Saarland University
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Bachelor's thesis

Computing quantum symmetries of graphs

submitted by

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I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

Ich versichere hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Saarbrücken, February 7, 2019

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Introduction

In this thesis, we study quantum automorphism groups of finite graphs. This is a generalization of the study of classical automorphism groups of graphs within the framework of compact matrix quantum groups, which were introduced by Woronowicz in [11]. The classical automorphism group is a subgroup of the symmetric group S_n given by

$$G_{aut}(\Gamma) = \{\sigma \in S_n | \sigma\varepsilon = \varepsilon\sigma\}$$

where $\Gamma = (V, E)$ is a finite graph with n vertices and no multiple edges, $\varepsilon \in M_n(\{0, 1\})$ is its adjacency matrix.
Based on this definition and Wang's definition of the quantum symmetric group in [10]

$$C(S_n^+) = C^*(u_{ij} | u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{ki} = 1)$$

we can define the quantum automorphism group of Γ as the compact matrix quantum group given by

$$C(G_{aut}^+(\Gamma)) = C^*(u_{ij} | u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{ki} = 1, u\varepsilon = \varepsilon u).$$

If we interpret $G_{aut}(\Gamma)$ as a compact matrix quantum group, we can see, that $G_{aut}(\Gamma) \subseteq G_{aut}^+(\Gamma)$ holds for all graphs Γ. The question one can ask now is, whether equality holds. If it does, we say that the graph does not have quantum symmetries, otherwise we say that it does have quantum symmetries.
Schmidt presented a criterion in [6] that answers this question for some graphs, namely if $G_{aut}(\Gamma)$ contains a pair of disjoint automorphisms, then $C(G_{aut}^+(\Gamma))$ is non-commutative, in particular $G_{aut}(\Gamma) \neq G_{aut}^+(\Gamma)$.
Moreover it can be interesting to be able to concretely compute this quantum automorphism group for a given graph. Fulton presented a lemma in [5] that helps significantly with this question, which states that $u_{ij} = 0$ if $\varepsilon_{ii}^l \neq \varepsilon_{jj}^l$ holds for some l.
After presenting these two criteria, the use of them is presented in some concrete examples in Section 2.3.
Since there are many graphs for which it is not known, whether they have quantum symmetries, we present some algorithms, that are able to answer this question on many graphs with small numbers of vertices. These algorithms were also implemented in python, GAP and Singular and the results for many connected graphs on 5, 6 and 7 vertices are presented. By analysing these results, we come to the main theorem of this thesis:

Main Theorem. *Let Γ be a graph on n vertices. If $n \leq 7$ then it holds:*

$$G_{aut}(\Gamma) = \mathbb{Z}_2 \Rightarrow G_{aut}^+(\Gamma) = \mathbb{Z}_2$$
$$G_{aut}(\Gamma) = \{e\} \Rightarrow G_{aut}^+(\Gamma) = \{e\}$$

1. Preliminaries on quantum groups of graphs

Let us mention some basic definitions and notations necessary for the following sections. We mainly follow [8].

1.1. **Finite graphs.**

1.1.1. **Definition.** A *finite graph* Γ is a pair (V, E), where the set of *vertices* V and the set of *edges* E are finite. By $r : E \to V$ denote the range map and by $s : E \to V$ denote the source map. It is called *undirected* if $\forall e \in E\ \exists f \in E$ such that $s(e) = r(f)$ and $r(e) = s(f)$.
A graph is *without multiple edges* if there are no $e, f \in E, e \neq f$ such that $s(e) = s(f)$ and $r(e) = r(f)$. If we have $e \in E$ with $s(e) = r(e)$, then e is called a *loop*.
For a finite graph $\Gamma = (V, E)$ with $V = \{1, \cdots, n\}$ define its *adjacency matrix* $\varepsilon \in M_n(\mathbb{N}_0)$ via $\varepsilon_{ij} := \#\{e \in E | s(e) = i, r(e) = j\}$. If Γ is without multiple edges, we thus have that $\varepsilon_{ij} \in \{0, 1\}$.
We define the *complement* of a finite graph without multiple edges $\Gamma = (V, E)$ as $\Gamma^c := (V, E')$, where $E' := (V \times V)\backslash E$. **In the following, we will only consider undirected finite graphs without multiple edges and without loops.**

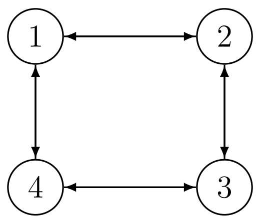
1.1.2. **Definition.** A *graph automorphism* of a finite graph without multiple edges $\Gamma = (V, E)$ is a bijective map $\sigma : V \to V$ such that $(\sigma(i), \sigma(j)) \in E \Leftrightarrow (i, j) \in E$. The set of all automorphisms of Γ is a group, called the *automorphism group* $Aut(\Gamma)$. If Γ has n vertices, we can view $Aut(\Gamma)$ as subgroup of S_n:

$$G_{aut}(\Gamma) = \{\sigma \in S_n | \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n$$

1.1.3. **Example.** Let $\Gamma = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (1, 4), (4, 1)\}$. Then Γ is a finite graph without multiple edges and without loops. It has the adjacency matrix

$$\varepsilon = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

We can also display Γ graphically:



It has the automorphism group $G_{aut}(\Gamma) = \{id, (1, 3), (2, 4), (1, 2)(3, 4), (1, 4)(2, 3), (1, 3)(2, 4), (1, 2, 3, 4), (1, 4, 3, 2)\} \subseteq S_4$.

1.2. **Compact matrix quantum groups.**

1.2.1. **Definition.** Compact matrix quantum groups were defined by Woronowicz [11, 12] in 1987. A *compact matrix quantum group* G is a unital C^*-algebra $C(G)$ equipped with a $*$-homomorphism $\Delta : C(G) \to C(G) \otimes C(G)$ and a unitary $u \in M_n(C(G))$, $n \in \mathbb{N}$, such that

(i) $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j

(ii) $\bar{u}$ is an invertible matrix

(iii) the elements u_{ij} $(1 \leq i, j \leq n)$ generate $C(G)$ (as a C^*-algebra).

The unitary u is called the *fundamental corepresentation (matrix)* of $(C(G), \Delta, u)$. Since (i) and (iii) uniquely determine Δ, one can also refer to the pair $(C(G), u)$ as a compact matrix quantum group.
If $G = (C(G), u)$ and $H = (C(H), v)$ are compact matrix quantum groups with $u \in M_n(C(G))$ and $v \in M_n(C(H))$, we say that G is a *compact matrix quantum subgroup* of H, if there is a surjective *-isomorphism from $C(H)$ to $C(G)$ mapping generators to generators. We then write $G \subseteq H$. If we have $G \subseteq H$ and $H \subseteq G$, they are said to be equal as compact matrix quantum groups.

1.2.2. **Example.** An example for a compact matrix quantum group is the *quantum symmetric group* $S_n^+ = (C(S_n^+), u)$, which was first defined by Wang [10] in 1998. It is the compact matrix quantum group given by

$$C\left(S_n^+\right) := C^*\left(u_{ij} | u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{ki} = 1 \forall i, j = 1, \cdots, n\right)$$

and the $*$-homomorphism Δ is given by

$$\Delta(u_{ij}) := u_{ij}^{'} := \sum_k u_{ik} \otimes u_{kj}.$$

That conditions (i) and (iii) from Definition 1.2.1 are fulfilled is obvious. To see, that Δ is in fact a $*$-homomorphism, we use that the projections u_{ij} and u_{ik} are orthogonal for $j \neq k$ (which can be easily deduced from the fact that $\sum_k u_{ik} = 1$) to check

$$u_{ij}^{'2} = \sum_{k,l} u_{ik} u_{il} \otimes u_{kj} u_{lj} = \sum_k u_{ik} \otimes u_{kj} = u_{ij}^{'}$$

and

$$\sum_k u_{ik}^{'} = \sum_k u_{kj}^{'} = 1 \otimes 1.$$

By the universal property of $C(S_n^+)$ we thus see that Δ is indeed a $*$-homomorphism from $C(S_n^+)$ to $C(S_n^+) \otimes C(S_n^+)$.

It can be shown that the quotient of $C\left(S_n^+\right)$ by the relation that all u_{ij} commute is exactly $C\left(S_n\right)$. Moreover, S_n can be seen as a compact matrix quantum group $S_n = \left(C\left(S_n\right), u\right)$, where $u_{ij} : S_n \to \mathbb{C}$ are the evaluation maps of the matrix entries. We then have $S_n \subseteq S_n^+$ as compact matrix quantum groups and have thus justified the name "quantum symmetric group".

1.2.3. **Remark.** For $n \geq 4$, the quantum symmetric group S_n^+ is really non-commutative. Indeed, we can construct a surjective $*$-homomorphism $\varphi : C(S_n^+) \to C^*(p, q | p = p^* = p^2, q = q^* = q^2) =: A$. For this, define the matrix

$$u' := \begin{bmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{bmatrix}$$

and define φ by mapping u (the generator matrix of S_n^+) to the $n \times n$ matrix, that has u' in the upper left corner and looks like the identity matrix everywhere else. Then u' obviously fulfills the relations of $C^*(S_n^+)$ and thus, by the universal property, φ is a $*$-homomorphism. Moreover φ is also surjective, since $\varphi(u_{11}) = p$ and $\varphi(u_{33}) = q$ and thus all generators of A are hit by the map. Since we now have a surjective

$*$-homomorphism from S_n^+ to a non-commutative C^*-algebra we thus see, that S_n^+ really is non-commutative for $n \geq 4$.
On the other hand, for $n \leq 3$, S_n^+ is commutative and thus it holds $S_n^+ = S_n$.

1.3. **Quantum automorphism groups of finite graphs.**

1.3.1. **Definition.** Given a finite graph without multiple edges $\Gamma = (V, E)$ with adjacency matrix ε, we define its *quantum automorphism group* $G_{aut}^+(\Gamma)$ as the compact matrix quantum group given by

$$C\left(G_{aut}^+(\Gamma)\right) := C^*\left(u_{ij} | u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{ki} = 1 \forall i,j = 1,\cdots,n, u\varepsilon = \varepsilon u\right).$$

It is justified to see this as the *quantum* automorphism group of Γ, since it has the same relations as $C(S_n^+)$ with the added relation $u\varepsilon = \varepsilon u$, compare with Definition 1.1.2 and Example 1.2.2.
One can show [9, Lemma 6.7] that the above definition is equivalent to u_{ij} fulfilling the following relations:

$$u_{ij} = u_{ij}^*, u_{ij}u_{ik} = \delta_{jk}u_{ij}, u_{ji}u_{ki} = \delta_{jk}u_{ji} \qquad 1 \leq i,j,k \leq n \tag{1.1}$$

$$\sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{ki} = 1 \qquad 1 \leq i \leq n \tag{1.2}$$

$$u_{ij}u_{kl} = u_{kl}u_{ij} = 0 \qquad (i,k) \in E, (j,l) \notin E \tag{1.3}$$

$$u_{ij}u_{kl} = u_{kl}u_{ij} = 0 \qquad (i,k) \notin E, (j,l) \in E \tag{1.4}$$

1.3.2. **Remark.** The above definition is the one given by Banica [1] in 2005. There is a different but related definition by Bichon [3] in 2003, but this one will not be considered here.

1.3.3. **Definition.** For a finite graph without multiple edges Γ, we say that Γ *has quantum symmetries* if $C(G_{aut}^+\{\Gamma\})$ is not commutative and that Γ *does not have quantum symmetries* otherwise.

1.3.4. **Example.** Let Γ be the full graph on n vertices, that is $E = V \times V \setminus \{(i,i) | i = 1,\cdots,n\}$. Then $G_{aut}^+(\Gamma) = S_n^+$. To see this, look at the relation $u\varepsilon = \varepsilon u$, which is nothing but $\sum_k u_{ik}\varepsilon_{kj} = \sum_k \varepsilon_{ik}u_{kj} \forall i,j$. Since $\varepsilon_{ij} = 1 - \delta_{ij}$, that is ε is 1 everywhere but on the diagonal, we have that

$$\sum_k u_{ik}\varepsilon_{kj} = \sum_{k,k\neq j} u_{ik} = \sum_k u_{ik} - u_{ij} = 1 - u_{ij} = \sum_k u_{kj} - u_{ij} = \sum_{k,k\neq i} u_{kj} = \sum_k \varepsilon_{ik}u_{kj}.$$

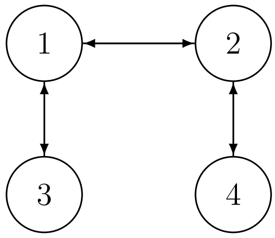
We thus see, that the relation $u\varepsilon = \varepsilon u$ is already implied by $\sum_k u_{ik} = \sum_k u_{kj} = 1$ $\forall i,j$ and thus we have $G_{aut}^+(\Gamma) = S_n^+$.
Since S_n^+ is non-commutative for $n \geq 4$ (see 1.2.3), we see that the full graph on 4 or more vertices is an example for a graph that does have quantum symmetries.

1.3.5. **Remark.** For a finite graph without multiple edges Γ, we have that

$$G_{aut}(\Gamma) \subseteq G_{aut}^+(\Gamma)$$

as compact matrix quantum groups, where equality holds, if and only if $C(G_{aut}^+(\Gamma))$ is commutative.

1.3.6. **Example.** Let $\Gamma =$



We want to show that $C(G^+_{aut}(\Gamma))$ is commutative. First note, that to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ it suffices to show $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$, since then we have

$$u_{kl}u_{ij} = (u_{ij}u_{kl})^* = (u_{ij}u_{kl}u_{ij})^* = u_{ij}u_{kl}u_{ij} = u_{ij}u_{kl}.$$

We have

$$u_{11}u_{33} = u_{11}u_{33}1 = u_{11}u_{33}\sum_{k=1}^{4} u_{1k} = u_{11}\sum_{k=1}^{4} u_{33}u_{1k} = u_{11}u_{33}\sum_{k;(3,k)\in E} u_{1k} = u_{11}u_{33}u_{11}$$

by relation (1.3) from Definition 1.3.1 since the only edge from vertex 3 is to 1. Moreover we have $u_{11}u_{34} = u_{34}u_{11} = 0$ and $u_{11}u_{43} = u_{43}u_{11} = 0$ by relations (1.3) and (1.4) and since there is an edge from 1 to 3 but none from 1 to 4. Now we can see

$$u_{11}u_{23} = u_{11}(1 - u_{13} - u_{33} - u_{43}) = (1 - u_{13} - u_{33} - u_{43})u_{11} = u_{23}u_{11}.$$

The same holds for $u_{11}u_{32} = u_{32}u_{11}$.
Let us now look at

$$u_{11}u_{44} = u_{11}u_{44}\sum_{k;(4,k)\notin E} u_{1k} = u_{11}u_{44}(u_{11} + u_{13}).$$

We can deduce from $u\varepsilon = \varepsilon u$ that $u_{13} = u_{42}$ and thus get

$$u_{11}u_{44} = u_{11}u_{44}u_{11} + u_{11}u_{44}u_{42}.$$

Since there is an edge from 4 to 2 but none from 4 to 4 we get $u_{44}u_{42} = 0$ and thus have that u_{11} and u_{44} commute and can now deduce, similar to the fact that u_{11} and u_{23} commute, the commutation of u_{11} with all the remaining generators. If we continue in a similar fashion, we eventually arrive at $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all i, j, k, l. Thus $G^+_{aut}(\Gamma) = G_{aut}(\Gamma) = \mathbb{Z}_2$.

