cases}$ *for all* $e \in E^1(\mathrm{H}\Delta)$*,*
  - $s_{\mathrm{H}\Delta}(e) = s_{\mathrm{H}\Gamma}(e)$ *for all* $e \in E^1(\mathrm{H}\Delta)$*,*
- source separation *if there is a non-empty set* $F \subset E^1(\mathrm{H}\Gamma)$*, a vertex* $w \in E^0(\mathrm{H}\Gamma)$ *and some vertex* $w' \in E^0(\mathrm{H}\Delta) \setminus E^0(\mathrm{H}\Gamma)$ *such that*
  - $F \subsetneq \{e \in E^1(\mathrm{H}\Gamma) : w \in s_{\mathrm{H}\Gamma}(e)\}$
  - $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma) \cup \{w'\}$
  - $E^1(\mathrm{H}\Delta) = E^1(\mathrm{H}\Gamma)$*,*
  - $r_{\mathrm{H}\Delta}(e) = \begin{cases} r_{\mathrm{H}\Gamma}(e), & w \notin r_{\mathrm{H}\Gamma}(e), \\ r_{\mathrm{H}\Gamma}(e) \cup \{w'\}, & \text{otherwise}, \end{cases}$ *for all* $e \in E^1(\mathrm{H}\Delta)$*,*
  - $s_{\mathrm{H}\Delta}(e) = \begin{cases} s_{\mathrm{H}\Gamma}(e), & e \notin F, \\ (s_{\mathrm{H}\Gamma}(e) \setminus \{w\}) \cup \{w'\}, & e \in F, \end{cases}$ *for all* $e \in E^1(\mathrm{H}\Delta)$*,*
- range decomposition *if there is an edge* f *in* $\mathrm{H}\Gamma$ *such that*
  - $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma)$*,*
  - $E^1(\mathrm{H}\Delta) = (E^1(\mathrm{H}\Gamma) \setminus \{f\}) \cup \{(f, v) : v \in r_{\mathrm{H}\Gamma}(f)\}$*,*
  - $r_{\mathrm{H}\Delta}(e) = \begin{cases} r_{\mathrm{H}\Gamma}(e), & e \notin \{(f, v) : v \in r_{\mathrm{H}\Gamma}(f)\}, \\ \{v\}, & e = (f, v), \end{cases}$ *for all* $e \in E^1(\mathrm{H}\Delta)$*,*
  - $s_{\mathrm{H}\Delta}(e) = s_{\mathrm{H}\Gamma}(e)$ *for all* $e \in E^1(\mathrm{H}\Delta)$*.*

The hypergraph $\mathrm{H}\Delta$ *is a* minor *of* $\mathrm{H}\Gamma$*, written* $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$*, if it is obtained from* $\mathrm{H}\Gamma$ *by any combination of these operations.*

Example 4.2. *The table below gives an example for every minor operation. We trust that the sketches are self-explanatory and do not give explicit definitions of the involved hypergraphs. Vertices or edges relevant to the respective minor operation are highlighted in red.*

vertex deletion	
edge deletion	
forward edge contraction	
backward edge contraction	
edge cutting	

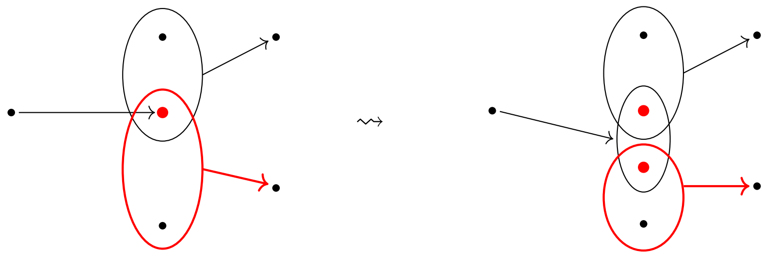
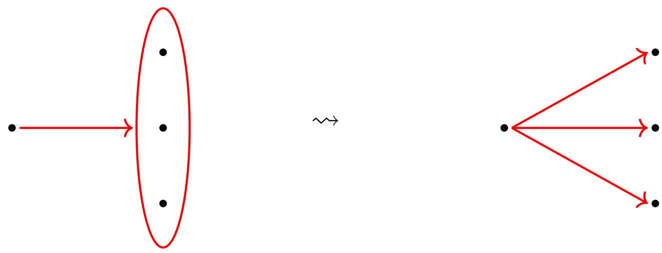
source separation	
range decomposition	

Table 4.1: Examples for the Minor Operations

The next theorem describes the effect of the minor operations on the associated C^*-algebra.

Theorem 4.3. *Let* $\mathrm{H}\Gamma$ *and* $\mathrm{H}\Delta$ *be two hypergraphs. The algebra* $C^*(\mathrm{H}\Delta)$ *is*

- *isomorphic to* $C^*(\mathrm{H}\Gamma)$ *if* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by range decomposition,*
- *a quotient of* $C^*(\mathrm{H}\Gamma)$ *if* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by source separation,*
- *a subalgebra of* $C^*(\mathrm{H}\Gamma)$ *if* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by edge cutting,*
- *a quotient of a subalgebra of* $C^*(\mathrm{H}\Gamma)$ *if* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by edge or vertex deletion,*
- *a full corner of* $C^*(\mathrm{H}\Gamma)$ *if* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by forward or backward edge contraction.*

In particular, $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$ *implies that* $C^*(\mathrm{H}\Delta)$ *is – up to Morita equivalence – obtained from* $C^*(\mathrm{H}\Gamma)$ *by alternatingly taking subalgebras and quotients. If* $C^*(\mathrm{H}\Gamma)$ *is exact then the same holds for* $C^*(\mathrm{H}\Delta)$*.*

Proof. For range decomposition we refer to the proof of Theorem 4.1 in [Tri22] which is easily adapted to the present situation. The other statements in the bullet points are proven separately in the next subsections, see Propositions 4.4, 4.7, 4.11 and 4.14. The last statement follows since the class of exact C^*-algebras is closed under taking quotients and subalgebras, see Proposition 2.30. □

4.2 Source Separation

In this section we prove that source separation corresponds to taking a quotient on the C^*-algebra side. Under special conditions, however, we can do better and the C^*-algebra remains unchanged.

Proposition 4.4 (source separation)**.** *Let* $\mathrm{H}\Delta$ *be obtained from* $\mathrm{H}\Gamma$ *by source separation of a nonempty set* $F \subset E^1(\mathrm{H}\Gamma)$ *at* $w \in \bigcap_{f\in F} s_{\mathrm{H}\Gamma}(f)$*. Then* $C^*(\mathrm{H}\Delta)$ *is a quotient of* $C^*(\mathrm{H}\Gamma)$*. Moreover, if*

$$w \in s_{\mathrm{H}\Gamma}(g) \implies \{w\} = s_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(g) \qquad \text{for all } g \notin F, \tag{$*$}$$

then $C^*(\mathrm{H}\Delta) = C^*(\mathrm{H}\Gamma)$.

Proof. Use the universal property of $C^*(\mathrm{H}\Gamma)$ to obtain a map $\varphi : C^*(\mathrm{H}\Gamma) \to C^*(\mathrm{H}\Delta)$ with

$$\varphi : \begin{cases} v \mapsto \hat{v}, & v \in E^0(\mathrm{H}\Gamma) \setminus \{w\}, \\ v \mapsto \hat{w} + \hat{w}', & v = w, \\ e \mapsto \hat{e}, & e \in E^1(\mathrm{H}\Gamma), \end{cases}$$

where we write $\hat{v}, \hat{e}$ for the generators of $C^*(\mathrm{H}\Delta)$ to avoid confusion with the elements of $C^*(\mathrm{H}\Gamma)$. Accordingly, we write $\hat{s}_{\mathrm{H}\Delta}(e)$ for $\sum_{v\in s_{\mathrm{H}\Delta}(e)} \hat{v}$ and $\hat{r}_{\mathrm{H}\Delta}(e)$ for $\sum_{v\in r_{\mathrm{H}\Delta}(e)} \hat{v}$. Clearly the $\varphi(v)$ are pairwise orthogonal projections and the $\varphi(e)$ are partial isometries. We check the hypergraph relations.

(HR1): For $e, e' \in E^1(\mathrm{H}\Gamma)$ it is

$$\varphi(e)^*\varphi(e') = \hat{e}^*\hat{e}' = \begin{cases} \delta_{ee'}\hat{r}_{\mathrm{H}\Delta}(e) = \delta_{ee'}\varphi(r_{\mathrm{H}\Gamma}(e)), & r_{\mathrm{H}\Gamma}(e) \neq \emptyset, \\ \delta_{ee'}\hat{e} = \delta_{ee'}\varphi(e), & \text{otherwise.} \end{cases}$$

(HR2): For $e \in E^1(\mathrm{H}\Gamma)$ we have

$$\varphi(e)\varphi(e)^* = \hat{e}\hat{e}^* \leq \hat{s}_{\mathrm{H}\Delta}(e) \leq \varphi(s_{\mathrm{H}\Gamma}(e)).$$

(HR3): Let $v \in E^0(\mathrm{H}\Gamma) \setminus \{w\}$ not be a sink in $\mathrm{H}\Gamma$. Then v is not a sink in $\mathrm{H}\Delta$ as well and therefore

$$\varphi(v) = \hat{v} \leq \sum_{e\in E^1(\mathrm{H}\Delta):v\in s_{\mathrm{H}\Delta}(e)} \hat{e}\hat{e}^* = \sum_{e\in E^1(\mathrm{H}\Gamma):v\in s_{\mathrm{H}\Gamma}(e)} \varphi(e)\varphi(e)^*.$$

For the vertex w observe that neither w nor w' is a sink in $\mathrm{H}\Delta$ and thus

$$\begin{aligned} \varphi(w) &= \hat{w} + \hat{w}' \\ &\leq \sum_{e\in E^1(\mathrm{H}\Delta):w\in s_{\mathrm{H}\Delta}(e)\vee w'\in s_{\mathrm{H}\Delta}(e)} \hat{e}\hat{e}^* \\ &= \sum_{e\in E^1(\mathrm{H}\Gamma):w\in s_{\mathrm{H}\Gamma}(e)} \varphi(e)\varphi(e)^*. \end{aligned}$$

To complete the proof of the first statement we show that φ is surjective. It suffices to observe that $\hat{w}'$ is in the image of φ since then $\varphi(C^*(\mathrm{H}\Gamma))$ contains all generators of $C^*(\mathrm{H}\Delta)$. Indeed, the edges in F are the only edges in $\mathrm{H}\Delta$ which have w' in their source and at the same time it is $w \notin s_{\mathrm{H}\Delta}(f)$ for all $f \in F$. Thus, $\hat{w}' \leq \sum_{f\in F} \hat{f}\hat{f}^* \perp \hat{w}$ and therefore

$$\hat{w}' = (\hat{w} + \hat{w}') \sum_{f\in F} \hat{f}\hat{f}^* = \varphi(w) \sum_{f\in F} \varphi(f)\varphi(f)^* \in \varphi(C^*(\mathrm{H}\Gamma)).$$

Finally, surjectivity of φ yields $C^*(\mathrm{H}\Delta) = C^*(\mathrm{H}\Gamma)/\ker(\varphi)$.

Now, assume $(*)$ and let $G := \{e \in E^1(\mathrm{H}\Gamma) : w \in s_{\mathrm{H}\Gamma}(e)\} \setminus F$. We show that the universal property of $C^*(\mathrm{H}\Delta)$ yields a map $\psi : C^*(\mathrm{H}\Delta) \to C^*(\mathrm{H}\Gamma)$ with

$$\psi : \begin{cases} \hat{v} \mapsto v, & v \in E^0(\mathrm{H}\Delta) \setminus \{w, w'\}, \\ \hat{v} \mapsto \left(\sum_{g\in G} gg^*\right) w \left(\sum_{g\in G} gg^*\right), & v = w, \\ \hat{v} \mapsto \left(\sum_{f\in F} ff^*\right) w \left(\sum_{f\in F} ff^*\right), & v = w', \\ \hat{e} \mapsto e, & e \in E^1(\mathrm{H}\Delta). \end{cases}$$

Indeed, observe for any $f \in F$ and $g \in G$

$$0 = ff^*gg^* = ff^* \left(\sum_{v\in s_{\mathrm{H}\Gamma}(f)} v\right) \left(\sum_{v\in s_{\mathrm{H}\Gamma}(g)} v\right) gg^* = ff^*wgg^*.$$

Therefore, we have

$$\begin{aligned} w &= \left(\sum_{e:w\in s_{\mathrm{H}\Gamma}(w)} ee^*\right) w \left(\sum_{e:w\in s_{\mathrm{H}\Gamma}(w)} ee^*\right) \\ &= \left(\sum_{f\in F} ff^*\right) w \left(\sum_{f\in F} ff^*\right) + \left(\sum_{g\in G} gg^*\right) w \left(\sum_{g\in G} gg^*\right) \end{aligned}$$

and

$$\begin{aligned} \psi(\hat{w}')^2 &= \left(\left(\sum_{f\in F} ff^*\right) w \left(\sum_{f\in F} ff^*\right)\right)^2 \\ &= \left(\sum_{f\in F} ff^*\right) w \left(\sum_{f\in F} ff^*\right) w \left(\sum_{f\in F} ff^*\right) \\ &= \left(\sum_{f\in F} ff^*\right) w \left(\sum_{e\in F\cup G} ee^*\right) w \left(\sum_{f\in F} ff^*\right) \\ &= \left(\sum_{f\in F} ff^*\right) w \left(\sum_{f\in F} ff^*\right) \\ &= \psi(\hat{w}') \end{aligned}$$

Similarly, one sees that $\psi(\hat{w})$ is a projection, and it is easily checked that $\psi(\hat{w})$ and $\psi(\hat{w}')$ are orthogonal. Thus, the $\psi(\hat{v})$ are pairwise orthogonal projections and evidently the $\psi(\hat{e})$ are partial isometries. We check the hypergraph relations.

(HR1): Since in $\mathrm{H}\Delta$ every edge e with $w \in r_{\mathrm{H}\Delta}(e)$ has also w' in its range we have for any edges $e, f \in E^1(\mathrm{H}\Delta)$

$$\begin{aligned}\psi(\hat{e})^*\psi(\hat{f}) &= e^* f \\ &= \begin{cases} \delta_{ef} r_{\mathrm{H}\Gamma}(e), & r_{\mathrm{H}\Gamma}(e) \neq \emptyset, \\ \delta_{ef} e, & \text{otherwise,} \end{cases} \\ &= \begin{cases} \delta_{ef} \psi(\hat{r}_{\mathrm{H}\Delta}(e)), & r_{\mathrm{H}\Delta}(e) \neq \emptyset, \\ \delta_{ef} \psi(\hat{e}), & \text{otherwise,} \end{cases}\end{aligned}$$

using $\psi(\hat{w}) + \psi(\hat{w}') = w$.

(HR2): For $e \in E^1(\mathrm{H}\Delta) \setminus (F \cup G)$ we have

$$\psi(\hat{e})\psi(\hat{e})^* = ee^* \leq s_{\mathrm{H}\Gamma}(e) = \psi(\hat{s}_{\mathrm{H}\Delta}(e)).$$

For $f \in F$ it is

$$\begin{aligned}\psi(\hat{f})\psi(\hat{f})^* &= ff^* \\ &= \left(1 - \sum_{g \in G} gg^*\right)(ff^*)\left(1 - \sum_{g \in G} gg^*\right) \\ &\leq \left(1 - \sum_{g \in G} gg^*\right) s_{\mathrm{H}\Gamma}(f) \left(1 - \sum_{g \in G} gg^*\right) \\ &= \sum_{v \in s_{\mathrm{H}\Gamma}(f) \setminus \{w\}} v + \left(1 - \sum_{g \in G} gg^*\right) w \left(1 - \sum_{g \in G} gg^*\right) \\ &= \sum_{v \in s_{\mathrm{H}\Gamma}(f) \setminus \{w\}} v + \left(\sum_{f \in F} ff^*\right) w \left(\sum_{f \in F} ff^*\right) \\ &= \psi(\hat{s}_{\mathrm{H}\Delta}(e)).\end{aligned}$$

Similarly, one sees $\psi(\hat{g})\psi(\hat{g})^* \leq \psi(\hat{s}_{\mathrm{H}\Delta}(g))$ for all $g \in G$.

(HR3): Let $v \in E^0(\mathrm{H}\Delta) \setminus \{w, w'\}$ not be a sink in $\mathrm{H}\Delta$. Then v is not a sink in $\mathrm{H}\Gamma$ neither and therefore

$$\psi(\hat{v}) = v \leq \sum_{e \in E^1(\mathrm{H}\Gamma): v \in s_{\mathrm{H}\Gamma}(e)} ee^* = \sum_{e \in E^1(\mathrm{H}\Delta): v \in s_{\mathrm{H}\Delta}(e)} \psi(\hat{e})\psi(\hat{e})^*.$$

For the vertex w we have

$$\begin{aligned}\psi(\hat{w}) &= \left(\sum_{g\in G} gg^*\right) w \left(\sum_{g\in G} gg^*\right)\\ &\leq \left(\sum_{g\in G} gg^*\right)\left(\sum_{e\in E^1(\mathrm{H}\Gamma):w\in s_{\mathrm{H}\Gamma}(e)} ee^*\right)\left(\sum_{g\in G} gg^*\right)\\ &= \left(\sum_{g\in G} gg^*\right)\left(\sum_{e\in F\cup G} ee^*\right)\left(\sum_{g\in G} gg^*\right)\\ &= \sum_{g\in G} gg^*\\ &= \sum_{e\in E^1(\mathrm{H}\Delta):w\in s_{\mathrm{H}\Delta}(e)} \psi(\hat{e})\psi(\hat{e})^*.\end{aligned}$$

Similarly, one obtains $\psi(\hat{w}') \leq \sum_{e\in E^1(\mathrm{H}\Delta):w'\in s_{\mathrm{H}\Delta}(e)} \psi(\hat{e})\psi(\hat{e})^*$.

Finally, observe that φ and ψ are inverse to each other. Indeed, $\psi\circ\varphi = \mathrm{id}_{C^*(\mathrm{H}\Gamma)}$ is easily checked on the generators of $C^*(\mathrm{H}\Gamma)$. For $\varphi\circ\psi = \mathrm{id}_{C^*(\mathrm{H}\Delta)}$ use that we have

$$\begin{aligned}\varphi(\psi(\hat{w})) &= \varphi\left(\left(\sum_{g\in G} gg^*\right) w \left(\sum_{g\in G} gg^*\right)\right)\\ &= \left(\sum_{g\in G} \hat{g}\hat{g}^*\right)(\hat{w}+\hat{w}')\left(\sum_{g\in G} \hat{g}\hat{g}^*\right)\\ &= \left(\sum_{g\in G} \hat{g}\hat{g}^*\right)\hat{w}\left(\sum_{g\in G} \hat{g}\hat{g}^*\right)\\ &= \hat{w}\end{aligned}$$

and similarly $\varphi(\psi(\hat{w}')) = \hat{w}'$. □

Remark 4.5. *We will say that* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by separating the source of an edge* $f\in E^1(\mathrm{H}\Gamma)$ *if* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by applying successively source separation on the non-empty set* $\{f\}\subset E^1(\mathrm{H}\Gamma)$ *at all vertices* $w\in s_{\mathrm{H}\Gamma}(f)$ *for which there is another edge* $e\in E^1(\mathrm{H}\Gamma)\setminus\{f\}$ *with* $w\in s_{\mathrm{H}\Gamma}(e)$.

The following is a simple corollary of the previous proposition.

Corollary 4.6. *Let* $\mathrm{H}\Delta$ *be obtained from* $\mathrm{H}\Gamma$ *by separating the source of an edge* $f\in E^1(\mathrm{H}\Gamma)$ *with* $|s_{\mathrm{H}\Gamma}(f)| = 1$. *Then* $C^*(\mathrm{H}\Delta) = C^*(\mathrm{H}\Gamma)$.

Proof. Let $w\in E^0(\mathrm{H}\Gamma)$ be the single element of $s_{\mathrm{H}\Gamma}(f)$. If there is no edge $e\in E^1(\mathrm{H}\Gamma)\setminus\{f\}$ with $w\in s_{\mathrm{H}\Gamma}(e)$, then $\mathrm{H}\Delta = \mathrm{H}\Gamma$ and there is nothing to show. Otherwise, one readily checks that the special condition $(*)$ from Proposition 4.4 holds true, and then the same proposition yields the claim. □

4.3 Edge Cutting

This section deals with hypergraphs obtained by edge cutting. On the C^*-algebra side this operation corresponds to taking a subalgebra.

Proposition 4.7 (edge cutting)**.** *Assume that* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by cutting an edge* f*. Then* $C^*(\mathrm{H}\Delta)$ *is a subalgebra of* $C^*(\mathrm{H}\Gamma)$.

Proof. Use the universal property of $C^*(\mathrm{H}\Delta)$ to obtain a map $\varphi : C^*(\mathrm{H}\Delta) \to C^*(\mathrm{H}\Gamma)$ with

$$\varphi : \begin{cases} v \mapsto \hat{v}, & v \in E^0(\mathrm{H}\Delta), \\ e \mapsto \hat{e}, & e \in E^1(\mathrm{H}\Delta) \setminus \{f\}, \\ e \mapsto \hat{f}\hat{f}^*, & e = f, \end{cases}$$

where we write again $\hat{v}$ and $\hat{e}$ for the generators of $C^*(\mathrm{H}\Gamma)$ to avoid confusion. Clearly the $\varphi(v)$ are pairwise disjoint projections and the $\varphi(e)$ are partial isometries. We check the hypergraph relations.

