Saarland University
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Master's Thesis

Quantum Isomorphic Graphs Constructed From Linear Binary Constraint Systems

submitted by

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Saarbrücken, 23. August 2021

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Introduction

In this thesis, we consider quantum isomorphisms of graphs. We study a class of examples of quantum isomorphic graphs that are not classically isomorphic. These examples were described by Lupini, Mančinska and Roberson in 2020 in [8]. In particular, we find the "witnesses" for these quantum isomorphisms, i. e. we give explicit magic unitaries implementing the quantum isomorphism between the given graphs.

The notion of quantum isomorphisms of graphs was first introduced in [2] as the existence of a perfect quantum strategy for the graph isomorphism game, which was also introduced there. In the same paper, the authors showed that the quantum isomorphism of two graphs G and H is equivalent to the existence of a magic unitary u with operators on a finite dimensional Hilbert space as entries, such that

$$M_G u = u M_H$$

holds for the adjacency matrices M_G and M_H of G and H respectively.

Recall that a matrix $u = (u_{ij})$ is a magic unitary, if the entries u_{ij} are projections on a Hilbert space and the rows and columns all sum up to one. If the Hilbert space is one dimensional, the magic unitary is just a permutation matrix and then G and H are isomorphic. Therefore all isomorphic graphs are also quantum isomorphic. Surprisingly, one can show however, that not all quantum isomorphic graphs are also isomorphic.

In [2] a procedure is described how to obtain graphs that are quantum isomorphic, but not classically isomorphic. More precisely, given a linear binary constraint system (BCS), two graphs are constructed which are quantum isomorphic but not isomorphic, if the binary constraint system is quantum satisfiable but not satisfiable. In [8], the authors note that a result of Arkhipov [1] yields a way to construct a linear binary constraint system which is quantum satisfiable but not satisfiable given a nonplanar graph. They used this result in connection with the results from [2] to state that for any nonplanar graph, there is a pair of graphs that are quantum isomorphic, but not isomorphic.

The constructions of these graphs however do not include an explicit construction for the quantum strategy or the magic unitary that witness the quantum isomorphism. In this thesis we trace back the construction of the graphs and the proof of the existence of a perfect quantum strategy to the result of Arkhipov. This result also states a construction for the quantum satisfying assignment for the binary constraint systems that are the basis for the construction of the graphs. Using this, one can construct a perfect quantum strategy for a BCS game that was designed in [8]. This game is similar to the game presented in [5] and the construction of the perfect quantum strategy is also described there. This strategy is in turn used to construct a perfect strategy for the graph isomorphism game, from which one can then construct the magic unitary implementing the quantum isomorphism.

The procedure to get the quantum isomorphic graphs and the magic unitary u witnessing the quantum isomorphism is thus as follows:

Step 1: Start with a nonplanar graph Γ

Step 2: Get a BCS $\mathcal{F}(\Gamma)$ that is quantum satisfiable but not satisfiable and its quantum satisfying assignment by Arkhipov's construction [1]

Step 3: Turn the quantum satisfying assignment of $\mathcal{F}(\Gamma)$ into a perfect quantum strategy for the BCS game in a similar way as in [5]

Step 4: Construct the two graphs $G_{\mathcal{F}(\Gamma)}$ and $G_{\mathcal{F}_1(\Gamma)}$ that are quantum isomorphic but not isomorphic using the construction in [2]

Step 5: Use the perfect quantum stategy for the BCS game of $\mathcal{F}(\Gamma)$ to build the perfect quantum strategy of the graph isomorphism game of $G_{\mathcal{F}(\Gamma)}$ and $G_{\mathcal{F}_1(\Gamma)}$

Step 6: Build the magic unitary u from the perfect quantum strategy of the graph isomorphism game

Since the smallest example one can get with this construction are graphs with 24 vertices and therefore the magic unitary has 24×24 entries, it is not desirable to calculate it by hand. Therefore, we implemented the algorithm to get the magic unitary given a binary constraint system with quantum satisfying assignment in Singular. In the appendix, we present the resulting magic unitary if one starts with the complete bipartite graph on 6 vertices, $K_{3,3}$.

In summary, we obtain the following result in our thesis:

Main Theorem. *Given a nonplanar graph* Γ, *we construct a magic unitary* u_Γ, *such that*

$$M_{G_{\mathcal{F}(\Gamma)}} u_\Gamma = u_\Gamma M_{G_{\mathcal{F}_1(\Gamma)}}.$$

This explicitly implements the quantum isomorphism of the nonisomorphic graphs $G_{\mathcal{F}(\Gamma)}$ *and* $G_{\mathcal{F}_1(\Gamma)}$.

1. Preliminaries

1.1. Finite Graphs.

1.1.1. **Definition.** A *finite graph* G is a pair (V, E), where the set of *vertices* V and the set of *edges* E are finite. By $r : E \to V$ denote the *range map* and by $s : E \to V$ denote the *source map*. It is called *undirected* if for all $e \in E$ there is an $f \in E$ such that $s(e) = r(f)$ and $r(e) = s(f)$. In this thesis, we only consider undirected graphs and the following definitions are also for undirected graphs.

Two vertices $u, v \in V$ are called *adjacent*, if there is an $e \in E$ such that $s(e) = u$ and $r(e) = v$. We also write $u \sim v$ if u and v are adjacent. A graph is *without multiple edges* if there are no $e, f \in E, e \neq f$ such that $s(e) = s(f)$ and $r(e) = r(f)$. If we have $e \in E$ with $s(e) = r(e)$, then e is called a *loop*.
A *path* $p = (v_1, v_2, \ldots, v_n)$ of G is a tuple of vertices from G such that $v_i \sim v_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. It is called *simple*, if all the vertices in it are distinct.
For a finite graph $G = (V, E)$ with $V = \{1, \ldots, n\}$, we define its *adjacency matrix* $M_G \in M_n(\mathbb{N}_0)$ via $(M_G)_{ij} := \#\{e \in E \mid s(e) = i, r(e) = j\}$, i. e. the (i, j)-entry of M_G is the number of edges from vertex i to vertex j. If G is without multiple edges, we have $(M_G)_{ij} \in \{0, 1\}$.
By $\overline{G} = (V, E')$ we denote the *complement* of G, which has the same vertex set, but the edges are such that if u and v are adjacent in G, then they are not adjacent in $\overline{G}$ and vice versa:

$$u \sim_G v \Longleftrightarrow u \not\sim_{\overline{G}} v.$$

Throughout this thesis we will only be considering graphs without multiple edges. In this case, we can identify every edge as the pair of vertices it connects: for $e \in E$ we can write $e = (s(e), r(e))$. Then we also have $E \subseteq V \times V$.

1.1.2. **Definition.** A *graph isomorphism* between two finite graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is a bijective map $\sigma : V_G \to V_H$ such that any two vertices $u, v \in V_G$ are adjacent in G if and only if they are adjacent in H, i. e.

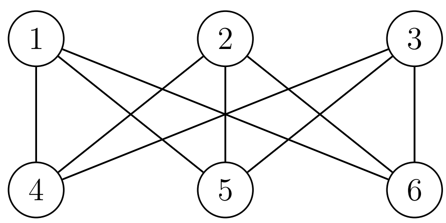
$$u \sim_G v \Longleftrightarrow \sigma(u) \sim_H \sigma(v).$$

If $G = H$, we call σ a *graph automorphism*. The set of all automorphisms of a finite graph without multiple edges G is a group, called the *automorphism group* $Aut(G)$. If G has n vertices and M_G is the adjacency matrix of G, we can view $Aut(G)$ as subgroup of S_n:

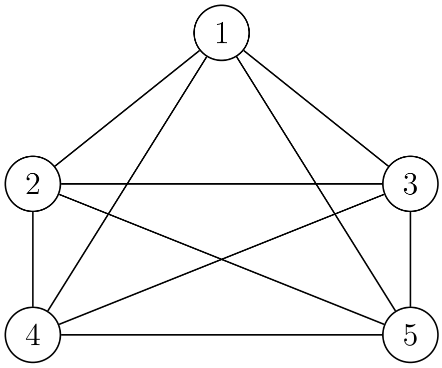
$$Aut(G) = \{\sigma \in S_n | \sigma M_G = M_G \sigma\} \subseteq S_n$$

1.1.3. **Definition.** A finite graph is *planar*, if it can be embedded into the Euclidian plane such that no two edges intersect.