1.4. **Known results about quantum automorphism groups of graphs.** All in all there are not many articles about quantum automorphism groups of graphs. In this section, some of the results of other articles are mentioned, for a more complete review of the literature on quantum automorphism groups of graphs however see Section 3.4 in [8].
Since $S_n^+ = S_n$ for $n \leq 3$, we also have $G^+_{aut}(\Gamma) = G_{aut}(\Gamma)$ if Γ has 3 or less vertices, which is why these graphs are not suited for the study of quantum automorphism groups. Moreover, Schmidt and Weber did a full classification of quantum automorphism groups of graphs with no multiple edges and no loops on 4 vertices in [8].
In [4], Bichon defined the free wreath product and showed, that the quantum automorphism group of n disjoint copies of a connected graph can be given using this free wreath product. In general for graphs that are not connected it usually holds that we can trace back in some way the quantum automorphism group of the entire graph to the quantum automorphism groups of the connected subgraphs. Moreover it holds, that $G^+_{aut}(\Gamma) = G^+_{aut}(\Gamma^c)$ where Γ^c is the complement of Γ, i.e. every non-edge of Γ is an edge of Γ^c and every edge of Γ is a non-edge of Γ^c. Since the

complement of a lot of not connected graphs is connected, we can thus also get the quantum automorphism group via the complement.
In general however, we can not easily see, whether a graph has quantum symmetries and if it does, what its quantum automorphism group is.

2. Examples of quantum automorphism groups of graphs

When talking about quantum symmetries of graphs, we distinguish between two main questions:

1. Does the graph have quantum symmetries, that is, is $C(G^+_{aut}(\Gamma))$ commutative or not?
2. What does the quantum automorphism group of the graph look like?

In this chapter, we will present two criteria that help with these questions and then we will answer them for some concrete graphs.

2.1. **The Schmidt criterion.**

The lemma in this section comes from an article by Simon Schmidt [6]. It states that a graph has quantum symmetry if the automorphism group of the graph contains a certain pair of permutations. In order to state it, we need the following definition.

2.1.1. **Definition.** Let $V = \{1, \cdots, r\}, r \in \mathbb{N}$. We say that two permutations $\sigma : V \to V$ and $\tau : V \to V$ are *disjoint*, if $\sigma(i) \neq i \Rightarrow \tau(i) = i$ and $\tau(i) \neq i \Rightarrow \sigma(i) = i$ for all $i \in V$.

2.1.2. **Lemma.** *Let $\Gamma = (V, E)$ be a finite graph without multiple edges, $V = \{1, \cdots, r\}, r \in \mathbb{N}$.*
*If there are two non-trivial, disjoint automorphisms $\sigma, \tau \in G_{aut}(\Gamma)$ then we get a surjective *-homomorphism $\varphi : C\left(G^+_{aut}(\Gamma)\right) \to C^*(p, q | p = p^* = p^2, q = q^* = q^2)$. In particular, Γ does have quantum symmetry.*

Proof. Let $\sigma, \tau \in G_{aut}(\Gamma)$ be non-trivial disjoint automorphisms. Define:

$$A := C^*\left(p, q | p = p^* = p^2, q = q^* = q^2\right)$$

Now, we want to use the universal property to get a surjective *-homomorphism $\varphi : C\left(G^+_{aut}(\Gamma)\right) \to A$. This yields the non-commutativity, since p and q do not have to commute.

Define

$$u' := \sigma \otimes p + \tau \otimes q + id_{\mathrm{M_r(\mathbb{C})}} \otimes (1 - q - p) \in \mathrm{M_r(\mathbb{C})} \otimes A \cong \mathrm{M_r(A)}$$

We thus have

$$u'_{ij} = \delta_{j\sigma(i)} \otimes p + \delta_{j\tau(i)} \otimes q + \delta_{ij} \otimes (1 - q - p) \in \mathbb{C} \otimes A \cong A$$

Now, we show that u' fulfills the relations of $G^+_{aut}(\Gamma)$. Since $\sigma, \tau \in G_{aut}(\Gamma)$, it holds that $\sigma\varepsilon = \varepsilon\sigma$ and $\tau\varepsilon = \varepsilon\tau$, where ε is the adjacency matrix of Γ. Therefore we have

$$\begin{aligned}
u'(\varepsilon \otimes 1) &= \left(\sigma \otimes p + \tau \otimes q + id_{\mathrm{M_r(\mathbb{C})}} \otimes (1 - q - p)\right)(\varepsilon \otimes 1) \\
&= \sigma\varepsilon \otimes p + \tau\varepsilon \otimes q + \varepsilon \otimes (1 - q - p) \\
&= \varepsilon\sigma \otimes p + \varepsilon\tau \otimes q + \varepsilon \otimes (1 - q - p) \\
&= (\varepsilon \otimes 1)u'
\end{aligned}$$

Furthermore, it holds

$$\begin{aligned}\sum_{i=1}^{r} u'_{ij} &= \sum_{i=1}^{r} \delta_{j\sigma(i)} \otimes p + \sum_{i=1}^{r} \delta_{j\tau(i)} \otimes q + \sum_{i=1}^{r} \delta_{ij} \otimes (1-q-p)\\ &= 1 \otimes p + 1 \otimes q + 1 \otimes (1-q-p)\\ &= 1 \otimes 1\end{aligned}$$

since $\delta_{j\sigma(i)} = 1$ for exactly one i and it is 0 for all other i. The same also holds for $\delta_{j\tau(i)}$ and δ_{ij}. A similar computation shows $\sum_{j=1}^{r} u'_{ij} = 1 \otimes 1$.
Let us now take a closer look at

$$u'_{ij} = \delta_{j\sigma(i)} \otimes p + \delta_{j\tau(i)} \otimes q + \delta_{ij} \otimes (1-q-p)$$

via a case-by-case analysis:

- if $i \neq j$ then $\delta_{ij} = 0$
  - if $\sigma(i) \neq i$ then $\tau(i) = i$ and thus $\delta_{j\tau(i)} = 0$ since σ and τ are disjoint. If now $\sigma(i) = j$ we have that $u'_{ij} = p$ and if $\sigma(i) \neq j$ we have $\delta_{j\sigma(i)} = 0$ and thus also $u'_{ij} = 0$.
  - if $\tau(i) \neq i$ we have $\sigma(i) = i$ and thus $\delta_{j\sigma(i)} = 0$. Similar to the above case, we thus have $u'_{ij} = q$ if $\tau(i) = j$ and $u'_{ij} = 0$ otherwise.
- if $i = j$ we have $\delta_{ij} = 1$
  - if $\sigma(i) = \tau(i) = i$ we have that $u'_{ij} = 1$ since $\delta_{j\sigma(i)} = 1 = \delta_{j\tau(i)}$.
  - if $\sigma(i) \neq i$ then $\tau(i) = i$ and then we have $\delta_{j\sigma(i)} = 0$ and $\delta_{j\tau(i)} = 1$ and thus $u'_{ij} = 0 \otimes p + 1 \otimes q + 1 \otimes (1-p-q) = 1 - p$.
  - if $\tau(i) \neq i$ then $\sigma(i) = i$ and thus $u'_{ij} = 1 - q$ similar to the above case.

Thus, all entries of u' are projections. By the universal property, we get a *-homomorphism $\varphi : C(G^+_{aut}(\Gamma)) \to A, u \mapsto u'$. This φ is also surjective, since p and q are in its image. □

2.2. **The Fulton criterion.** The following three lemmata can be found in the PhD thesis of Fulton [5].

2.2.1. **Lemma.** *Let Γ be a finite graph without multiple edges, $\varepsilon \in M_n(\{0,1\})$ be its adjacency matrix and $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Then:*

(i) $(1 \cdots 1)$ *is a left eigenvector of u with eigenvalue 1.*
(ii) $(1 \cdots 1)\,\varepsilon^l$ *is a left eigenvector of u with eigenvalue 1 for $l \in \mathbb{N}$.*
(iii) $(1 \cdots 1)^t$ *is a right eigenvector of u with eigenvalue 1.*
(iv) $\varepsilon^l\,(1 \cdots 1)^t$ *is a right eigenvector of u with eigenvalue 1.*

Proof. (i) It holds

$$((1 \cdots 1)\,u)_i = \sum_{k=1}^{n} u_{ki} = 1$$

for all $1 \leq i \leq n$.

(ii) Since $u\varepsilon = \varepsilon u$ holds, we also get $u\varepsilon^l = \varepsilon^l u$. By using this and (i), we deduce

$$(1 \cdots 1)\,\varepsilon^l u = (1 \cdots 1)\,u\varepsilon^l = (1 \cdots 1)\,\varepsilon^l$$

That (iii) and (iv) hold can be seen similarly as in (i) and (ii). □

2.2.2. **Lemma.** *Let Γ be a finite graph without multiple edges, $\varepsilon \in M_n(\{0,1\})$ be its adjacency matrix and $(u_{ij})_{1\leq i,j\leq n}$ be the generators of $C(G^+_{aut}(\Gamma))$. Denote by $\varepsilon^{(l)}_{ij}$ the (i,j) entry of $\varepsilon^{(l)}$. Then it holds:*

(i) *If $\sum_{k=1}^{n} \varepsilon^{(l)}_{ki} \neq \sum_{k=1}^{n} \varepsilon_{kj}$ for some $l \in \mathbb{N}$, then $u_{ij} = 0$.*

(ii) *If* $\sum_{k=1}^{n} \varepsilon_{ik}^{(l)} \neq \sum_{k=1}^{n} \varepsilon_{jk}$ *for some* $l \in \mathbb{N}$*, then* $u_{ij} = 0$*.*

Proof. (i) Assume $\sum_{k=1}^{n} \varepsilon_{ki}^{(l)} \neq \sum_{k=1}^{n} \varepsilon_{kj}^{(l)}$ for some $l \in \mathbb{N}$. We know that $(1 \cdots 1)\, \varepsilon^l u = (1 \cdots 1)\, \varepsilon^l$ and by comparing the j-th column, we get

$$\sum_{r=1}^{n} \sum_{k=1}^{n} \varepsilon_{kr}^{(l)} u_{rj} = \sum_{k=1}^{n} \varepsilon_{kj}^{(l)}$$

Multiplying this by u_{ij} yields

$$\sum_{k=1}^{n} \varepsilon_{ki}^{(l)} u_{ij} = \sum_{k=1}^{n} \varepsilon_{kj}^{(l)} u_{ij}$$

or equivalently

$$\left(\sum_{k=1}^{n} \varepsilon_{ki}^{(l)} - \sum_{k=1}^{n} \varepsilon_{kj}^{(l)} \right) u_{ij} = 0.$$

Since $\sum_{k=1}^{n} \varepsilon_{ki}^{(l)} \neq \sum_{k=1}^{n} \varepsilon_{kj}^{(l)}$ holds, we get $u_{ij} = 0$.

(ii) Similar to (i), using

$$u\varepsilon^l \,(1 \cdots 1)^t = \varepsilon^l \,(1 \cdots 1)^t .$$

□

2.2.3. **Lemma** (The Fulton criterion)**.** *Let* Γ *be a finite graph without multiple edges,* $\varepsilon \in M_n(\{0,1\})$ *be its adjacency matrix and* $(u_{ij})_{1 \leq i,j \leq n}$ *be the generators of* $C(G_{aut}^{+}(\Gamma))$*. Denote by* $\varepsilon_{ij}^{(l)}$ *the* (i,j) *entry of* $\varepsilon^{(l)}$*. If* $\varepsilon_{ii}^{(l)} \neq \varepsilon_{jj}^{(l)}$ *for some* $l \in \mathbb{N}$*, then* $u_{ij} = 0$*.*

Proof. Assume $\varepsilon_{ii}^{(l)} \neq \varepsilon_{jj}^{(l)}$ for some $l \in \mathbb{N}$. It holds that $u\varepsilon^l = \varepsilon^l u$ or equivalently

$$\sum_{k=1}^{n} \varepsilon_{ik}^{(l)} u_{kj} = \sum_{k=1}^{n} u_{ik} \varepsilon_{kj}^{(l)}.$$

Multiplying this equation by u_{ij} yields

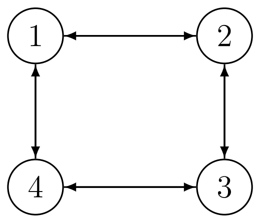
$$\varepsilon_{ii}^{(l)} u_{ij} = \varepsilon_{jj}^{(l)} u_{ij}.$$

Since $\varepsilon_{ii}^{(l)} \neq \varepsilon_{jj}^{(l)}$, we get $u_{ij} = 0$. □

2.2.4. **Remark.** To clarify, why the Fulton criterion makes sense, look at ε^l: for a graph with adjacency matrix ε, the entry ε_{ij}^{l} denotes the number of different ways to get from the node i to the node j in l steps. Thus, if we have $\varepsilon_{ii}^{l} \neq \varepsilon_{jj}^{l}$ for some l, we know, that no automorphism of Γ will map i to j and thus also u_{ij} should be 0. In practice, we use the Fulton criterion by computing ε^2, ε^3, $\cdots$ and look on each of their diagonals. This then often yields $u_{ij} = 0$ for at least some i, j and thus simplifies the process of computing $C(G_{aut}^{+}(\Gamma))$, see the examples in the next section.

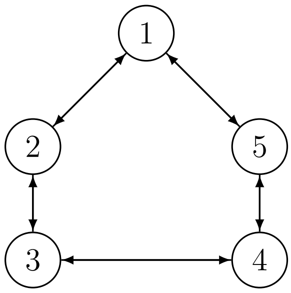
2.3. **Examples of quantum automorphism groups of finite graphs.**

2.3.1. **Example.** Let Γ be the circle on 4 vertices, that is Γ looks like:



Then by the Schmidt criterion 2.1.2 it has quantum symmetries, since $\sigma := (1,3) \in G_{aut}(\Gamma)$ and $\tau := (2,4) \in G_{aut}(\Gamma)$ (see Example 1.1.3) and σ and τ are disjoint. Banica, Benoît and Collins showed in [2] that $G^+_{aut}(\Gamma) = H_2^+$, where H_2^+ is the hyperoctahedral quantum group as defined by Bichon in [4].