(HR1): For $e, g \in E^1(\mathrm{H}\Delta) \setminus \{f\}$ with $r_{\mathrm{H}\Gamma}(e) \neq \emptyset$ it is

$$\varphi(e)^*\varphi(g) = \hat{e}^*\hat{g} = \delta_{eg}\hat{r}_{\mathrm{H}\Gamma}(e) = \delta_{eg}\varphi(r_{\mathrm{H}\Delta}(e))$$

If $r_{\mathrm{H}\Gamma}(e) = \emptyset$, then

$$\varphi(e)^*\varphi(g) = \hat{e}^*\hat{g} = \delta_{eg}\hat{e} = \delta_{eg}\varphi(e).$$

Further, we have

$$\varphi(f)^*\varphi(e) = \hat{f}\hat{f}^*\hat{e} = 0 = \varphi(e)^*\varphi(f)$$

and

$$\varphi(f)^*\varphi(f) = \hat{f}\hat{f}^*\hat{f}\hat{f}^* = \hat{f}\hat{f}^* = \varphi(f).$$

(HR2): For $e \in E^1(\mathrm{H}\Delta) \setminus \{f\}$, we have

$$\varphi(e)\varphi(e)^* = \hat{e}\hat{e}^* \leq \hat{s}_{\mathrm{H}\Gamma}(e) = \varphi(s_{\mathrm{H}\Delta}(e))$$

Moreover, it is

$$\varphi(f)\varphi(f)^* = \hat{f}\hat{f}^*\hat{f}\hat{f}^* = \hat{f}\hat{f}^* \leq \hat{s}_{\mathrm{H}\Gamma}(f) = \varphi(s_{\mathrm{H}\Delta}(f))$$

(HR3): For every $v \in E^0(\mathrm{H}\Delta)$ that is not a sink in $\mathrm{H}\Delta$ we have

$$\varphi(v) = \hat{v} \leq \sum_{e \in E^1(\mathrm{H}\Gamma): v \in s_{\mathrm{H}\Gamma}(e)} \hat{e}\hat{e}^* = \sum_{e \in E^1(\mathrm{H}\Delta): v \in s_{\mathrm{H}\Delta}(e)} \varphi(e)\varphi(e)^*.$$

It remains to show that φ is injective. Let ρ be the universal representation of $C^*(\mathrm{H}\Delta)$ on a Hilbert space $\mathcal{H}$ given by the GNS-construction. Further, let κ be a cardinal larger than the dimension of $\mathcal{H}$ and let

$\sigma : C^*(\mathrm{H}\Delta) \to B(\mathcal{H}^\kappa)$ be κ times the representation ρ. Then $\sigma(f)\mathcal{H}^\kappa$ and $\sigma(r_{\mathrm{H}\Delta}(f))\mathcal{H}^\kappa$ have the same dimension and therefore $B(\mathcal{H}^\kappa)$ contains a partial isometry V with $VV^* = \sigma(f)$ and $V^*V = \sigma(r_{\mathrm{H}\Delta}(f))$.

Now, one readily checks that the universal property of $C^*(\mathrm{H}\Gamma)$ yields a representation τ of $C^*(\mathrm{H}\Gamma)$ on $\mathcal{H}^\kappa$ with

$$\tau : \begin{cases} \hat{v} \mapsto \sigma(v), & v \in E^0(\mathrm{H}\Gamma), \\ \hat{e} \mapsto \sigma(e), & e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}, \\ \hat{e} \mapsto V, & e = f, \end{cases}$$

Evidently, $\sigma = \tau \circ \varphi$, so $x \in \ker(\varphi) \implies x \in \ker(\sigma) = \ker(\rho)$. Since ρ is faithful, the latter entails $x = 0$ and therefore φ is injective. □

4.4 Edge or Vertex Deletion

Let us say that $\mathrm{H}\Delta$ is obtained from $\mathrm{H}\Gamma$ by deleting a set $S \subset E^0(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma)$ if one gets $\mathrm{H}\Delta$ from $\mathrm{H}\Gamma$ by successively deleting the edges and vertices in S. Clearly, the order in which these operations are performed is irrelevant. The next lemma shows that for suitably "nice" sets S the algebra $C^*(\mathrm{H}\Delta)$ is a quotient of $C^*(\mathrm{H}\Gamma)$. First, we need a definition.

Definition 4.8 (ideally closed set)**.** *Let* $\mathrm{H}\Gamma$ *be a hypergraph. A subset* $S \subset E^0(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma)$ *is called* ideally closed *if*

- *whenever an edge* e *is in* S*, then* $r(e) \subset S$*,*
- *whenever an edge* $e \in E^1(\mathrm{H}\Gamma)$ *satisfies* $s(e) \subset S$ *or* $\emptyset \neq r(e) \subset S$*, then* $e \in S$*,*
- *whenever a vertex* $v \in E^0(\mathrm{H}\Gamma)$ *is not a sink and satisfies* $v \in s(e) \implies e \in S$ *for all edges* $e \in E^1(\mathrm{H}\Gamma)$*, then* $v \in S$*.*

Lemma 4.9. *If* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by deleting an ideally closed set* $S \subset E^0(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma)$*, then* $C^*(\mathrm{H}\Delta)$ *is isomorphic to the quotient* $C^*(\mathrm{H}\Gamma)/(S)$*.*

Proof. Step 1 Use the universal property of $C^*(\mathrm{H}\Gamma)$ to obtain a map $\varphi : C^*(\mathrm{H}\Gamma) \to C^*(\mathrm{H}\Delta)$ with

$$\varphi : \begin{cases} v \mapsto \hat{v}, & v \in E^0(\mathrm{H}\Gamma) \setminus S, \\ v \mapsto 0, & v \in E^0(\mathrm{H}\Gamma) \cap S, \\ e \mapsto \hat{e}, & e \in E^1(\mathrm{H}\Gamma) \setminus S, \\ e \mapsto 0, & e \in E^1(\mathrm{H}\Gamma) \cap S, \end{cases}$$

where we write $\hat{v}$ and $\hat{e}$ for the generators of $C^*(\mathrm{H}\Delta)$ to avoid confusion. Clearly the $\varphi(v)$ are pairwise orthogonal projections and the $\varphi(e)$ are partial isometries. We check the hypergraph relations.

(HR1): Let $e, f \in E^1(\mathrm{H}\Gamma) \setminus S$. If e has nonempty range in HΔ, then the same holds in HΓ since otherwise the edge e would have been deleted, too. Thus, we have

$$\begin{aligned}\varphi(e)^*\varphi(f) &= \hat{e}^*\hat{f} \\ &= \begin{cases}\delta_{ef}\hat{r}_{\mathrm{H}\Delta}(e), & r_{\mathrm{H}\Delta}(e) \neq \emptyset, \\ \delta_{ef}\hat{e}, & \text{otherwise},\end{cases} \\ &= \begin{cases}\delta_{ef}\varphi(r_{\mathrm{H}\Gamma}(e)), & r_{\mathrm{H}\Gamma}(e) \neq \emptyset, \\ \delta_{ef}\varphi(e), & \text{otherwise}.\end{cases}\end{aligned}$$

For $e \in E^1(\mathrm{H}\Gamma) \cap S$ and $f \in E^1(\mathrm{H}\Gamma)$ it is

$$\varphi(e)^*\varphi(f) = 0 = \begin{cases}\delta_{ef}\varphi(r_{\mathrm{H}\Gamma}(e)), & r_{\mathrm{H}\Gamma}(e) \neq \emptyset, \\ \delta_{ef}\varphi(e), & \text{otherwise},\end{cases}$$

since together with e all vertices in $r_{\mathrm{H}\Gamma}(e)$ are deleted. By taking adjoints one obtains the same equality for $e \in E^1(\mathrm{H}\Gamma)$ and $f \in E^1(\mathrm{H}\Gamma) \cap S$.

(HR2): For $e \in E^1(\mathrm{H}\Gamma) \setminus S$ observe

$$\varphi(e)\varphi(e)^* = \hat{e}\hat{e}^* \leq \hat{s}_{\mathrm{H}\Delta}(e) = \varphi(s_{\mathrm{H}\Gamma}(e)).$$

If $e \in E^1(\mathrm{H}\Gamma) \cap S$, then $\varphi(e)\varphi(e)^* = 0 \leq \varphi(s_{\mathrm{H}\Gamma}(e))$ is trivial.

(HR3): If $v \in E^0(\mathrm{H}\Gamma) \setminus S$ were a sink in HΔ but not in HΓ then it would have been deleted since S is ideally closed. Thus, we have for every $v \in E^0(\mathrm{H}\Gamma) \setminus S$ that is not a sink in HΓ

$$\begin{aligned}\varphi(v) = \hat{v} &\leq \sum_{e \in E^1(\mathrm{H}\Delta): v \in s_{\mathrm{H}\Delta}(e)} \hat{e}\hat{e}^* \\ &= \sum_{e \in E^1(\mathrm{H}\Delta): v \in s_{\mathrm{H}\Delta}(e)} \varphi(e)\varphi(e)^* \\ &= \sum_{e \in E^1(\mathrm{H}\Gamma): v \in s_{\mathrm{H}\Gamma}(e)} \varphi(e)\varphi(e)^*.\end{aligned}$$

If $v \in E^0(\mathrm{H}\Gamma) \cap S$, then $\varphi(v) = 0 \leq \sum_{e \in E^1(\mathrm{H}\Gamma): v \in s_{\mathrm{H}\Gamma}(e)} \varphi(e)\varphi(e)^*$ is trivial.

Observe that $\varphi(C^*(\mathrm{H}\Gamma))$ contains all generators of $C^*(\mathrm{H}\Delta)$ and therefore the map φ is surjective.

Step 2 We show that $C^*(\mathrm{H}\Delta)$ satisfies the universal property of the quotient $C^*(\mathrm{H}\Gamma)/(S)$ where the quotient map is given by φ. Indeed, let A be any C^*-algebra and $\chi : C^*(\mathrm{H}\Gamma) \to A$ a $*$-homomorphism with $S \subset \ker(\chi)$. The universal property of $C^*(\mathrm{H}\Delta)$ yields a map $\psi : C^*(\mathrm{H}\Delta) \to A$ with

$$\psi : \begin{cases}\hat{v} \mapsto \chi(v), & v \in E^0(\mathrm{H}\Delta), \\ \hat{e} \mapsto \chi(e), & e \in E^1(\mathrm{H}\Delta).\end{cases}$$

Evidently, the $\psi(\hat{v})$ are pairwise orthogonal projections and the $\psi(\hat{e})$ are partial isometries. We check the hypergraph relations.

(HR1): For $e, f \in E^1(\mathrm{H}\Delta)$ we have

$$\begin{aligned}
\psi(\hat{e})^*\psi(\hat{f}) &= \chi(e)^*\chi(f) \\
&= \chi(e^*f) \\
&= \begin{cases} \chi(\delta_{ef} r_{\mathrm{H}\Gamma}(e)) = \delta_{ef} \sum_{v \in E^0(\mathrm{H}\Gamma) \cap S \cap r_{\mathrm{H}\Gamma}(e)} \chi(v), & r_{\mathrm{H}\Gamma}(e) \neq \emptyset, \\ \chi(\delta_{ef} e), & \text{otherwise}, \end{cases} \\
&= \begin{cases} \delta_{ef}\psi(\hat{r}_{\mathrm{H}\Delta}(e)), & r_{\mathrm{H}\Delta}(e) \neq \emptyset, \\ \delta_{ef}\psi(\hat{e}), & \text{otherwise}, \end{cases}
\end{aligned}$$

using $\chi(v) = 0$ for $v \in E^0(\mathrm{H}\Gamma) \cap S$.

(HR2): For $e \in E^1(\mathrm{H}\Delta)$ observe

$$\begin{aligned}
\psi(\hat{e})\psi(\hat{e})^* &= \chi(e)\chi(e)^* \\
&= \chi(ee^*) \\
&\leq \chi(s_{\mathrm{H}\Gamma}(e)) \\
&= \sum_{v \in E^0(\mathrm{H}\Gamma) \cap S \cap s_{\mathrm{H}\Gamma}(e)} \chi(v) \\
&= \psi(\hat{s}_{\mathrm{H}\Delta}(e))
\end{aligned}$$

(HR3): If $v \in E^0(\mathrm{H}\Delta)$ is not a sink, then v is not a sink in $\mathrm{H}\Gamma$ neither, and we have

$$\begin{aligned}
\psi(\hat{v}) &= \chi(v) \\
&\leq \chi\left(\sum_{e \in E^1(\mathrm{H}\Gamma): v \in s_{\mathrm{H}\Gamma}(e)} ee^*\right) \\
&= \sum_{e \in E^1(\mathrm{H}\Gamma) \cap S: v \in s_{\mathrm{H}\Gamma}(e)} \chi(ee^*) \\
&= \sum_{e \in E^1(\mathrm{H}\Delta): v \in s_{\mathrm{H}\Delta}(e)} \psi(\hat{e})\psi(\hat{e})^*.
\end{aligned}$$

One readily checks $\chi = \psi \circ \varphi$. Clearly, ψ is the unique map from $C^*(\mathrm{H}\Delta)$ into A with this property. It follows $C^*(\mathrm{H}\Delta) = C^*(\mathrm{H}\Gamma)/(S)$ as desired. □

The next preparatory lemma allows to remove a vertex from the source of an edge without changing the associated hypergraph C^*-algebra.

Lemma 4.10. *Let $\mathrm{H}\Gamma$ be a hypergraph and let $w \in E^0(\mathrm{H}\Gamma)$, $f \in E^1(\mathrm{H}\Gamma)$ with $w \in s_{\mathrm{H}\Gamma}(f)$ and $r_{\mathrm{H}\Gamma}(f) = \emptyset$. Assume that f is the only edge starting from w, i.e. for all $e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$ it is $w \notin s_{\mathrm{H}\Gamma}(e)$, and obtain $\mathrm{H}\Delta$ from $\mathrm{H}\Gamma$ by removing w from the source of f, possibly deleting the edge f if $\{w\} = s_{\mathrm{H}\Gamma}(f)$, i.e.*

- $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma)$,
- $E^1(\mathrm{H}\Delta) = \begin{cases} E^1(\mathrm{H}\Gamma) \setminus \{f\}, & \text{if } \{w\} = s_{\mathrm{H}\Gamma}(f), \\ E^1(\mathrm{H}\Gamma), & \text{otherwise}, \end{cases}$

- $s_{\mathrm{H}\Delta}(e) = \begin{cases} s_{\mathrm{H}\Gamma}(e), & e \neq f, \\ s_{\mathrm{H}\Gamma}(e) \setminus \{w\}, & e = f, \end{cases}$ *for all* $e \in E^1(\mathrm{H}\Delta)$*,*
- $r_{\mathrm{H}\Delta}(e) = r_{\mathrm{H}\Gamma}(e)$ *for all* $e \in E^1(\mathrm{H}\Delta)$.

Then $C^*(\mathrm{H}\Delta) = C^*(\mathrm{H}\Gamma)$*. In particular, if for all edges* $e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$ *we have* $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset$*, then we can delete the edge* f *from* $\mathrm{H}\Gamma$ *without changing the associated* C^**-algebra.*

Proof. If $\{w\} = s_{\mathrm{H}\Gamma}(f)$, then set $\hat{f} = 0 \in C^*(\mathrm{H}\Delta)$. Using the universal property of $C^*(\mathrm{H}\Gamma)$ and $C^*(\mathrm{H}\Delta)$, respectively, one obtains maps $\varphi : C^*(\mathrm{H}\Gamma) \to C^*(\mathrm{H}\Delta)$ and $\psi : C^*(\mathrm{H}\Delta) \to C^*(\mathrm{H}\Gamma)$ with

$$\varphi : \begin{cases} v \mapsto \hat{v}, & v \in E^0(\mathrm{H}\Gamma), \\ e \mapsto \hat{e}, & e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}, \\ e \mapsto \hat{f} + \hat{w}, & e = f, \end{cases}$$

and

$$\psi : \begin{cases} \hat{v} \mapsto v, & v \in E^0(\mathrm{H}\Delta), \\ \hat{e} \mapsto e, & e \in E^1(\mathrm{H}\Delta) \setminus \{f\}, \\ \hat{e} \mapsto f - w, & e = f. \end{cases}$$

As usually we write $\hat{v}$ and $\hat{e}$ for the generators of $C^*(\mathrm{H}\Delta)$ to avoid confusion with the elements of $C^*(\mathrm{H}\Gamma)$. For both φ and ψ the hypergraph relations are checked by routine calculations. Moreover, one easily checks that φ and ψ are inverse to each other. Thus, $C^*(\mathrm{H}\Delta) = C^*(\mathrm{H}\Gamma)$.

Finally, if for all edges $e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$ it is $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset$, then we can use the previous result to successively remove every vertex $w \in s_{\mathrm{H}\Gamma}(f)$ from the source of f without changing the associated C^*-algebra. In the end, this deletes the edge f. □

Proposition 4.11 (edge/vertex deletion). *Assume that* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by*

1. *deleting an edge* f *or*
2. *deleting a vertex* w*.*

Then $C^*(\mathrm{H}\Delta)$ *is the quotient of a subalgebra of* $C^*(\mathrm{H}\Gamma)$*.*

Proof. Ad (1): First obtain $\mathrm{H}\Gamma'$ from $\mathrm{H}\Gamma$ by cutting the edge f. Then we have $C^*(\mathrm{H}\Gamma') \subset C^*(\mathrm{H}\Gamma)$ by Proposition 4.7. Let $v_1, \ldots, v_n \in E^1(\mathrm{H}\Gamma')$ be those vertices in $s_{\mathrm{H}\Gamma'}(f)$ which have only f as outgoing edge and obtain $\mathrm{H}\Gamma''$ from $\mathrm{H}\Gamma'$ by removing the v_i from the source of f and leaving everything else invariant. By Lemma 4.10 this does not change the associated C^*-algebra, i.e. we have $C^*(\mathrm{H}\Gamma'') = C^*(\mathrm{H}\Gamma')$. If f has been deleted in the process we are done. Otherwise, observe that in $\mathrm{H}\Gamma''$ the set $\{f\} \subset E^0(\mathrm{H}\Gamma'') \cup E^1(\mathrm{H}\Gamma'')$ is ideally closed. Since $\mathrm{H}\Delta$ is obtained from $\mathrm{H}\Gamma''$ by deleting the edge f, it follows from Lemma 4.9 that $C^*(\mathrm{H}\Delta)$ is a quotient of $C^*(\mathrm{H}\Gamma'') \subset C^*(\mathrm{H}\Gamma)$.

Ad (2): First, obtain $\mathrm{H}\Gamma'$ from $\mathrm{H}\Gamma$ by cutting all edges e with $s_{\mathrm{H}\Gamma}(e) = \{v\}$ or $r_{\mathrm{H}\Gamma}(e) = \{v\}$. Then $C^*(\mathrm{H}\Gamma')$ is a subalgebra of $C^*(\mathrm{H}\Gamma)$ by Lemma 4.7. One readily verifies that $\mathrm{H}\Delta$ is obtained from $\mathrm{H}\Gamma'$ by deleting the vertex v together with all edges e with $s_{\mathrm{H}\Gamma'}(e) = \{v\}$. Fortunately, the set $S = \{v\} \cup \{e \in E^1(\mathrm{H}\Gamma') : s_{\mathrm{H}\Gamma'}(e) = \{v\}\}$ is ideally closed in $\mathrm{H}\Gamma'$ and therefore $C^*(\mathrm{H}\Delta)$ is a quotient of $C^*(\mathrm{H}\Gamma') \subset C^*(\mathrm{H}\Gamma)$ by Lemma 4.9. □

4.5 Edge Contraction

In this section, we prove that if $\mathrm{H}\Delta$ is obtained from $\mathrm{H}\Gamma$ by forward or backward edge contraction, then $C^*(\mathrm{H}\Delta)$ is a full corner in $C^*(\mathrm{H}\Gamma)$. In a first step, the following two lemmas show how to obtain an intermediate hypergraph $\mathrm{H}\Gamma'$ from $\mathrm{H}\Gamma$ by rearranging the edges such that $C^*(\mathrm{H}\Gamma) = C^*(\mathrm{H}\Gamma')$. Afterwards, it is easy to show the result for forward edge contraction and the statement for backward edge contraction follows immediately.

Lemma 4.12. *Assume that* $\mathrm{H}\Gamma$ *is a hypergraph and* $f \in E^1(\mathrm{H}\Gamma)$ *an edge with nonempty range such that*

- $r_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset$,
- $r_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) \neq \emptyset \implies r_{\mathrm{H}\Gamma}(e) = s_{\mathrm{H}\Gamma}(f)$ *for all* $e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$ *and*
- f *is the only edge starting from* $s_{\mathrm{H}\Gamma}(f)$*, i.e.* $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset$ *for all* $e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$.

Further, let $\mathrm{H}\Delta$ *be given by*

- $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma)$,
- $E^1(\mathrm{H}\Delta) = E^1(\mathrm{H}\Gamma)$,
- $r_{\mathrm{H}\Delta}(e) = \begin{cases} r_{\mathrm{H}\Gamma}(e), & r_{\mathrm{H}\Gamma}(e) \neq s_{\mathrm{H}\Gamma}(f), \\ r_{\mathrm{H}\Gamma}(f), & \text{otherwise}, \end{cases}$
- $s_{\mathrm{H}\Delta}(e) = s_{\mathrm{H}\Gamma}(e)$,

for all edges $e \in E^1(\mathrm{H}\Delta)$*. Then* $C^*(\mathrm{H}\Delta) = C^*(\mathrm{H}\Gamma)$.