1.1.4. **Example.** Two examples of nonplanar graphs are the complete bipartite graph on 6 vertices $K_{3,3}$:



and the complete graph on 5 vertices K_5:



In fact, as we will see with the next theorem, one might say that these are *the* two nonplanar graphs.

1.1.5. **Definition.** A graph G is said to be a *topological minor* of a graph H, if there exists a map $\Phi : G \to H$ consisting of $\Phi_V : V(G) \to V(H)$ and $\Phi_E : E(G) \to \{p \,|\, p \text{ is a path in } H\}$ such that:

- Φ_V is injective;
- for an edge (u, v) of G, $\Phi_E((u, v))$ is a simple path from $\Phi_V(u)$ to $\Phi_V(v)$;
- for two distinct edges (u_1, v_1) and (u_2, v_2) of G, the paths $\Phi_E((u_1, v_1))$ and $\Phi_E((u_2, v_2))$ do not share any vertices except potentially one of the endpoints $\Phi_V(u_1)$, $\Phi_V(v_1)$, $\Phi_V(u_2)$ or $\Phi_V(v_2)$.

1.1.6. **Theorem** (Kuratowski)**.** *A graph is nonplanar if and only if it contains $K_{3,3}$ or K_5 as a topological minor.*

1.1.7. **Definition.** For a graph $G = (V, E)$, we call a set $S \subseteq V$ *independent*, if no two vertices in S are connected by an edge. The size of the largest independent set in G is denoted by $\alpha(G)$.

1.2. **C^*-Algebras and Magic Unitaries.** We first define Hilbert spaces, which are the underlying spaces for our discussions in this thesis.

1.2.1. **Definition.** An *inner product* on a complex vector space H is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$, that satisfies for all $x, y, z \in H$, $\mu, \nu \in \mathbb{C}$:

(i) $\langle \mu x + \nu y, z \rangle = \mu \langle x, z \rangle + \nu \langle y, z \rangle$
(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$
(iii) $\langle x, x \rangle \geq 0$
(iv) $\langle x, x \rangle = 0 \Rightarrow x = 0$

A complex vector space together with such an inner product is called *pre-Hilbert space*. If the pre-Hilbert space is closed with respect to the norm induced by the inner product $\|x\| := \sqrt{\langle x, x \rangle}$, it is a *Hilbert space*.

Now we give an abstract definition of C^*-algebras.

1.2.2. **Definition.** (i) A $\mathbb{C}$-*algebra* A is a complex vector space together with a bilinear associative multiplication $\cdot : A \times A \to$ such that for all $x, y \in A$, $\lambda \in \mathbb{C}$ we have $\lambda(xy) = (\lambda x)y = x(\lambda y)$. The algebra is *unital* if it contains a multiplicative unit $1 \in A$, i. e. $1x = x1 = x$ for all $x \in A$.

(ii) A *normed algebra* A is a $\mathbb{C}$-algebra with a norm, that is *submultiplicative*, i. e. $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

(iii) A *Banach algebra* is a normed algebra that is complete.

(iv) An *involution* on an algebra A is an antilinear map $* : A \to A$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in A$.

(v) A C^*-*algebra* is a Banach algebra A with an involution, where the norm fulfills the C^*-*identity*: $\|x^*x\| = \|x\|^2$ for all $x \in A$.

1.2.3. **Example.** A trivial example for a C^*-algebra are the complex numbers $\mathbb{C}$. Here, the involution is given by complex conjugation and the norm is the absolute value.

Another class of examples for C^*-algebras are the bounded linear operators on a Hilbert space. Before we get to that example, let us define adjoint operators.

1.2.4. **Definition.** Let H_1, H_2 be Hilbert spaces and $T : H_1 \to H_2$ be a linear bounded operator. We denote by $T^* : H_2 \to H_1$ the *adjoint* of T which fulfills

$$\langle Tx, y\rangle_{H_2} = \langle x, T^*y\rangle_{H_1} \text{ for all } x \in H_1 \text{ and } y \in H_2.$$

It exists by the Riesz Representation Theorem and can be shown to be unique.

If $T = (t_{ij})$ is an operator between the two Hilbert spaces $H_1 = \mathbb{C}^{d_1}$ and $H_2 = \mathbb{C}^{d_2}$, the adjoint of T is of the form $T^* = (\overline{t_{ji}})$.

If $H_1 = H_2$ and $T = T^*$, we call T *selfadjoint*.

1.2.5. **Definition.** If $u \in \mathbb{C}^d$ is a complex vector of some dimension d, one can also consider u^*, the adjoint of u, as the linear functional

$$\begin{aligned} u^* : \mathbb{C}^d &\longrightarrow \mathbb{C} \\ v &\longmapsto \langle u, v\rangle. \end{aligned}$$

1.2.6. **Example.** Let H be a Hilbert space. We denote by

$$B(H) := \{T : H \to H \mid T \text{ is linear and bounded}\}$$

the space of all bounded linear operators on H. In fact, for linear operators, being bounded is the same as being continuous. The involution on $B(H)$ is given by the adjoint map $^* : B(H) \to B(H)$ from Definition 1.2.4. In the case where H is finite dimensional, i. e. $H = \mathbb{C}^d$ for some dimension d, the adjoint is just the conjugate transpose.

$B(H)$ can be equipped with a norm that satisfies the C^*-identity:

$$\|T\| := inf\,\{C > 0 \mid \|Tx\| \le Cx \text{ for all } x \in H\} = sup\,\{\|Tx\| \mid \|x\| = 1\}$$

One can see, that with this *operator norm*, $B(H)$ becomes a C^*-algebra. Actually, there is a deeper connection between algebras of bounded linear operators on Hilbert spaces and C^*-algebras, which has been shown by Gelfand and Naimark. The second Gelfand-Naimark Theorem states that any C^*-algebra A is isomorphic to a C^*-subalgebra of $B(H)$ for some Hilbert space H.

1.2.7. **Definition.** If A is a C^*-algebra and $p, v \in A$ we call p a *projection* if $p = p^* = p^2$ and we call v a *symmetry* if v is selfadjoint and $v^2 = 1$.

1.2.8. **Remark.** In C^*-algebras, projections and symmetries play an important role. A projection p and a symmetry v in a C^*-algebra A can be seen as generalized versions of $p \in \{0, 1\}$ and $v \in \{-1, 1\}$ if $A = \mathbb{C}$.

1.2.9. **Lemma.** *There is a bijection from the set of projections to the set of symmetries in a C^*-algebra given by*

$$p \mapsto 2p - 1.$$

Proof. If p is a projection, then $(2p-1)^* = 2p^* - 1^* = 2p - 1$ and $(2p-1)^2 = 4p - 4p + 1 = 1$ and thus $2p - 1$ is a symmetry. On the other hand, if v is a symmetry, we have $(\frac{1}{2}(v+1))^* = \frac{1}{2}(v^* + 1^*) = \frac{1}{2}(v+1)$ and $(\frac{1}{2}(v+1))^2 = \frac{1}{4}(v^2 + 2v + 1) = \frac{1}{4}(2v + 2 \cdot 1) = \frac{1}{2}(v+1)$ and therefore $\frac{1}{2}(v+1)$ is a projection. □

1.2.10. **Lemma.** *If A is a C^*-algebra with $A \subseteq B(H)$ for a finite dimensional Hilbert space H, i. e. A consists of complex matrices, the following statements hold:*

- *$p \in A$ is a projection if and only if p diagonalizes orthogonally with eigenvalues $\{0,1\}$.*
- *$v \in A$ is a symmetry if and only if v diagonalizes orthogonally with eigenvalues $\{-1,1\}$.*

Moreover, if v is a symmetry, then $p = \frac{1}{2}(v+1)$ is the projection on its eigenspace to the eigenvalue 1 and $1-p$ is the projection on its eigenspace to the eigenvalue -1.