2.3.2. **Example.** In contrast to the circle on 4 vertices, the circle on n vertices with $n \geq 5$ does not have quantum symmetries. Look for example at $n = 5$, $\Gamma =$



We then see that

$$u_{11}u_{22} = u_{11}u_{22}\sum_{k=1}^{5} u_{1k} = u_{11}u_{22}\sum_{k;(2,k)\in E} = u_{11}u_{22}(u_{11} + u_{13}).$$

But we also have that

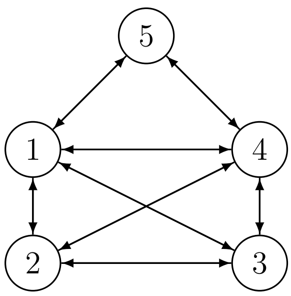
$$u_{11}u_{22}u_{13} = u_{11}(1 - u_{21} - u_{23} - u_{24} - u_{25})u_{13} = 0$$

since $u_{11}u_{21} = u_{11}u_{13} = u_{11}u_{24} = 0$ and $u_{23}u_{13} = u_{25}u_{13} = 0$ by relations (1.1), (1.3) and (1.4) from Definition 1.3.1. We thus have that $u_{11}u_{22} = u_{11}u_{22}u_{11}$ and as in Example 1.3.6 it follows that $u_{11}u_{22} = u_{22}u_{11}$. The rest of the commutations can be shown similar to the proof that the Petersen graph has no quantum symmetries, see Theorems 3.2 and 3.3 in [7]. The proofs rely on the fact that the circle on 5 vertices is strongly regular.
For the general case on n vertices it was shown by Banica in [1] that it does not have quantum symmetries. We thus have that $G^+_{aut}(\Gamma) = G_{aut}(\Gamma) = D_n$, where D_n is the dihedral group, given by

$$D_n = \langle x, y | x^n = y^2 = (xy)^2 = 1\rangle.$$

Moreover for $n \leq 3$ we have by Remark 1.2.3 that the circle on n vertices does not have quantum symmetries. The circle on 4 vertices is thus really an exception.

2.3.3. **Example.** Let $\Gamma =$



We have that $(2,3) \in G_{aut}(\Gamma)$ and $(1,4) \in G_{aut}(\Gamma)$. The Schmidt criterion 2.1.2 thus yields that Γ does have quantum symmetry. Now we want to compute the quantum automorphism group of Γ. We have that

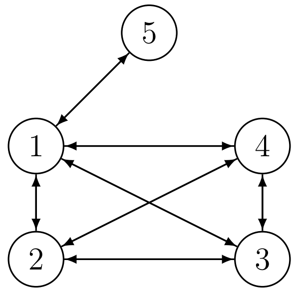
$$\varepsilon = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \varepsilon^2 = \begin{bmatrix} 4 & 2 & 2 & 3 & 1 \\ 2 & 3 & 2 & 2 & 2 \\ 2 & 2 & 3 & 2 & 2 \\ 3 & 2 & 2 & 4 & 1 \\ 1 & 2 & 2 & 1 & 2 \end{bmatrix}$$

Thus, by the Fulton criterion 2.2.3 and since $\sum_{k=1}^{5} u_{ik} = \sum_{k=1}^{5} u_{ki} = 1$ we have that the generator matrix of $C(G^+_{aut}(\Gamma))$ looks like

$$u = \begin{bmatrix} u_{11} & 0 & 0 & 1-u_{11} & 0 \\ 0 & u_{22} & 1-u_{22} & 0 & 0 \\ 0 & 1-u_{22} & u_{22} & 0 & 0 \\ 1-u_{11} & 0 & 0 & u_{11} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If u_{11} or u_{22} were 0 or if they were equal, $C(G^+_{aut}(\Gamma))$ would be commutative. Since we know that Γ does have quantum symmetry however, we see that the u above is the final generator matrix. We thus conclude that $G^+_{aut}(\Gamma) = \widehat{\mathbb{Z}_2 * \mathbb{Z}_2}$, see [8] before Theorem (3.8).

2.3.4. **Example.** If we now look at the above graph after taking away the edge $(4, 5)$, we get $\Gamma =$



and thus

$$\varepsilon = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \varepsilon^2 = \begin{bmatrix} 4 & 2 & 2 & 2 & 0 \\ 2 & 3 & 2 & 2 & 1 \\ 2 & 2 & 3 & 2 & 1 \\ 2 & 2 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

By the Fulton criterion 2.2.3 we get

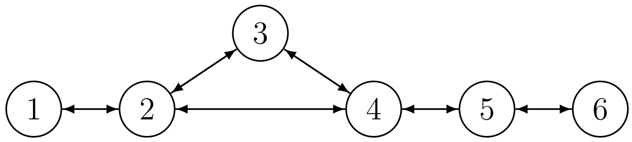
$$u = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & u_{22} & u_{23} & u_{24} & 0 \\ 0 & u_{32} & u_{33} & u_{34} & 0 \\ 0 & u_{42} & u_{43} & u_{44} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If we look at the 3×3 matrix in the middle that is neither 0 nor 1, we can see that it fulfills all the relations of $C(S_3^+)$, namely $\sum_{k=2}^{4} u_{ik} = \sum_{k=2}^{4} u_{ki} = 1$ for $i = 2, 3, 4$ and all u_{ij} are projections, see Example 1.2.2. As we know, that $S_3^+ = S_3$, that is $C(S_3^+)$ is commutative, we can conclude, that $G^+_{aut}(\Gamma) = G_{aut}(\Gamma) = S_3$.

2.3.5. **Lemma.** *Let Γ be a graph and u_{ij} be the generators of its quantum automorphism group. If $u_{ij} = \delta_{ij}$ then $G^+_{aut}(\Gamma) = G_{aut}(\Gamma) = \{e\}$.*

Proof. If $u_{ij} = \delta_{ij}$ then $G^+_{aut}(\Gamma)$ is generated only by 1, since every u_{ij} is either 1 or 0. Thus $dim(C\left(G^+_{aut}(\Gamma)\right)) = 1 \Rightarrow C(G^+_{aut}(\Gamma))$ is commutative and thus also $G^+_{aut}(\Gamma) = G_{aut}(\Gamma)$ and moreover, since $|G^+_{aut}(\Gamma)| = 1$, $G^+_{aut}(\Gamma) = G_{aut}(\Gamma) = \{e\}$. □

2.3.6. **Example.** Let $\Gamma =$



We thus have that

$$\varepsilon = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \varepsilon^4 = \begin{bmatrix} 3 & 2 & 4 & 5 & 1 & 1 \\ 2 & 12 & 7 & 7 & 6 & 1 \\ 4 & 7 & 8 & 7 & 5 & 1 \\ 5 & 7 & 7 & 13 & 2 & 4 \\ 1 & 6 & 5 & 2 & 6 & 0 \\ 1 & 1 & 1 & 4 & 0 & 2 \end{bmatrix}$$

(we use ε^4 here since this yields the most information) and see by the Fulton criterion 2.2.3 that $u_{ij} = \delta_{ij}$, since $\varepsilon^4_{ii} \neq \varepsilon^4_{jj}$ for all $i \neq j$ and since $\sum_{k=1}^{6} u_{ik} = 1$ for all i. By Lemma 2.3.5 we thus see that $G^+_{aut}(\Gamma) = G_{aut}(\Gamma) = \{e\}$.

3. Calculating quantum symmetries on the computer

In this chapter, the software tools written to calculate the quantum symmetries of graphs are presented in an abstracted version. The source code of all scripts written will be made available electronically.

3.1. GAP graph constructor.

The first algorithm is used to provide the data set on which the other algorithm is used, i.e. it constructs the graphs of which quantum symmetries are supposed to be calculated.

Algorithm 1: graph constructor

```
input : n - the number of vertices,
        g - a subgroup of the symmetric group on n points (optional)
output: a list of all connected graphs on n vertices that have g as
        automorphism group (if g is given), together with the generators of
        the automorphism group of the graphs
generate a list matrixList of all symmetric matrices containing ones and
  zeros;
initialize an empty graphList;
for matrix in matrixList do
    construct a graph G that has matrix as adjacency matrix;
    if G is connected then
        put G in graphList
compute isomorphism classes of the graphs in graphList and only keep one
  representative of each class in graphList;
if g is given then
    for G in graphList do
        if G does not have g as autgroup then
            remove G from graphList
print all graphs from graphList to a file;
```

The above algorithm is in fact a simplified version, since in the original it consists of a python and a GAP part. The python script will execute the first for-loop and then print the GAP-code, that consists of the remaining instructions applied to the graphs of *graphList*, to a file. This file will then have to be executed by GAP. Since GAP's representation of graphs already includes the generators of the graph's automorphism group, there is no special instruction necessary to print them.

The instruction, that is the most time and memory consuming in this algorithm, is the computation of isomorphism classes. This instruction is also the reason, why I have only considered graphs on up to 7 vertices and on 7 vertices even only those with automorphism groups $\mathbb{Z}_2$ or $\{e\}$ - if I want to calculate the isomorphism classes of all graphs on 7 vertices, my computer is working for several days before finally crashing. One way to improve the program might thus be to find some easy-to-check (i.e. not computationally expensive) criterion, that implies that two graphs are isomorphic and to sort out some of the isomorphic graphs already in the python part. This will probably not be a criterion that is equivalent to the two graphs being isomorphic, since for determining graph isomorphism there is no known algorithm that works in polynomial time.

3.2. **Quantum Symmetry calculator.** The next algorithm was written in Singular and is supposed to compute, whether the graph has quantum symmetry. It has mostly been written by Christian Eder and Andreas Steenpass and has been improved by Viktor Levandovskyy and myself.

Algorithm 2: QSym Calculator

input : E - the adjacency matrix of a graph G on N vertices
output: 1 - if the graph does not have quantum symmetries
0 or does not terminate - if the algorithm can not find out, whether the graph does have quantum symmetries

1 let r be a ring with generators $0, 1, u_{ij}, i, j = 1, \cdots, N$;
2 let I be an empty ideal in r;
3 let J be an empty ideal in r;
4 **for** $i = 1..N$ **do**
5   add the relation $\sum_{j=1}^{N} u_{ij} = 1$ to the ideal I;
6   add the relation $\sum_{j=1}^{N} u_{ji} = 1$ to the ideal I;
7 **for** $i, j, k = 1..N$ **do**
8   add the relation $u_{ik}u_{jk} = \delta_{ij}u_{ik}$ to the ideal I;
9   add the relation $u_{ki}u_{kj} = \delta_{ij}u_{ki}$ to the ideal I;
10 **for** $i, j, k, l = 1..N$ **do**
11   **if** $E[i, j] = 1$ **then**
12   **if** $E[k, l] \neq 1$ **then**
13   add the relation $u_{ik}u_{jl} = 0$ to the ideal I;
14   **else**
15   **if** $E[k, l] = 1$ **then**
16   add the relation $u_{ik}u_{jl} = 0$ to the ideal I;
17 **for** $k = 1..N^2$ **do**
18   **for** $i, j = 1..N$ **do**
19   **if** $E^k[i, i] \neq E^k[j, j]$ **then**
20   add the relation $u_{ij} = 0$ to the ideal I;
21 **for** $i, j, k, l = 1..N$ **do**
22   add the relation $u_{ij}u_{kl} = u_{kl}u_{ij}$ to the ideal J ;
23 **for** t *in* J **do**
24   **if** t *is not in* I **then**
25   return 0 and exit;
26 return 1;

This algorithm takes a graph and computes the ideal of all relations of the generators of its quantum automorphism group. However it does not know anything about the C^*-structure on $C(G_{aut}^{+}(\Gamma))$. This is why, if the algorithm says that the graph does have quantum symmetry (i.e. it returns 0), this is not necessarily true. The ideal of the quantum automorphism relations is stored in the variable I. First, relation (1.2) from Definition 1.3.1 is added in lines 4-6. Then, relation (1.1) is added in lines 7-9. In lines 10 - 16, the final relations (1.3) and (1.4) are added. Note, that for relations (1.3) and (1.4) we do not need to add the other combinations of i, j, k and l within the loop, since every possible combination will be passed by the loop. After that, in lines 17 - 20, the Fulton criterion 2.2.3 is used, to add some

more information to the ideal.
Then, the ideal J is computed in lines 21 and 22, which consists of all commutation relations between the generators of the quantum automorphism group u_{ij}.

To see, whether the quantum automorphism group is commutative, the algorithm then checks for every element of J if it is also in I. If that is the case, then I is obviously commutative and thus also $C(G^+_{aut}(\Gamma))$, since I then contains all commutation relations of $C(G^+_{aut}(\Gamma))$ and since $I \subseteq C(G^+_{aut}(\Gamma))$. If this is not the case however, we do not know, whether the graph has quantum symmetry, as the C^*-structure that is missing in I might yield the missing commutation relations.
For checking the inclusion of J in I, the implementation uses Gröbner bases. In some cases however, the Gröbner basis of the ideal I might be infinite, in which case the program will not terminate. This happened three times on the graphs, that I used the QSym calculator on, namely on the graph on 5 vertices with index 20 and the graphs on 6 vertices with indices 0 and 111.

3.3. **The produced data and its interpretation.** Not all of the data produced is included here, since that would be more than 200 pages. Instead I will only show parts of the results here, the rest will be made available separately. Since Schmidt and Weber have already calculated the quantum automorphism groups of all graphs on 4 vertices in [8], only larger graphs are considered here. In the appendix, we can see the tables of all connected graphs on 5 vertices and the tables of graphs on 5 and 6 vertices that have the classical automorphism group $\mathbb{Z}_2$ or the trivial group. The information displayed there for each graph consists of

- a picture of the graph
- whether or not said graph is regular
- the (classical) automorphism group of the graph given by its generators
- the order of the automorphism group
- whether or not the QSym calculator says, that the graph has quantum symmetries
- and whether or not the graph has disjoint automorphisms.

Here is a table that gives an overview over the number of connected graphs for a given size of the automorphism group and the number of those graphs that have quantum symmetries ("5 vertices qsym" and "6 vertices qsym").