Proof. We use induction over the number n of edges $g \in E^1(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(g) = s_{\mathrm{H}\Gamma}(f)$. If $n = 0$, then $\mathrm{H}\Delta = \mathrm{H}\Gamma$ and there is nothing to do. For the induction step let g be an edge with $r_{\mathrm{H}\Gamma}(g) = s_{\mathrm{H}\Gamma}(f)$ and obtain $\mathrm{H}\Gamma'$ from $\mathrm{H}\Gamma$ by changing the range of g to $r_{\mathrm{H}\Gamma}(f)$ and leaving everything else invariant. Then the universal property of $C^*(\mathrm{H}\Gamma)$ yields a map $\varphi : C^*(\mathrm{H}\Gamma) \to C^*(\mathrm{H}\Gamma')$ with

$$\varphi : \begin{cases} v \mapsto \hat{v}, & v \in E^0(\mathrm{H}\Gamma), \\ e \mapsto \hat{e}, & e \in E^1(\mathrm{H}\Gamma) \setminus \{g\}, \\ e \mapsto \hat{g}\hat{f}^*, & e = g, \end{cases}$$

where we write $\hat{v}$ and $\hat{e}$ for the generators of $C^*(\mathrm{H}\Gamma')$ to avoid confusion. Evidently, the $\varphi(v)$ are pairwise orthogonal projections and since

$$\varphi(g)\varphi(g)^*\varphi(g) = \hat{g}\,(\hat{f}^*\hat{f})\,\hat{g}^*\hat{g}\hat{f}^* = \hat{g}\,(\hat{g}^*\hat{g})\,\hat{g}^*\hat{g}\hat{f}^* = \hat{g}\hat{f}^* = \varphi(g)$$

the $\varphi(e)$ are partial isometries. Further, we have

$$\varphi(g)^*\varphi(g) = \hat{f}\hat{g}^*\hat{g}\hat{f}^* = \hat{f}\hat{f}^*\hat{f}\hat{f}^* = \hat{f}\hat{f}^* = \hat{s}_{\mathrm{H}\Gamma'}(f) = \varphi(r_{\mathrm{H}\Gamma}(g))$$

and

$$\varphi(g)\varphi(g)^* = \hat{g}\hat{f}^*\hat{f}\hat{g}^* = \hat{g}\hat{g}^*\hat{g}\hat{g}^* = \hat{g}\hat{g}^* \leq \hat{s}_{\mathrm{H}\Gamma'}(g) = \varphi(s_{\mathrm{H}\Gamma}(g)).$$

The other hypergraph relations are checked by routine calculations. At the same time, one obtains a map $\psi : C^*(\mathrm{H}\Gamma') \to C^*(\mathrm{H}\Gamma)$ with

$$\psi : \begin{cases} \hat{v} \mapsto v, & v \in E^0(\mathrm{H}\Gamma'), \\ \hat{e} \mapsto e, & e \in E^1(\mathrm{H}\Gamma') \setminus \{g\} \\ \hat{e} \mapsto gf, & e = g. \end{cases}$$

Indeed, $\psi(\hat{g})$ is a partial isometry since

$$\psi(\hat{g})\psi(\hat{g})^*\psi(\hat{g}) = g(ff^*)g^*gf = g(g^*g)g^*gf = gf = \psi(\hat{g}).$$

Moreover,

$$\psi(\hat{g})^*\psi(\hat{g}) = f^*(g^*g)f = f^*(ff^*)f = f^*f = r_{\mathrm{H}\Gamma}(f) = \psi(\hat{r}_{\mathrm{H}\Gamma'}(g))$$

and

$$\psi(\hat{g})\psi(\hat{g})^* = g(ff^*)g^* = g(g^*g)g^* = gg^* \leq s_{\mathrm{H}\Gamma}(g) = \psi(\hat{s}_{\mathrm{H}\Gamma'}(g)).$$

Again the other hypergraph relations are checked by routine calculations. As

$$\varphi(\psi(\hat{g})) = \varphi(gf) = \hat{g}\hat{f}^*\hat{f} = \hat{g}\hat{g}^*\hat{g} = \hat{g}$$

and

$$\psi(\varphi(g)) = \psi(\hat{g}\hat{f}^*) = gff^* = gg^*g = g$$

the maps φ and ψ are inverse to each other. Thus, $C^*(\mathrm{H}\Gamma) = C^*(\mathrm{H}\Gamma')$ and we may apply the induction hypothesis to obtain $C^*(\mathrm{H}\Gamma) = C^*(\mathrm{H}\Delta)$. □

Lemma 4.13. *Assume that* $\mathrm{H}\Gamma$ *is a hypergraph and* $f \in E^1(\mathrm{H}\Gamma)$ *an edge with nonempty range such that*

- $r_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset$,
- $r_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(e) \neq \emptyset \implies r_{\mathrm{H}\Gamma}(f) \subset s_{\mathrm{H}\Gamma}(e)$ *for all* $e \in E^1(\mathrm{H}\Gamma)$,
- f *is the only edge starting from* $s_{\mathrm{H}\Gamma}(f)$, *i.e.* $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset$ *for all* $e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$.

Further, let $\mathrm{H}\Delta$ *be given by*

- $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma)$,
- $E^1(\mathrm{H}\Delta) = E^1(\mathrm{H}\Gamma)$,
- $r_{\mathrm{H}\Delta}(e) = \begin{cases} r_{\mathrm{H}\Gamma}(e), & e \neq f, \\ s_{\mathrm{H}\Gamma}(f), & e = f, \end{cases}$
- $s_{\mathrm{H}\Delta}(e) = \begin{cases} s_{\mathrm{H}\Gamma}(e), & e \neq f \wedge r_{\mathrm{H}\Gamma}(f) \not\subset s_{\mathrm{H}\Gamma}(e), \\ r_{\mathrm{H}\Gamma}(f), & e = f, \\ (s_{\mathrm{H}\Gamma}(e) \setminus r_{\mathrm{H}\Gamma}(f)) \cup s_{\mathrm{H}\Gamma}(f), & \textit{otherwise}, \end{cases}$

for all edges $e \in E^1(\mathrm{H}\Delta)$. *Then* $C^*(\mathrm{H}\Delta) = C^*(\mathrm{H}\Gamma)$.

Proof. Let $g_1, \dots, g_n \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$ be those edges with $r_{\mathrm{H}\Gamma}(f) \subset s_{\mathrm{H}\Gamma}(g_i)$ and use the universal property of $C^*(\mathrm{H}\Gamma)$ to obtain a map $\varphi : C^*(\mathrm{H}\Gamma) \to C^*(\mathrm{H}\Delta)$ with

$$\varphi : \begin{cases} v \mapsto \hat{v}, & v \in E^0(\mathrm{H}\Gamma), \\ e \mapsto \hat{e}, & e \in E^1(\mathrm{H}\Gamma) \setminus \{f, g_1, \dots, g_n\}, \\ e \mapsto \hat{f}^*, & e = f, \\ e \mapsto (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i, & e = g_i \text{ and } r_{\mathrm{H}\Gamma}(g_i) \neq \emptyset, \\ e \mapsto (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i)), & e = g_i \text{ and } r_{\mathrm{H}\Gamma}(g_i) = \emptyset, \end{cases}$$

where we write $\hat{v}$ and $\hat{e}$ for the generators of $C^*(\mathrm{H}\Delta)$ to avoid confusion. Here, we set $\hat{s}_{\mathrm{H}\Gamma}(g_i) := \sum_{v \in s_{\mathrm{H}\Gamma}(g_i)} \hat{v}$ and $\hat{s}_{\mathrm{H}\Delta}(g_i) := \sum_{v \in s_{\mathrm{H}\Delta}(g_i)} \hat{v}$. Evidently, the $\varphi(v)$ are pairwise orthogonal projections. Using

$$\hat{f}^* \hat{s}_{\mathrm{H}\Gamma}(g_i) = \hat{f}^* \hat{f} \hat{f}^* \hat{s}_{\mathrm{H}\Gamma}(g_i) = \hat{f}^* \hat{s}_{\mathrm{H}\Delta}(f) \hat{s}_{\mathrm{H}\Gamma}(g_i) = \hat{f}^* \hat{s}_{\mathrm{H}\Delta}(f) = \hat{f}^* \hat{f} \hat{f}^* = \hat{f}^*$$

and

$$\hat{f}^* \hat{g}_i = \hat{f}^* \hat{s}_{\mathrm{H}\Delta}(f) \hat{s}_{\mathrm{H}\Delta}(g_i) \hat{g}_i = 0$$

one obtains

$$\begin{aligned} &\varphi(g_i)\varphi(g_i)^*\varphi(g_i) \\ &\quad = (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i\hat{g}_i^*(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i))(\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i \\ &\quad = (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i\hat{g}_i^*(\hat{f}^*\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i) + \hat{f}^* + \hat{f})\hat{g}_i \\ &\quad = (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i\hat{g}_i^*(\hat{f}^*\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i \\ &\quad = (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i\hat{g}_i^*\hat{s}_{\mathrm{H}\Delta}(g_i)(\hat{r}_{\mathrm{H}\Delta}(f) + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i \\ &\quad = (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i\hat{g}_i^*\hat{g}_i \\ &\quad = (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i \\ &\quad = \varphi(g_i) \end{aligned}$$

for edges g_i with nonempty range. Otherwise, the same calculation can be used to show that $\varphi(g_i)$ is a projection. Thus, the $\varphi(e)$ are partial isometries. We check the hypergraph relations.

(HR1): For $e, e' \notin \{f, g_1, \dots, g_n\}$ we have

$$\varphi(e^*e') = \hat{e}^*\hat{e}' = \begin{cases} \delta_{ee'}\hat{r}_{\mathrm{H}\Delta}(e) = \delta_{ee'}\varphi(r_{\mathrm{H}\Gamma}(e)), & r_{\mathrm{H}\Gamma}(e) \neq \emptyset, \\ \delta_{ee'}\hat{e} = \delta_{ee'}\varphi(e), & \text{otherwise}, \end{cases}$$

and for $e = f$ it is

$$\varphi(f)^*\varphi(f) = \hat{f}\hat{f}^* = \hat{s}_{\mathrm{H}\Delta}(f) = \varphi(r_{\mathrm{H}\Gamma}(f))$$

since f is the only edge in $\mathrm{H}\Delta$ that starts from $r_{\mathrm{H}\Gamma}(f)(= s_{\mathrm{H}\Delta}(f))$. Further, one obtains for g_i with nonempty range the equality

$$\begin{aligned} \varphi(g_i)^*\varphi(g_i) &= \hat{g}_i^*(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i))(\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i \\ &= \hat{g}_i^*\hat{g}_i \\ &= \hat{r}_{\mathrm{H}\Delta}(g_i) \\ &= \varphi(r_{\mathrm{H}\Gamma}(g_i)), \end{aligned}$$

and for g_i with empty range the equality

$$\varphi(g_i)^*\varphi(g_i) = \varphi(g_i)$$

similarly as in the calculation of $\varphi(g_i)\varphi(g_i)^*\varphi(g_i)$ above. For g_i with nonempty range and $e \notin \{g_1, \ldots, g_n, f\}$ observe

$$\begin{aligned}
\varphi(e)^*\varphi(g_i) &= \hat{e}^*(\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i \\
&= \hat{e}^*\hat{f} + \hat{e}^*\hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{g}_i \\
&= \hat{e}^*\hat{f} + \hat{e}^*\hat{s}_{\mathrm{H}\Delta}(e)\hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{s}_{\mathrm{H}\Delta}(g_i)\hat{g}_i \\
&= \hat{e}^*\hat{f} + \hat{e}^*\left(\sum_{v \in s_{\mathrm{H}\Delta}(e) \cap s_{\mathrm{H}\Gamma}(g_i) \cap s_{\mathrm{H}\Delta}(g_i)} \hat{v}\right)\hat{g}_i \\
&= \hat{e}^*\hat{f} + \hat{e}^*\left(\sum_{v \in s_{\mathrm{H}\Delta}(e) \cap s_{\mathrm{H}\Delta}(g_i)} \hat{v}\right)\hat{g}_i \\
&= \hat{e}^*\hat{f} + \hat{e}^*\hat{g}_i \\
&= 0
\end{aligned}$$

and for $g_j \neq g_i$ with nonempty range

$$\begin{aligned}
\varphi(g_i)^*\varphi(g_j) &= \hat{g}_i^*(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i))(\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_j))\hat{g}_j \\
&= \hat{g}_i^*(\hat{f}^*\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{s}_{\mathrm{H}\Gamma}(g_j) + \hat{f}^*\hat{s}_{\mathrm{H}\Gamma}(g_j) + \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{f})\hat{g}_j \\
&= \hat{g}_i^*(\hat{r}_{\mathrm{H}\Delta}(f) + \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{s}_{\mathrm{H}\Gamma}(g_j) + \hat{f}^* + \hat{f})\hat{g}_j \\
&= \hat{g}_i^*(\hat{r}_{\mathrm{H}\Delta}(f) + \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{s}_{\mathrm{H}\Gamma}(g_j))\hat{g}_j \\
&= \hat{g}_i^*\hat{g}_j \\
&= 0.
\end{aligned}$$

Similar calculations apply if g_i and/or g_j has empty range. Further, for any edge $e \notin \{g_1, \ldots, g_n, f\}$ use $s_{\mathrm{H}\Delta}(e) \cap r_{\mathrm{H}\Delta}(f) = \emptyset$ to obtain

$$\varphi(e)^*\varphi(f) = \hat{e}^*\hat{f}^* = \hat{e}^*\hat{e}\hat{e}^*\hat{f}^*\hat{f}\hat{f}^* = 0.$$

Finally, observe for g_i with nonempty range

$$\begin{aligned}
\varphi(g_i)^*\varphi(f) &= \hat{g}_i^*(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{f}^* \\
&= \hat{g}_i^*(\hat{f}^*\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{f}^*) \\
&= \hat{g}_i^*(\hat{f}^*\hat{f}\hat{f}^*\hat{f}^*\hat{f}\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{f}^*\hat{f}\hat{f}^*) \\
&= \hat{g}_i^*(\hat{f}^*\hat{s}_{\mathrm{H}\Delta}(f)\hat{r}_{\mathrm{H}\Delta}(f)\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{r}_{\mathrm{H}\Delta}(f)\hat{f}^*) \\
&= 0
\end{aligned}$$

using $s_{\mathrm{H}\Delta}(f) \cap r_{\mathrm{H}\Delta}(f) = \emptyset$ and $s_{\mathrm{H}\Gamma}(g_i) \cap r_{\mathrm{H}\Delta}(f) = \emptyset$. Similarly, one obtains $\varphi(g_i)^*\varphi(f) = 0$ for g_i with empty range.

(HR2): For $e \notin \{f, g_1, \dots, g_n\}$ the second hypergraph relation is easily checked. For the remaining edges observe

$$
\begin{aligned}
&\varphi(g_i)\varphi(g_i)^* \\
&= (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i\hat{g}_i^*(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i)) \\
&\leq (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{s}_{\mathrm{H}\Delta}(g_i)(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i)) \\
&= (\hat{f}\hat{f}^*\hat{f}\hat{s}_{\mathrm{H}\Delta}(g_i) + \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{s}_{\mathrm{H}\Delta}(g_i))\,(\hat{s}_{\mathrm{H}\Delta}(g_i)\hat{f}^*\hat{f}\hat{f}^* + \hat{s}_{\mathrm{H}\Delta}(g_i)\hat{s}_{\mathrm{H}\Gamma}(g_i)) \\
&= (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i) - \hat{f}\hat{f}^*)\,(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i) - \hat{f}\hat{f}^*) \\
&= \hat{f}\hat{f}^* + \hat{f}\hat{s}_{\mathrm{H}\Gamma}(g_i) - \hat{f}\hat{f}\hat{f}^* \\
&\qquad + \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i) - \hat{s}_{\mathrm{H}\Gamma}(g_i)\hat{f}\hat{f}^* \\
&\qquad - \hat{f}\hat{f}^*\hat{f}^* - \hat{f}\hat{f}^*\hat{s}_{\mathrm{H}\Gamma}(g_i) + \hat{f}\hat{f}^* \\
&= \hat{s}_{\mathrm{H}\Gamma}(g_i) \\
&= \varphi(s_{\mathrm{H}\Gamma}(g_i))
\end{aligned}
$$

using $\hat{f}\hat{f}^* = \hat{s}_{\mathrm{H}\Delta}(f) = \hat{r}_{\mathrm{H}\Gamma}(f) \leq \hat{s}_{\mathrm{H}\Gamma}(g_i) \perp \hat{s}_{\mathrm{H}\Gamma}(f) = \hat{r}_{\mathrm{H}\Delta}(f) = \hat{f}^*\hat{f} \leq \hat{s}_{\mathrm{H}\Delta}(g_i) \perp \hat{s}_{\mathrm{H}\Delta}(f) = \hat{f}\hat{f}^*$. Note that these (in)equalities hold for g_i with empty or nonempty range at the same time. Moreover, it is

$$\varphi(f)\varphi(f)^* = \hat{f}^*\hat{f} = \hat{r}_{\mathrm{H}\Delta}(f) = \hat{s}_{\mathrm{H}\Gamma}(f) = \varphi(s_{\mathrm{H}\Gamma}(f)).$$

(HR3): If $v \notin r_{\mathrm{H}\Gamma}(f) \cup s_{\mathrm{H}\Gamma}(f)$ is not a sink in $\mathrm{H}\Gamma$, then with

$$\hat{f}\hat{v} = \hat{f}\hat{f}^*\hat{f}\hat{v} = 0 = \hat{f}\hat{r}_{\mathrm{H}\Delta}(f)\hat{s}_{\mathrm{H}\Delta}(e)\hat{e}\hat{e}^* = \hat{f}\hat{f}^*\hat{f}\hat{e}\hat{e}^* = \hat{f}\hat{e}\hat{e}^*$$

for all $e \notin \{g_1, \dots, g_n, f\}$ one obtains

$$
\begin{aligned}
\varphi(v) &= \hat{v} \\
&= (\hat{f} + 1 - \hat{f}^*\hat{f})\hat{v}(\hat{f}^* + 1 - \hat{f}^*\hat{f}) \\
&\leq (\hat{f} + 1 - \hat{f}^*\hat{f}) \left(\sum_{e \in E^1(\mathrm{H}\Delta): v \in s_{\mathrm{H}\Delta}(e)} \hat{e}\hat{e}^* \right) (\hat{f}^* + 1 - \hat{f}^*\hat{f}) \\
&= \sum_{e \in E^1(\mathrm{H}\Delta): e \neq g_i \wedge v \in s_{\mathrm{H}\Gamma}(e)} \hat{e}\hat{e}^* \\
&\qquad + \sum_{g_i : v \in s_{\mathrm{H}\Gamma}(g_i)} (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i\hat{g}_i^*(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i)) \\
&= \sum_{e \in E^1(\mathrm{H}\Delta): v \in s_{\mathrm{H}\Gamma}(e)} \varphi(e)\varphi(e)^*.
\end{aligned}
$$

Further, if $v \in s_{\mathrm{H}\Gamma}(f)$, then f is the only edge in $\mathrm{H}\Gamma$ starting from v and therefore

$$\varphi(v) = \hat{v} \leq \hat{s}_{\mathrm{H}\Gamma}(f) = \hat{r}_{\mathrm{H}\Delta}(f) = \hat{f}^*\hat{f} = \sum_{e \in E^1(\mathrm{H}\Gamma): v \in s_{\mathrm{H}\Gamma}(e)} \varphi(f)\varphi(f)^*.$$

Finally, let $v \in r_{\mathrm{H}\Gamma}(f)$ not be a sink in $\mathrm{H}\Gamma$. Then in $\mathrm{H}\Gamma$ the g_i are all edges that start from v while in $\mathrm{H}\Delta$ the g_i are all edges that start from $s_{\mathrm{H}\Gamma}(f)(= r_{\mathrm{H}\Delta}(f))$. Thus, we get

$$\begin{aligned}
\varphi(v) = \hat{v} \leq \hat{s}_{\mathrm{H}\Delta}(f) \\
&= \hat{f}\hat{f}^* \\
&= \hat{f}\hat{f}^*\hat{f}\hat{f}^* \\
&= (\hat{f} + 1 - \hat{f}^*\hat{f})\hat{f}^*\hat{f}(\hat{f}^* + 1 - \hat{f}^*\hat{f}) \\
&\leq (\hat{f} + 1 - \hat{f}^*\hat{f}) \left(\sum_{i=1}^{n} \hat{g}_i\hat{g}_i^* \right) (\hat{f}^* + 1 - \hat{f}^*\hat{f}) \\
&= \sum_{i=1}^{n} (\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i\hat{g}_i^*(\hat{f}^* + \hat{s}_{\mathrm{H}\Gamma}(g_i)) \\
&= \sum_{e \in E^1(\mathrm{H}\Gamma): v \in s_{\mathrm{H}\Gamma}(e)} \varphi(e)\varphi(e)^*.
\end{aligned}$$

It remains to find an inverse map $\psi : C^*(\mathrm{H}\Delta) \to C^*(\mathrm{H}\Gamma)$. A close inspection of $\mathrm{H}\Delta$ reveals that it satisfies the assumptions required from $\mathrm{H}\Gamma$ and indeed we get back $\mathrm{H}\Gamma$ from $\mathrm{H}\Delta$ by the same procedure that gave us $\mathrm{H}\Delta$ from $\mathrm{H}\Gamma$. Thus, by the very same arguments as above one obtains a map ψ with

$$\psi : \begin{cases}
\hat{v} \mapsto v, & v \in E^0(\mathrm{H}\Delta), \\
\hat{e} \mapsto e, & e \in E^1(\mathrm{H}\Delta) \setminus \{f, g_1, \dots, g_n\}, \\
\hat{e} \mapsto f^*, & e = f, \\
\hat{e} \mapsto (f + s_{\mathrm{H}\Delta}(g_i))g_i, & e = g_i \text{ and } r_{\mathrm{H}\Delta}(g_i) \neq \emptyset, \\
\hat{e} \mapsto (f + s_{\mathrm{H}\Delta}(g_i))g_i(f^* + s_{\mathrm{H}\Delta}(g_i)), & e = g_i \text{ and } r_{\mathrm{H}\Delta}(g_i) = \emptyset.
\end{cases}$$

Using

$$\begin{aligned}
\varphi(\psi(\hat{g}_i)) &= \varphi((f + s_{\mathrm{H}\Delta}(g_i))g_i) \\
&= (\hat{f}^* + \hat{s}_{\mathrm{H}\Delta}(g_i))(\hat{f} + \hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i \\
&= (\hat{f}^*\hat{f} + \hat{f}^*\hat{s}_{\mathrm{H}\Gamma}(g_i) + \hat{s}_{\mathrm{H}\Delta}(g_i)\hat{f} + \hat{s}_{\mathrm{H}\Delta}(g_i)\hat{s}_{\mathrm{H}\Gamma}(g_i))\hat{g}_i \\
&= (\hat{f}^*\hat{f} + \hat{f}^* + 0 + \hat{s}_{\mathrm{H}\Delta}(g_i) - \hat{f}^*\hat{f})\hat{g}_i \\
&= \hat{g}_i
\end{aligned}$$

for g_i with nonempty range and a similar calculation for g_i with empty range, one readily checks that φ and ψ are inverse to each other. This concludes the proof. □

Proposition 4.14 (edge contraction)**.** *Assume that* $\mathrm{H}\Delta$ *is obtained from* $\mathrm{H}\Gamma$ *by*

1. *forward contracting an edge* f *with* $s_{\mathrm{H}\Gamma}(f) = \{w\}$*, or*
2. *backward contracting an edge* f *with* $r_{\mathrm{H}\Gamma}(f) = \{w\}$*.*

Then $C^*(\mathrm{H}\Delta) = (1 - w)C^*(\mathrm{H}\Gamma)(1 - w)$ *and the latter is a full corner in* $C^*(\mathrm{H}\Gamma)$*.*

Proof. Ad (1): Because of the range decomposition operation we may assume without loss of generality that

$$w \in r_{\mathrm{H}\Gamma}(e) \implies \{w\} = r_{\mathrm{H}\Gamma}(e) \tag{$*$}$$

holds for all edges $e \in E^1(\mathrm{H}\Gamma)$. Otherwise, first decompose the ranges of all edges e with $w \in r_{\mathrm{H}\Gamma}(e)$ and then apply forward contraction on f. Since $w \in r_{\mathrm{H}\Gamma}(e)$ implies $r_{\mathrm{H}\Gamma}(e) \cap r_{\mathrm{H}\Gamma}(f) = \emptyset$ by assumption, we may apply range decomposition backwards to obtain the desired hypergraph $\mathrm{H}\Delta$.