Proof. If p is diagonalizable and can be written as $p = udu^{-1}$, where d is a diagonal matrix, it holds that

$$p = p^2 \iff udu^{-1} = udu^{-1}udu^{-1} = ud^2u^{-1} \iff d = d^2.$$

If p is now a projection, it is in particular selfadjoint and therefore diagonalizable. Since $p = p^2$ holds, we thus have $d = d^2$. As d is a diagonal matrix, it must hold that all its entries are in $\{0,1\}$ and thus the eigenvalues of p are also from $\{0,1\}$.

On the other hand, if $p = udu^{-1}$ is diagonalizable with eigenvalues $\{0,1\}$ then p is selfadjoint as it is diagonalizable. Moreover we have $d = d^2$, and therefore also $p = p^2$, which means that p is a projection.

If now v is a symmetry, then it is also selfadjoint and we can write again $v = udu^{-1}$. By $v^2 = 1$ we get that $v^2 = udu^{-1}udu^{-1} = ud^2u^{-1} = 1$ and thus $d^2 = 1$. Again, since d is diagonal, its entries have to be from $\{-1,1\}$ and thus so do the eigenvalues of v. If we have again a diagonalizable matrix $v = udu^{-1}$ with eigenvalues $\{-1,1\}$ then v is selfadjoint as above and $1 = d^2 = ud^2u^{-1} = udu^{-1}udu^{-1} = v^2$.

Lastly, if v is a symmetry we know that p is indeed a projection by Lemma 1.2.9 and therefore also $1-p = \frac{1}{2}(1-v)$ is a projection. Let now x be an eigenvector of v to the value 1: $x \in Eig_1(v)$. Then we have

$$x = \frac{1}{2}2x = \frac{1}{2}(1+v)x = px$$

and thus $x \in pH$. On the other hand if we have $x \in H$ and consider

$$v(px) = \frac{1}{2}v(1+v)x = \frac{1}{2}(v+v^2)x = \frac{1}{2}(v+1)x = px$$

we see that $px \in Eig_1(v)$ and thus p is the projection onto $Eig_1(v)$ as desired.

If $x \in Eig_{-1}(v)$ then it holds that

$$x = \frac{1}{2}2x = \frac{1}{2}(1-v)x = (1-p)x$$

and thus $x \in (1-p)H$. If $x \in H$, then

$$v((1-p)x) = \frac{1}{2}v(1-v)x = \frac{1}{2}(v-v^2)x = \frac{1}{2}(v-1)x = -\frac{1}{2}(1-v)x = -(1-p)x$$

and therefore $(1-p)x \in Eig_{-1}(v)$. □

We saw in Example 1.2.6 that we can get C^*-algebras by considering operators on Hilbert spaces. Sometimes it is however useful to stay more abstract. We will now introduce *universal C^*-algebras* as a way to define C^*-algebras that fulfill certain relations one wants to study.

1.2.11. **Definition.** Let $E = \{x_i \mid i \in I\}$ be a set of generators, where I is some index set, and denote by $P(E)$ the *free *-algebra* on the generators set E.

Given some set of polynomials $R \subseteq P(E)$, we denote by $J(R)$ the two sided *-ideal generated by R. The *universal *-algebra* on *generators* E with *relations* R is then defined as

$$A(E \mid R) := P(E)/J(R).$$

If we want a C^*-algebra, we also need a norm, so we define

$$\|x\| := sup\,\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E \mid R)\}$$

If $\|x\| < \infty$ it is easy to see, that $\|\cdot\|$ is also a C^*-seminorm and that

$$\{x \in A(E \mid R) \mid \|x\| = 0\}$$

is a two sided ideal in $A(E \mid R)$. We then define the *universal C^*-algebra* on *generators* E with *relations* R as

$$C^*(E \mid R) := \overline{A(E \mid R)/\{x \in A(E \mid R) \mid \|x\| = 0\}}^{\|\cdot\|}.$$

Note however, that $\|x\| = \infty$ may sometimes happen and in that case we cannot define $C^*(E \mid R)$.

With universal C^*-algebras, we can define abstract analogues of classical structures. In order to present an example, we will first define compact matrix quantum groups, which were first introduced by Woronowicz [15, 16] in 1987.

1.2.12. **Definition.** A *compact matrix quantum group* G^+ is a unital, potentially non-commutative C^*-algebra $C(G^+)$ equipped with a $*$-homomorphism $\Delta : C(G^+) \to C(G^+) \otimes C(G^+)$ and a unitary $u \in M_n(C(G^+))$, $n \in \mathbb{N}$, such that

(i) $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j,
(ii) $\bar{u}$ is an invertible matrix,
(iii) the elements u_{ij} $(1 \leq i, j \leq n)$ generate $C(G^+)$ (as a C^*-algebra).

The unitary u is called the *fundamental corepresentation (matrix)* of $(C(G^+), \Delta, u)$. Since (i) and (iii) uniquely determine Δ, one can also refer to the pair $(C(G^+), u)$ as a compact matrix quantum group.
If $G^+ = (C(G^+), u)$ and $H^+ = (C(H^+), v)$ are compact matrix quantum groups with $u \in M_n(C(G^+))$ and $v \in M_n(C(H^+))$, we say that G^+ is a *compact matrix quantum subgroup* of H^+, if there is a surjective *-homomorphism from $C(H^+)$ to $C(G^+)$ mapping generators to generators. We then write $G^+ \subseteq H^+$. If we have $G^+ \subseteq H^+$ and $H^+ \subseteq G^+$, they are said to be equal as compact matrix quantum groups.

1.2.13. **Example.** Let us now take a look at the classical automorphism group of a graph G from Definition 1.1.2. One can show, that the algebra of continuous functions from $Aut(G)$ to $\mathbb{C}$ can be written as a universal C^*-algebra as follows:

$$C(Aut(G)) = C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1,\ u_{ij}u_{kl} = u_{kl}u_{ij} \\ \text{for all } \ i, j, k, l \in [n]\,,\ uM_G = M_G u)$$

where M_G is the adjacency matrix of G and u is the matrix of all the u_{ij} and by $[n]$ we denote the set $\{1, \ldots, n\} \subseteq \mathbb{N}$.

It gets interesting, when we drop the commutation relation and define the so-called *quantum automorphism group* of G as the compact matrix quantum group defined by

$$C(Aut(G)^+) = C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1 \text{ for all } i,j \in [n]\,,\, uM_G = M_G u)$$

because one can show that this does not always coincide with the classical automorphism group. This definition was first introduced by Banica [3] in 2005. See e. g. [6] or [13] for more discussion on the subject of quantum automorphism groups of graphs.

In the above example, one might note the similarity between requiring that a permutation matrix commutes with the adjacency matrix of a graph for the classical automorphism group and the commutation relation $uM_G = M_G u$ in the definition for the quantum automorphism group. In fact, these matrices of operators that fulfill special conditions can be seen as a generalisation of permutation matrices, which we call magic unitaries.

1.2.14. **Definition.** A matrix $u = (u_{ij})_{i,j\in[n]}$ with entries from a unital C^*-algebra is a *magic unitary*, if it fulfills that all u_{ij} are projections, and that $\sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1$.

1.2.15. **Lemma.** *Any magic unitary u as in Definition 1.2.14 will be unitary, i. e. $uu^* = u^*u = 1$.*

Proof. This can easily be seen by looking at the entries of uu^*:

$$(uu^*)_{ij} = \sum_{k=1}^n u_{ik}(u^*)_{kj} = \sum_{k=1}^n u_{ik}u_{jk} = \begin{cases} \sum_{k=1}^n u_{ik} = 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

where $u_{ik}u_{jk} = 0$ for $i \neq j$ since u_{ik} and u_{jk} are projections that sum up to 1 and such projections are always pairwise orthogonal. Seeing that $u^*u = 1$ holds works similarly. □

If the C^*-algebra from which the entries of the magic unitary are coming is $\mathbb{C}$, the magic unitary is nothing else than a permutation matrix. Indeed, the only projections in $\mathbb{C}$ are 0 and 1 and in order for each row and each column to sum up to 1, there has to be exactly one 1 in each row and column and 0 everywhere else.