Order(AutGroup)	5 vertices	5 vertices qsym	6 vertices	6 vertices qsym
720	0	0	1	1
120	1	1	1	1
72	0	0	1	1
48	0	0	4	4
36	0	0	1	1
24	1	1	1	1
16	0	0	3	3
12	3	3	10	8
10	1	0	1	0
8	2	2	9	9
6	1	0	7	0
4	3	3	28	26
2	9	0	37	0
1	0	0	8	0
all	21	10	112	55

Looking at the table we note a few things:
Firstly, the ratio of graphs with quantum symmetry and without seems to be around 50 : 50. Since it is known however that almost all graphs have trivial quantum automorphism group and thus also no quantum symmetries, this is just a distortion for small n and does not hold in general. An interesting question might be, from which n onwards it can be seen, that this ratio is in fact not 50 : 50. Moreover, for graphs whose automorphism group has order 1 or 2, we see that no graphs have quantum symmetry, see also Theorem 3.3.4. An open question is, whether this does hold for general n. Graphs with automorphism groups of order 6 and 10 on 5 and 6 vertices do not have quantum symmetries, while on the other hand, graphs with automorphism groups of order 4, 8, 12 and 16 seem to often have quantum symmetries, which leads to the question, whether this holds in general. Lastly, note that the graph on 5 vertices with $order(G_{aut}) = 120$ and the graph on 6 vertices with

$order(G_{aut}) = 720$ are the full graphs and we have shown in Example 1.3.4 that they have the quantum automorphism group S_n^+ and thus have quantum symmetries.

We now present summarizing tables of all the graphs in the appendix, that each state

- the index of the graph in the corresponding section
- its regularity
- the classical automorphism group by generators
- the output of the QSym calculator (where ? means that the program did not terminate)
- and whether or not the graph has disjoint automorphisms.

Note that by the Schmidt criterion 2.1.2 disjoint automorphisms already imply that the graph does have quantum symmetries.

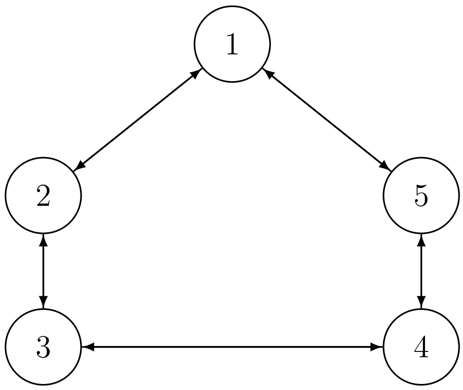
3.3.1 (**Graphs on 5 vertices**). Here is an overview of the graphs on 5 vertices.

Table 1: Graphs on 5 vertices

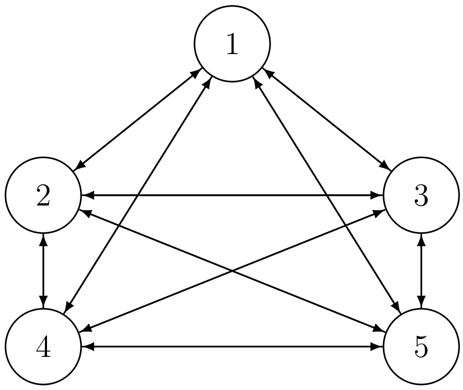
index	regular	AutGroup	Order(AutGroup)	QSym	disj auts
0	not regular	Group([(3,4), (2,3), (1,2)])	24	yes	yes
1	not regular	Group([(1,2)])	2	no	no
2	not regular	Group([(3,4), (1,2)])	4	yes	yes
3	not regular	Group([(1,2)(4,5)])	2	no	no
4	not regular	Group([(1,2)(4,5)])	2	no	no
5	not regular	Group([(2,3)])	2	no	no
6	not regular	Group([(2,3)])	2	no	no
7	not regular	Group([(2,3)])	2	no	no
8	not regular	Group([(3,4)])	2	no	no
9	not regular	Group([(3,4), (2,3)])	6	no	no
10	not regular	Group([(4,5), (2,3), (1,2)])	12	yes	yes
11	not regular	Group([(4,5), (2,3), (1,2)])	12	yes	yes
12	not regular	Group([(2,3), (1,2)(3,4)])	8	yes	yes
13	2-regular	Group([(2,3)(4,5), (1,2)(3,4)])	10	no	no
14	not regular	Group([(2,3)(4,5)])	2	no	no
15	not regular	Group([(1,2)(3,4)])	2	no	no
16	not regular	Group([(2,3), (4,5)])	4	yes	yes
17	not regular	Group([(4,5), (2,3)])	4	yes	yes
18	not regular	Group([(3,4), (1,2), (1,3)(2,4)])	8	yes	yes
19	not regular	Group([(4,5), (3,4), (1,2)])	12	yes	yes
20	4-regular	Group([(4,5), (3,4), (2,3), (1,2)])	120	?	yes

Looking at this table, we note that on 5 vertices there is an equivalence between the output of the Qsym calculator and the Schmidt criterion. This does not hold in general however, see Remark 3.3.7.

As we can also see from the table, there are 2 regular graphs on 5 vertices. One is the circle on 5 vertices, which is 2-regular:



The other one is the full graph on 5 vertices:



Index: 20
AutGroup = Group([(4,5), (3,4), (2,3), (1,2)])

3.3.2 (**Graphs on 5 vertices with autgroup $\mathbb{Z}_2$**). To make it easier to deduce something for graphs that have automorphism group $\mathbb{Z}_2$, here is a table of only those graphs.

Table 2: Graphs on 5 vertices

index	regular	AutGroup	Order(AutGroup)	QSym	disj auts
0	not regular	Group([(1,2)])	2	no	no
1	not regular	Group([(1,2)(4,5)])	2	no	no
2	not regular	Group([(1,2)(4,5)])	2	no	no
3	not regular	Group([(2,3)])	2	no	no
4	not regular	Group([(2,3)])	2	no	no
5	not regular	Group([(2,3)])	2	no	no
6	not regular	Group([(3,4)])	2	no	no
7	not regular	Group([(2,3)(4,5)])	2	no	no
8	not regular	Group([(1,2)(3,4)])	2	no	no

As can be seen here, all graphs that have automorphism group $\mathbb{Z}_2$ do not have quantum symmetries.

3.3.3 (**Graphs on 6 vertices with autgroup $\{e\}$ and $\mathbb{Z}_2$**). Since the table of all graphs on 6 vertices would be too large to fit in here, only the tables of graphs with automorphism groups $\mathbb{Z}_2$ and $\{e\}$ are presented.

Table 3: Graphs on 6 vertices

index	regular	AutGroup	Order(AutGroup)	QSym	disj auts
0	not regular	Group(())	1	no	no
1	not regular	Group(())	1	no	no
2	not regular	Group(())	1	no	no
3	not regular	Group(())	1	no	no
4	not regular	Group(())	1	no	no
5	not regular	Group(())	1	no	no
6	not regular	Group(())	1	no	no
7	not regular	Group(())	1	no	no

Table 4: Graphs on 6 vertices

index	regular	AutGroup	Order(AutGroup)	QSym	disj auts
0	not regular	Group([(1,2)])	2	no	no
1	not regular	Group([(1,2)])	2	no	no
2	not regular	Group([(2,3)(4,5)])	2	no	no
3	not regular	Group([(3,4)])	2	no	no
4	not regular	Group([(1,2)(3,4)(5,6)])	2	no	no
5	not regular	Group([(1,2)(3,4)(5,6)])	2	no	no
6	not regular	Group([(2,3)(4,5)])	2	no	no
7	not regular	Group([(2,3)(4,5)])	2	no	no
8	not regular	Group([(3,4)])	2	no	no
9	not regular	Group([(3,4)])	2	no	no
10	not regular	Group([(3,4)])	2	no	no
11	not regular	Group([(3,4)])	2	no	no
12	not regular	Group([(2,3)])	2	no	no
13	not regular	Group([(2,3)])	2	no	no
14	not regular	Group([(2,4)(3,5)])	2	no	no
15	not regular	Group([(4,5)])	2	no	no
16	not regular	Group([(4,5)])	2	no	no
17	not regular	Group([(4,5)])	2	no	no
18	not regular	Group([(2,3)])	2	no	no
19	not regular	Group([(1,2)])	2	no	no
20	not regular	Group([(1,2)])	2	no	no
21	not regular	Group([(2,3)(5,6)])	2	no	no
22	not regular	Group([(1,2)(4,5)])	2	no	no
23	not regular	Group([(1,2)(4,5)])	2	no	no
24	not regular	Group([(1,2)(4,5)])	2	no	no
25	not regular	Group([(1,2)(4,5)])	2	no	no
26	not regular	Group([(2,3)])	2	no	no
27	not regular	Group([(2,3)])	2	no	no
28	not regular	Group([(1,2)(3,5)(4,6)])	2	no	no
29	not regular	Group([(2,3)])	2	no	no
30	not regular	Group([(2,3)])	2	no	no
31	not regular	Group([(2,3)(5,6)])	2	no	no
32	not regular	Group([(2,3)(5,6)])	2	no	no
33	not regular	Group([(3,4)])	2	no	no
34	not regular	Group([(3,4)])	2	no	no
35	not regular	Group([(1,2)(3,4)(5,6)])	2	no	no

36	not regular	Group([(2,3)(4,5)])	2	no	no

As with the graphs on 5 vertices, we again see, that all graphs with automorphism group $\mathbb{Z}_2$ and moreover also those with trivial automorphism group (of which there are no graphs on 5 vertices) do not have quantum symmetries. In fact this also holds for graphs on 7 vertices. Since there are so many graphs on 7 vertices however (317 with automorphism group $\mathbb{Z}_2$ and 144 with trivial automorphism group), these tables are not included here.
We summarize this information in the main theorem of this thesis:

3.3.4. **Theorem.** *Let Γ be a connected graph on n vertices. If $n \leq 7$ then it holds:*

$$G_{aut}(\Gamma) = \mathbb{Z}_2 \Rightarrow G^+_{aut}(\Gamma) = \mathbb{Z}_2$$
$$G_{aut}(\Gamma) = \{e\} \Rightarrow G^+_{aut}(\Gamma) = \{e\}$$

Proof. For $n \in \{1, 2, 3\}$ it holds generally, that $G_{aut}(\Gamma) = G^+_{aut}(\Gamma)$, see Remark 1.2.3. For $n = 4$, we can see in [8] that the statement holds. For $n \in \{5, 6, 7\}$, the QSym calculator has computed, that all graphs with automorphism group $\mathbb{Z}_2$ or $\{e\}$ do not have quantum symmetries. □

3.3.5. **Remark.** It remains an open question for now, whether the above statement holds for all $n \in \mathbb{N}$.

Moreover, when analysing the produced data, we note the following correspondence between disjoint automorphisms and quantum symmetries:

3.3.6. **Proposition.** *Let Γ be a connected graph on n vertices. If $n \leq 6$ then it holds:*

$$\exists\ \sigma, \tau \in G_{aut}(\Gamma) \text{ disjoint} \Leftrightarrow \Gamma \text{ has quantum symmetries}$$

Proof. The one direction was shown in the proof of the Schmidt criterion 2.1.2. For the other direction check the produced data. □

3.3.7. **Remark.** The above Proposition only covers graphs with up to 6 vertices, since the produced data does not cover all connected graphs on 7 vertices but only those that have the automorphism groups $\mathbb{Z}_2$ or $\{e\}$.
Moreover, there is a counterexample for the general statement, as revealed in private communication with Simon Schmidt.

References

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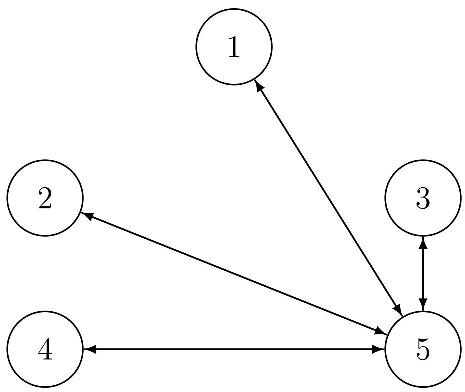
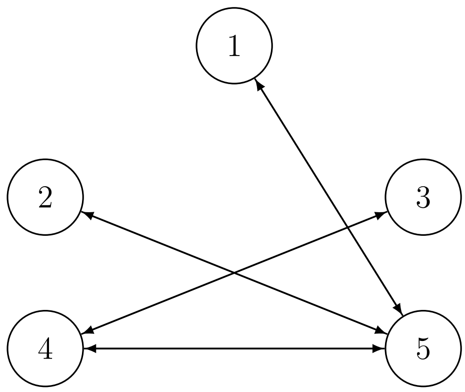
Appendices

A. Connected graphs on 5 points

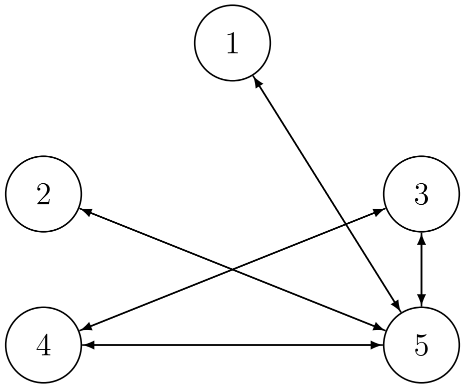
Here is the produced data for connected graphs on 5 vertices, consisting of a picture of each graph, information about its automorphism group, its complement and its quantum automorphism group, where the field q_aut is 1, if the graph does not have quantum symmetries, 0, if the qsym calculator says, that it does have quantum symmetries and ? if the qsym calculator did not terminate. There are 21 connected graphs on 5 vertices, 2 of which are regular.



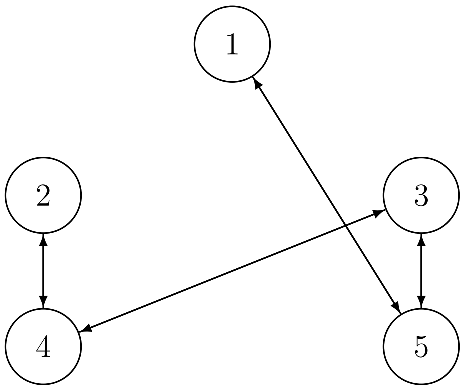
Index: 0
AutGroup = Group([(3,4), (2,3), (1,2)])
Order(AutGroup) = 24
The complement of this graph is not connected.
q_aut: 0
This graph has disjoint automorphisms and thus quantum symmetries.



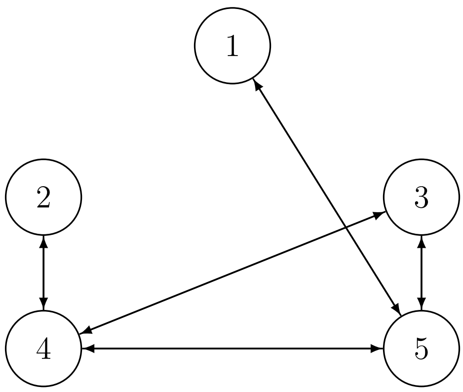
Index: 1
AutGroup = Group([(1,2)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



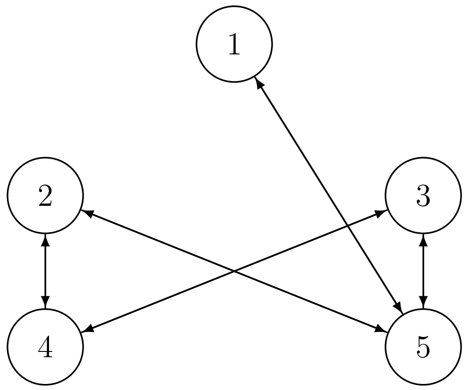
Index: 2
AutGroup = Group([(3,4), (1,2)])
Order(AutGroup) = 4
The complement of this graph is not connected.
q_aut: 0
This graph has disjoint automorphisms and thus quantum symmetries.