Now, assume that $(*)$ is true and obtain $\mathrm{H}\Gamma'$ from $\mathrm{H}\Gamma$ by changing the range of every edge e with $\{w\} = r_{\mathrm{H}\Gamma}(e)$ to $r_{\mathrm{H}\Gamma}(f)$. Then Lemma 4.12 yields $C^*(\mathrm{H}\Gamma') = C^*(\mathrm{H}\Gamma)$. It remains to show that $C^*(\mathrm{H}\Delta)$ is a full corner of $C^*(\mathrm{H}\Gamma')$.

First, observe that $\mathrm{H}\Delta$ is obtained from $\mathrm{H}\Gamma'$ by first cutting the edge f and then deleting the ideally closed set $\{w, f\}$. Let $\mathrm{H}\Gamma''$ be the hypergraph obtained after the first operation. Then $C^*(\mathrm{H}\Gamma'')$ is a subalgebra of $C^*(\mathrm{H}\Gamma')$ by a previous proposition and since w has no incoming or outgoing edges in $\mathrm{H}\Gamma''$ one readily checks $C^*(\mathrm{H}\Gamma'') = C^*(\mathrm{H}\Delta) \oplus \mathbb{C}w$. Thus, $C^*(\mathrm{H}\Delta) \subset C^*(\mathrm{H}\Gamma'') \subset C^*(\mathrm{H}\Gamma') = C^*(\mathrm{H}\Gamma)$. A closer look at the embeddings reveals that $C^*(\mathrm{H}\Delta)$ is the subalgebra generated by the projections v for $v \neq w$ and the partial isometries e for $e \neq f$. All of these elements are in the corner $(1-w)C^*(\mathrm{H}\Gamma')(1-w)$, and therefore we have $C^*(\mathrm{H}\Delta) \subset (1-w)C^*(\mathrm{H}\Gamma')(1-w)$.

To show equality, use that a dense subset of $C^*(\mathrm{H}\Gamma')$ is spanned by words of the form $x = x_1 \dots x_n$ with $x_i \in E^0(\mathrm{H}\Gamma') \cup E^1(\mathrm{H}\Gamma') \cup E^1(\mathrm{H}\Gamma')^*$. We claim that if x is in $(1-w)C^*(\mathrm{H}\Gamma')(1-w)$, then it is also in $C^*(v, e : v \neq w, e \neq f) = C^*(\mathrm{H}\Delta)$. This implies immediately

$$(1-w)C^*(\mathrm{H}\Gamma')(1-w) \subset C^*(\mathrm{H}\Delta).$$

We prove the claim by induction over the number N of occurrences of f or w in the word x. Without loss of generality, however, x does not contain the letter w since we could replace it with ff^*. If $N = 0$ there is nothing to show. For the induction step distinguish the following cases:

<u>Case 1</u> $x = fx'$ for some x'. Then $x \in (1-w)C^*(\mathrm{H}\Gamma')(1-w)$ implies

$$x = (1-w)x = (1-w)fx' = (1-w)ff^*fx' = (1-w)wfx' = 0.$$

<u>Case 2</u> $x = x'efx''$ for some x', x'' and an edge $e \in E^1(\mathrm{H}\Gamma')$. The properties of $\mathrm{H}\Gamma'$ imply that the intersection $r_{\mathrm{H}\Gamma'}(e) \cap s_{\mathrm{H}\Gamma'}(f)$ is empty. Therefore, $x = 0$.

<u>Case 3</u> $x = x'e^*fx''$ for some x', x'' and an edge $e \in E^1(\mathrm{H}\Gamma)$. Unless x is zero the intersection $s_{\mathrm{H}\Gamma'}(e) \cap s_{\mathrm{H}\Gamma'}(f)$ must not be empty. This leaves only the possibility $e = f$. Then

$$x = x'e^*fx'' = x'f^*fx'' = x'r_{\mathrm{H}\Gamma'}(f)x'' = \sum_{v \in r_{\mathrm{H}\Gamma'}(f)} x'vx''.$$

On the last term we may apply the induction hypothesis.

<u>Case 4</u> $x = x'vfx''$ for some x', x'' and a vertex $v \in E^0(\mathrm{H}\Gamma)$. By assumption, we have without loss of generality $v \neq w$. Thus,

$$x = x'vfx'' = 0.$$

<u>Case 5</u> $x = x'f^*$, $x = x'f^*ex''$, $x = x'f^*e^*x''$ or $x = x'f^*vx''$. By passing to the adjoint one of the previous cases applies.

Ad (2): One checks that the edge f satisfies the assumptions from Lemma 4.13. Therefore, we may obtain $\mathrm{H}\Gamma'$ by changing the source of every edge e with $w \in s_{\mathrm{H}\Gamma}(e)$ from $s_{\mathrm{H}\Gamma}(e)$ to $(s_{\mathrm{H}\Gamma}(e) \setminus \{w\}) \cup s_{\mathrm{H}\Gamma}(f)$ and invert the direction of the edge f without changing the associated C^*-algebra. Then one readily checks that $\mathrm{H}\Delta$ is obtained from $\mathrm{H}\Gamma'$ by forward contracting the edge f and the claim follows from (1). □

5 Nuclearity in the Absence of Forbidden Minors

This chapter develops tools for proving nuclearity of a hypergraph C^*-algebra $C^*(\mathrm{H}\Gamma)$ if $\mathrm{H}\Gamma$ has none of the forbidden minors, i.e. if $\mathrm{H}\Gamma_i \not\leq \mathrm{H}\Gamma$ holds for all $i \leq 4$. To that end, we present a reduction procedure which allows transforming a hypergraph $\mathrm{H}\Gamma$ into an "easier" hypergraph $\mathrm{H}\Delta$ such that $C^*(\mathrm{H}\Gamma)$ is nuclear if, and only if, the same holds for $C^*(\mathrm{H}\Delta)$. For the reduced hypergraph $\mathrm{H}\Delta$ we are able to show that $\mathrm{H}\Delta$ has one of the forbidden minors as soon as it contains an edge with nonempty range.

More precisely, in Subsection 5.1 we prove Lemma 3.3 which says that $\mathrm{H}\Gamma$ can be normalized without changing the associated C^*-algebra up to Morita-equivalence. In Subsection 5.2, we show a useful lemma which is then used in Subsections 5.3, 5.4 and 5.5 to eliminate so-called "easy edges", "easy cycles" and edges ending in a so-called "simple quasisink" from $\mathrm{H}\Gamma$ without changing nuclearity of the associated C^*-algebra. Taking the normalization and these eliminations together, in Subsection 5.6 we obtain a reduction procedure with the desired properties. Finally, in Subsection 5.7 it turns out that any reduced hypergraph $\mathrm{H}\Delta$ has one of the forbidden minors as soon as it contains an edge with nonempty range.

5.1 Normal Hypergraphs

In this section we prove Lemma 3.3. First, let us recall the definition of a normal hypergraph.

Definition (Definition 3.2)**.** *A hypergraph* $\mathrm{H}\Gamma$ *is called* normal *if it has the following properties.*

1. $|r(e)| \leq 1$ *for all edges* $e \in E^1(\mathrm{H}\Gamma)$.
2. *For every edge* e *there exists another edge* f *with* $s(e) \cap s(f) \neq \emptyset$ *or* $\emptyset \neq r(e) \subset s(e)$.
3. *Whenever* (e, f) *is a pair of distinct edges with* $|s(e) \cap s(f)| = 1$*, then one of the following holds:*
   a) $|s(e)| = |s(f)| = 1$.
   b) *There is an edge* $g \neq e$ *with* $s(e) \cap s(f) \subsetneq s(e) \cap s(g)$.

Example 5.1. *In the figure below we sketch examples of normal and not normal hypergraphs, respectively.*

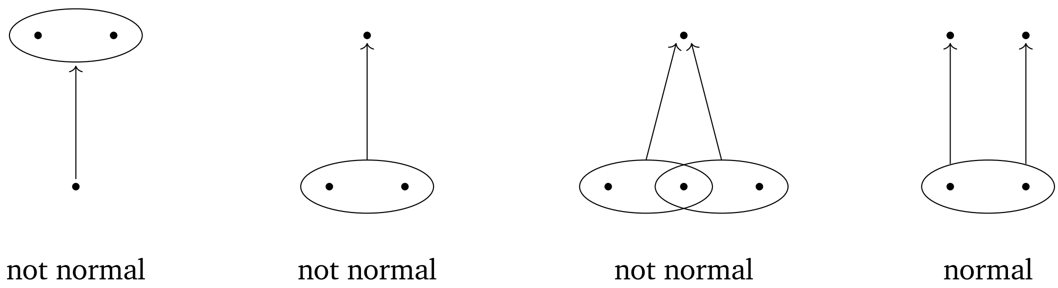


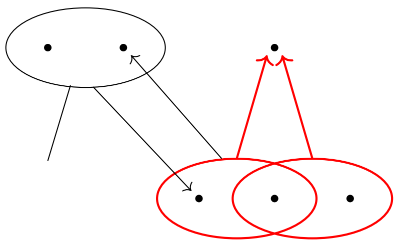
Figure 5.1: Normal and Not Normal Hypergraphs

Now, let us recall Lemma 3.3 from Chapter 3.

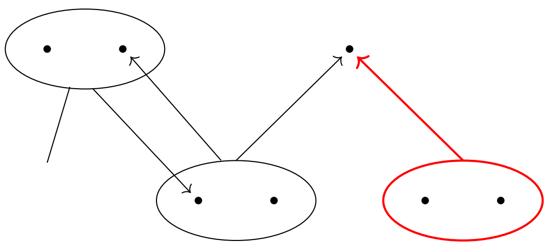
Lemma (Lemma 3.3)**.** *Let* $\mathrm{H}\Gamma$ *by a hypergraph. Then there is a normal hypergraph* $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$ *such that* $C^*(\mathrm{H}\Delta)$ *is Morita equivalent to* $C^*(\mathrm{H}\Gamma)$*. We call* $\mathrm{H}\Delta$ *a* normalized version *of* $\mathrm{H}\Gamma$.

The basic idea behind the lemma is to take an arbitrary hypergraph $\mathrm{H}\Gamma$ and to apply range decomposition, backward contraction and the special variant of source separation from Proposition 4.4 as often as possible. First, let us illustrate this procedure in an example.

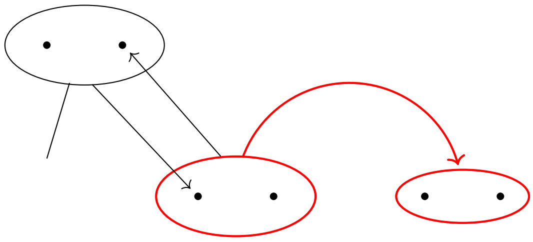
Example 5.2. *Let us consider the hypergraph* $\mathrm{H}\Gamma$ *given by the sketch below.*



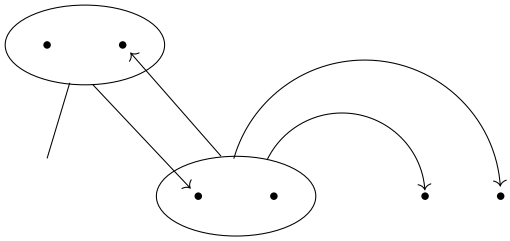
Going through all edges one checks that $\mathrm{H}\Gamma$ *satisfies conditions (1) and (2) of Definition 3.2. However, condition (3) from the same definition is violated since the sources of the two edges on the right-hand side (marked in red) overlap in exactly one vertex, but there is no third edge as required. On the other hand, one readily checks that exactly this property allows to invoke the special case from Proposition 4.4, i.e. we may separate the source of the right red edge without changing the associated* C^**-algebra. This operation yields the hypergraph* $\mathrm{H}\Gamma'$ *given by the sketch below.*



Now, condition (1) and (3) are satisfied, but the edge on the right-hand side (marked in red) violates condition (2). To get rid of this, let us apply backward contraction on that edge. By Theorem 4.3 this does not change the associated C^*-algebra up to Morita equivalence. We obtain the hypergraph $\mathrm{H}\Gamma''$ as in the sketch below.



Clearly, $\mathrm{H}\Gamma''$ violates condition (1) because the edge on the right-hand side (in red) has two vertices in its range. To change this let us apply range decomposition on that edge. By Theorem 4.3 the associated C^*-algebra remains unchanged, and we obtain the hypergraph $\mathrm{H}\Delta$ depicted below.



One readily checks that $\mathrm{H}\Delta$ is a normal hypergraph, $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$ and $C^*(\mathrm{H}\Gamma) =_M C^*(\mathrm{H}\Delta)$.

The following proof of Lemma 3.3 generalizes the steps that were used in the previous example. Essentially, we will first use successive source separations to eliminate all pairs of edges that violate condition (3) from Definition 3.2, and then apply successively backward contraction on all edges that violate condition (2) from the same definition. In between these steps, range decomposition will be used to guarantee condition (1) from Definition 3.2.

Proof of Lemma 3.3. Consider the sets $S_i := S_i(\mathrm{H}\Gamma)$ given by

$$S_1 := \{e \in E^1(\mathrm{H}\Gamma) \,|\, e \text{ violates condition (2) from Definition 3.2}\},$$
$$S_2 := \{(e,f) \in E^1(\mathrm{H}\Gamma) \times E^1(\mathrm{H}\Gamma) \,|\, e \neq f \text{ and the pair } (e,f) \text{ violates condition (3) from Definition 3.2}\},$$

and set $n_i := n_i(\mathrm{H}\Gamma) := |S_i|$ for $i = 1, 2$.

Step 1 First, let us assume $n_2(\mathrm{H}\Gamma) = 0$. By applying range decomposition on all edges we may further assume that every edge $e \in E^1(\mathrm{H}\Gamma)$ satisfies $|r_{\mathrm{H}\Gamma}(e)| \leq 1$. We prove the claim by induction over the number of vertices.

If $\mathrm{H}\Gamma$ has no vertices or $n_1(\mathrm{H}\Gamma) = 0$, then $\mathrm{H}\Gamma$ is normal and there is nothing to do. Otherwise, choose some edge $e \in S_1(\mathrm{H}\Gamma)$. If $r_{\mathrm{H}\Gamma}(e) = \emptyset$ then we can delete the edge e without changing the associated C^*-algebra, see Lemma 4.10. Hence, without loss of generality we may assume $r_{\mathrm{H}\Gamma}(e) \neq \emptyset$. Then by assumption we have $r_{\mathrm{H}\Gamma}(e) \not\subset s_{\mathrm{H}\Gamma}(e)$, $|r_{\mathrm{H}\Gamma}(e)| = 1$ and e is the only edge starting from $s_{\mathrm{H}\Gamma}(e)$, i.e. it is $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset$ for all $f \in E^1(\mathrm{H}\Gamma) \setminus \{e\}$.

Therefore, we may construct a hypergraph $\mathrm{H}\Gamma'$ by applying backward contraction on e. Further, obtain another hypergraph $\mathrm{H}\Gamma''$ from $\mathrm{H}\Gamma'$ by applying range decomposition on all edges. Evidently, $\mathrm{H}\Gamma'' \leq \mathrm{H}\Gamma$ and $\mathrm{H}\Gamma''$ has fewer vertices than $\mathrm{H}\Gamma$. Moreover, we have $C^*(\mathrm{H}\Gamma'') =_M C^*(\mathrm{H}\Gamma)$ by Theorem 4.3.

In order to apply the induction hypothesis we need to check $n_2(\mathrm{H}\Gamma'') = 0$. Assume that there is a pair $(e'', f'') \in S_2(\mathrm{H}\Gamma'')$, i.e. e'' and f'' are distinct edges in $\mathrm{H}\Gamma''$ with

- $|s_{\mathrm{H}\Gamma''}(e'') \cap s_{\mathrm{H}\Gamma''}(f'')| = 1$,
- $|s_{\mathrm{H}\Gamma''}(e'')| > 1$ or $|s_{\mathrm{H}\Gamma''}(f'')| > 1$ and
- there is no edge g'' in $\mathrm{H}\Gamma''$ with $s_{\mathrm{H}\Gamma''}(e'') \cap s_{\mathrm{H}\Gamma''}(f'') \subsetneq s_{\mathrm{H}\Gamma''}(e'') \cap s_{\mathrm{H}\Gamma''}(g'')$.

Clearly, it is not $s_{\mathrm{H}\Gamma''}(e'') = s_{\mathrm{H}\Gamma''}(f'')$. A moment's thought shows that there are edges e', f' in $\mathrm{H}\Gamma'$ with $s_{\mathrm{H}\Gamma'}(e') = s_{\mathrm{H}\Gamma''}(e'')$ and $s_{\mathrm{H}\Gamma'}(f') = s_{\mathrm{H}\Gamma''}(f'')$ such that $(e', f') \in S_2(\mathrm{H}\Gamma')$. Recall that $\mathrm{H}\Gamma'$ is obtained from $\mathrm{H}\Gamma$ by deleting the edge e together with the vertex in $r_{\mathrm{H}\Gamma}(e)$ and by replacing $r_{\mathrm{H}\Gamma}(e)$ with $s_{\mathrm{H}\Gamma}(e)$ in the range or source of every edge different from e. With this in mind, it is not difficult to see $|s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f')| = 1$ as well as $|s_{\mathrm{H}\Gamma}(e')| > 1$ or $|s_{\mathrm{H}\Gamma}(f')| > 1$. Assume that there is an edge g in $\mathrm{H}\Gamma$ with

$$s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f') \subsetneq s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(g).$$

Let w be the unique vertex in $r_{\mathrm{H}\Gamma}(e)$. One checks

$$\begin{aligned} s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(f') &= \begin{cases} ((s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f')) \setminus \{w\}) \cup s_{\mathrm{H}\Gamma}(e), & w \in s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f'), \\ s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f'), & \text{otherwise}, \end{cases} \\ &\subsetneq \begin{cases} ((s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(g)) \setminus \{w\}) \cup s_{\mathrm{H}\Gamma}(e), & w \in s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f'), \\ s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(g), & \text{otherwise}, \end{cases} \\ &= \begin{cases} s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(g), & w \in s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f'), \\ ((s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(g)) \setminus s_{\mathrm{H}\Gamma}(e)) \cup \{w\}, & w \in (s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(g)) \setminus s_{\mathrm{H}\Gamma}(f'), \\ s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(g), & \text{otherwise}. \end{cases} \end{aligned}$$

Using that the intersection on the left-hand side does not contain w, it follows

$$s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(f') \subsetneq s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(g).$$

This contradicts the assumption $(e', f') \in S_2(\mathrm{H}\Gamma')$. Thus, there is no such edge g, and we have $(e', f') \in S_2(\mathrm{H}\Gamma)$. However, we assumed $S_2(\mathrm{H}\Gamma) = \emptyset$. By contradiction, $S_2(\mathrm{H}\Gamma'') = \emptyset$ and $n_2(\mathrm{H}\Gamma'') = 0$.

Since in $\mathrm{H}\Gamma''$ every edge has at most one vertex in its range we may apply the induction hypothesis on $\mathrm{H}\Gamma''$ and conclude.

Step 2 In the general case, use induction over $n_2(\mathrm{H}\Gamma)$. If $n_2(\mathrm{H}\Gamma) = 0$, the previous step applies. Otherwise, choose a pair $(e, f) \in S_2(\mathrm{H}\Gamma)$ and let $\{w\} = s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f)$. The negation of condition (3) from Definition 3.2 entails

$$w \in s_{\mathrm{H}\Gamma}(g) \implies \{w\} = s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(g) \quad \text{for all } g \in E^1(\mathrm{H}\Gamma) \setminus \{e\}. \tag{$*$}$$

Let us construct a hypergraph $\mathrm{H}\Gamma'$ by applying source separation on $\{e\}$ at w, i.e. $\mathrm{H}\Gamma'$ is given by

- $E^0(\mathrm{H}\Gamma') = E^0(\mathrm{H}\Gamma) \cup \{w'\}$
- $E^1(\mathrm{H}\Gamma') = E^1(\mathrm{H}\Gamma)$,

- $r_{\mathrm{H}\Gamma'}(g) = \begin{cases} r_{\mathrm{H}\Gamma}(g), & w \notin r_{\mathrm{H}\Gamma}(g), \\ r_{\mathrm{H}\Gamma}(g) \cup \{w'\}, & \text{otherwise}, \end{cases}$ for all $g \in E^1(\mathrm{H}\Gamma')$,
- $s_{\mathrm{H}\Gamma'}(g) = \begin{cases} s_{\mathrm{H}\Gamma}(g), & g \neq e, \\ (s_{\mathrm{H}\Gamma}(g) \setminus \{w\}) \cup \{w'\}, & g = e, \end{cases}$ for all $g \in E^1(\mathrm{H}\Gamma')$.