1.3. **Quantum Measurements.** We will give a short introduction into quantum mechanics here with the goal of defining what a quantum measurement is. This will be needed to define quantum strategies for nonlocal games. The definitions are mainly taken from the preliminary section of [2] and from [11] and in the latter you can also find a more detailed introduction into the field.

1.3.1. **Definition.** The postulates of quantum mechanics state that to any isolated physical system, we can associate a Hilbert space H which is called the *state space* of the system. We will refer to systems described like this as *quantum systems*. In this thesis we will be considering only finite dimensional state spaces $H = \mathbb{C}^d$.

The *state* of the system is described by a positive semidefinite matrix with trace 1. A special case of this notion are so-called *pure states*, which can be described by a unit vector $\psi \in H$, where ψ is called the *state vector*. Pure states are sufficient for our purposes in the setting of quantum strategies for games, which is why we restrict to those.

1.3.2. **Remark.** Note that while in most of the mathematical literature, the orthonormal basis of a Hilbert space is written as $\{e_i\}$, in most of the physics literature it is written as $\{|i\rangle\}$ in the so-called bra-ket notation. In this thesis, we will follow the mathematical convention and write it as $\{e_i\}$.

1.3.3. **Definition.** In order to gain information about a quantum system, we can apply *quantum measurements*. A quantum measurement is a collection $\{M_m\}$ of *measurement operators* that act on the state space of the system. Here the index m of the operators refers to the possible outcomes of the measurements. If the system is in state ψ immediately before the measurement, then the probability that the result m occurs is given by

$$p(m) = \langle \psi, M_m^* M_m \psi \rangle$$

which can be written as $p(m) = \psi^* M_m^* M_m \psi$ if the underlying Hilbert space is $\mathbb{C}^d$ for some dimension d. Note that $p(m) = \langle M_m \psi, M_m \psi \rangle \geq 0$ since $\langle \cdot, \cdot \rangle$ is an inner product. If the outcome m does occur, then in particular $p(m) > 0$, and the state of the system after the measurement is

$$\frac{M_m \psi}{\sqrt{\langle \psi, M_m^* M_m \psi \rangle}} = \frac{M_m \psi}{\sqrt{p(m)}}.$$

The measurement operators have to satisfy the *completeness equation*

$$\sum_m M_m^* M_m = 1$$

which implies that the probabilities for the outcomes sum up to one:

$$\sum_m p(m) = 1.$$

1.3.4. **Lemma** (Exercise 2.57 in [11]). *Let two sets of measurement operators $\{L_l\}$ and $\{M_m\}$ on the finite dimensional Hilbert space $\mathbb{C}^d$ be given and let $\psi \in \mathbb{C}^d$ be a state vector. Then the measurement defined by first measuring $\{L_l\}$ on ψ and afterwards measuring $\{M_m\}$ on the resulting state is equivalent to a single measurement defined by measurement operators $\{N_{lm}\}$ where $N_{lm} = M_m L_l$.*

Proof. When measuring the operators $\{L_l\}$ on ψ, the probability of receiving the result l_0 is

$$p(l_0) = \psi^* L_{l_0}^* L_{l_0} \psi.$$

If the result of the measurement is l_0, then $p(l_0) > 0$ and the state after the measurement is

$$\psi_1 = \frac{L_{l_0} \psi}{\sqrt{p(l_0)}}.$$

If we now measure the operators $\{M_m\}$ on ψ_1, the probability of getting m_0 as the result is

$$p(m_0) = \psi_1^* M_{m_0}^* M_{m_0} \psi_1 = \frac{(L_{l_0} \psi)^*}{\sqrt{p(l_0)}} M_{m_0}^* M_{m_0} \frac{L_{l_0} \psi}{\sqrt{p(l_0)}} = \frac{\psi^* L_{l_0}^* M_{m_0}^* M_{m_0} L_{l_0} \psi}{p(l_0)}$$

from which it follows that

$$p(l_0) p(m_0) = \psi^* L_{l_0}^* M_{m_0}^* M_{m_0} L_{l_0} \psi.$$

The state after performing the second measurement with result m_0 is

$$\psi_2 = \frac{M_{m_0} \psi_1}{\sqrt{p(m_0)}} = \frac{M_{m_0} L_{l_0} \psi}{\sqrt{p(l_0) p(m_0)}} = \frac{M_{m_0} L_{l_0} \psi}{\sqrt{\psi^* L_{l_0}^* M_{m_0}^* M_{m_0} L_{l_0} \psi}}.$$

On the other hand, considering the measurement of $\{N_{lm}\}$ on ψ, we see that the probability of getting the result l_0m_0 is

$$p(l_0m_0) = \psi^* N^*_{l_0m_0} N_{l_0m_0} \psi = \psi^* L^*_{l_0} M^*_{m_0} M_{m_0} L_{l_0} \psi$$

and if the result l_0m_0 occurs, the state after the measurement is

$$\psi_3 = \frac{N_{l_0m_0}}{\sqrt{p(l_0m_0)}} = \frac{M_{m_0} L_{l_0} \psi}{\sqrt{\psi^* L^*_{l_0} M^*_{m_0} M_{m_0} L_{l_0} \psi}}.$$

We thus see that ψ_2 and ψ_3 are equal. Moreover, the probability $p(l_0m_0)$ of receiving the result l_0m_0 when measuring $\{N_{l_0m_0}\}$ is the same as the probability $p(l_0)p(m_0)$ of first measuring $\{L_l\}$ and then measuring $\{M_m\}$ with results l_0 and m_0 respectively. Therefore, the two measurements are equivalent. □

We will be using a special case of quantum measurements, namely so-called projective measurements.

1.3.5. **Definition.** A *projective measurement* is described by an *observable* M, which is a selfadjoint operator on the state space being observed. This observable has a spectral decomposition

$$M = \sum_{m \in sp(M)} m P_m$$

where P_m is the projection on the eigenspace of M belonging to the eigenvalue m. Here, the possible outcomes of the measurement correspond to the eigenvalues m of the observable M. This is a special case of Definition 1.3.3, since here the $\{P_m\}$ fulfill the role of the collection of measurement operators. This implies that the probability of getting the result m after measuring the state ψ is given by

$$p(m) = \langle \psi, P_m \psi \rangle$$

and the state of the system after measuring the outcome m is given by

$$\frac{P_m \psi}{\sqrt{p(m)}}.$$

Another way to look at a quantum measurement is via a *Positive Operator-Valued Measure* (POVM).

1.3.6. **Definition.** A *POVM* $\mathcal{M}$ is a family of selfadjoint, positive semidefinite matrices $\{M_i : i \in [n]\}$ such that $\sum_{i=1}^n M_i = 1$. The probability of receiving the result m when measuring a POVM is given by $p(m) = \langle \psi, M_m \psi \rangle$ where ψ is again the current state vector.

1.3.7. **Lemma.** *Given a collection of measurement operators $\{M_m\}$ as in Definition 1.3.3, the corresponding POVM is $\{M^*_m M_m\}$. Moreover, if all the matrices in a POVM are projections, then it corresponds to a projective measurement.*

Proof. It is easy to see, that the $\{M^*_m M_m\}$ form a POVM, as the condition for being a POVM, namely $\sum_m M^*_m M_m = 1$ is exactly the completeness equation, that the operators M_m have to satisfy in order to be a valid measurement. Furthermore, also the probability to receive the result m matches between the two definitions.

If we now have a POVM $\{P_m\}$ that consists only of projections, we see that $\{P_m\}$ is also a valid collection of measurement operators in the sense of Definition 1.3.3, as $\sum_m P^*_m P_m = \sum_m P_m = 1$ by the POVM condition and since all the P_m are projections. □

1.3.8. **Remark.** The difference between describing a quantum measurement using measurement operators and using a POVM is, that using the measurement operators, we have a description of the state of the system after the measurement. If we are not interested in this state, we can therefore work with POVMs, which we will do for the rest of this thesis.