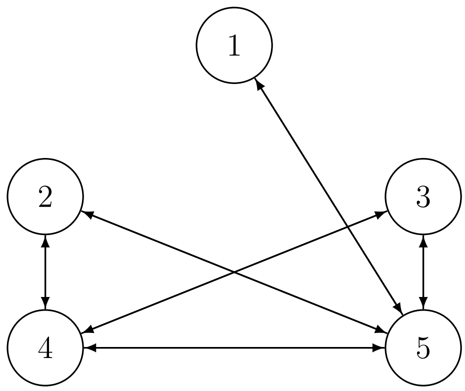
Index: 3
AutGroup = Group([(1,2)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



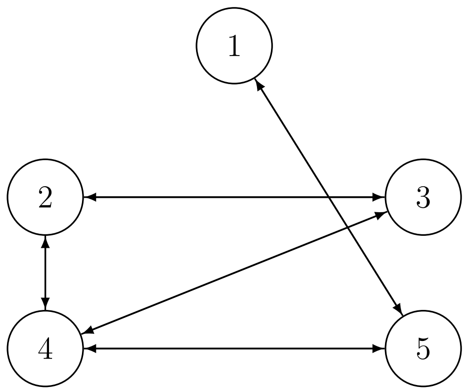
Index: 4
AutGroup = Group([(1,2)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



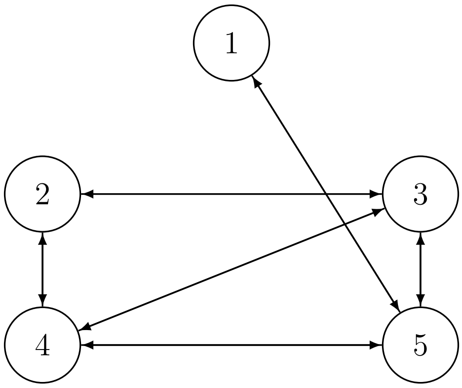
Index: 5
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



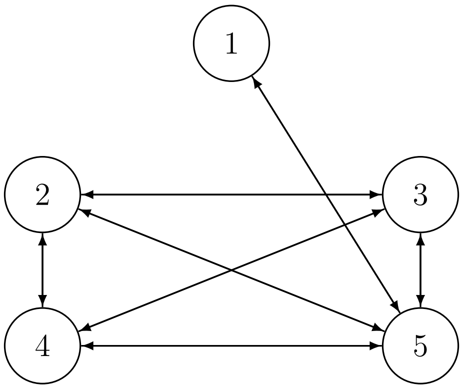
Index: 6
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



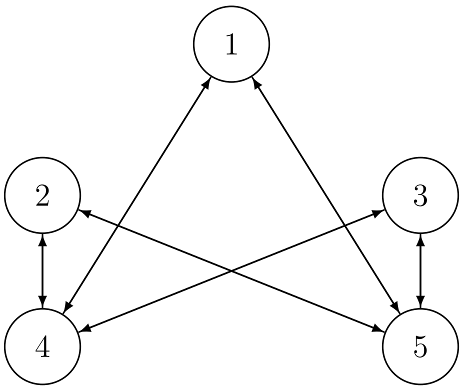
Index: 7
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



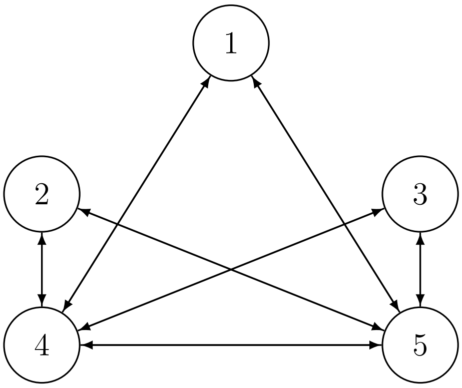
Index: 8
AutGroup = Group([(3,4)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



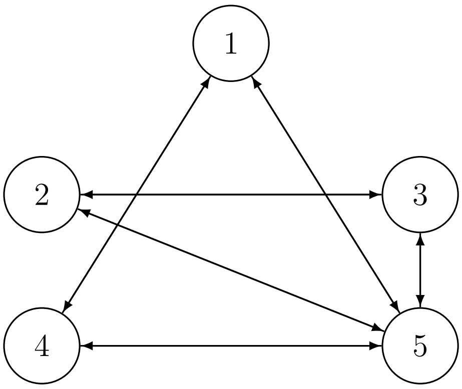
Index: 9
AutGroup = Group([(3,4), (2,3)])
Order(AutGroup) = 6
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



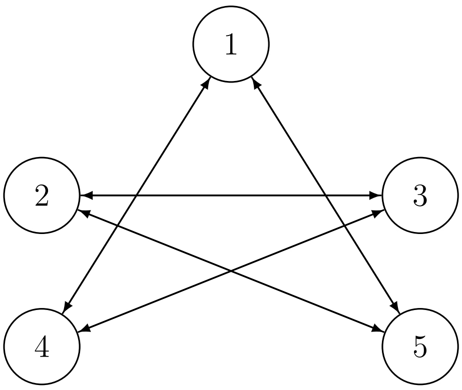
Index: 10
AutGroup = Group([(4,5), (2,3), (1,2)])
Order(AutGroup) = 12
The complement of this graph is not connected.
q_aut: 0
This graph has disjoint automorphisms and thus quantum symmetries.



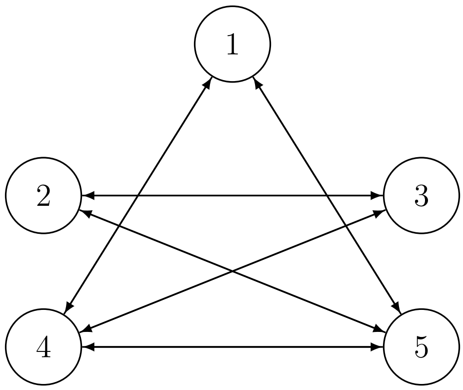
Index: 11
AutGroup = Group([(4,5), (2,3), (1,2)])
Order(AutGroup) = 12
The complement of this graph is not connected.
q_aut: 0
This graph has disjoint automorphisms and thus quantum symmetries.



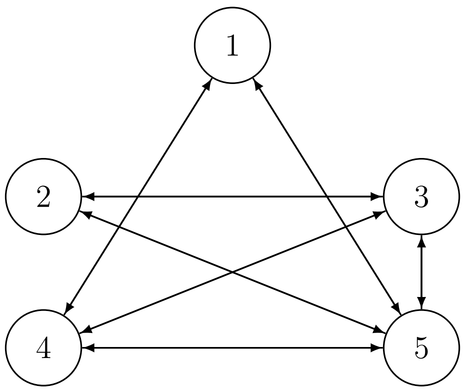
Index: 12
AutGroup = Group([(2,3), (1,2)(3,4)])
Order(AutGroup) = 8
The complement of this graph is not connected.
q_aut: 0
This graph has disjoint automorphisms and thus quantum symmetries.



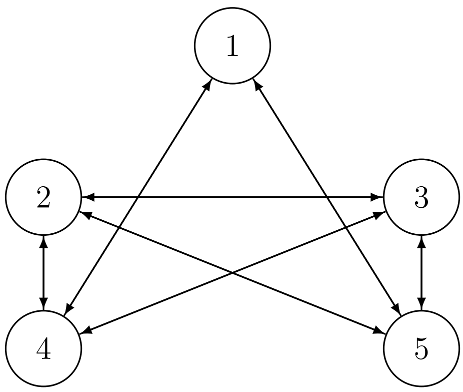
Index: 13
AutGroup = Group([(2,3)(4,5), (1,2)(3,4)])
Order(AutGroup) = 10
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



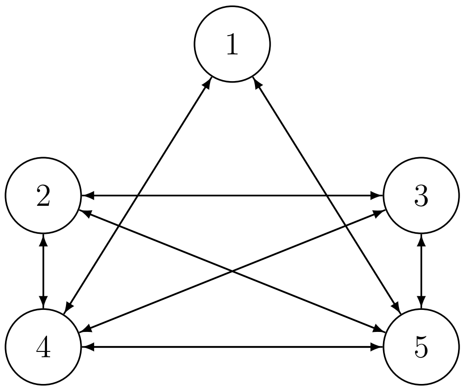
Index: 14
AutGroup = Group([(2,3)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



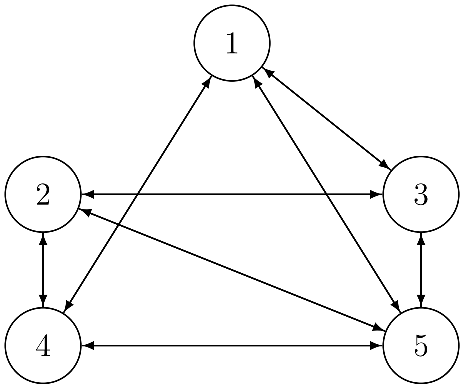
Index: 15
AutGroup = Group([(1,2)(3,4)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



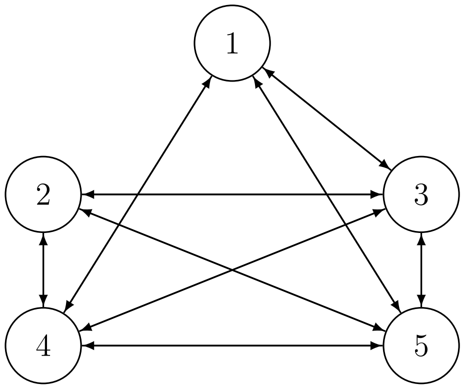
Index: 16
AutGroup = Group([(2,3), (4,5)])
Order(AutGroup) = 4
The complement of this graph is not connected.
q_aut: 0
This graph has disjoint automorphisms and thus quantum symmetries.



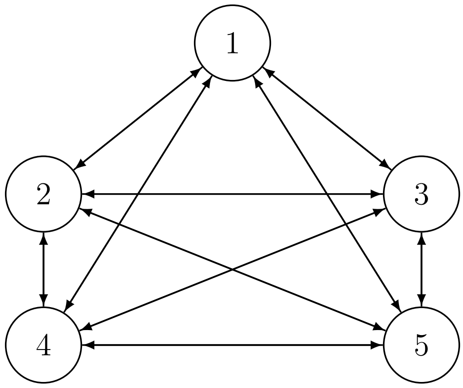
Index: 17
AutGroup = Group([(4,5), (2,3)])
Order(AutGroup) = 4
The complement of this graph is not connected.
q_aut: 0
This graph has disjoint automorphisms and thus quantum symmetries.



Index: 18
AutGroup = Group([(3,4), (1,2), (1,3)(2,4)])
Order(AutGroup) = 8
The complement of this graph is not connected.
q_aut: 0
This graph has disjoint automorphisms and thus quantum symmetries.



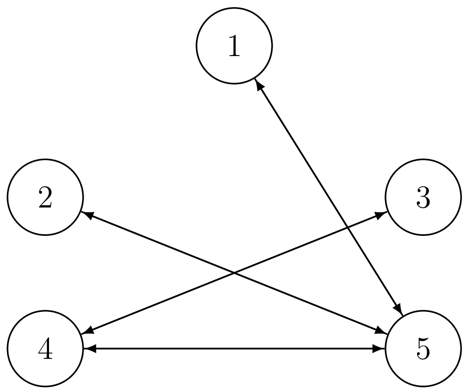
Index: 19
AutGroup = Group([(4,5), (3,4), (1,2)])
Order(AutGroup) = 12
The complement of this graph is not connected.
q_aut: 0
This graph has disjoint automorphisms and thus quantum symmetries.



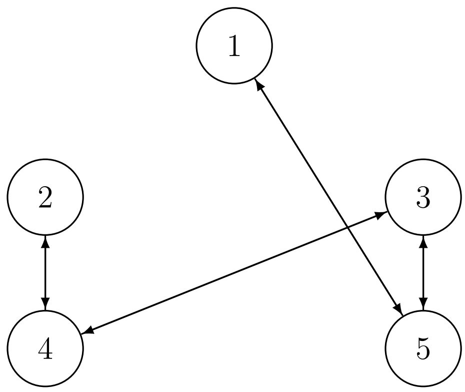
Index: 20
AutGroup = Group([(4,5), (3,4), (2,3), (1,2)])
Order(AutGroup) = 120
The complement of this graph is not connected.
q_aut: ?
This graph has disjoint automorphisms and thus quantum symmetries.

B. Connected graphs on 5 points with automorphism group $\mathbb{Z}_2$

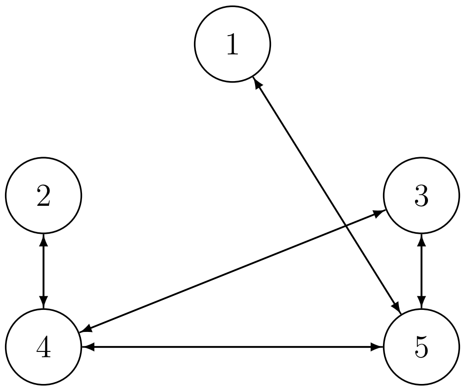
Here is the produced data for connected graphs on 5 vertices with automorphism group $\mathbb{Z}_2$, consisting of a picture of each graph, information about its automorphism group, its complement and its quantum automorphism group, where the field q_aut is 1, if the graph does not have quantum symmetries, 0, if the qsym calculator says, that it does have quantum symmetries and ? if the qsym calculator did not terminate. There are 9 connected graphs on 5 vertices with automorphism group $\mathbb{Z}_2$, none of which are regular.