In view of Proposition 4.4 and $(*)$ we have $C^*(\mathrm{H}\Gamma') = C^*(\mathrm{H}\Gamma)$. In order to apply the induction hypothesis we need to show $n_2(\mathrm{H}\Gamma') < n_2(\mathrm{H}\Gamma)$. To do this, let us first show $S_2(\mathrm{H}\Gamma') \subset S_2(\mathrm{H}\Gamma)$. For that, assume $(e', f') \notin S_2(\mathrm{H}\Gamma)$. We show $(e', f') \notin S_2(\mathrm{H}\Gamma')$. As $(e', f') \notin S_2(\mathrm{H}\Gamma)$ one of the following cases applies.

i) $|s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f')| \neq 1$. We show that then $|s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(f')| \neq 1$ holds as well. Indeed, when passing from $\mathrm{H}\Gamma$ to $\mathrm{H}\Gamma'$ the vertex w in the source of the edge e is replaced with w', but the sources of all other edges remain unchanged. Thus, the only possibility that $|s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(f')| = 1$, is that either $e' = e$ or $f' = e$, and $w \in s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f')$. Without loss of generality, assume $e' = e$. Then we have

$$s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f) = \{w\} \subsetneq s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f')$$

which contradicts the assumption $(e, f) \in S_2(\mathrm{H}\Gamma)$. Thus, $|s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(f')| \neq 1$.

ii) $|s_{\mathrm{H}\Gamma}(e')| = |s_{\mathrm{H}\Gamma}(f')| = 1$. Since passing from $\mathrm{H}\Gamma$ to $\mathrm{H}\Gamma'$ does not change the cardinalities of the sources of the edges, it follows directly

$$|s_{\mathrm{H}\Gamma'}(e')| = |s_{\mathrm{H}\Gamma'}(f')| = 1.$$

iii) There is an edge $g' \in E^1(\mathrm{H}\Gamma) \setminus \{e'\}$ such that $s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f') \subsetneq s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(g')$. Without loss of generality, the intersection on the left-hand side has cardinality 1 since otherwise case (i) applies. Then, $|s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(g')| \geq 2$. We show

$$s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(f') \subsetneq s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(g'). \tag{+}$$

If $e', f', g' \neq e$, we have

$$s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(f') = s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(f') \subsetneq s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(g') = s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(g')$$

since the involved source sets remain unchanged when passing from $\mathrm{H}\Gamma$ to $\mathrm{H}\Gamma'$. However, if one of the edges e', f', g' equals e, then one of the two intersections might lose the vertex w and get smaller. This matters only, if it happens on the right-hand side of (+) but not on the left-hand side. In this case, one has $g' = e$ or $e' = e$ and $w \in s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(g')$. It follows

$$s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = \{w\} \subsetneq s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(g') = \begin{cases} s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(g'), & \text{or} \\ s_{\mathrm{H}\Gamma}(e') \cap s_{\mathrm{H}\Gamma}(e), \end{cases}$$

contradicting the assumption $(e, f) \in S_2(\mathrm{H}\Gamma)$. Altogether, we get

$$s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(f') \subsetneq s_{\mathrm{H}\Gamma'}(e') \cap s_{\mathrm{H}\Gamma'}(g').$$

In any of the above cases it follows $(e', f') \notin S_2(\mathrm{H}\Gamma')$. Thus, we have the implication

$$(e', f') \notin S_2(\mathrm{H}\Gamma) \implies (e', f') \notin S_2(\mathrm{H}\Gamma')$$

which is equivalent to $S_2(\mathrm{H}\Gamma') \subset S_2(\mathrm{H}\Gamma)$. At the same time, one readily checks that the pair (e, f) is in $S_2(\mathrm{H}\Gamma)$ but not in $S_2(\mathrm{H}\Gamma')$. Thus, the subset relation is strict, and we have $n_2(\mathrm{H}\Gamma') < n_2(\mathrm{H}\Gamma)$. Now, we may apply the induction hypothesis and conclude. □

The previous proof translates directly into an algorithm for constructing a normalized version of a given hypergraph $\mathrm{H}\Gamma$.

Algorithm 1 Hypergraph Normalization

```
procedure NORMALIZE(hypergraph HΓ)
    HΓ ← take HΓ and apply range decomposition on all edges
    while True do
        if there is a pair (e, f) that violates condition (3) from Definition 3.2 then
            HΓ ← take HΓ and separate the source of {e} at s(e) ∩ s(f)
        else if there is an edge e that violates condition (2) from Definition 3.2 then
            if r(e) = ∅ then
                HΓ ← take HΓ and delete the edge e
            else
                HΓ ← take HΓ and apply backward contraction on e
            end if
            HΓ ← take HΓ and apply range decomposition on all edges
        else if none of the previous cases applies then
            break
        end if
    end while
    return HΓ
end procedure
```

5.2 Entry-/Exit-Closed Edge Sets

Let $\mathrm{H}\Gamma$ and $\mathrm{H}\Delta$ be hypergraphs and let $p \in C^*(\mathrm{H}\Gamma), q \in C^*(\mathrm{H}\Delta)$ be projections. Sometimes one observes $pC^*(\mathrm{H}\Gamma)p = qC^*(\mathrm{H}\Delta)q$ although the hypergraphs themselves are different. In the following, we find two situations where this is true.

Definition 5.3. *Let $\mathrm{H}\Gamma$ be a hypergraph and let $F \subset E^1(\mathrm{H}\Gamma)$ be a set of edges in $\mathrm{H}\Gamma$. Then F is* closed under source entries *if*

$$\forall f \in F, e \in E^1(\mathrm{H}\Gamma) : s(f) \cap r(e) \neq \emptyset \implies e \in F.$$

Similarly, F is closed under range exits *if*

$$\forall f \in F, e \in E^1(\mathrm{H}\Gamma) : r(f) \cap s(e) \neq \emptyset \implies e \in F.$$

Lemma 5.4. *Let $\mathrm{H}\Gamma$ be a hypergraph, $F \subset E^1(\mathrm{H}\Gamma)$ and let $p \in C^*(\mathrm{H}\Gamma)$ be a projection. Further, obtain $\mathrm{H}\Delta$ from $\mathrm{H}\Gamma$ by cutting all edges in F, and assume that one of the following holds:*

1. *F is closed under source entries, and $|s_{\mathrm{H}\Gamma}(f)| = 1$, $pf = 0$ hold for all $f \in F$.*
2. *F is closed under range exits, and $|r_{\mathrm{H}\Gamma}(f)| = 1$, $fp = 0$ hold for all $f \in F$. Further, we have*

$$\forall f \in F, e \in E^1(\mathrm{H}\Gamma) : r_{\mathrm{H}\Gamma}(f) \cap r_{\mathrm{H}\Gamma}(e) \neq \emptyset \implies e = f. \qquad (*)$$

Then $pC^*(\mathrm{H}\Gamma)p = pC^*(\mathrm{H}\Delta)p$.

Proof. Recall from Theorem 4.3 that $C^*(\mathrm{H}\Delta)$ is a subalgebra of $C^*(\mathrm{H}\Gamma)$. Further, recall from Lemma 2.14 that a dense subset of $C^*(\mathrm{H}\Gamma)$ is spanned by words of the form

$$x = x_1 \dots x_n \qquad \text{with } n \in \mathbb{N}, x_i \in E^0(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma) \cup (E^1(\mathrm{H}\Gamma))^*,$$

where for every $i < n$ neither of the following is true:

a) $x_i x_{i+1} = e^* f$ for some edges $e, f \in E^1(\mathrm{H}\Gamma)$.

b) $x_i x_{i+1} = ev$ or $x_i x_{i+1} = ve^*$ for some $e \in E^1(\mathrm{H}\Gamma), v \in E^0(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(e) \neq \emptyset$, and either $v \notin r_{\mathrm{H}\Gamma}(e)$ or $\{v\} = r_{\mathrm{H}\Gamma}(e)$.

c) $x_i x_{i+1} = ev$ or $x_i x_{i+1} = ve^*$ for some $e \in E^1(\mathrm{H}\Gamma), v \in E^0(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(e) = \emptyset$, and either $v \notin s_{\mathrm{H}\Gamma}(e)$ or $\{v\} = s_{\mathrm{H}\Gamma}(e)$.

d) $x_i x_{i+1} = ve$ or $x_i x_{i+1} = e^* v$ for some $e \in E^1(\mathrm{H}\Gamma), v \in E^0(\mathrm{H}\Gamma)$ with $v \notin s_{\mathrm{H}\Gamma}(e)$ or $\{v\} = s_{\mathrm{H}\Gamma}(e)$.

e) $x_i x_{i+1} = ef$ or $x_i x_{i+1} = f^* e^*$ for some $e, f \in E^1(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset$.

f) $x_i x_{i+1} = ef^*$ for some $e, f \in E^1(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(e) \neq \emptyset$ and $r_{\mathrm{H}\Gamma}(e) \cap r_{\mathrm{H}\Gamma}(f) = \emptyset$.

g) $x_i x_{i+1} = ef^*$ for some $e, f \in E^1(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(e) = \emptyset$ and $s_{\mathrm{H}\Gamma}(e) \cap r_{\mathrm{H}\Gamma}(f) = \emptyset$.

Let $x = x_1 \dots x_n$ be such a word with $pxp \neq 0$. It suffices to show $x \in C^*(\mathrm{H}\Delta)$ in the situations (1) and (2).

Ad (1): We show $x_i \notin F \cup F^*$ for all $i \leq n$. Indeed, for any $f \in F$ and all $e \in E^1(\mathrm{H}\Gamma), v \in E^0(\mathrm{H}\Gamma)$ one observes the following:

- $r_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) \neq \emptyset$ implies $e \in F$, since F is closed under source entries.
- $v \in s_{\mathrm{H}\Gamma}(f)$ implies $\{v\} = s_{\mathrm{H}\Gamma}(f)$ since $|s_{\mathrm{H}\Gamma}(f)| = 1$.

Combining these observations with the properties of x, one checks that as soon as x_i is in F for some $i > 1$, then x_{i-1} is in F as well. Inductively, it follows that either $x_1 \in F$ or $x_i \notin F$ for all $i \leq n$. However, in the first case we have $pxp = (px_1)x_2 \dots x_n p = 0$ since $pf = 0$ holds for all $f \in F$. Hence, it is $x_i \notin F$ for all $i \leq n$. By symmetry, $x_i \notin F^*$ for all $i \leq n$ holds as well. As $(E^0(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma)^*) \setminus (F \cup F^*)$ is a subset of $C^*(\mathrm{H}\Delta)$, we obtain $x \in C^*(\mathrm{H}\Delta)$ as desired.

Ad (2): Let x be as above and assume $pxp \neq 0$. This time, we show that every occurrence of some $f \in F$ in the product x is followed by f^*. Indeed, for all $e \in E^1(\mathrm{H}\Gamma)$ and $v \in E^0(\mathrm{H}\Gamma)$ one observes the following:

- $r_{\mathrm{H}\Gamma}(f) \neq \emptyset$ since $|r_{\mathrm{H}\Gamma}(f)| = 1$.
- $r_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(e) \neq \emptyset$ implies $e \in F$ since F is closed under range exits.
- $r_{\mathrm{H}\Gamma}(f) \cap r_{\mathrm{H}\Gamma}(e) \neq \emptyset$ implies $e = f$ due to $(*)$.
- $v \in r_{\mathrm{H}\Gamma}(f)$ implies $\{v\} = r_{\mathrm{H}\Gamma}(f)$ since $|r_{\mathrm{H}\Gamma}(f)| = 1$.

Combining these observations with the properties of x, one checks that for all $i < n$, $x_i = f \in F$ entails $x_{i+1} = f^*$ or $x_{i+1} \in F$. Assume $x_i = f \in F$ and $x_{i+1} \neq f^*$ for some $i < n$. Without loss of generality $i < n$ is maximal with this property. Then $x_{i+1} \in F$ and by induction one obtains $x_n \in F$ as well using maximality of i. Then, however, we have $pxp = px_1 \dots x_{n-1}(x_n p) = 0$ contradicting the assumption that $pxp \neq 0$. Thus, every occurrence of some $f \in F$ as a factor in the product x is followed by f^*. By symmetry, every occurrence of $f^* \in F^*$ in x is preceded by f as well. Altogether, we get $x \in C^*(\mathrm{H}\Delta)$ since $C^*(\mathrm{H}\Delta)$ contains ff^* for all $f \in F$ as well as $(E^0(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma)^*) \setminus (F \cup F^*)$. □

5.3 Elimination of Easy Edges

In this subsection, we introduce the notion of an easy edge, and prove that in a hypergraph $\mathrm{H}\Gamma$ certain edges around an easy edge can be cut without changing nuclearity of the associated hypergraph C^*-algebra.

Definition 5.5 (easy edge)**.** *Let* $\mathrm{H}\Gamma$ *be a hypergraph and let* $f_0 \in E^1(\mathrm{H}\Gamma)$. *Set*

$$F := \{f_0\} \cup \{f \in E^1(\mathrm{H}\Gamma) | \exists n \in \mathbb{N}_0, e_1, \dots, e_n \in E^1(\mathrm{H}\Gamma) : f e_1 \dots e_n f_0 \text{ is a path in } \mathrm{H}\Gamma\}.$$

The edge f_0 *is called* easy *if for all* $f \in F$ *it is* $|s_{\mathrm{H}\Gamma}(f)| = |r_{\mathrm{H}\Gamma}(f)| = 1$. *In this case, we call* F *the* easy edge set *generated by* f_0.

Example 5.6. *In the hypergraph* $\mathrm{H}\Gamma$ *below the edge* f *is easy, and the edges colored in red form the easy edge set generated by* f.

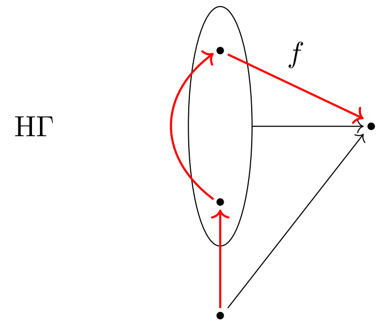


Figure 5.2: Example of an easy edge

Lemma 5.7. *Let* $\mathrm{H}\Gamma$ *be a hypergraph with an edge* $f \in E^1(\mathrm{H}\Gamma)$ *such that* $|s_{\mathrm{H}\Gamma}(f)| = 1$ *and assume that* $s_{\mathrm{H}\Gamma}(f)$ *is a source, i.e.*

$$\forall e \in E^1(\mathrm{H}\Gamma) : \; r_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset.$$

Obtain $\mathrm{H}\Delta$ *from* $\mathrm{H}\Gamma$ *by cutting the edge* f. *Then* $C^*(\mathrm{H}\Gamma)$ *is nuclear iff the same holds for* $C^*(\mathrm{H}\Delta)$.

Proof. Without loss of generality, the edge f has non-empty range. If there is an edge $e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$ with $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) \neq \emptyset$, then obtain $\mathrm{H}\Gamma'$ from $\mathrm{H}\Gamma$ by separating the source of f, i.e.

- $E^0(\mathrm{H}\Gamma') = E^0(\mathrm{H}\Gamma) \cup \{w_f\}$,
- $E^1(\mathrm{H}\Gamma') = E^1(\mathrm{H}\Gamma)$,
- $r_{\mathrm{H}\Gamma'}(e) = r_{\mathrm{H}\Gamma}(e)$ for all $e \in E^1(\mathrm{H}\Gamma')$,
- $s_{\mathrm{H}\Gamma'}(e) = \begin{cases} s_{\mathrm{H}\Gamma}(e), & e \neq f, \\ \{w_f\}, & e = f, \end{cases}$ for all $e \in E^1(\mathrm{H}\Gamma')$.

Otherwise, set $\mathrm{H}\Gamma' := \mathrm{H}\Gamma$ and let $w_f \in E^0(\mathrm{H}\Gamma')$ be the vertex with $s_{\mathrm{H}\Gamma}(f) = \{w_f\}$. By Proposition 4.4 we have in any case $C^*(\mathrm{H}\Gamma') = C^*(\mathrm{H}\Gamma)$. Further, in $C^*(\mathrm{H}\Gamma')$, f is the only edge starting from w_f. Hence, applying forward contraction on f does not change the associated C^*-algebra up to Morita equivalence, see Proposition 4.14. Let $\mathrm{H}\Gamma''$ be the obtained hypergraph. One readily checks

$$C^*(\mathrm{H}\Delta) = \mathbb{C} \oplus \mathbb{C}^*(\mathrm{H}\Gamma'') =_M \mathbb{C} \oplus C^*(\mathrm{H}\Gamma)$$

and this yields the claim. □

Lemma 5.8. *Let $\mathrm{H}\Gamma$ be a hypergraph that contains an easy edge f_0. Further, let F be the easy edge set generated by f_0 and obtain $\mathrm{H}\Delta$ from $\mathrm{H}\Gamma$ by cutting all edges in F. Then $C^*(\mathrm{H}\Gamma)$ is nuclear if, and only if, the same holds for $C^*(\mathrm{H}\Delta)$.*

Proof. Recall from Theorem 4.3 that $C^*(\mathrm{H}\Delta)$ is a subalgebra of $C^*(\mathrm{H}\Gamma)$.

Step 1 First, assume

$$r_{\mathrm{H}\Gamma}(f_0) = s_{\mathrm{H}\Gamma}(f) \qquad \text{for some } f \in F. \tag{$*$}$$

Let $S \subset E^0(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma)$ be given by

$$S := (E^1(\mathrm{H}\Gamma) \setminus F) \cup \left(E^0(\mathrm{H}\Gamma) \setminus \bigcup_{f \in F} s_{\mathrm{H}\Gamma}(f) \right).$$

We show that S is ideally closed in the sense of Definition 4.8. Indeed, we have the following:

- Whenever an edge e is in S, then $r_{\mathrm{H}\Gamma}(e)$ is a subset of S. Otherwise, there would be an edge $f \in F$ with $r_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) \neq \emptyset$. By definition of F this implies $e \in F$, i.e. $e \notin S$.
- Whenever an edge $e \in E^1(\mathrm{H}\Gamma)$ satisfies $s_{\mathrm{H}\Gamma}(e) \subset S$ or $\emptyset \neq r_{\mathrm{H}\Gamma}(e) \subset S$, then $e \in S$. Indeed, if e is not in S, then we have $e \in F$, and this implies $s_{\mathrm{H}\Gamma}(e) \cap S = \emptyset = r_{\mathrm{H}\Gamma}(e) \cap S$. For the latter equality, use $(*)$ to obtain that for every $f \in F$ there is an $f' \in F$ with $r_{\mathrm{H}\Gamma}(f) = s_{\mathrm{H}\Gamma}(f')$.
- Whenever a vertex $v \in E^0(\mathrm{H}\Gamma)$ is not a sink and satisfies $v \in s_{\mathrm{H}\Gamma}(e) \implies e \in S$ for all edges $e \in E^1(\mathrm{H}\Gamma)$, then $v \in S$. Indeed, if $v \notin S$, then there is an edge $f \in F$ with $v \in s_{\mathrm{H}\Gamma}(f)$.

Step 2 As S is ideally closed, Lemma 4.9 yields the short exact sequence

$$0 \to (S) \to C^*(\mathrm{H}\Gamma) \to C^*(\Phi) \to 0,$$

where Φ is obtained from $\mathrm{H}\Gamma$ by deleting all edges and vertices in S. Since all edges that are not in S have exactly one vertex in their range and source, respectively, one verifies that Φ is an ordinary graph. Thus, $C^*(\Phi)$ is nuclear. With Proposition 2.28 it follows that $C^*(\mathrm{H}\Gamma)$ is nuclear iff the same holds for (S).

Step 3 Set,

$$p := 1 - \sum_{f \in F} ff^* = \sum_{e \in S \cap E^1(\mathrm{H}\Gamma)} ee^* + \sum_{v \in S \cap E^0(\mathrm{H}\Gamma): v \text{ is a sink}} v.$$

Let us show $(S) = (pC^*(\mathrm{H}\Gamma)p)$. Evidently, $p \in (S)$ and therefore $pC^*(\mathrm{H}\Gamma)p \subset (S)$. Further, for vertices v in S one has that v is a sink or every edge $e \in E^1(\mathrm{H}\Gamma)$ with $v \in s_{\mathrm{H}\Gamma}(e)$ is in S. In any case, $v = pv = vp = pvp$. On the other hand, for every edge e in S with non-empty range, the range $r_{\mathrm{H}\Gamma}(e)$ contains only vertices from S. Thus, one checks

$$pep = ee^*er_{\mathrm{H}\Gamma}(e)p = er_{\mathrm{H}\Gamma}(e) = e \qquad \text{for all } e \in S.$$

If $e \in S \cap E^1(\mathrm{H}\Gamma)$ has empty range, then

$$pep = (ee^*)e(ee^*) = e.$$

Altogether, we have $S \subset pC^*(\mathrm{H}\Gamma)p \subset (S)$, and this entails $(S) = (pC^*(\mathrm{H}\Gamma)p)$.

Step 4 Next, use Lemma 5.4 to obtain $pC^*(\mathrm{H}\Gamma)p = pC^*(\mathrm{H}\Delta)p$. Indeed, by definition the set F is closed under source entries and for every edge $f \in F$ it is $|s_{\mathrm{H}\Gamma}(f)| = |r_{\mathrm{H}\Gamma}(f)| = 1$. Further, for every $f \in F$ one has

$$pf = \left(1 - \sum_{f\in F} ff^*\right) ff^*f = 0.$$

Hence, the conditions for Lemma 5.4(1) are satisfied and therefore $pC^*(\mathrm{H}\Gamma)p = pC^*(\mathrm{H}\Delta)p$.