In the POVM framework, one will sometimes say that the result of measuring the POVM $\{P_m\}$ on the state ψ is the operator P_{m_0}, or also that one "measures P_{m_0}". This means that the result of the measurement is m_0 and serves as a reminder of the operator determining the probability of m_0 occuring as the result of the measurement:

$$p(m_0) = \langle psi, P_m \psi \rangle$$

1.3.9. **Definition.** If we have two quantum systems S_1 and S_2 with corresponding state spaces $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, then the state space of the *joint system* (S_1, S_2) is given by the tensor product $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. If S_1 is in state $\psi_1 \in \mathbb{C}^{d_1}$ and S_2 is in state $\psi_2 \in \mathbb{C}^{d_2}$ then the joint system (S_1, S_2) is in state $\psi_1 \otimes \psi_2$. However not every state in (S_1, S_2) can be written as a tensor product. The states that cannot be written as a tensor product are called *entangled* states. Given quantum measurements by the POVMs $\{M_i\}$ and $\{N_j\}$ on S_1 and S_2 respectively, the *product measurement* on the joint system (S_1, S_2) is given by $\{M_i \otimes N_j\}$ where the probability of getting the outcome (i, j) when measuring the state ψ is equal to $\langle \psi, M_i \otimes N_j \psi \rangle = \psi^*(M_i \otimes N_j)\psi$.

It is a useful fact, that any state $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$ admits a so-called *Schmidt decomposition* $\psi = \sum_{i=1}^d \lambda_i \alpha_i \otimes \beta_i$ where $\{\alpha_i\}$ and $\{\beta_i\}$ are orthonormal bases of $\mathbb{C}^d$ and $\lambda_i \geq 0$. One can also define the Schmidt decomposition for states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ where $d_1 \neq d_2$, but we do not need this case in this thesis. A state is said to have *full Schmidt rank* if $\lambda_i > 0$ for all i. Moreover a state is *maximally entangled* if all Schmidt coefficients are the same.

1.3.10. **Definition.** We now introduce two useful maps. First, the *vectorization* map $vec : \mathbb{C}^{d_1 \times d_2} \to \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, which is the linear extension of the map that maps the matrix uv^* to $u \otimes \overline{v}$, where $\overline{v}$ is the entrywise complex conjugate of v and $\otimes$ denotes the Kronecker product. In other words, the matrix $M = (m_{i,j})_{i=1,\ldots,d_1, j=1,\ldots,d_n}$ is mapped to the column vector, that consists of the stacked columns of M:

$$vec(M) = (m_{1,1}, \ldots, m_{d_1,1}, m_{1,2}, \ldots, m_{d_1,2}, \ldots, m_{1,d_2}, \ldots, m_{d_1,d_2})^T.$$

We denote the inverse of the vectorization map by $mat : \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \to \mathbb{C}^{d_1 \times d_2}$.

We will now list a few properties of the *vec* and *mat* maps from Definition 1.3.10.

1.3.11. **Lemma.** *First, we have that:*

$$vec(A)^* vec(B) = Tr(A^* B) \text{ for all } A, B.$$

Moreover it holds that $vec(AXB^T) = (A \otimes B)vec(X)$ for selfadjoint operators of appropriate sizes and from this and the above identity it follows for $\rho = mat(\psi)$:

$$\psi^*(A \otimes B)\psi = vec(\rho)^*(A \otimes B)vec(\rho) = vec(\rho)^* vec(A\rho B^T) = Tr(\rho^* A \rho B^T)$$

where again A and B are selfadjoint.

A useful statement when measuring the canonical maximally entangled state is the following.

1.3.12. **Lemma.** *Let $\psi_d := \frac{1}{\sqrt{d}} \sum_{i=1}^d e_i \otimes e_i \in \mathbb{C}^d \otimes \mathbb{C}^d$, where e_i is the i-th standard basis vector, be the so-called* canonical maximally entangled state. *It holds that*

$$\psi_d^*(A \otimes B)\psi_d = \frac{1}{d} Tr(AB^T)$$

for all A, $B \in \mathbb{C}^{d\times d}$ that are selfadjoint.

Proof. Using the *vec* and *mat* maps from Definition 1.3.10, the proof is fairly straightforward. Putting $\rho_d := mat(\psi_d)$, we note that $\rho_d = \frac{1}{\sqrt{d}}1$ and therefore, by the identity introduced in Lemma 1.3.11, we see

$$\psi_d^*(A \otimes B)\psi_d = Tr(\rho_d^* A \rho_d B^T) = Tr(\frac{1}{d}1A1B^T) = \frac{1}{d}Tr(AB^T).$$

□

1.4. **Nonlocal Games.** The definition for nonlocal games we use is the same as the one used in [2].

1.4.1. **Definition.** A two-party *nonlocal game* includes a verifier and two players, who are by convention called Alice and Bob, who devise a cooperative strategy. The game is defined by finite input sets X_A, X_B and finite output sets Y_A, Y_B, which are associated to Alice and Bob respectively, a Boolean predicate $V : X_A \times X_B \times Y_A \times Y_B \to \{0, 1\}$ and a distribution π on $X_A \times X_B$.

In the game, the verifier samples an input $(x_A, x_B) \in X_A \times X_B$ using the distribution π and sends x_A to Alice and x_B to Bob. The players then respond with y_A and y_B respectively. The game is said to be won if $V(x_A, x_B, y_A, y_B) = 1$.

In a nonlocal game, the players can devise a strategy beforehand, but cannot communicate after receiving the input. In our setting, only one round of the game is played, but we want to consider strategies that win with certainty, i. e. the probability of winning is equal to 1. Such strategies are called *winning* or *perfect strategies.*

1.4.2. **Definition.** In a *classical strategy*, the responses of Alice and Bob are either completely determined by their input or they can use some shared randomness.

What we will mainly consider in this thesis however are *quantum strategies* for games. A quantum strategy consists of a joint system and an entangled state ψ that is shared between Alice and Bob. Moreover, both Alice and Bob have a family of POVMs, $\{\mathcal{A}_x\}$ and $\{\mathcal{B}_x\}$ respectively, for every possible input x, that they each measure upon receiving x as input and then they respond based on their measurement. Commonly, we will write $\mathcal{A}_x = \{A_{xy}\}_{y\in Y_A}$ and $\mathcal{B}_x = \{B_{xy}\}_{y\in Y_B}$.

For any strategy, both quantum and classical, we denote by $p(y_A, y_B|x_A, x_B)$ the joint conditional probability of Alice and Bob responding with y_A and y_B on input x_A and x_B. Such a probability distribution is called a *correlation.*

1.4.3. **Lemma.** *Let S be a strategy for a nonlocal game as defined in Definition 1.4.1 and let p_S be the corresponding correlation. It is easy to see, that S is a perfect strategy if and only if $p_S(y_A, y_B|x_A, x_B) = 0$ whenever $\pi(x_A, x_B) > 0$ and $V(x_A, x_B, y_A, y_B) = 0$.*

1.4.4. **Lemma.** *It is a well-known fact, that if we have a quantum strategy given by a state $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$ and measurements $\mathcal{A}_x = \{A_{xy}\}_y$ for each $x \in X_A$ and $\mathcal{B}_x = \{B_{xy}\}_y$ for each $x \in X_B$, we may assume, that ψ has full Schmidt rank.*

Proof. To see this, we assume that the Schmidt decomposition of ψ has $d' \leq d$ non-zero coefficients, i. e. it has a Schmidt decomposition of the form $\psi = \sum_{i=1}^{d'} \lambda_i \alpha_i \otimes \beta_i$, where $\{\alpha_i\}_i^d$ and $\{\beta_i\}_i^d$ are orthonormal bases of $\mathbb{C}^d$, and where $\lambda_i > 0$ for all $i \in [d']$. Here, we sorted the bases such that the non-zero Schmidt coefficients are associated to the first d' basis elements, and the coefficients that are 0 come later. We now show briefly, that we can "move" the entire strategy to $\mathbb{C}^{d'} \otimes \mathbb{C}^{d'}$, where ψ will have full Schmidt rank. For this, consider the isometries $U_A := \sum_{i=1}^{d'} e_i \alpha_i^*$ and $U_B := \sum_{i=1}^{d'} e_i \beta_i^*$, where $\{e_i\}_i^{d'}$ is the canonical orthonormal basis of $\mathbb{C}^{d'}$, and we put

$$\tilde{\psi} := (U_A \otimes U_B)\,\psi, \quad \tilde{A}_{xy} := U_A A_{xy} U_A^* \text{ for all } x, y \text{ and } \tilde{B}_{xy} := U_B B_{xy} U_B^* \text{ for all } x, y.$$

One can show, that the $\left\{\tilde{A}_{xy}\right\}_y$ and the $\left\{\tilde{B}_{xy}\right\}_y$ still form valid quantum measurements and that $\tilde{\psi} = \sum_{i=1}^{d'} \lambda_i e_i \otimes e_i$ is a valid quantum state with full Schmidt rank. Moreover, the quantum strategy given by $\tilde{\psi}$, $\left\{\tilde{A}_{xy}\right\}_y$ and $\left\{\tilde{B}_{xy}\right\}_y$ generates the same quantum correlation as the strategy given by ψ, $\{A_{xy}\}_y$ and $\{B_{xy}\}_y$. □

Sometimes we will state that we are *working in the Schmidt basis of* ψ, by which we mean that the above transformation has been applied and ψ therefore has full Schmidt rank. In this case, it also holds that $\rho = mat(\psi)$ is a diagonal matrix with positive entries.