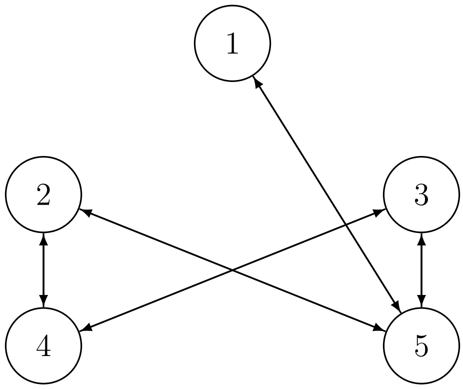
Index: 0
AutGroup = Group([(1,2)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



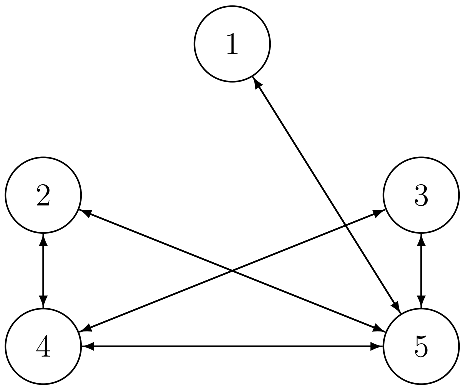
Index: 1
AutGroup = Group([(1,2)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



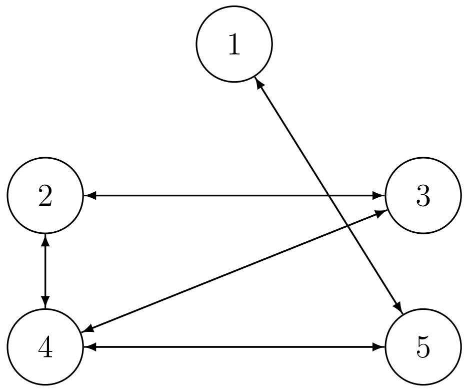
Index: 2
AutGroup = Group([(1,2)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



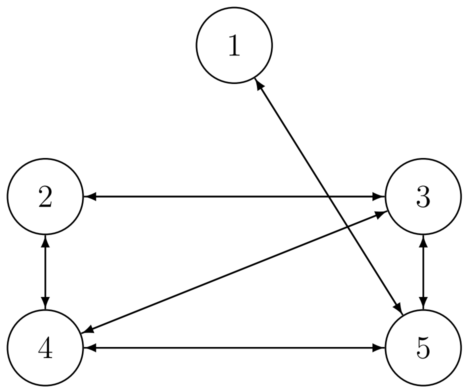
Index: 3
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



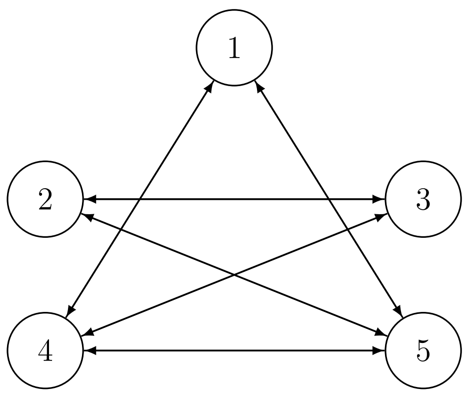
Index: 4
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



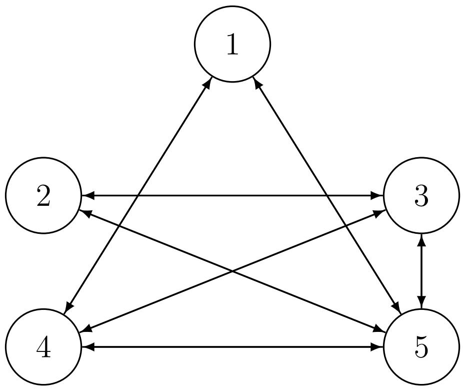
Index: 5
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



Index: 6
AutGroup = Group([(3,4)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



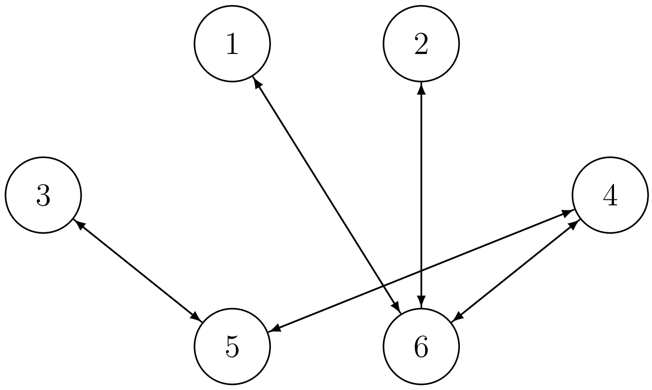
Index: 7
AutGroup = Group([(2,3)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



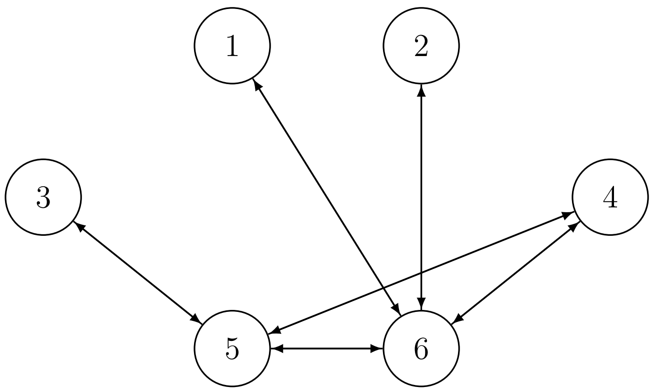
Index: 8
AutGroup = Group([(1,2)(3,4)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.

C. Connected graphs on 6 points with automorphism group $\mathbb{Z}_2$

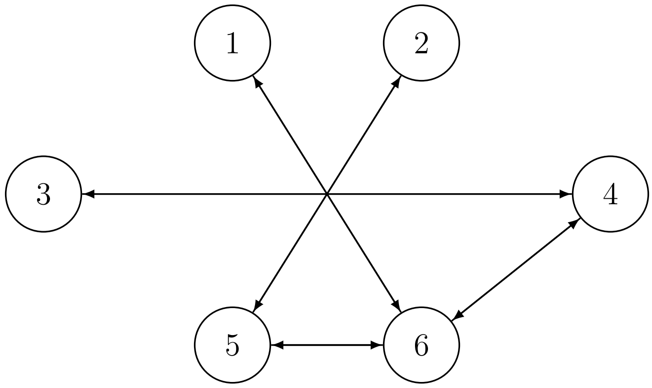
Here is the produced data for connected graphs on 6 vertices with automorphism group $\mathbb{Z}_2$, consisting of a picture of each graph, information about its automorphism group, its complement and its quantum automorphism group, where the field q_aut is 1, if the graph does not have quantum symmetries, 0, if the qsym calculator says, that it does have quantum symmetries and ? if the qsym calculator did not terminate. There are 37 connected graphs on 6 vertices, none of which are regular.



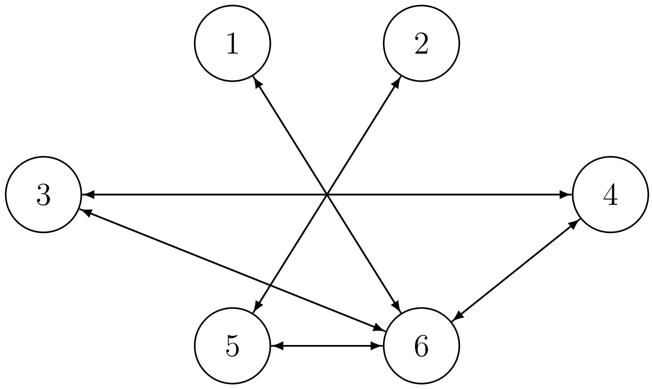
Index: 0
AutGroup = Group([(1,2)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



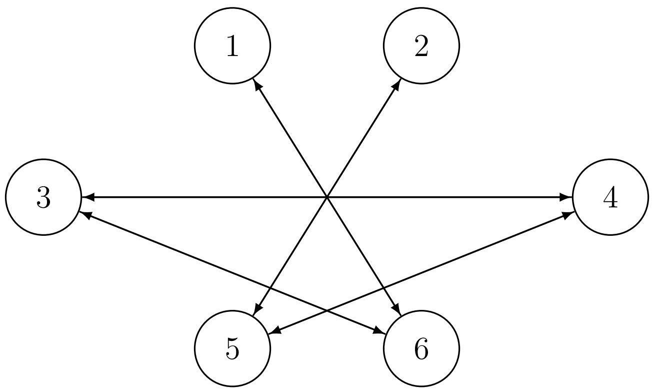
Index: 1
AutGroup = Group([(1,2)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



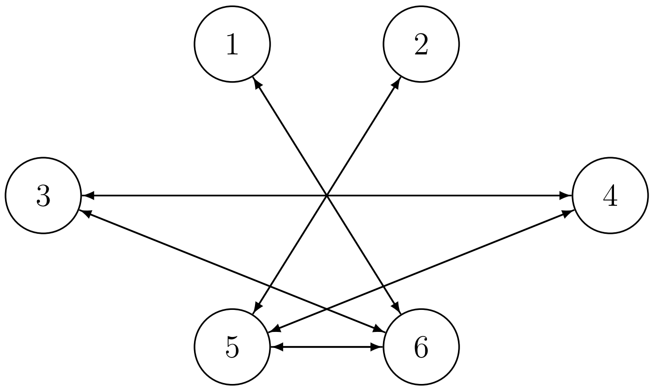
Index: 2
AutGroup = Group([(2,3)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



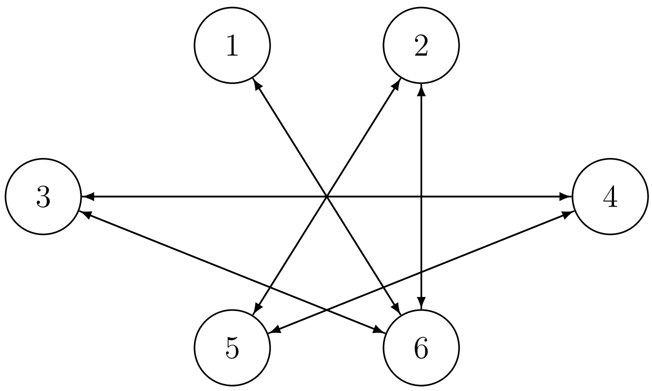
Index: 3
AutGroup = Group([(3,4)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



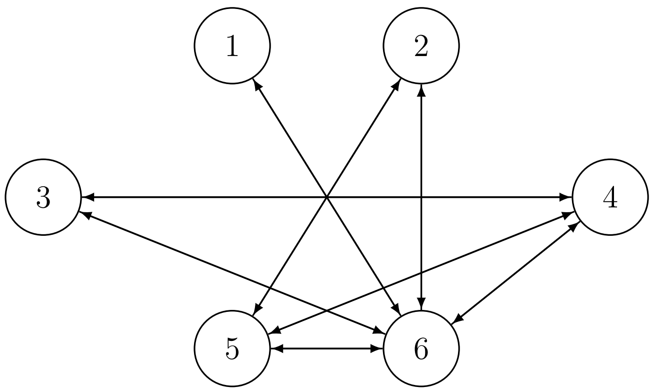
Index: 4
AutGroup = Group([(1,2)(3,4)(5,6)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



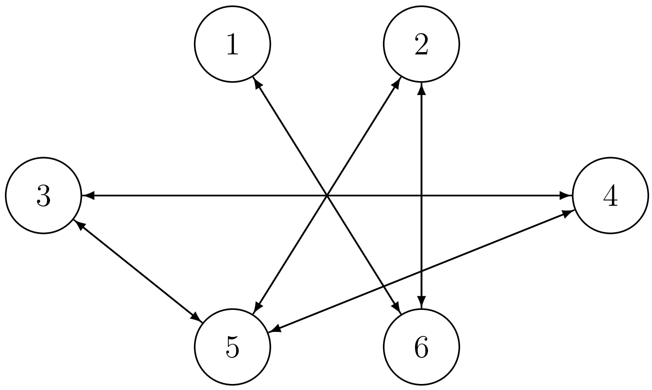
Index: 5
AutGroup = Group([(1,2)(3,4)(5,6)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



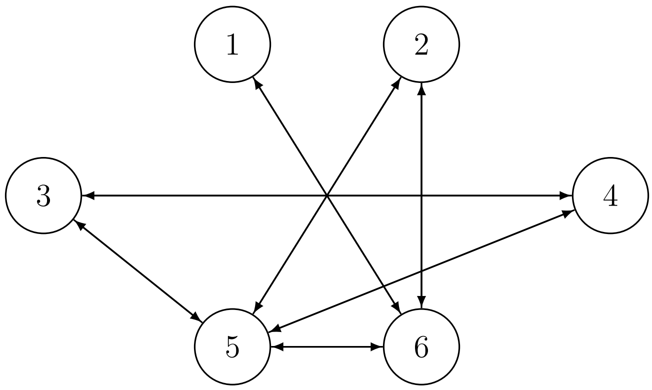
Index: 6
AutGroup = Group([(2,3)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



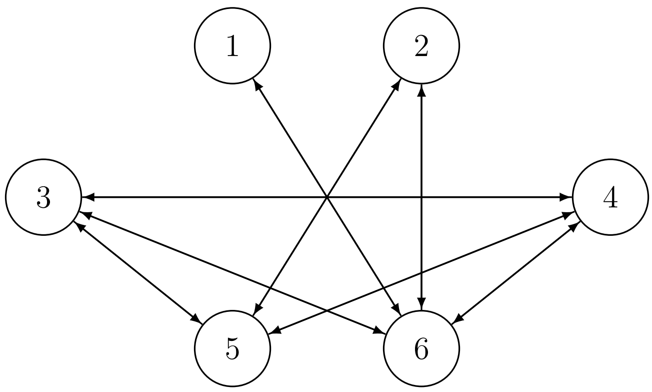
Index: 7
AutGroup = Group([(2,3)(4,5)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



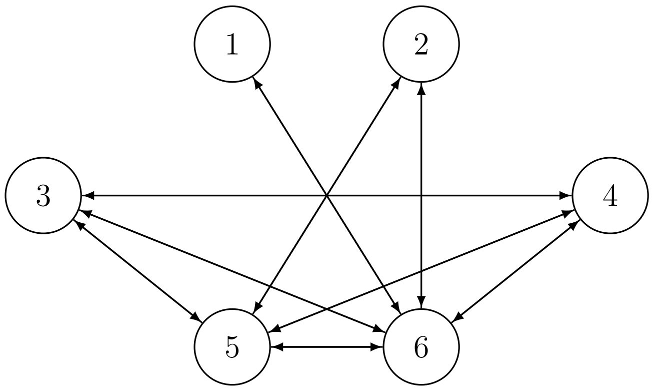
Index: 8
AutGroup = Group([(3,4)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



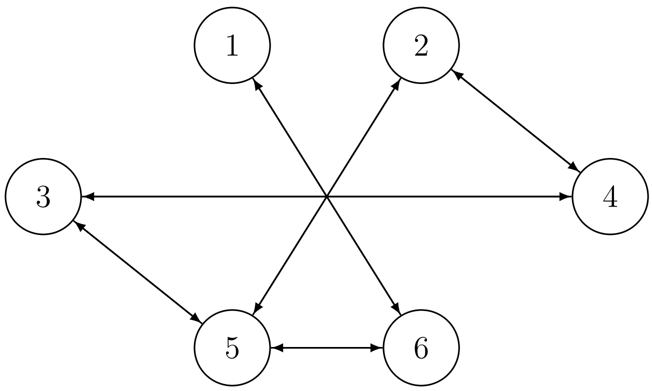
Index: 9
AutGroup = Group([(3,4)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