Step 5 Let us show $C^*(\mathrm{H}\Delta) = \mathbb{C}^{|F|} \oplus pC^*(\mathrm{H}\Delta)p$. Indeed, from the proof of Proposition 4.7 it is clear that $C^*(\mathrm{H}\Delta)$ is the subalgebra of $C^*(\mathrm{H}\Gamma)$ generated by the elements in

$$\left(E^0(\mathrm{H}\Gamma) \setminus \bigcup_{f\in F} s_{\mathrm{H}\Gamma}(f)\right) \cup (E^1(\mathrm{H}\Gamma)\setminus F) \subset C^*(\mathrm{H}\Gamma) \qquad \text{and} \qquad \{ff^* : f \in F\} \subset C^*(\mathrm{H}\Gamma).$$

One checks that the elements in the first set are contained in $pC^*(\mathrm{H}\Delta)p$ while the elements in the latter set are pairwise orthogonal projections. Further, for every $f \in F$ we have $ff^* \perp p$. From that one gets immediately

$$\begin{aligned} C^*(\mathrm{H}\Delta) &= C^*(ff^* : f \in F) \oplus C^*\left(v, e : v \in E^0(\mathrm{H}\Gamma) \setminus \bigcup_{f\in F} s_{\mathrm{H}\Gamma}(f), e \in E^1(\mathrm{H}\Gamma)\setminus F\right) \\ &= \mathbb{C}^{|F|} \oplus pC^*(\mathrm{H}\Delta)p. \end{aligned}$$

In particular, $pC^*(\mathrm{H}\Delta)p$ is nuclear iff the same holds for $C^*(\mathrm{H}\Delta)$. Putting everything together and using $pC^*(\mathrm{H}\Delta)p =_M (pC^*(\mathrm{H}\Delta)p)$, we obtain the following equivalences:

$$\begin{aligned} & C^*(\mathrm{H}\Gamma) \text{ is nuclear} \\ \Leftrightarrow\quad & (S) = (pC^*(\mathrm{H}\Gamma)p) = (pC^*(\mathrm{H}\Delta)p) \text{ is nuclear} \\ \Leftrightarrow\quad & pC^*(\mathrm{H}\Delta)p \text{ is nuclear} \\ \Leftrightarrow\quad & C^*(\mathrm{H}\Delta) \text{ is nuclear.} \end{aligned}$$

Step 6 Finally, let us remove the assumption $(*)$ that there is an edge $f \in F$ with $r_{\mathrm{H}\Gamma}(f_0) = s_{\mathrm{H}\Gamma}(f)$. We show the general claim by induction over $|F|$. If $|F| = 1$ then the claim follows from Lemma 5.7. For $|F| > 1$, there are two possibilities: If $(*)$ is true, then the claim follows from the previous steps. Otherwise, let $f_1, \dots, f_k \in F$ be the edges with $r_{\mathrm{H}\Gamma}(f_i) = s_{\mathrm{H}\Gamma}(f_0)$. Then the f_i for $i \le k$ are easy edges in $\mathrm{H}\Gamma$ and their generated easy edge sets F_i do not contain f_0 since otherwise $(*)$ would be true. Thus, $|F_i| < |F|$ holds for all $i \le k$. By induction, we may cut all edges in the sets F_i. Afterwards, $s_{\mathrm{H}\Gamma}(f_0)$ is a source. By Lemma 5.7 we may cut the edge f_0 without changing nuclearity of the associated C^*-algebra, and this yields the claim. □

Example 5.9. *Let us consider the hypergraph* $\mathrm{H}\Gamma$ *from the previous example. Below we sketch* $\mathrm{H}\Gamma$ *with the easy edge set* F *generated by the edge* f *colored in red, together with the hypergraph* $\mathrm{H}\Delta$ *obtained by cutting all edges in* F*.*

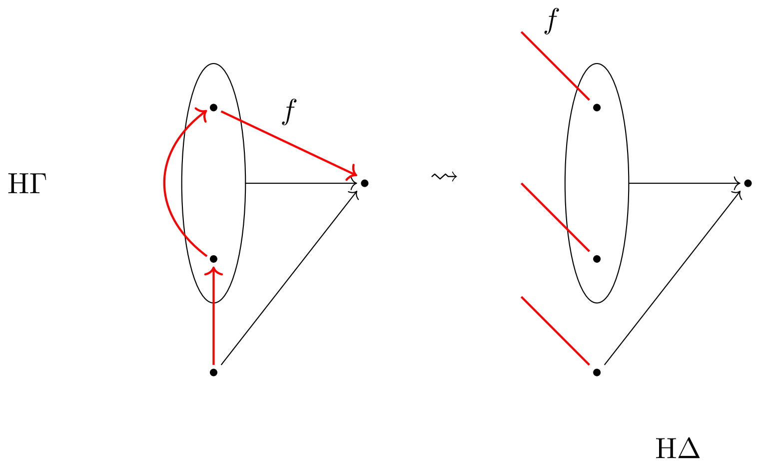


Figure 5.3: Cutting an easy edge set

Corollary 5.10. *Let* $\mathrm{H}\Gamma$ *be a normal hypergraph that contains no easy edge. Then, for all edges* e *in* $\mathrm{H}\Gamma$ *with non-empty range one of the following holds:*

1. $|s_{\mathrm{H}\Gamma}(e)| > 1$.
2. *There are edges* $e_1, \ldots, e_n \in E^1(\mathrm{H}\Gamma)$ *with* $n \geq 2$ *and* $e_n = e$ *such that* $e_1 \ldots e_n$ *is a path in* $\mathrm{H}\Gamma$ *and*

$$|s_{\mathrm{H}\Gamma}(e_1)| > 1 = |s_{\mathrm{H}\Gamma}(e_2)| = |s_{\mathrm{H}\Gamma}(e_3)| = \cdots = |s_{\mathrm{H}\Gamma}(e_n)|.$$

Proof. Let $e \in E^1(\mathrm{H}\Gamma)$ have non-empty range and assume that neither (1) nor (2) is true. Then, every edge f in the set F given by

$$F :=_{\mathrm{H}\Gamma} \{e\} \cup \{e_1 \in E^1(\mathrm{H}\Gamma) | \, \exists n \in \mathbb{N}_0, e_2, \ldots, e_n \in E^1(\mathrm{H}\Gamma) : e_1 e_2 \ldots e_n e \text{ is a path in } \mathrm{H}\Gamma\}$$

satisfies $|s_{\mathrm{H}\Gamma}(f)| = 1$. Since $\mathrm{H}\Gamma$ is normal we have further $|r_{\mathrm{H}\Gamma}(f)| = 1$ for all $f \in F$. Hence, e is an easy edge, and the claim follows by contradiction. □

5.4 Elimination of Easy Cycles

Let us define the notion of an easy cycle in a hypergraph. We will see that in a hypergraph $\mathrm{H}\Gamma$ all edges that are part of an easy cycle can be cut without losing the information about nuclearity of $C^*(\mathrm{H}\Gamma)$.

Definition 5.11. *Let* $\mathrm{H}\Gamma$ *be a hypergraph. A cycle* $\mu = f_1 \ldots f_n$ *is called* easy *if for all* $i \leq n$ *and all* $e \in E^1(\mathrm{H}\Gamma)$ *we have*

- $r_{\mathrm{H}\Gamma}(f_i) = \{w_i\}$ *for suitable vertices* w_i*,*
- $\{w_i\} \cap r_{\mathrm{H}\Gamma}(e) \neq \emptyset \implies e = f_i$*,*

- $\{w_i\} \cap s_{\mathrm{H}\Gamma}(e) \neq \emptyset \implies e = f_{i+1}$ *if* $i < n$,
- $\{w_n\} \cap s_{\mathrm{H}\Gamma}(e) \neq \emptyset \implies e = f_1$,

i.e. the edges f_i *have exactly one vertex* w_i *in their range and every vertex* w_i *has exactly one incoming and exactly one outgoing edge.*

Example 5.12. *Below we present two hypergraphs* $\mathrm{H}\Delta_1$ *and* $\mathrm{H}\Delta_2$. *While in* $\mathrm{H}\Delta_1$ *the edges* f_1 *and* f_2 *form an easy cycle, this is not true in* $\mathrm{H}\Delta_2$.

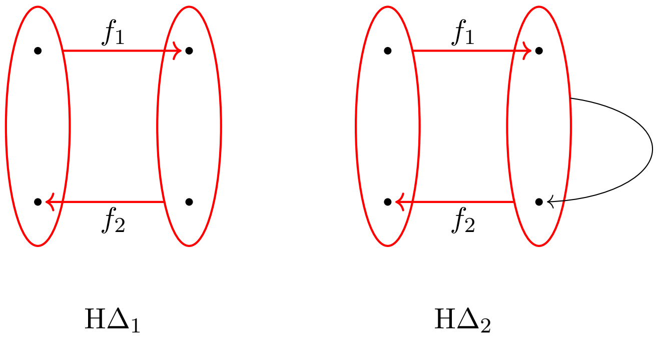


Figure 5.4: (Non-)Example of an easy cycle

The following lemma shows that edges on an easy cycle can be cut without losing the information about nuclearity of $C^*(\mathrm{H}\Gamma)$.

Lemma 5.13. *Let* $\mathrm{H}\Gamma$ *contain an easy cycle* $\mu = f_1 \dots f_n$ *and obtain* $\mathrm{H}\Delta$ *from* $\mathrm{H}\Gamma$ *by cutting all edges* f_i. *Then* $C^*(\mathrm{H}\Gamma)$ *is nuclear if, and only if, the same holds for* $C^*(\mathrm{H}\Delta)$.

Proof. Let $r_{\mathrm{H}\Gamma}(f_i) = \{w_i\}$ for all $i \leq n$ and recall from Theorem 4.3 that $C^*(\mathrm{H}\Delta)$ is a subalgebra of $C^*(\mathrm{H}\Gamma)$.

Step 1 Define $S := (E^0(\mathrm{H}\Gamma) \cup E^1(\mathrm{H}\Gamma)) \setminus \bigcup_i \{f_i, w_i\}$. Let us check that S is ideally closed in the sense of Definition 4.8.

- If $e \in S$, then $\{w_1, \dots, w_n\} \cap r_{\mathrm{H}\Gamma}(e) = \emptyset$ since μ is an easy cycle. Thus, $r_{\mathrm{H}\Gamma}(e) \subset S$.
- If $\emptyset \neq r_{\mathrm{H}\Gamma}(e) \subset S$ or $s_{\mathrm{H}\Gamma}(e) \subset S$ holds for an edge e, then e cannot be any of the f_i. Thus $e \in S$.
- If v is not a sink and every edge that starts from v is in S, then v cannot be any of the w_i since f_i starts from w_i and is not in S. Hence $v \in S$.

Step 2 As S is ideally closed, Lemma 4.9 yields the short exact sequence

$$0 \to (S) \to C^*(\mathrm{H}\Gamma) \to C^*(\Phi) \to 0,$$

where Φ is obtained from $\mathrm{H}\Gamma$ by deleting all edges and vertices in S. It is not hard to verify, that Φ is an ordinary graph, i.e. $|s_\Phi(e)| = |r_\Phi(e)| = 1$ for all $e \in E^1(\Phi)$. Indeed, $E^1(\Phi) = \{f_1, \dots, f_n\}$ and $E^0(\Phi) = \{w_1, \dots, w_n\}$. Therefore, $C^*(\Phi)$ is nuclear as a graph C^*-algebra. With Proposition 2.28 it follows that $C^*(\mathrm{H}\Gamma)$ is nuclear iff the same holds for the ideal (S).

Step 3 Set $p := 1 - \sum_{i=1}^{n} w_i \in C^*(\mathrm{H}\Gamma)$. One readily checks $pv = v$ and $pe = ep = e$ for all vertices $v \in E^0(\mathrm{H}\Gamma) \cap S$ and all edges $e \in E^1(\mathrm{H}\Gamma) \cap S$. Furthermore, we have

$$p = 1 - \sum_{i=1}^{n} w_i = \sum_{v \in E^0(\mathrm{H}\Gamma)} v - \sum_{i=1}^{n} w_i = \sum_{v \in E^0(\mathrm{H}\Gamma) \cap S} v \in S.$$

Combining both observations, one obtains $S \subset pC^*(\mathrm{H}\Gamma)p \subset (S)$, and therefore $(pC^*(\mathrm{H}\Gamma)p) = (S)$. In particular, (S) and $pC^*(\mathrm{H}\Gamma)p$ are Morita-equivalent, so that nuclearity of the former C^*-algebra is equivalent to nuclearity of the latter C^*-algebra.

Step 4 We show $pC^*(\mathrm{H}\Gamma)p = pC^*(\mathrm{H}\Delta)p$ using Lemma 5.4(2). Evidently, the set $\{f_i\}$ is closed under range exits. Moreover, we have $|r_{\mathrm{H}\Gamma}(f_i)| = 1$ and $f_i p = f_i w_i p = 0$ for all $i \leq n$. Finally, we have for all $i \leq n$ and $e \in E^1(\mathrm{H}\Gamma)$, $r_{\mathrm{H}\Gamma}(f_i) \cap s_{\mathrm{H}\Gamma}(e) = \{w_i\} \cap s_{\mathrm{H}\Gamma}(e) \neq \emptyset \implies e = f_i$. As $\mathrm{H}\Delta$ is obtained from $\mathrm{H}\Gamma$ by cutting all edges f_i, the claim follows from Lemma 5.4(2).

Putting the previous steps together, $C^*(\mathrm{H}\Gamma)$ is nuclear iff the same holds for $pC^*(\mathrm{H}\Gamma)p = pC^*(\mathrm{H}\Delta)p$.

Step 5 It remains to show that $pC^*(\mathrm{H}\Delta)p$ is nuclear iff the same holds for $C^*(\mathrm{H}\Delta)$. However, since in $\mathrm{H}\Delta$ the vertices w_i have no incoming edge and the only outgoing edge f_i has empty range, it is not hard to check

$$C^*(\mathrm{H}\Delta) = \mathbb{C}^n \oplus pC^*(\mathrm{H}\Delta)p.$$

The claim follows immediately. □

5.5 Elimination of Simple Quasisinks

Let us define the notion of a simple quasisink. We will see that in a hypergraph $\mathrm{H}\Gamma$ all edges which end in a simple quasisink can be cut without losing the information about nuclearity of $C^*(\mathrm{H}\Gamma)$.

Definition 5.14. *Let $\mathrm{H}\Gamma$ be a hypergraph. A vertex $w \in E^0(\mathrm{H}\Gamma)$ is called a* simple quasisink *if*

- *there is at most one edge $e \in E^1(\mathrm{H}\Gamma)$ with $w \in s_{\mathrm{H}\Gamma}(e)$ and in this case we have $r_{\mathrm{H}\Gamma}(e) = \emptyset$, and*
- *there is at most one edge $e \in E^1(\mathrm{H}\Gamma)$ with $w \in r_{\mathrm{H}\Gamma}(e)$.*

We say that an edge f ends in a simple quasisink if we have $r_{\mathrm{H}\Gamma}(f) = \{w\}$ for a simple quasisink w.

Example 5.15. *The figure below presents three hypergraphs $\mathrm{H}\Delta_1, \mathrm{H}\Delta_2, \mathrm{H}\Delta_3$. While the vertex w is not a simple quasisink in the first two hypergraphs, it is a simple quasisink in $\mathrm{H}\Delta_3$.*

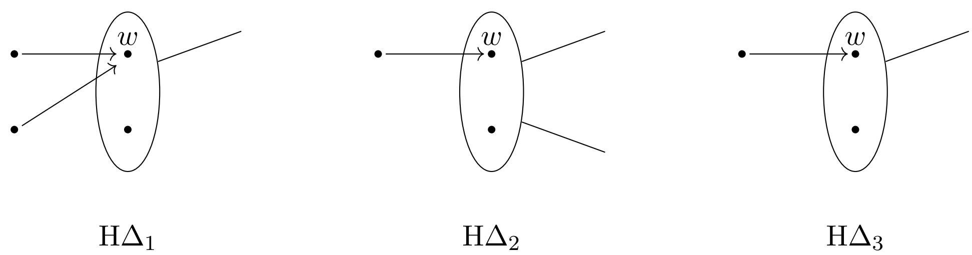


Figure 5.5: (Non-)Examples of a simple quasisink

Lemma 5.16. *Let* $\mathrm{H}\Gamma$ *be a hypergraph. Assume that* $w \in C^*(\mathrm{H}\Gamma)$ *is a simple quasisink in* $\mathrm{H}\Gamma$ *with* $\{w\} = r_{\mathrm{H}\Gamma}(f)$ *for some* $f \in E^1(\mathrm{H}\Gamma)$ *and let* $\mathrm{H}\Delta$ *be obtained from* $\mathrm{H}\Gamma$ *by cutting the edge* f*. Then* $C^*(\mathrm{H}\Gamma)$ *is nuclear iff the same holds for* $C^*(\mathrm{H}\Delta)$*.*

Proof. First, assume that there is an edge $e \in E^1(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(e) = \emptyset$ and $w \in s_{\mathrm{H}\Gamma}(e)$. Then e is the only edge which has w in its source. It follows from Lemma 4.10 that we can remove w from the source of e without changing the associated C^*-algebra. If $\{w\} = s_{\mathrm{H}\Gamma}(e)$, then this means to delete the edge e.

Thus, without loss of generality w is a sink in $\mathrm{H}\Gamma$. Evidently, the set $\{f\}$ is closed under range exits in the sense of Definition 5.3. Now, let

$$p := 1 - w \in C^*(\mathrm{H}\Gamma).$$

One readily checks $fp = fwp = 0$. Moreover, it is $|r_{\mathrm{H}\Gamma}(f)| = |\{w\}| = 1$ and f is the only edge in $\mathrm{H}\Gamma$ which has w in its range. Therefore, the conditions for Lemma 5.4(2) are satisfied, and we obtain

$$pC^*(\mathrm{H}\Gamma)p = pC^*(\mathrm{H}\Delta)p.$$

The corner on the left-hand side is a full corner in $C^*(\mathrm{H}\Gamma)$ since

$$\begin{aligned} w \notin s_{\mathrm{H}\Gamma}(f) &\implies s_{\mathrm{H}\Gamma}(f) \in pC^*(\mathrm{H}\Gamma)p \\ &\implies f = s_{\mathrm{H}\Gamma}(f)f \in (pC^*(\mathrm{H}\Gamma)p) \\ &\implies w = f^*f \in (pC^*(\mathrm{H}\Gamma)p) \\ &\implies 1 = p + w \in (pC^*(\mathrm{H}\Gamma)p). \end{aligned}$$

Hence, $C^*(\mathrm{H}\Gamma)$ and $pC^*(\mathrm{H}\Gamma)p$ are Morita-equivalent, and $C^*(\mathrm{H}\Gamma)$ is nuclear iff the same holds for $pC^*(\mathrm{H}\Gamma)p = pC^*(\mathrm{H}\Delta)p$.

Finally, in $\mathrm{H}\Delta$ the vertex w has neither an incoming nor an outgoing edge. Therefore, it is not hard to check that

$$C^*(\mathrm{H}\Delta) = \mathbb{C} \oplus pC^*(\mathrm{H}\Delta)p.$$

In particular, $pC^*(\mathrm{H}\Delta)p$ is nuclear iff the same holds for $C^*(\mathrm{H}\Delta)$. This concludes the proof. □

5.6 Hypergraph Reduction

Let us use the results from the previous subsections in order to obtain a reduction procedure which transforms a hypergraph $\mathrm{H}\Gamma$ into another hypergraph $\mathrm{H}\Delta$ such that $C^*(\mathrm{H}\Gamma)$ is nuclear if, and only if, the same holds for $C^*(\mathrm{H}\Delta)$.

Algorithm 2 Hypergraph Reduction

1: **procedure** REDUCE(hypergraph $\mathrm{H}\Gamma$)
2: $\quad\mathrm{H}\Gamma \leftarrow$ normalize $\mathrm{H}\Gamma$
3: $\quad$**while** True **do**
4: $\quad\quad$**if** there is an easy edge f in $\mathrm{H}\Gamma$ **then**
5: $\quad\quad\quad\mathrm{H}\Gamma \leftarrow$ take $\mathrm{H}\Gamma$ and cut all edges in the easy edge set generated by f
6: $\quad\quad$**else if** there is an easy cycle $f_1 \dots f_n$ in $\mathrm{H}\Gamma$ **then**
7: $\quad\quad\quad\mathrm{H}\Gamma \leftarrow$ take $\mathrm{H}\Gamma$ and cut the edges $f_1, \dots, f_n$
8: $\quad\quad$**else if** there is an edge f in $\mathrm{H}\Gamma$ with $r_{\mathrm{H}\Gamma}(f) = \{w\}$ for a simple quasisink w **then**
9: $\quad\quad\quad\mathrm{H}\Gamma \leftarrow$ take $\mathrm{H}\Gamma$ and cut the edge f
10: $\quad\quad$**else if** there is an edge f in $\mathrm{H}\Gamma$ with $r_{\mathrm{H}\Gamma}(f) = \emptyset = s_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(e)$ for all $e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$ **then**
11: $\quad\quad\quad\mathrm{H}\Gamma \leftarrow$ take $\mathrm{H}\Gamma$ and delete the edge f
12: $\quad\quad$**else if** none of the previous cases applies **then**
13: $\quad\quad\quad$**break**
14: $\quad\quad$**end if**
15: $\quad$**end while**
16: $\quad$**return** $\mathrm{H}\Gamma$
17: **end procedure**

Theorem 5.17. *Algorithm 2 terminates for every hypergraph* $\mathrm{H}\Gamma$*. The obtained hypergraph* $\mathrm{H}\Delta := \mathrm{reduce}(\mathrm{H}\Gamma)$ *is a normal hypergraph minor of* $\mathrm{H}\Gamma$ *which contains no easy edge, no easy cycle and no edge that ends in a simple quasisink. Further,* $C^*(\mathrm{H}\Gamma)$ *is nuclear if, and only if, the same holds for* $C^*(\mathrm{H}\Delta)$*.*

Proof. Evidently, $\mathrm{H}\Delta$ is a hypergraph minor of $\mathrm{H}\Gamma$. Moreover, the algorithm terminates since in each application of lines 4 – 14 either an edge with nonempty range is cut, or an edge is deleted, or the loop breaks. As there are only finitely many edges and vertices in $\mathrm{H}\Gamma$, at some point neither of the first two cases applies. Then the algorithm terminates.

Evidently, $\mathrm{H}\Delta$ contains no easy edge, no easy cycle and no edge that ends in a simple quasisink since the "while" loop only breaks if in all three of these cases the involved edges had been cut. Let us show that $\mathrm{H}\Delta$ is normal. Evidently, $\mathrm{H}\Gamma$ is normal after line 2. The operations in lines 4 – 9 only cut some edges and otherwise leave $\mathrm{H}\Gamma$ unchanged. Looking at the conditions for normality from Definition 3.2, there is only one way how this operation can destroy normality of $\mathrm{H}\Gamma$: If one cuts an edge f which satisfies $\emptyset \neq r_{\mathrm{H}\Gamma}(f) \subset s_{\mathrm{H}\Gamma}(f)$ and where there is no edge $e \in E^1(\mathrm{H}\Gamma) \setminus \{f\}$ with $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) \neq \emptyset$. However, in this case the operation in line 11 ensures that the edge f is deleted later on which restores normality of $\mathrm{H}\Gamma$. By the conditions in line 10, the edge deletion operation in line 11 never destroys normality of $\mathrm{H}\Gamma$. Thus, $\mathrm{H}\Delta$ is normal.