1.4.5. **Definition.** A nonlocal game is called *synchronous*, if the input sets for both players and the question posed to both players are the same, i. e. if $X = X_A = X_B$, $Y = Y_A = Y_B$ and $V(y_1, y_2, x, x) = 0$ for all $x \in X$ and $y_1 \neq y_2 \in Y$.

Synchronous games are interesting, since their perfect quantum strategies are of a special form, namely the one specified in the following lemma, which has been shown in multiple papers in this form or a similar one: [4, 9, 10, 12]. We present the version also chosen in [2].

1.4.6. **Lemma.** *Let* $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$, $\mathcal{A}_x = \{A_{xy} | y \in Y\}$, *and* $\mathcal{B}_x = \{B_{xy} | y \in Y\}$ *be a perfect quantum strategy for a synchronous game with input set* X *and output set* Y*. If* ψ *and all* A_{xy} *and* B_{xy} *are expressed in the Schmidt basis of* ψ*, and* $\rho = mat(\psi)$*, then*

(i) $A_{xy} = B_{xy}^T$ *for all* $x \in X$ *and* $y \in Y$*,*
(ii) $\mathcal{A}_x$ *and* $\mathcal{B}_x$ *are projective measurements for all* $x \in X$*, i. e. the operators* A_{xy} *and* B_{xy} *are projections for all* $x \in X$ *and* $y \in Y$*,*
(iii) $A_{xy}\rho = \rho A_{xy}$ *and* $B_{xy}\rho = \rho B_{xy}$ *for all* $x \in X$ *and* $y \in Y$*,*
(iv) $p(y, y'|x, x') = \psi^*(A_{xy} \otimes B_{x'y'})\psi = 0 \Longleftrightarrow A_{xy} A_{x'y'} = 0.$

2. Quantum Isomorphism of Graphs

There are two equivalent possibilities to characterize quantum isomorphisms of graphs, the first one using quantum strategies for the so-called graph isomorphism game, the second one being similar to the definition of quantum automorphism groups of graphs. In this chapter we will present both and show their equivalence.

2.1. **The Graph Isomorphism Game.**

In [2], a nonlocal game is introduced, that captures the notion of graph isomorphism when using classical strategies and therefore allows a natural extension of the notion of graph isomorphism, by considering quantum strategies.

2.1.1. **Definition.** Given two graphs, G and H, the (G, H)*-isomorphism game* is defined as follows: The verifier uniformly samples two vertices $x_A, x_B \in V(G) \cup V(H)$ and gives them to Alice and Bob respectively. Here, we assume that $V(G)$ and $V(H)$ are disjoint in order to make sure that Alice and Bob always know from which graph they received their vertex. Alice and Bob then respond with vertices $y_A, y_B \in V(G) \cup V(H)$. Alice and Bob win the game, if two conditions are met:

(1) Having received a vertex from one graph, each player must respond with a vertex from the other graph, i. e.

$$x_A \in V(G) \iff y_A \in V(H) \text{ and } x_B \in V(G) \iff y_B \in V(H)$$

(2) We denote by g_A the unique vertex from $\{x_A, y_A\}$ which is in $V(G)$ and similarly g_B, h_A and h_B. The second condition then is, that the relationship between g_A and g_B must be the same as the one between h_A and h_B:

$$rel(g_A, g_B) = rel(h_A, h_B)$$

where the relationship is either equality, adjacency or distinct non-adjacency. In other words, it must hold that

$$g_A = g_B \iff h_A = h_B$$

and if $g_A \neq g_B$, and therefore also $h_A \neq h_B$,

$$g_A \sim g_B \iff h_A \sim h_B$$

must hold.

Given a strategy S for the (G, H) isomorphism game, we will denote the corresponding correlation by p_S. Recall from Lemma 1.4.3 that S is a perfect strategy if and only if $p_S(y_A, y_B|x_A, x_B) = 0$ whenever y_A, y_B, x_A and x_B do not fulfill the winning conditions. In [2] the following proposition is shown, justifying the name "isomorphism game".

2.1.2. **Proposition.** *Given two graphs G and H, there exists a perfect classical strategy for the (G, H)-isomorphism game if and only if G and H are isomorphic.*

Proof. "⇒":Seeing that any isomorphism $\varphi : V_G \longrightarrow V_H$ of G and H yields a perfect classical strategy is straightforward, since φ induces a strategy as follows: given vertex $x \in V_G$, the player will return $\varphi(x)$, and given $x \in V_H$, the player will return $\varphi^{-1}(x)$. That condition (1) is fulfilled is obvious, and also that $g_A = g_B \iff h_A = h_B$ hold follows immediately from φ being a bijection. The rest of (2) follows from the fact that an isomorphism fulfills $u \sim v \iff \varphi(u) \sim \varphi(v)$: assume $g_A \sim g_B$ then we have that $\varphi(g_A) = h_A \sim h_B = \varphi(g_B)$ and similarly for $g_A \not\sim g_B$.

"⇐": Going from a perfect strategy to an isomorphism is a little bit more work. Let S be a perfect classical strategy for the (G, H)-isomorphism game. Since in a classical strategy, Alice and Bob can make use of some local randomness, the corresponding correlation is of the form $p_S = \sum_i \lambda_i p_i$, where the p_i are correlations corresponding to deterministic strategies, i. e. $p_i(y_A, y_B\,|x_A, x_B) \in \{0, 1\}$ for all $x_A, x_B, y_A, y_B \in V(G) \cup V(H)$, and the λ_i encode the randomness and satisfy $\lambda_i > 0$ and $\sum_i \lambda_i = 1$. However, since a strategy is only perfect if $p(y_A, y_B\,|x_A, x_B) = 0$ for x_A, x_B, y_A and y_B not fulfilling the winning conditions, every p_i needs to belong to a perfect strategy in order for S to be a perfect strategy. Therefore, we only need to show the statement for deterministic strategies, i. e. the strategy consists of two functions, $f_A, f_B : V(G) \cup V(H) \longrightarrow V(G) \cup V(H)$, that map inputs to outputs for Alice and Bob respectively. However, condition (2) from Definition 2.1.1 implies that $f_A(x) = f_B(x)$ for all $x \in V(G) \cup V(H)$, as Alice and Bob have to return the same output given the same input, and thus we have a single function $f = f_A = f_B$ defining the strategy. Moreover, by condition (1), the restrictions of f to $V(G)$ and $V(H)$ are of the form $f_{|V(G)} : V(G) \longrightarrow V(H)$ and $f_{|V(H)} : V(H) \longrightarrow V(G)$. Since condition (2) requires for $g, g' \in V(G)$ that $rel(g, g') = rel(f(g), f(g'))$ it follows that $g \sim g' \iff f(g) \sim f(g')$ and therefore $f_{|V(G)}$ is an isomorphism between G and a subgraph of H. However, by a similar argument, $f_{|V(H)}$ is an isomorphism between

H and a subgraph of G, which is only possible, if G and H are isomorphic and $f_{|V(G)}$ and $f_{|V(H)}$ are isomorphisms. In fact the two functions are inverse to each other: if Alice is sent the vertex $g^* \in V(G)$ and Bob is sent $f(g^*) := h^* \in V(H)$, it must hold that $rel(g^*, f(h^*)) = rel(f(g^*), h^*)$ since the strategy defined by f is perfect. But the relationship between $f(g^*)$ and h^* is equality, and therefore $g^* = f(h^*)$. □

Now that we know, that classical strategies correspond to classical isomorphisms of graphs, it is natural to ask, what kind of isomorphisms quantum strategies yield. We will see, that using quantum strategies, the isomorphism game can be won perfectly even for graphs that are not isomorphic, which leads to the following definition, which was first given in [2].