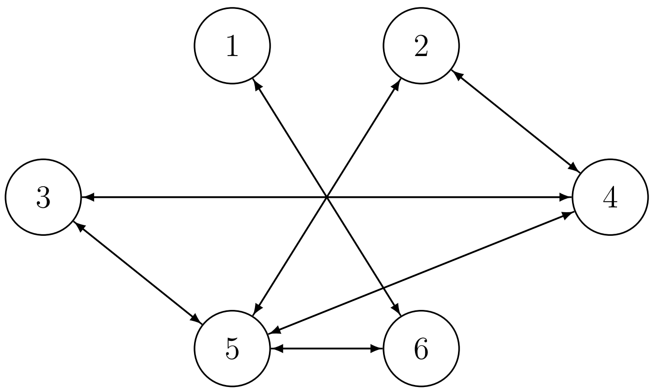
Index: 10
AutGroup = Group([(3,4)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



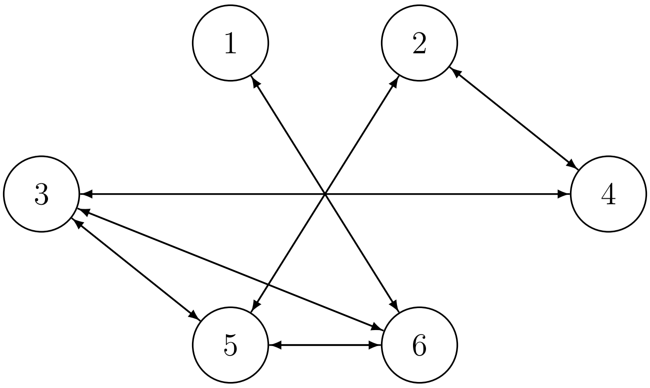
Index: 11
AutGroup = Group([(3,4)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



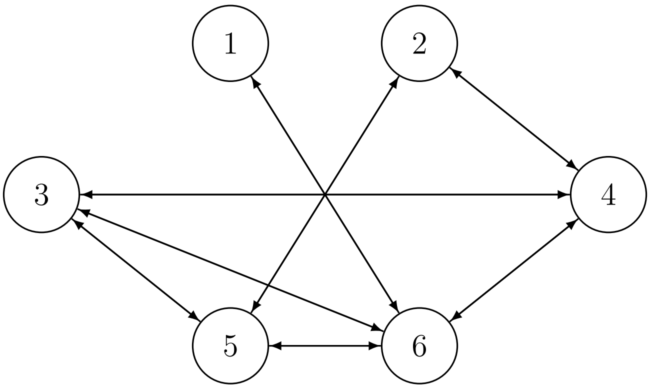
Index: 12
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



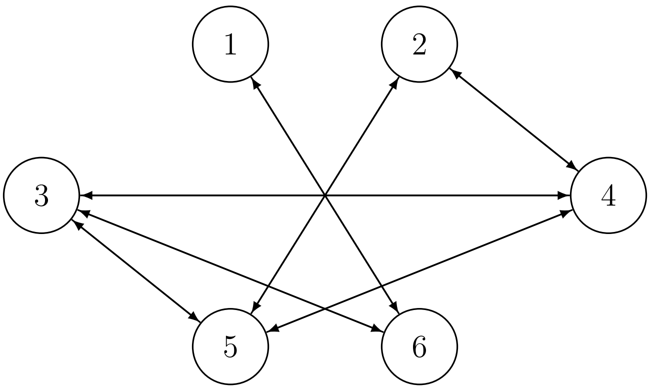
Index: 13
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



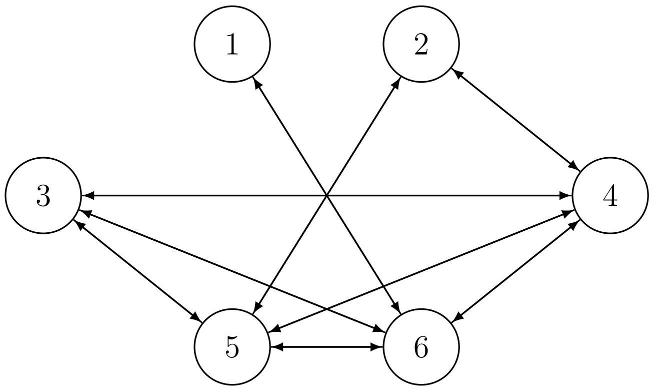
Index: 14
AutGroup = Group([(2,4)(3,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



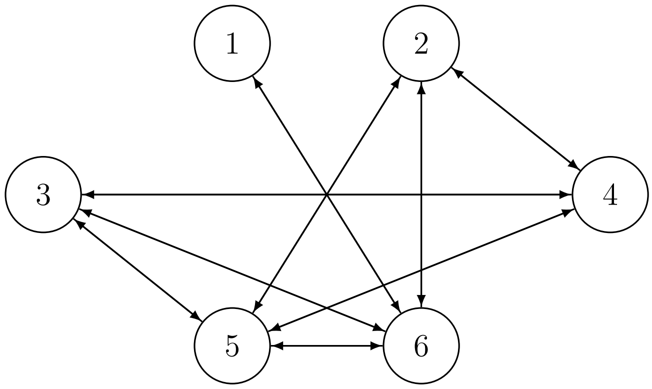
Index: 15
AutGroup = Group([(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



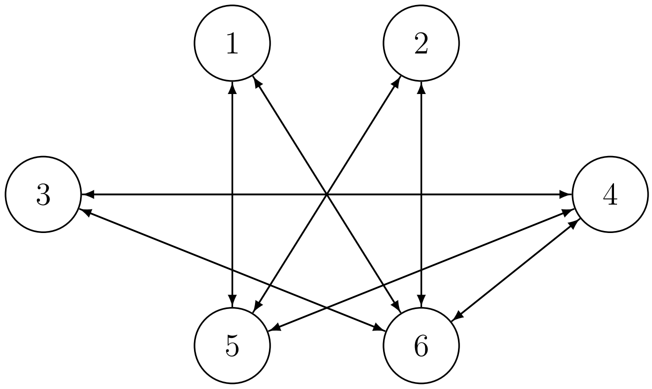
Index: 16
AutGroup = Group([(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



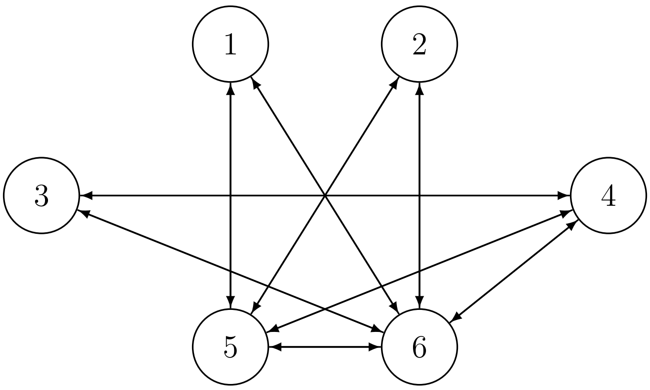
Index: 17
AutGroup = Group([(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



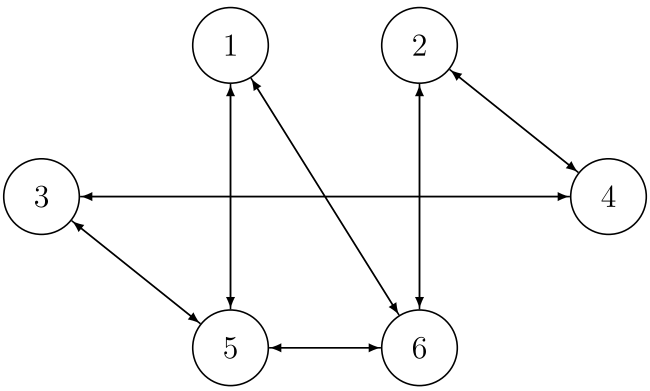
Index: 18
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



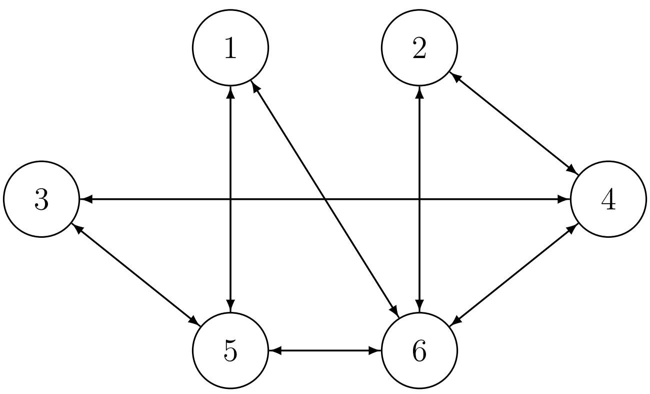
Index: 19
AutGroup = Group([(1,2)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



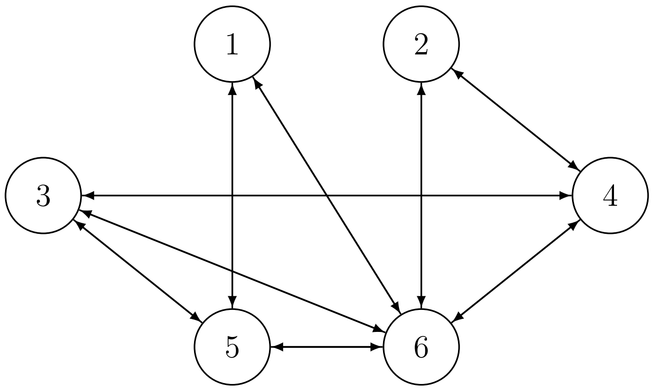
Index: 20
AutGroup = Group([(1,2)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



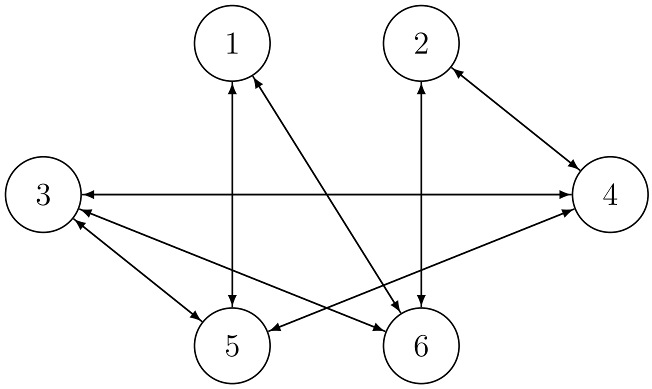
Index: 21
AutGroup = Group([(2,3)(5,6)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



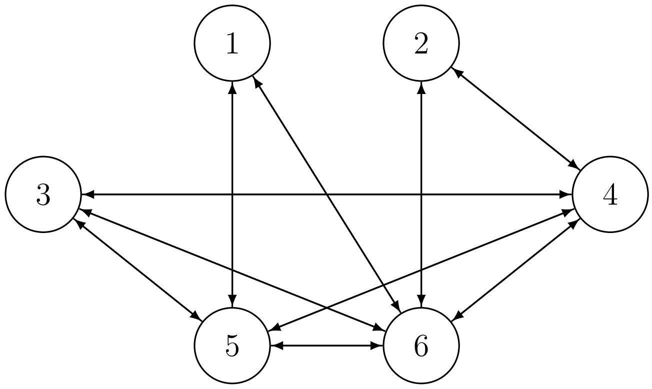
Index: 22
AutGroup = Group([(1,2)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



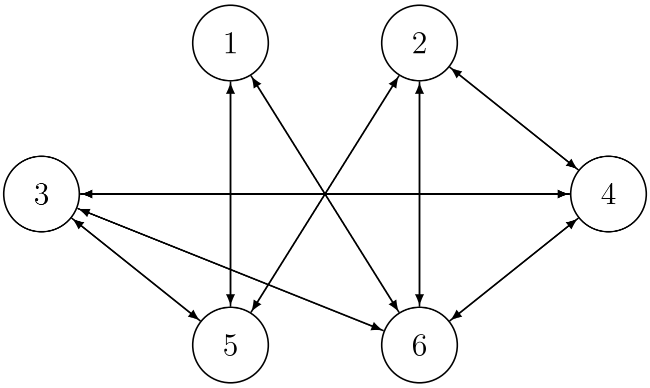
Index: 23
AutGroup = Group([(1,2)(4,5)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



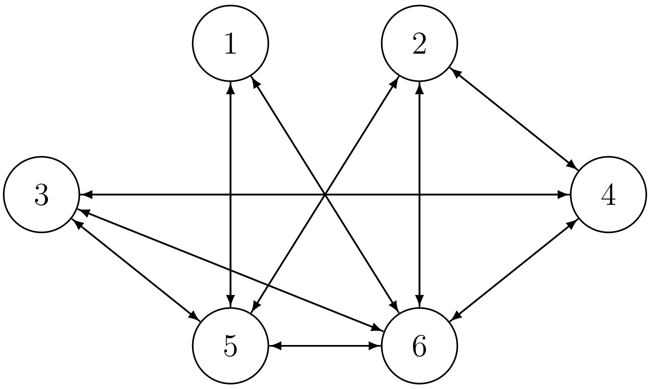
Index: 24
AutGroup = Group([(1,2)(4,5)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



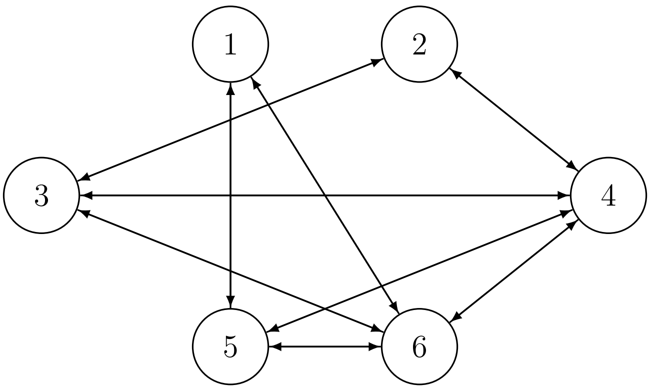
Index: 25
AutGroup = Group([(1,2)(4,5)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