Finally, $C^*(\mathrm{H}\Gamma)$ is nuclear iff the same holds for $C^*(\mathrm{H}\Delta)$. Indeed, by Lemmas 5.8, 5.13, and 5.16 the operations in lines 5, 7, and 9 change the hypergraph $\mathrm{H}\Gamma$ in such a way that nuclearity for the original and the modified hypergraph C^*-algebra are equivalent. By Lemma 4.10 the operation in line 11 does not change the associated hypergraph C^*-algebra at all. This concludes the proof. □

5.7 Reduced Hypergraphs

After applying the reduction procedure from the previous subsection, the hypergraph $\mathrm{H}\Gamma$ is in a very special form: It is a normal hypergraph that contains no easy edge, no easy cycle and no edge ending in a simple quasisink. In Theorem 5.19 below, it turns out that whenever $\mathrm{H}\Gamma$ contains an edge with nonempty range, then $\mathrm{H}\Gamma$ immediately has one of the forbidden minors $\mathrm{H}\Gamma_1, \mathrm{H}\Gamma_2, \mathrm{H}\Gamma_3, \mathrm{H}\Gamma_4$ from Chapter 3. Put differently, if $\mathrm{H}\Gamma$ has none of the forbidden minors, then all edges in $\mathrm{H}\Gamma$ have empty range, i.e. $\mathrm{H}\Gamma$ is an undirected hypergraph. Furthermore, in Proposition 5.20 we show the following: If $\mathrm{H}\Gamma$ has the minor $\mathrm{H}\Gamma_4$ but $\mathrm{H}\Gamma_i \not\leq \mathrm{H}\Gamma$ holds for all $i \leq 3$, then $\mathrm{H}\Gamma_4$ can be obtained from $\mathrm{H}\Gamma$ using only two operations which preserve nuclearity of the associated C^*-algebra.

For the proof of these statements the following lemma will be useful.

Lemma 5.18. *Let* $\mathrm{H}\Gamma$ *be a normal hypergraph. Further, let* $f_1 \dots f_n$ *be a path in* $\mathrm{H}\Gamma$ *with* $|s_{\mathrm{H}\Gamma}(f_i)| = 1$ *for all* $i \geq 2$. *Then the hypergraph* $\mathrm{H}\Delta$ *given by*

- $E^0(\mathrm{H}\Delta) = E^0(\mathrm{H}\Gamma)$,
- $E^1(\mathrm{H}\Delta) = E^1(\mathrm{H}\Gamma) \setminus \{f_2, \dots, f_n\}$,
- $s_{\mathrm{H}\Delta}(e) = s_{\mathrm{H}\Gamma}(e)$ *for all* $e \in E^1(\mathrm{H}\Delta)$,
- $r_{\mathrm{H}\Delta}(e) = \begin{cases} r_{\mathrm{H}\Gamma}(e), & e \neq f_1, \\ r_{\mathrm{H}\Gamma}(f_n), & e = f_1, \end{cases}$ *for all* $e \in E^1(\mathrm{H}\Delta)$,

is a hypergraph minor of $\mathrm{H}\Gamma$.

Proof. Without loss of generality, we have

$$f_i f_{i+1} \dots f_j \text{ is not a cycle for all } 2 \leq i \leq j \leq n, \tag{$*$}$$

i.e. the path $f_2 \dots f_n$ does not contain a cycle. Indeed, assume that the statement holds under this additional assumption $(*)$, and let $\mu = f_2 \dots f_n$ contain a cycle. In this case, obtain a shorter path $f_{i_1} \dots f_{i_m}$ by removing all edges from μ that are part of a cycle, and construct the hypergraph $\mathrm{H}\Delta'$ by applying the statement on the path $f_1 f_{i_1} \dots f_{i_m}$. Then, $\mathrm{H}\Delta$ is obtained from $\mathrm{H}\Delta'$ by deleting all edges in $\{f_2, \dots, f_n\} \setminus \{f_{i_1} \dots f_{i_m}\}$. Hence, we have $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$ as desired.

Let us prove the claim under the assumption $(*)$ by induction over n. If $n = 1$, then we have $\mathrm{H}\Delta = \mathrm{H}\Gamma$ and there is nothing to show. For the induction step, let $f_1, \dots, f_n$ be a path in $\mathrm{H}\Gamma$ with $n \geq 2$ and $|s_{\mathrm{H}\Gamma}(f_i)| = 1$ for all $i \geq 2$ such that $f_2 \dots f_n$ does not contain a cycle. The induction hypothesis applied on the path $f_2 \dots f_n$ yields a minor $\mathrm{H}\Gamma^{(1)} \leq \mathrm{H}\Gamma$ with

- $E^0(\mathrm{H}\Gamma^{(1)}) = E^0(\mathrm{H}\Gamma)$,
- $E^1(\mathrm{H}\Gamma^{(1)}) = E^1(\mathrm{H}\Gamma) \setminus \{f_3, \dots, f_n\}$,
- $s_{\mathrm{H}\Gamma^{(1)}}(e) = s_{\mathrm{H}\Gamma}(e)$ for all $e \in E^1(\mathrm{H}\Gamma^{(1)})$,
- $r_{\mathrm{H}\Gamma^{(1)}}(e) = \begin{cases} r_{\mathrm{H}\Gamma}(e), & e \neq f_2, \\ r_{\mathrm{H}\Gamma}(f_n), & e = f_2, \end{cases}$ for all $e \in E^1(\mathrm{H}\Gamma^{(1)})$.

Let $w \in E^0(\mathrm{H}\Gamma^{(1)})$ be the vertex with $\{w\} = s_{\mathrm{H}\Gamma^{(1)}}(f_2)$. Now, consider the following constructions:

1. Obtain $\mathrm{H}\Gamma^{(2)}$ from $\mathrm{H}\Gamma^{(1)}$ by separating the source of f_2 in the sense of Remark 4.5. Note that f_2 must not be a cycle, and therefore due to condition (2) from the definition of normality (Definition 3.2) there is an edge $e \neq f_2$ in $\mathrm{H}\Gamma$ with $w \in s_{\mathrm{H}\Gamma}(e)$. As the path $f_2 \dots f_n$ does not contain a cycle, we have $e \notin \{f_2, \dots, f_n\}$. Therefore, it is $e \in E^1(\mathrm{H}\Gamma^{(1)})$. Since e is an edge different from f_2 with $w \in s_{\mathrm{H}\Gamma^{(1)}}(e)$, the hypergraph $\mathrm{H}\Gamma^{(2)}$ contains a new edge $w' \in E^0(\mathrm{H}\Gamma^{(2)}) \setminus E^0(\mathrm{H}\Gamma^{(1)})$ such that f_2 is the only edge with $w' \in s_{\mathrm{H}\Gamma^{(2)}}(f_2)$.
2. Obtain $\mathrm{H}\Gamma^{(3)}$ from $\mathrm{H}\Gamma^{(2)}$ by applying range decomposition on all edges in the set

   $$F := \{e \in E^1(\mathrm{H}\Gamma^{(2)}) |\ w' \in r_{\mathrm{H}\Gamma^{(2)}}(e)\}.$$

   Since in $\mathrm{H}\Gamma^{(1)}$ every edge has exactly one vertex in its range, we can write

   $$E^1(\mathrm{H}\Gamma^{(3)}) = E^1(\mathrm{H}\Gamma^{(2)}) \cup \{e' : e \in F\}$$

   where for every $e \in F$ it is $r_{\mathrm{H}\Gamma^{(3)}}(e') = \{w'\}$ and $r_{\mathrm{H}\Gamma^{(3)}}(e) = \{w\}$.
3. Obtain $\mathrm{H}\Gamma^{(4)}$ from $\mathrm{H}\Gamma^{(3)}$ by deleting all edges in the set

   $$\{e' : e \in F \setminus \{f_1\}\} \cup \{f_1\}.$$

4. Finally, obtain $\mathrm{H}\Delta$ from $\mathrm{H}\Gamma^{(4)}$ by applying forward contraction on the edge f_2.

In this way, we get $\mathrm{H}\Delta$ from $\mathrm{H}\Gamma^{(1)} \leq \mathrm{H}\Gamma$ by applying suitable hypergraph minor operations. It follows $\mathrm{H}\Delta \leq \mathrm{H}\Gamma$. ☐

Now, let us prove the main theorem of this subsection.

Theorem 5.19. *Let $\mathrm{H}\Gamma$ be a normal hypergraph which contains no easy edge, no easy cycle and no edge that ends in a simple quasisink. Then one of the following is true:*

1. *$\mathrm{H}\Gamma_i \leq \mathrm{H}\Gamma$ for some $i = 1, 2, 3, 4$, where the $\mathrm{H}\Gamma_i$ are the forbidden minors from Chapter 3.*
2. *Every edge in $\mathrm{H}\Gamma$ has empty range.*

Proof. Assume that $\mathrm{H}\Gamma$ contains an edge e with nonempty range $\{w\}$ for some $w \in E^0(\mathrm{H}\Gamma)$. As $\mathrm{H}\Gamma$ does not contain an easy edge, by Corollary 5.10 we may assume without loss of generality that $|s_{\mathrm{H}\Gamma}(e)| \geq 2$. By assumption, the vertex w is not a simple quasisink. Therefore, one of the following cases applies:

A) It is $w \in s_{\mathrm{H}\Gamma}(e)$.

B) There is an edge $f \neq e$ with nonempty range and $w \in s_{\mathrm{H}\Gamma}(f)$.

C) There are two edges $f, g \in E^1(\mathrm{H}\Gamma) \setminus \{e\}$ with empty range and $w \in s_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(g)$.

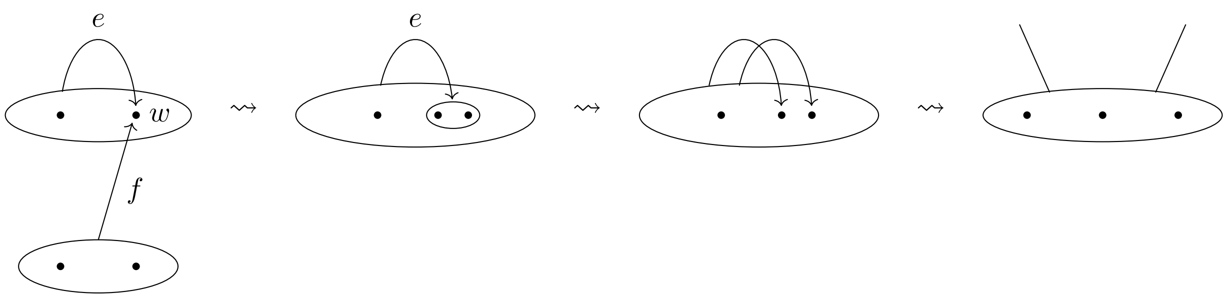
D) There is an edge $f \neq e$ with $r_{\mathrm{H}\Gamma}(f) = \{w\}$.

We discuss each of these cases separately. It will be suitable to consider Case (B) last.

Case A Assume that (A) holds and observe that the edge e must not be an easy cycle in $\mathrm{H}\Gamma$. Therefore, one of the following three cases (A1) – (A3) applies.

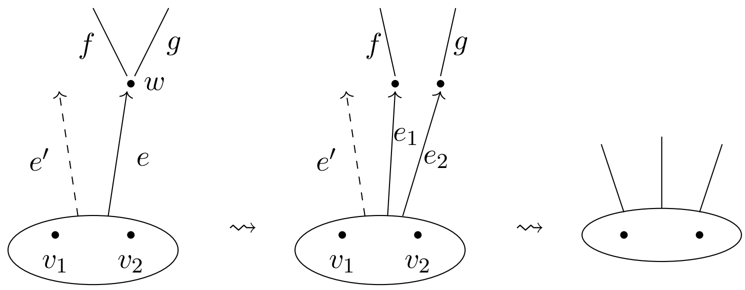
Case A1. There is an edge $f \neq e$ with $w \in s_{\mathrm{H}\Gamma}(f)$. By condition (3) from the definition of normality (Definition 3.2), without loss of generality there is at least one vertex v different from w in the intersection $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f)$. Now, cut the edge f and delete all edges and vertices except for e, f, v and w. This yields the minor $\mathrm{H}\Gamma_3$.

Case A2. There is an edge $f \neq e$ with $\{w\} = r_{\mathrm{H}\Gamma}(f)$ and $|s_{\mathrm{H}\Gamma}(f)| \geq 2$. In this case, separate the source of the edge f and then delete all edges and vertices except for e, f as well as two vertices in $s(e)$ and $s(f)$, respectively. Afterwards, apply backward contraction on the edge f, followed by range decomposition of e. Cutting all resulting edges leaves us with the minor $\mathrm{H}\Gamma_1$. Below we sketch the involved operations schematically.



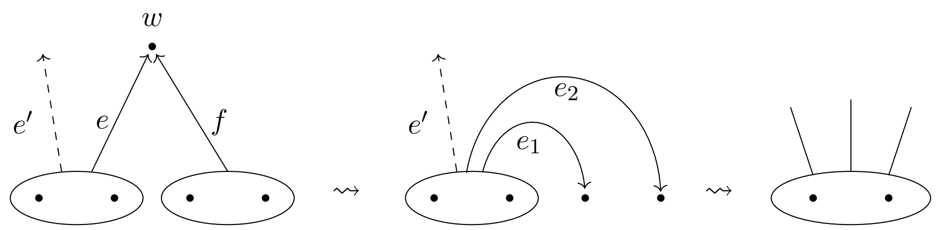
Case A3. There is an edge $f \neq e$ with $\{w\} = r_{\mathrm{H}\Gamma}(f)$ and $|s_{\mathrm{H}\Gamma}(f)| = 1$. The edge f must not be easy and therefore by Corollary 5.10 there is a path $f_1 \ldots f_n$ with $f_n = f$ and $|s_{\mathrm{H}\Gamma}(f_1)| > 1 = |s_{\mathrm{H}\Gamma}(f_i)|$ for all $i \geq 2$. Using that $\mathrm{H}\Gamma$ is normal one checks $f_i \neq e$ for all $i \leq n$. Use Lemma 5.18 with the path $f_1 \ldots f_n$ to obtain a minor $\mathrm{H}\Gamma'$ where $|s(f_1)| \geq 2$ and $r(f_1) = \{w\}$ hold. Then the construction from Case (A2) applied on the hypergraph $\mathrm{H}\Gamma'$ yields the minor $\mathrm{H}\Gamma_1 \leq \mathrm{H}\Gamma' \leq \mathrm{H}\Gamma$.

Case C Assume that (C) holds. Without loss of generality case (A) does not apply. Using condition (2) from the definition of normality (Definition 3.2) one finds an edge $e' \neq e$ with $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(e') \neq \emptyset$. Combining with condition (3) from the same definition, without loss of generality there are at least two vertices v_1 and v_2 in the intersection $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(e')$. Now, separate the source of $\{f\}$ at w and afterwards apply range decomposition on the edge e. This operation replaces the edge e with two edges e_1, e_2 that have the same source as e. Finally, cut all edges, and then delete all edges and vertices except for v_1, v_2, e_1, e_2 and e'. This yields the minor $\mathrm{H}\Gamma_2$. Note that $e' = f$ or $e' = g$ is allowed. Below we sketch the involved operations schematically.



Case D Assume that (D) holds and observe that f must not be an easy edge. Further, we may assume $w \notin s_{\mathrm{H}\Gamma}(e)$ since otherwise Case (A) applies. This leaves the following three possibilities (D1) – (D3).

Case D1. It is $|s_{\mathrm{H}\Gamma}(f)| \geq 2$ and $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) = \emptyset$. We may assume $w \notin s_{\mathrm{H}\Gamma}(f)$ since otherwise Case (A) applies for f in the place of e. By conditions (2) and (3) from the definition of normality (Definition 3.2) there is another edge $e' \neq e$ with $|s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(e')| \geq 2$. Now, transform the hypergraph $\mathrm{H}\Gamma$ as follows: First, delete all edges and vertices except for e, e', f and two vertices in $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(e')$ and $s_{\mathrm{H}\Gamma}(f)$, respectively. Afterwards, apply backward contraction on the edge f, and then decompose the range of e. This replaces the edge e with two new edges e_1, e_2 that have the same source as e. Cutting all edges and deleting all vertices except for those in $s(e) \cap s(e')$ gives the minor $\mathrm{H}\Gamma_2$. Below we sketch the involved operations schematically.



Case D2. It is $|s_{\mathrm{H}\Gamma}(f)| \geq 2$ and $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f) \neq \emptyset$. Again we may assume $w \notin s_{\mathrm{H}\Gamma}(f)$. By condition (3) from the definition of normality (Definition 3.2) there are two possibilities:

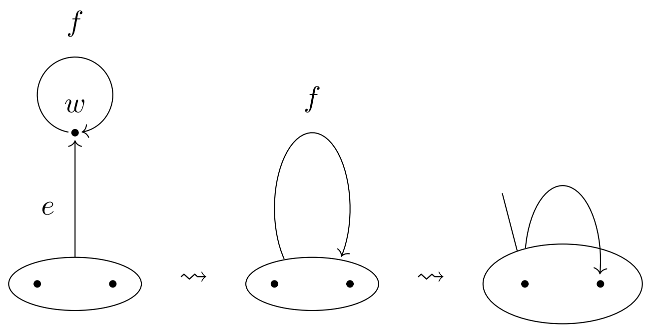
- It is $|s_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(e)| \geq 2$.
- There is an edge $g \neq e, f$ such that $s_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(e) \subsetneq s_{\mathrm{H}\Gamma}(g) \cap s_{\mathrm{H}\Gamma}(e)$.

In the latter case, separate the source of f and then use the same construction as in Case (D1) to obtain the minor $\mathrm{H}\Gamma_2$. In the first case, there are at least two vertices v_1, v_2 in the intersection $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f)$. Delete all edges and vertices except for e, f, v_1, v_2 and w. This yields the minor $\mathrm{H}\Gamma_4$.

Case D3. It is $|s_{\mathrm{H}\Gamma}(f)| = 1$. Since f must not be an easy edge, there is a path $f_1 \ldots f_n$ in $\mathrm{H}\Gamma$ with $n \geq 2$, $f_n = f$ and $|s_{\mathrm{H}\Gamma}(f_1)| > 1 = |s_{\mathrm{H}\Gamma}(f_i)|$ for all $i \geq 2$ (see Corollary 5.10). Let us distinguish two cases:

Case D3.1. We have $s_{\mathrm{H}\Gamma}(f_1) \cap s_{\mathrm{H}\Gamma}(e) = \emptyset$. One easily checks, that then $f_i \neq e$ holds for all $i \leq n$. Use Lemma 5.18 to obtain a minor $\mathrm{H}\Gamma'$ where $r(f_1) = \{w\}$. Then Case (D1) applies and yields the minor $\mathrm{H}\Gamma_2$.

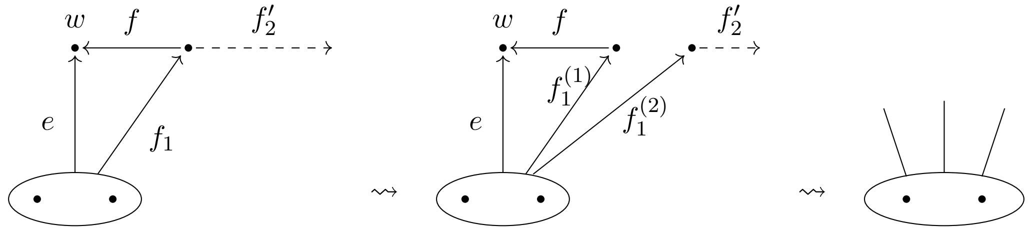
Case D3.2. We have $f_1 = e$. Then Lemma 5.18 applied on the path $f_2 \ldots f_n$ yields a hypergraph minor $\mathrm{H}\Gamma' \leq \mathrm{H}\Gamma$ where $\{w\} = s(f_2) = r(f_2)$. Delete all edges and vertices except for e, f_2, w and two vertices in $s(e)$. Finally, apply backward contraction on the edge e, range decomposition on the edge f_2 and afterwards cut one of the obtained edges. This yields the minor $\mathrm{H}\Gamma_3$. Below we sketch the involved operations schematically.



Case D3.3. It is $f_1 \neq e$ and $s_{\mathrm{H}\Gamma}(f_1) \cap s_{\mathrm{H}\Gamma}(e) \neq \emptyset$. Due to condition (3) from the definition of normality (Definition 3.2) there are two possibilities:

- It is $|s_{\mathrm{H}\Gamma}(f_1) \cap s_{\mathrm{H}\Gamma}(e)| \geq 2$.
- There is an edge $g \neq e, f_1$ such that $s_{\mathrm{H}\Gamma}(f_1) \cap s_{\mathrm{H}\Gamma}(e) \subsetneq s_{\mathrm{H}\Gamma}(g) \cap s_{\mathrm{H}\Gamma}(e)$.

In the latter case, separate the source of f_1 and then use the same construction as in Case (D3.1) to obtain the minor $\mathrm{H}\Gamma_2$. Otherwise, there are at least two vertices in the intersection $s_{\mathrm{H}\Gamma}(f_1) \cap s_{\mathrm{H}\Gamma}(e)$. Moreover, without loss of generality $r_{\mathrm{H}\Gamma}(f_1) \cap s_{\mathrm{H}\Gamma}(f_1) = \emptyset$ since otherwise Case (A) applies for the edge f_1. Similarly, we may assume without loss of generality that $r_{\mathrm{H}\Gamma}(f_2) \cap s_{\mathrm{H}\Gamma}(f_2) = \emptyset$ since otherwise Case (D3.2) applies for the edge f_1 in the place of e. Now, by conditions (2) and (3) from the definition of normality, there is an edge $f_2' \neq f_2$ in $\mathrm{H}\Gamma$ with $s_{\mathrm{H}\Gamma}(f_2) = s_{\mathrm{H}\Gamma}(f_2')$. Separate the source of f_2' and afterwards apply range decomposition on f_1. This operation replaces f_1 with two new edges $f_1^{(1)}$ and $f_1^{(2)}$. Finally, delete all edges and vertices except for $e, f_1^{(1)}, f_1^{(2)}$ and two vertices in $s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(f_1)$. This yields the minor $\mathrm{H}\Gamma_2$. Below we sketch the involved operations schematically.



Case B Finally, assume that (B) holds and distinguish the following two cases (B1) – (B2).

Case B1. It is $|s_{\mathrm{H}\Gamma}(f)| = 1$. Then there are two possibilities. If $s_{\mathrm{H}\Gamma}(f) = r_{\mathrm{H}\Gamma}(f)$, then Case (D) applies. Otherwise, by conditions (2) and (3) from the definition of normality (Definition 3.2), there is another edge $f' \neq f$ with $s_{\mathrm{H}\Gamma}(f') = \{w\}$. After cutting the edges f and f' one is in the same situation as in Case (C). Similarly as above, one obtains the minor $\mathrm{H}\Gamma_2$.