2.1.3. **Definition.** Two graphs G and H are called *quantum isomorphic*, if there exists a perfect quantum strategy for the (G, H)-isomorphism game.

2.1.4. **Remark.** In [2] there is another notion of quantum isomorphism which is called *quantum commuting isomorphism*, which is defined similar as quantum isomorphism, only requiring a so-called perfect quantum commuting strategy. Moreover, in [8], what we call quantum isomorphism is called *quantum tensor isomorphism* and they call two graphs quantum isomorphic if a perfect C^*-strategy as defined in [7] exists for the isomorphism game. In [8] they also show that their version of *quantum isomorphism* is the same as *quantum commuting isomorphism.* As we will see later in this section in Theorem 2.2.1, the quantum isomorphism from Definition 2.1.3 of two graphs G and H is equivalent to the existence of a magic unitary u with entries from a finite dimensional Hilbert space such that $M_G u = u M_H$ for the adjacency matrices M_G and M_H of G and H. As is shown in [8], the notion of *quantum commuting isomorphism* and therefore also their version of *quantum isomorphism* is equivalent to the existence of a magic unitary u with entries from $B(H)$ for any Hilbert space H such that the same property $M_G u = u M_H$ holds.

Since we will however only consider the version from Definition 2.1.3, we do not consider these other notions in more depth.

Recall from Definition 1.4.2 that a quantum strategy for the (G, H)-isomorphism game consists of a shared entangled state ψ and two families of POVMs

$$\{\mathcal{A}_x = \{A_{xy} | y \in V(G) \cup V(H)\} \mid x \in V(G) \cup V(H)\}$$

and

$$\{\mathcal{B}_x = \{B_{xy} | y \in V(G) \cup V(H)\} \mid x \in V(G) \cup V(H)\}.$$

Upon input x_A Alice measures $\mathcal{A}_{x_A}$, and the result of the measurement will yield some y_A, which will be her output. The same holds for Bob with input x_B and output y_B. By Definition 1.3.9, the probability of Alice and Bob responding with y_A and y_B when receiving input x_A and x_B is given by

$$p(y_A, y_B | x_A, x_B) = \psi^*(A_{x_A y_A} \otimes B_{x_B y_B})\psi.$$

Therefore, any perfect quantum strategy for the graph isomorphism game must fulfill that $\psi^*(A_{x_A y_A} \otimes B_{x_B y_B})\psi = 0$ whenever conditions (1) or (2) of Definition 2.1.1 fail.

Since any classical strategy can be seen as a quantum strategy with one dimensional observables, we see that any two isomorphic graphs are also quantum isomorphic. A natural question is to ask whether there are graphs that are quantum isomorphic but not isomorphic, and indeed there are, as was shown e. g. in [2]. The main result of this thesis will give an explicit construction for the quantum strategies for these examples from [2]. For now, we can state the following lemma.

2.1.5. **Lemma.** *If two graphs G and H are isomorphic, then they are also quantum isomorphic. However the other direction does not hold.*

2.2. **Quantum Graph Isomorphism via Magic Unitaries.** The motivation for the other characterization of quantum isomorphisms between graphs is closer to the classical case. Recall from Example 1.2.13 the quantum analogy of automorphisms of graphs: since the automorphism group of a graph G with adjacency matrix M_G can be written as $Aut(G) = \{\sigma \in S_n | \sigma M_G = M_G \sigma\}$, we defined the quantum automorphism group of G via

$$C(Aut(G)^+) = \left\{ u_{ij} \mid u_{ij} \text{ are projections, } \sum_{k=1}^{n} u_{ik} = \sum_{k=1}^{n} u_{kj} = 1\,,\ uM_G = M_G u \right\}.$$

Here, u is the magic unitary that defines the compact matrix quantum group $C(Aut(G)^+)$.

Now, we want to consider isomorphisms between two different graphs, rather than isomorphisms of a graph with itself. In the classical case, an isomorphism between two graphs (that necessarily have the same number of vertices) is just a permutation σ such that $M_G \sigma = \sigma M_H$, where M_G and M_H are the adjacency matrices of the graphs. Taking inspiration from the case of quantum automorphisms, one can therefore ask, what happens if we replace the permutation matrices by magic unitaries as defined in Definition 1.2.14. One can show, as has been done in [2], that this yields the notion of quantum isomorphism from Definition 2.1.3.

2.2.1. **Theorem.** *Given two graphs G and H with adjacency matrices M_G and M_H, the following holds: G and H are quantum isomorphic if and only if there exists a magic unitary u with entries that are operators on a finite dimensional Hilbert space, such that*

$$M_G u = u M_H.$$

One can convert between the two equivalent characterizations as follows: Given a perfect quantum strategy for the (G, H)-isomorphism game with shared entangled state ψ, POVMs $\mathcal{A}_x = \{A_{xy}\}_y$ for Alice and $\mathcal{B}_x = \{B_{xy}\}_y$ for Bob for each vertex x of G and H, the magic unitary u that fulfills $M_G u = u M_H$ is constructed as

$$u := (A_{gh})_{g \in V(G), h \in V(H)}.$$

On the other hand, given such a magic unitary $u = (A_{gh})_{g \in V(G), h \in V(H)}$, the perfect quantum strategy for the isomorphism game is of the form $\psi := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes e_i$, $A_{hg} := A_{gh}$ for $g \in V(G), h \in V(h)$ and $A_{g'g} = 0 = A_{h'h}$ for $g, g' \in V(G)$ and $h, h' \in V(H)$.

Before we prove the above theorem, we will introduce some lemmas, that will help with the proof. First, we note that the graph isomorphism game is synchronous. Indeed, both Alice and Bob can get vertices from either graph and respond with vertices from either graph. Moreover, if they receive the same vertex, they need to respond with the same vertex to win the game. Therefore, Lemma 1.4.6 about quantum strategies for synchronous games holds for the graph isomorphism game. For strategies for the graph isomorphism however, additional properties hold, as was shown in Theorem 5.3 in [2].

2.2.2. **Lemma.** *Let two graphs G and H and a perfect quantum strategy for the (G, H)-isomorphism game consisting of the shared entangled state $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$ and the POVMs $\mathcal{A}_x = \{A_{xy} \mid y \in V\}$ and $\mathcal{B}_x = \{B_{xy} \mid y \in V\}$ for all $x \in V :=$*

$V(G) \cup V(H)$ be given. If $\rho = mat(\psi)$ and if all A_{xy} and B_{xy} are expressed in the Schmidt basis of ψ then the following holds:

(i) $A_{xy} = B_{xy}^T$ *for all* $x, y \in V$
(ii) $\mathcal{A}_x$ *and* $\mathcal{B}_x$ *are projective measurements for all* $x \in V$, *i. e. the operators* A_{xy} *and* B_{xy} *are projections for all* $x, y \in V$
(iii) $A_{xy}\rho = \rho A_{xy}$ *and* $B_{xy}\rho = \rho B_{xy}$ *for all* $x, y \in Y$
(iv) $p(y, y'|x, x') = \psi^*(A_{xy} \otimes B_{x'y'})\psi = 0 \Leftrightarrow A_{xy}B_{x'y'} = 0$
(v) $A_{xy} = 0$ *if* $x, y \in V(G)$ *or* $x, y \in V(H)$
(vi) $A_{xy} = A_{yx}$ *for all* $x, y \in V$

With this lemma, we can state some properties of the operators from a perfect quantum strategy of the isomorphism game, as was also shown in [2].