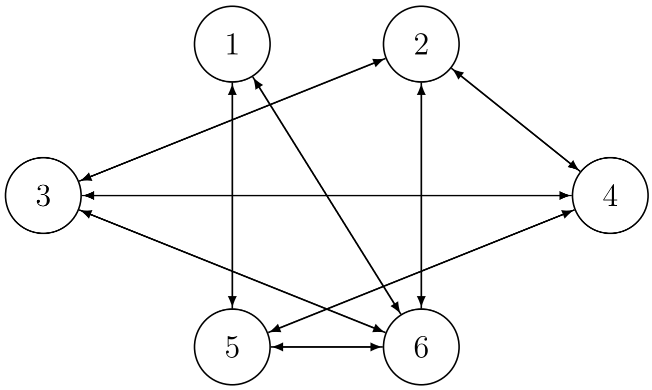
Index: 26
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



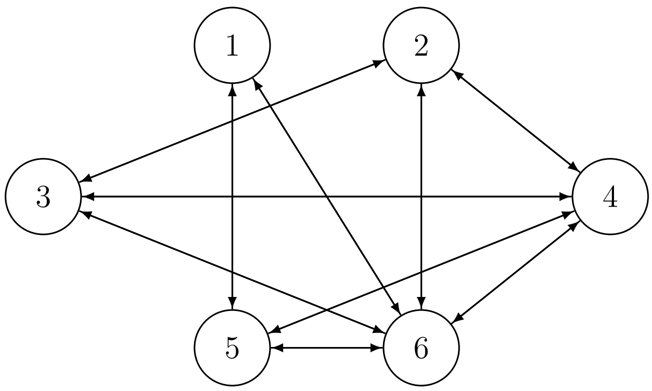
Index: 27
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



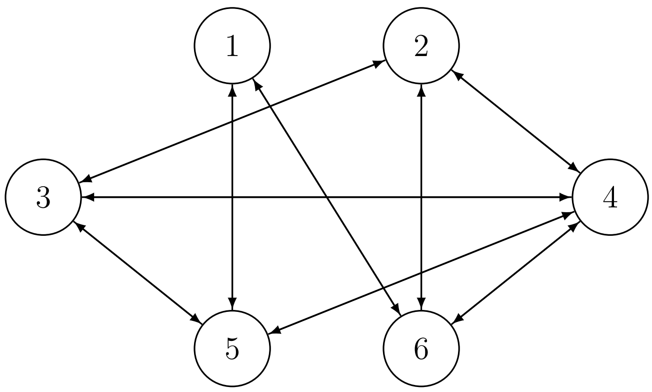
Index: 28
AutGroup = Group([(1,2)(3,5)(4,6)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



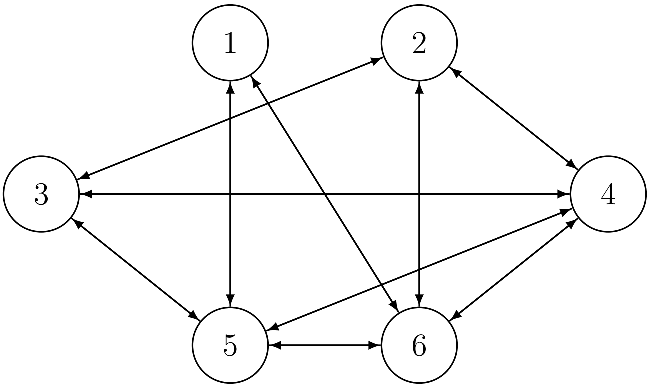
Index: 29
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



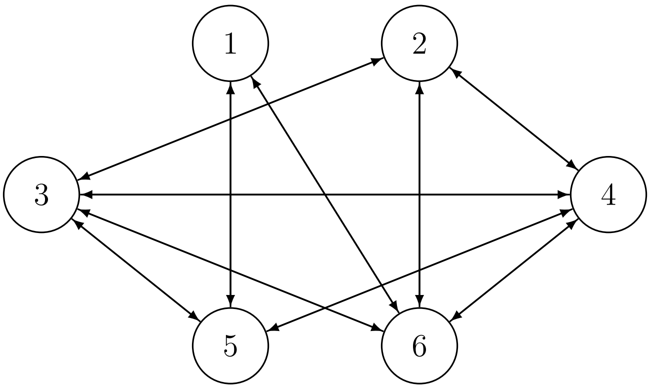
Index: 30
AutGroup = Group([(2,3)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



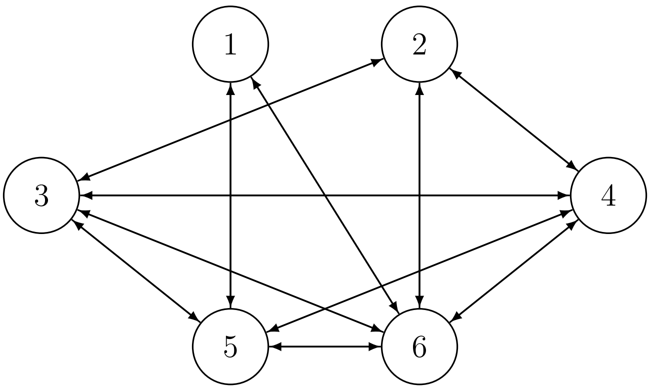
Index: 31
AutGroup = Group([(2,3)(5,6)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



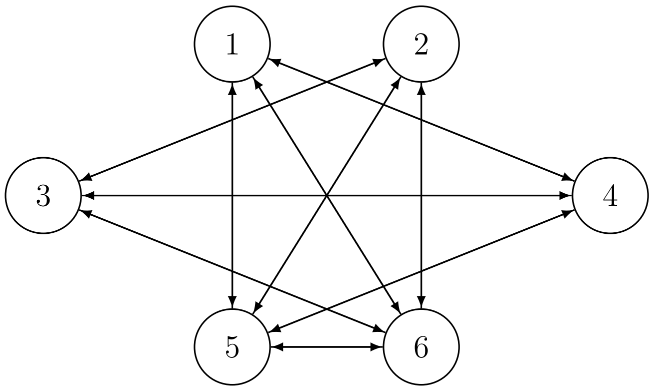
Index: 32
AutGroup = Group([(2,3)(5,6)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



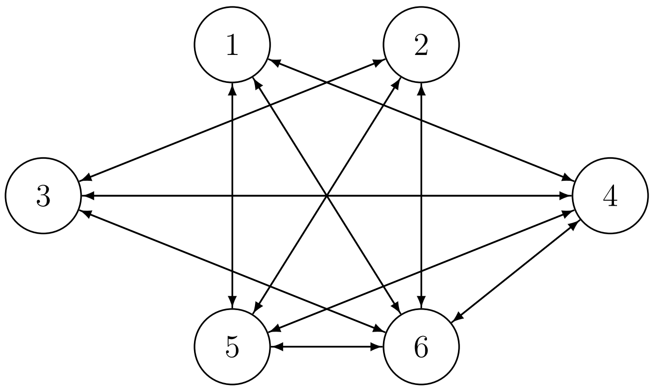
Index: 33
AutGroup = Group([(3,4)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



Index: 34
AutGroup = Group([(3,4)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.



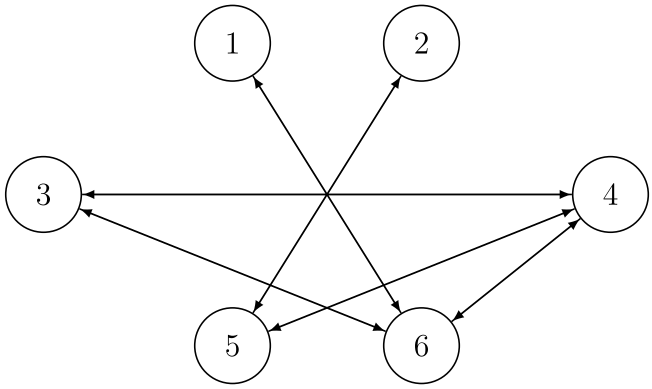
Index: 35
AutGroup = Group([(1,2)(3,4)(5,6)])
Order(AutGroup) = 2
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



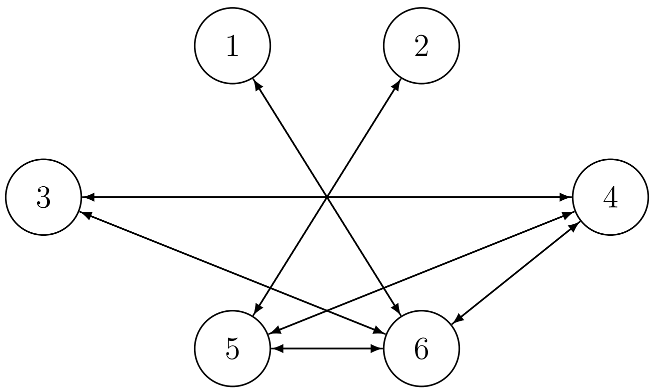
Index: 36
AutGroup = Group([(2,3)(4,5)])
Order(AutGroup) = 2
The complement of this graph is not connected.
q_aut: 1
This graph does not have disjoint automorphisms.

D. Connected graphs on 6 points with trivial automorphism group

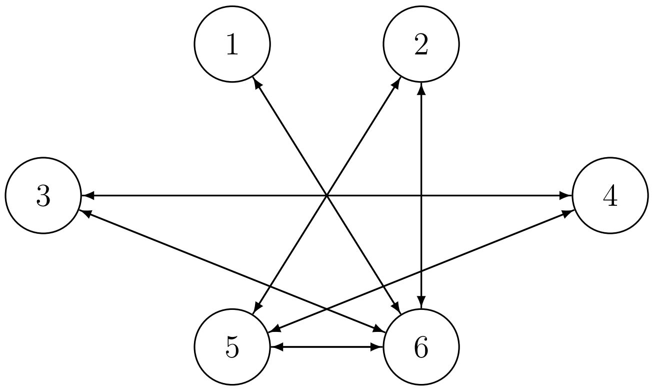
Here is the produced data for connected graphs on 6 vertices with trivial automorphism group, consisting of a picture of each graph, information about its automorphism group, its complement and its quantum automorphism group, where the field q_aut is 1, if the graph does not have quantum symmetries, 0, if the qsym calculator says, that it does have quantum symmetries and ? if the qsym calculator did not terminate. There are 8 connected graphs on 6 vertices with trivial automorphism group, none of which are regular.



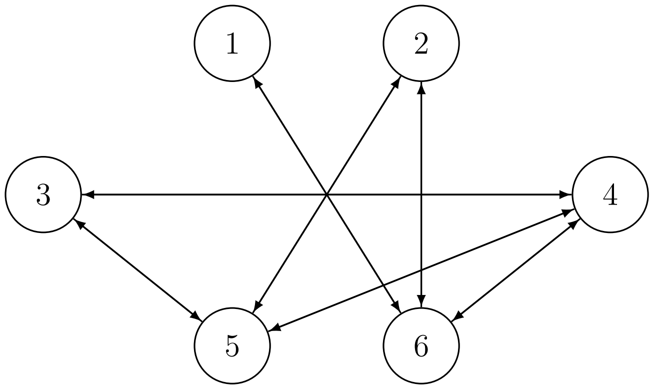
Index: 0
AutGroup = Group(())
Order(AutGroup) = 1
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



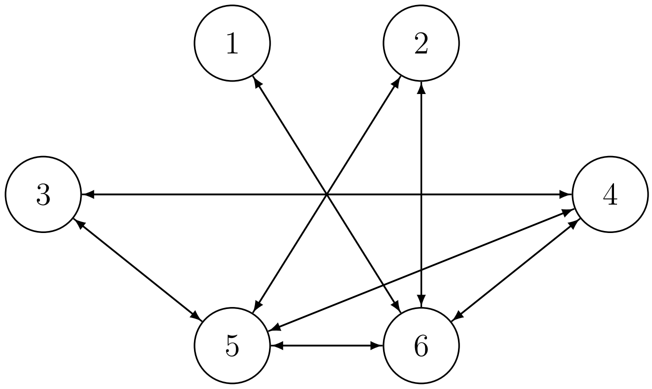
Index: 1
AutGroup = Group(())
Order(AutGroup) = 1
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



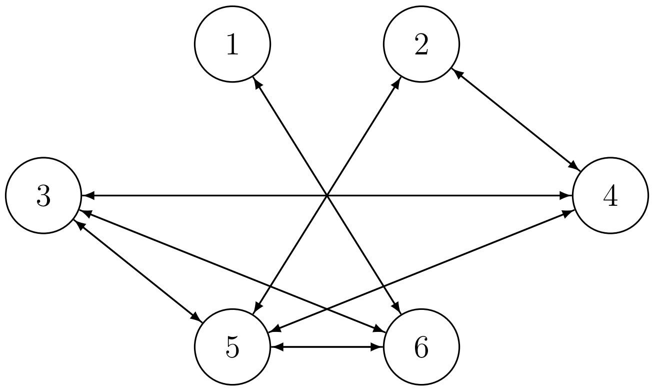
Index: 2
AutGroup = Group(())
Order(AutGroup) = 1
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



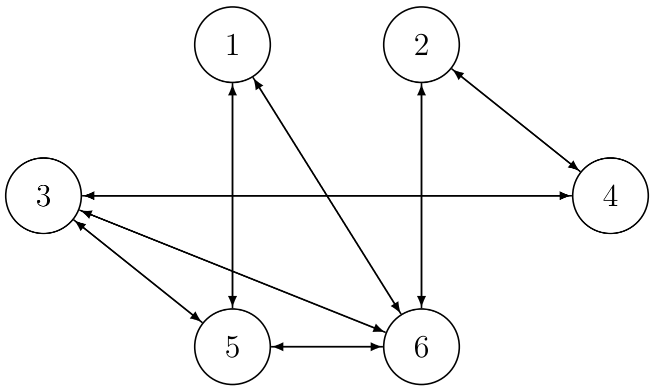
Index: 3
AutGroup = Group(())
Order(AutGroup) = 1
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



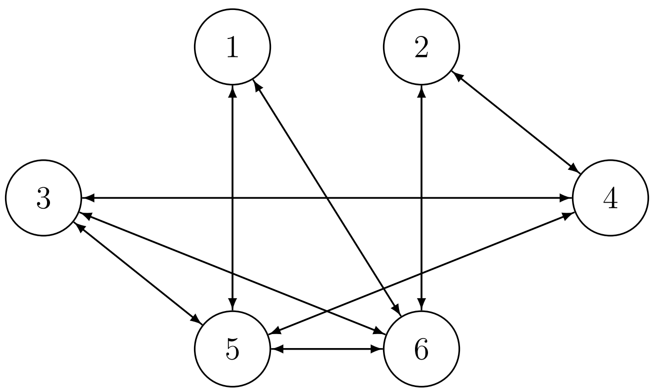
Index: 4
AutGroup = Group(())
Order(AutGroup) = 1
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



Index: 5
AutGroup = Group(())
Order(AutGroup) = 1
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



Index: 6
AutGroup = Group(())
Order(AutGroup) = 1
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.



Index: 7
AutGroup = Group(())
Order(AutGroup) = 1
The complement of this graph is connected.
q_aut: 1
This graph does not have disjoint automorphisms.