Case B2. None of the previous cases (A), (C), (D), (B1) applies for any edge with nonempty range. Then there is an edge e_2 with $\{w\} \subsetneq s_{\mathrm{H}\Gamma}(e_2)$ and $r_{\mathrm{H}\Gamma}(e_2) \neq \emptyset$. Let $\{w_2\} := r_{\mathrm{H}\Gamma}(e_2)$. Due to the fact that none of the cases (A), (C), (D), (B1) applies for e_2, there is an edge e_3 and a vertex w_3 with $\{w_2\} \subsetneq s_{\mathrm{H}\Gamma}(e_3)$ and $\{w_3\} = r_{\mathrm{H}\Gamma}(e_2) \neq \emptyset$. Inductively repeating this argument and using that $\mathrm{H}\Gamma$ has only finitely many edges, one finds a cycle $f_1 \dots f_n \in \mathrm{H}\Gamma$ and vertices $v_1, \dots, v_n$ such that

$$\begin{aligned} r_{\mathrm{H}\Gamma}(f_n) &= \{v_n\} \subsetneq s_{\mathrm{H}\Gamma}(f_1), \\ r_{\mathrm{H}\Gamma}(f_1) &= \{v_1\} \subsetneq s_{\mathrm{H}\Gamma}(f_2), \\ &\dots, \\ r_{\mathrm{H}\Gamma}(f_{n-1}) &= \{v_{n-1}\} \subsetneq s_{\mathrm{H}\Gamma}(f_n). \end{aligned}$$

As $f_1, \dots, f_n$ must not be an easy cycle, there is an $i \leq n$ such that the vertex v_i has two different incoming edges or v_i has two different outgoing edges. Without loss of generality, v_1 has this property. However, v_1 must not have an incoming edge different from f_1 since then Case (D) would apply for the edge f_1. Hence, v_1 has an outgoing edge different from f_2 which we call f_2'. After cutting the edges f_2 and f_2' the edge f_1 has the same property as the edge e in Case (C). Therefore, the argument from the discussion of Case (C) yields the minor $\mathrm{H}\Gamma_2$. □

Note that in the previous proof the minor $\mathrm{H}\Gamma_4$ is obtained only in Case (D2). The next proposition investigates this situation more closely.

Proposition 5.20. *Let* $\mathrm{H}\Gamma$ *be a normal hypergraph that contains no easy edge, no easy cycle and no edge that ends in a simple quasisink. Assume that* $\mathrm{H}\Gamma_4$ *is a minor of* $\mathrm{H}\Gamma$ *and that* $\mathrm{H}\Gamma_i \not\leq \mathrm{H}\Gamma$ *holds for* $i \leq 3$*. Then* $\mathrm{H}\Gamma_4$ *can be obtained from* $\mathrm{H}\Gamma$ *using only the following operations:*

- *deletion of an ideally closed set in the sense of Definition 4.8*
- *removing a vertex from the source of an edge as in Lemma 4.10*

Both operations preserve nuclearity of the associated C^**-algebra.*

Proof. Step 1 If every edge $e \in E^1(\mathrm{H}\Gamma)$ has empty range, then $\mathrm{H}\Gamma$ cannot have the minor $\mathrm{H}\Gamma_4$. Using Corollary 5.10, there is an edge e with $r_{\mathrm{H}\Gamma}(e) \neq \emptyset$ and $|s_{\mathrm{H}\Gamma}(e)| \geq 2$. A close investigation of the case distinction from the proof of Theorem 5.19 reveals that Case (D2) must apply since in all other cases $\mathrm{H}\Gamma$ has one of the hypergraphs $\mathrm{H}\Gamma_1, \mathrm{H}\Gamma_2, \mathrm{H}\Gamma_3$ as a minor. Hence, for every edge e with $r_{\mathrm{H}\Gamma}(e) \neq \emptyset$ and $|s_{\mathrm{H}\Gamma}(e)| \geq 2$ there is another edge $e' \neq e$ with $r_{\mathrm{H}\Gamma}(e) = r_{\mathrm{H}\Gamma}(e')$ and $|s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(e')| \geq 2$.

Step 2 Let $F := \{f \in E^1(\mathrm{H}\Gamma) : |s_{\mathrm{H}\Gamma}(f)| \geq 2 \text{ and } r_{\mathrm{H}\Gamma}(f) \neq \emptyset\}$. By the previous step, the set F is nonempty. We show that there is an edge $f \in F$ such that $r_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(e) = \emptyset$ holds for all edges $e \in E^1(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(e) \neq \emptyset$. Indeed, assume that this is not true, and let $f_1 \in F$. By assumption there is another edge $f_2 \in E^1(\mathrm{H}\Gamma) \setminus \{f_1\}$ such that $r_{\mathrm{H}\Gamma}(f_1) \subset s_{\mathrm{H}\Gamma}(f_2)$ and $r_{\mathrm{H}\Gamma}(f_2) \neq \emptyset$. We prove $f_2 \in F$. First, assume $|s_{\mathrm{H}\Gamma}(f_2)| = 1$. There are two possibilities:

- It is $r_{\mathrm{H}\Gamma}(f_2) = s_{\mathrm{H}\Gamma}(f_2)$. Then it is not difficult to obtain the minor $\mathrm{H}\Gamma_3$ similarly as in Case (D3.2) from the proof of the previous theorem.
- It is not $r_{\mathrm{H}\Gamma}(f_2) = s_{\mathrm{H}\Gamma}(f_2)$. Then conditions (2) and (3) from the definition of normality (Definition 3.2) yield another edge f_2' with $s_{\mathrm{H}\Gamma}(f_2) = s_{\mathrm{H}\Gamma}(f_2')$. Using the construction from Case (C) in the proof of Theorem 5.19 we get the minor $\mathrm{H}\Gamma_2$.

In any event, this contradicts the assumption $\mathrm{H}\Gamma_i \not\leq \mathrm{H}\Gamma$ for $i \leq 3$. Hence, $|s_{\mathrm{H}\Gamma}(f_2)| \geq 2$ and f_2 is in the set F. It follows that there is a path $f_1 \dots f_{|E^1(\mathrm{H}\Gamma)|+1}$ in $\mathrm{H}\Gamma$ which contains only edges from F. Clearly, this path has a closed subpath. By removing superfluous edges one obtains a cycle $g_1 \dots g_n$ with $g_i \in F$ for all $i \leq n$. Now, it is not difficult to obtain the hypergraph minor $\mathrm{H}\Gamma_3$ from $\mathrm{H}\Gamma$. By contradiction this proves that there is an edge $f \in F$ such that $r_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(e) = \emptyset$ holds for all edges $e \in E^1(\mathrm{H}\Gamma)$ with nonempty range.

Step 3 By the previous steps there are $v_1, v_2, w \in E^0(\mathrm{H}\Gamma)$ and $f, f' \in E^1(\mathrm{H}\Gamma)$ such that

$$\{v_1, v_2\} \subset s_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(f'), \quad \{w\} = r_{\mathrm{H}\Gamma}(f) = r_{\mathrm{H}\Gamma}(f'), \quad r_{\mathrm{H}\Gamma}(f) \cap s_{\mathrm{H}\Gamma}(e) = \emptyset$$

hold for all $e \in E^1(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(e) \neq \emptyset$. We show that there is no edge $e \in E^1(\mathrm{H}\Gamma) \setminus \{f, f'\}$ such that $r_{\mathrm{H}\Gamma}(e) = \{w\}$. Assume that this is not true and let $e \in E^1(\mathrm{H}\Gamma)$ have range $\{w\}$. There are two possibilities: If $|s_{\mathrm{H}\Gamma}(e)| \geq 2$, then the construction from Case (D1) in the proof of Theorem 5.19 yields the minor $\mathrm{H}\Gamma_2$. Otherwise, the argument from Case (D3.1) yields the same minor. However, by assumption $\mathrm{H}\Gamma_2 \not\leq \mathrm{H}\Gamma$, and therefore we obtain the claim by contradiction.

Step 4 Let us show that there is no edge $e \in E^1(\mathrm{H}\Gamma)$ with $r_{\mathrm{H}\Gamma}(e) \subset \{v_1, v_2\}$. Assume the opposite and, without loss of generality, let $e \neq f, f'$ be an edge with $r_{\mathrm{H}\Gamma'}(e) = \{v_1\}$. Distinguish the following cases:

Case 1. It is $|s_{\mathrm{H}\Gamma'}(e)| \geq 2$. In this case, separate the source of e, and delete all edges and vertices except for e, f, f', v_1, v_2, w as well as two vertices in $s(e)$. Afterwards, apply backward contraction on the edge e. This yields the minor $\mathrm{H}\Gamma_1$.

Case 2. It is $|s_{\mathrm{H}\Gamma'}(e)| = 1$. Since the edge e must not be easy, by Corollary 5.10 there is a path $e_1 \dots e_n$ in $\mathrm{H}\Gamma'$ with $e_n = e$ and $|s_{\mathrm{H}\Gamma'}(e_1)| > 1 = |s_{\mathrm{H}\Gamma'}(e_i)|$ for all $i \geq 2$. Apply Lemma 5.18 to obtain a minor where $|s_{\mathrm{H}\Gamma'}(e_1)| > 1$ and $s_{\mathrm{H}\Gamma'}(e_1) = r_{\mathrm{H}\Gamma'}(e_1) = \{v_1\}$. Now, the construction from Case (1) yields the minor $\mathrm{H}\Gamma_1$.

Summarizing, as soon as there is an edge $e \neq f, f'$ with $r_{\mathrm{H}\Gamma'}(e) \subset \{v_1, v_2\}$, then $\mathrm{H}\Gamma_1$ is a minor of $\mathrm{H}\Gamma$. By contradiction, it follows that there are no edges e with $r_{\mathrm{H}\Gamma'}(e) \subset \{v_1, v_2\}$.

Step 5 Next, let us show that there is at most one edge $e \in E^1(\mathrm{H}\Gamma)$ with $w \in s_{\mathrm{H}\Gamma}(e)$. Assume the opposite, and let $e, e' \in E^1(\mathrm{H}\Gamma)$ be edges with $w \in s_{\mathrm{H}\Gamma}(e) \cap s_{\mathrm{H}\Gamma}(e')$. Then a similar construction as in Case (C) of the proof of Theorem 5.19 yields the minor $\mathrm{H}\Gamma_2$. This proves the claim by contradiction.

Assume that $e \in E^1(\mathrm{H}\Gamma)$ is an edge with $w \in s_{\mathrm{H}\Gamma}(e)$. By Step (2) we know that e has empty range. Construct a hypergraph $\mathrm{H}\Gamma'$ by removing the vertex w from the source of e as in Lemma 4.10. One easily checks that the assumptions for this lemma are satisfied. Hence, we have $C^*(\mathrm{H}\Gamma') = C^*(\mathrm{H}\Gamma)$. In $\mathrm{H}\Gamma'$ the vertex w is a sink.

Step 6 Set

$$S := (E^0(\mathrm{H}\Gamma') \cup E^1(\mathrm{H}\Gamma')) \setminus \{v_1, v_2, w, f, f'\}.$$

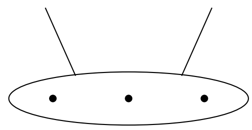
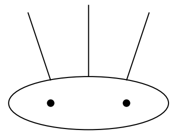
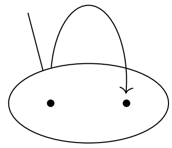
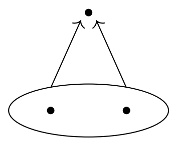
We show that S is ideally closed. We check the three conditions from Definition 4.8.

- Assume that e is an edge in S. Then it is $e \notin \{f, f'\}$. Combining Steps (3) and (4) one observes $r_{\mathrm{H}\Gamma'}(e) \subset E^0(\mathrm{H}\Gamma') \setminus \{v_1, v_2, w\} \subset S$.
- Assume that $e \in E^1(\mathrm{H}\Gamma')$ satisfies $s_{\mathrm{H}\Gamma'}(e) \subset S$ or $\emptyset \neq r_{\mathrm{H}\Gamma'}(e) \subset S$. Both claims are not true for f, f'. Therefore, $e \in E^1(\mathrm{H}\Gamma) \setminus \{f, f'\} \subset S$.
- Finally, assume that $v \in E^0(\mathrm{H}\Gamma')$ is not a sink and satisfies $v \in s_{\mathrm{H}\Gamma'}(e) \implies e \in S$ for all edges $e \in E^1(\mathrm{H}\Gamma')$. Clearly, this is not true for neither v_1, v_2 nor w and therefore $v \in E^0(\mathrm{H}\Gamma') \setminus \{v_1, v_2, w\} \subset S$.

Evidently, $\mathrm{H}\Gamma_4$ is obtained from $\mathrm{H}\Gamma'$ by deleting the set S. This concludes the proof. □

6 The Forbidden Minors

Recall the forbidden minors $\mathrm{H}\Gamma_1, \mathrm{H}\Gamma_2, \mathrm{H}\Gamma_3, \mathrm{H}\Gamma_4$ from Section 3:

$\mathrm{H}\Gamma_1$	$E^0 = \{v_1, v_2, v_3\},$ $E^1 = \{e, f\},$	$s(e) = s(f) = E^0,$	$r(e) = r(f) = \emptyset$	
$\mathrm{H}\Gamma_2$	$E^0 = \{v_1, v_2\},$ $E^1 = \{e, f, g\},$	$s(e) = s(f) = s(g) = E^0,$	$r(e) = r(f) = r(g) = \emptyset$	
$\mathrm{H}\Gamma_3$	$E^0 = \{v, w\},$ $E^1 = \{e, f\},$	$s(e) = s(f) = E^0,$	$r(e) = \emptyset,$ $r(f) = \{w\}$	
$\mathrm{H}\Gamma_4$	$E^0 = \{v_1, v_2, w\},$ $E^1 = \{e, f\},$	$s(e) = s(f) = \{v_1, v_2\},$	$r(e) = r(f) = \{w\}$	

We will now prove Proposition 3.1. Let us first recall the statement.

Proposition (Proposition 3.1)**.** *We have*

1. $C^*(\mathrm{H}\Gamma_1) = C^*(\mathrm{H}\Gamma_2) = \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3$,
2. $C^*(\mathrm{H}\Gamma_3)$ *is the universal unital C^*-algebra generated by one partial isometry,*
3. $C^*(\mathrm{H}\Gamma_4) =_M M_2 *_{\mathbb{C}} \mathbb{C}^2$.

In particular, the C^*-algebras $C^*(\mathrm{H}\Gamma_1), C^*(\mathrm{H}\Gamma_2), C^*(\mathrm{H}\Gamma_3)$ are not exact while $C^*(\mathrm{H}\Gamma_4)$ is not nuclear.

Proof. Ad (1): Recall that edges with empty range correspond to projections. One readily checks

$$C^*\left(v_1, v_2, v_3, e, f \;\middle|\; \begin{array}{l} v_1, v_2, v_3 \text{ pairwise orthogonal projections,} \\ e, f \text{ orthogonal projections,} \\ v_1 + v_2 + v_3 = e + f \end{array} \right) = \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3.$$

Analogously, one sees $C^*(\mathrm{H}\Gamma_2) = \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^3$.

Ad (2): Let $\mathcal{P}_1$ be the universal unital C^*-algebra generated by one partial isometry S. The respective universal properties yield unital maps $\varphi : C^*(\mathrm{H}\Gamma_3) \to \mathcal{P}_1$ and $\psi : \mathcal{P}_1 \to C^*(\mathrm{H}\Gamma_3)$ with

$$\varphi : \begin{cases} v \mapsto 1 - S^*S, \\ w \mapsto S^*S, \\ e \mapsto 1 - SS^*, \\ f \mapsto S, \end{cases}$$

and

$$\psi : \begin{cases} S \mapsto f, \\ 1_{\mathcal{P}_1} \mapsto 1_{C^*(\mathrm{H}\Gamma_3)}. \end{cases}$$

Indeed, $\psi(S)$ is a partial isometry, $\varphi(v), \varphi(w)$ are orthogonal projections and $\varphi(e), \varphi(f)$ are partial isometries. We check the hypergraph relations.

(HR1): Clearly, $\varphi(e)^*\varphi(f) = 0$ and $\varphi(e)$ is a projection. Furthermore, we have

$$\varphi(f)^*\varphi(f) = S^*S = \varphi(w). \tag{6.1}$$

(HR2): In $C^*(\mathrm{H}\Gamma_3)$ it is $s(e) = s(f) = 1$ and therefore the inequalities

$$\varphi(e)\varphi(e)^* \leq \varphi(s(e)), \quad \varphi(f)\varphi(f)^* \leq \varphi(s(f))$$

are trivial.

(HR3): Observe

$$\varphi(v) = 1 - S^*S = \varphi(e)\varphi(e)^* \leq \varphi(e)\varphi(e)^* + \varphi(f)\varphi(f)^*$$

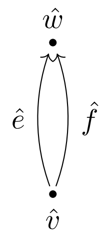
and

$$\varphi(w) = S^*S = \varphi(f)\varphi(f)^* \leq \varphi(e)\varphi(e)^* + \varphi(f)\varphi(f)^*.$$

One readily verifies that φ and ψ are inverse to each other and this yields the claim.

Ad (3): Let $\mathrm{H}\Delta$ be the hypergraph given by

- $E^0(\Delta) = \{\hat{v}, \hat{w}\}$,
- $E^1(\Delta) = \{\hat{e}, \hat{f}\}$,
- $s_\Delta(\hat{e}) = s_\Delta(\hat{f}) = \{\hat{v}\}$,
- $r_\Delta(\hat{e}) = r_\Delta(\hat{f}) = \{\hat{w}\}$.



On the right-hand side above, we sketch the hypergraph Δ. Evidently, Δ is an ordinary graph and from Proposition 2.5 we get $C^*(\Delta) = M_3$ with the identification $\hat{w} = E_{11}, \hat{v} = E_{22} + E_{33}, \hat{e} = E_{21}, \hat{f} = E_{31}$. Now, use Proposition 2.21 to show $C^*(\mathrm{H}\Gamma_4) = M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2$. Indeed, one verifies

$$\begin{aligned}
C^*(\mathrm{H}\Gamma_4) &= C^*\left(v_1, v_2, w, e, f \;\middle|\; \begin{array}{l} v_1, v_2, w \text{ are pairwise orthogonal projections,} \\ e, f \text{ are partial isometries,} \\ e^*e = f^*f = w, \\ e^*f = 0, \\ v_1 + v_2 = ee^* + ff^* \end{array} \right) \\
&= C^*\left(v, v_1, v_2, w, e, f \;\middle|\; \begin{array}{l} v, w \text{ are orthogonal projections,} \\ v_1, v_2 \text{ are orthogonal projections,} \\ e, f \text{ are partial isometries,} \\ e^*e = f^*f = w, \\ e^*f = 0, \\ v = ee^* + ff^*, \\ v = v_1 + v_2 \end{array} \right) \\
&= C^*\left(\hat{v}, \hat{w}, \hat{e}, \hat{f} \;\middle|\; \begin{array}{l} \hat{v}, \hat{w} \text{ are orthogonal projections,} \\ \hat{e}, \hat{f} \text{ are partial isometries,} \\ \hat{e}^*\hat{e} = \hat{f}^*\hat{f} = w, \\ \hat{e}^*\hat{f} = 0, \\ \hat{v} = \hat{e}\hat{e}^* + \hat{f}\hat{f}^* \end{array} \right) \\
&\quad *_{\hat{v}=v_1+v_2} C^*\left(v_1, v_2 \;\middle|\; v_1, v_2 \text{ are orthogonal projections} \right) \\
&= C^*(\Delta) *_{\hat{v}=1} \mathbb{C}^2 \\
&= M_3 *_{E_{22}+E_{33}=1} \mathbb{C}^2 \\
&=_M M_2 *_{\mathbb{C}} \mathbb{C}^2,
\end{aligned}$$

where we use Proposition 2.38 for the last step. □

Proposition 6.1. *Let* $\mathrm{H}\Gamma$ *be a hypergraph and let* $\mathrm{H}\Delta := \mathrm{reduce}(\mathrm{H}\Gamma)$ *be the reduced version of* $\mathrm{H}\Gamma$ *obtained by Algorithm 2.*

1. *If* $\mathrm{H}\Gamma_i \leq \mathrm{H}\Delta$ *for some* $i \leq 3$*, then* $C^*(\mathrm{H}\Gamma)$ *is not exact.*
2. *If* $\mathrm{H}\Gamma_4 \leq \mathrm{H}\Delta$*, then* $C^*(\mathrm{H}\Gamma)$ *is not nuclear.*

Proof. Ad (1): Observe $\mathrm{H}\Gamma_i \leq \mathrm{H}\Delta \leq \mathrm{H}\Gamma$. We know from Theorem 4.3 that exactness transfers to hypergraph minors and this yields the claim.

Ad (2): By Theorem 5.17, $C^*(\mathrm{H}\Gamma)$ is nuclear iff the same holds for $C^*(\mathrm{H}\Delta)$. If $\mathrm{H}\Gamma_i \leq \mathrm{H}\Delta$ holds for some $i \leq 3$, then the claim follows from (1). Otherwise, Proposition 5.20 yields that $\mathrm{H}\Gamma_4$ is obtained from $\mathrm{H}\Delta$ using only the following operations:

- deletion of an ideally closed set
- removing a vertex from the source of an edge as in Lemma 4.10

The first operation corresponds to taking a quotient on the C^*-algebra side by Theorem 4.3, while the second operation does not change the C^*-algebra at all by Lemma 4.10. As $C^*(\mathrm{H}\Gamma_4)$ is not nuclear, the same holds for $C^*(\mathrm{H}\Delta)$, and this concludes the proof. □

Finally, we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. Let $\mathrm{H}\Delta := \mathrm{reduce}(\mathrm{H}\Gamma)$ be the reduced version of $\mathrm{H}\Gamma$ obtained by Algorithm 2. By Theorem 5.17, $\mathrm{H}\Delta$ is a normal hypergraph minor of $\mathrm{H}\Gamma$, and $C^*(\mathrm{H}\Gamma)$ is nuclear if, and only if, the same holds for $C^*(\mathrm{H}\Delta)$.

Ad (1) and (2): This follows immediately from the previous Proposition 6.1.

Ad (3): The hypergraph $\mathrm{H}\Delta$ satisfies the conditions for Theorem 5.19. Thus, if $\mathrm{H}\Delta$ has none of the forbidden minors, then it must be an undirected hypergraph. $\square$

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