2.2.3. **Lemma.** *Let G and H be two graphs. Then G and H are quantum isomorphic if and only if there exist projections A_{gh} for $g \in V(G)$ and $h \in V(H)$ such that the equations*

(i) $\sum_{h \in V(H)} A_{gh} = I$ *for all* $g \in V(G)$,
(ii) $\sum_{g \in V(G)} A_{gh} = I$ *for all* $h \in V(H)$,
(iii) $A_{gh}A_{g'h'} = 0$ *if* $rel(g, g') \neq rel(h, h')$.

Proof. We put $V := V(G) \cup V(H)$. With Lemma 2.2.2, it is relatively easy to see, that Alices operators A_{gh} from a perfect quantum strategy S fulfill properties (i) to (iii). First, by (v) from Lemma 2.2.2, it holds that $A_{gg'} = 0$ for g and g' coming from the same graph. Therefore, and since $\mathcal{A}_g = \{A_{gh} \,|h \in V\}$ is a POVM for $g \in V(G)$, we get

$$1 = \sum_{h \in V} A_{gh} = \sum_{h \in V(H)} A_{gh}$$

which is property (i). Moreover, we have by (vi) from Lemma 2.2.2 that $A_{gh} = A_{hg}$ for all g and h. We thus have $\sum_{g \in V(G)} A_{gh} = \sum_{g \in V(G)} A_{hg}$ and since $\mathcal{A}_h = \{A_{hg} \,|g \in V\}$ is a POVM and again $A_{hh'} = 0$ if $h, h' \in V(H)$ we get

$$1 = \sum_{g \in V} A_{hg} = \sum_{g \in V(G)} A_{hg} = \sum_{g \in V(G)} A_{gh}$$

which is property (ii). Lastly, let $g, g' \in V(G)$ and $h, h' \in V(H)$ such that $rel(g, g') \neq rel(h, h')$. Then $p_S(h, h' \,|g, g') = 0$, as S is a perfect strategy. But then, by (iv) from Lemma 2.2.2, $A_{gh}A_{g'h'} = 0$, i. e. (iii) holds. Lastly, all the A_{gh} are projections by (ii) of Lemma 2.2.2.

Let now operators A_{gh} that fulfill properties (i) to (iii) be given. We will build a perfect quantum strategy for the (G, H)-isomorphism game. For this, we put $A_{hg} := A_{gh}$ for $g \in V(G), h \in V(H)$ and $A_{g'g} = 0 = A_{h'h}$ for $g, g' \in V(G)$ and $h, h' \in V(H)$. Lastly put $B_{gh} := A_{gh}^T$ for all $g, h \in V$. Now we just have to show that the quantum strategy consisting of the canonical maximally entangled state as shared state $\psi = \frac{1}{\sqrt{d}} \sum_{i=1}^d e_i \otimes e_i$ and POVMs $\mathcal{A}_x = \{A_{xy} \,|\, y \in V\}$ for Alice and $\mathcal{B}_x = \{B_{xy} \,|\, y \in V\}$ for Bob, for all $x \in V$, is perfect. We know by Lemma 1.3.12 that

$$p(y, y' \,|\, x, x') = \psi^*(A_{xy} \otimes B_{x'y'})\psi = \frac{1}{d} Tr \left(A_{xy}B_{x'y'}^T\right) = \frac{1}{d} Tr \left(A_{xy}A_{x'y'}\right)$$

and by (iii) it follows from this, that $p(y, y' \,|\, x, x') = 0$ whenever condition (2) of Definition 2.1.1 from the isomorphism game is violated. Moreover, since $A_{gg'}$ and

$A_{hh'}$ are 0 whenever $g, g' \in V(G)$ or $h, h' \in V(H)$, condition (1) of Definition 2.1.1 will always be satisfied. Thus, the strategy is perfect.

$\square$

A similar version of the following lemma was shown Lemma 6.7 in [14], however there it was shown for the special case $G = H$.

2.2.4. **Lemma.** *Given two graphs G and H with adjacency matrices M_G and M_H and a magic unitary $u = (u_{gh})_{g\in V(G), h\in V(H)}$, the following are equivalent:*

(i) for all $g, g' \in V(G)$, $h, h' \in V(H)$: $u_{gh}u_{g'h'} = 0$ if $rel(g, g') \neq rel(h, h')$.

(ii) $M_G u = uM_H$,

Proof. $(i) \Rightarrow (ii)$: Looking at the (g, h)-entries of $M_G u$ and uM_H, we note that

$$(M_G u)_{gh} = \sum_{g':g'\sim g} u_{g'h} \text{ and } (uM_H)_{gh} = \sum_{h':h'\sim h} u_{gh'}$$

since the (g, g')-entry of M_G is equal to 1 if g and g' are adjacent and equal to 0 otherwise, and similarly for M_H. Since rows and columns of a magic unitary sum to one, we get by (i) that

$$\sum_{g':g'\sim g} u_{g'h} = \sum_{g'\sim g} u_{g'h} \sum_{h'} u_{gh'} = \sum_{g'\sim g} u_{g'h} \sum_{h'\sim h} u_{gh'} = \sum_{g'} u_{g'h} \sum_{h'\sim h} u_{gh'} = \sum_{h'\sim h} u_{gh'}$$

and therefore $M_G u = uM_H$ as desired.

$(ii) \Rightarrow (i)$: We want to show that u_{gh} and $u_{g'h'}$ are orthogonal, if g and g' do not have the same relation as h and h'. For this, note that projections summing up to one are always orthogonal, and in particular all projections that appear in the same row or column of u are orthogonal. Therefore, if one of $rel(g, g')$ and $rel(h, h')$ is "equality", but the other one is not, u_{gh} and $u_{g'h'}$ appear in the same row or column of u but are distinct and are therefore orthogonal. Next, we show orthogonality if one of the relations is "adjacency" and the other one is not. We see that $M_G u = uM_H$ implies

$$\sum_{g'\sim g} u_{g'h} = \sum_{h'\sim h} u_{gh'} \text{ for all } g \in V(G),\ h \in V(H)$$

by comparing $M_G u$ and uM_H entrywise as above. Next, since the projections u_{gh} are pairwise orthogonal for fixed h, we see that

$$\left(\sum_{g':g'\sim g} u_{g'h}\right)^2 = \sum_{g':g'\sim g} u_{g'h}$$

and putting this together with the entrywise comparison above, we get

$$\sum_{g'\sim g} u_{g'h} \sum_{h'\sim h} u_{gh'} = \left(\sum_{g'\sim g} u_{g'h}\right)^2 = \sum_{g'\sim g} u_{g'h} = \sum_{g'\sim g} u_{g'h} \sum_{h'\in V(H)} u_{gh'}.$$

From this it follows, that

$$\sum_{g'\sim g} \sum_{h'\not\sim h} u_{g'h} u_{gh'} = 0.$$

If we take traces in the above equation, we get that

$$\sum_{g'\sim g, h'\not\sim h} Tr\left(u_{g'h} u_{gh'}\right) = 0$$

and since all u_{gh} are projections and therefore positive semidefinite, it follows that $Tr\,(u_{g'h}u_{gh'}) = 0$ for all $g' \sim g$ and $h' \not\sim h$ which in turn implies that $u_{g'h}u_{gh'} = 0$, again, since they are positive semidefinite. In a similar manner, one can show, that $u_{g'h}u_{gh'} = 0$ if $g' \not\sim g$ and $h' \sim h$ hold. We thus have that if one of the relations is "adjacency" and the other one is not the operators are orthogonal, which we wanted to show. The only thing missing for (iii) to hold, is the case where one of the relations is "distinct non-adjacency" and the other one is not, however, in this case, the other relation will be one of "adjacency" or "equality" and we are again in one of the earlier cases. □

We will now go on to prove Theorem 2.2.1.

Proof (of Thm 2.2.1). Put again $V := V(G) \cup V(H)$. We will first show, how to get the magic unitary from the perfect quantum strategy. By Lemma 2.2.3, we get projections A_{gh} for $g \in V(G)$ and $h \in V(H)$ such that

(i) $\sum_{h \in V(H)} A_{gh} = I$ for all $g \in V(G)$
(ii) $\sum_{g \in V(G)} A_{gh} = I$ for all $h \in V(H)$
(iii) $A_{gh}A_{g'h'} = 0$ if $rel(g, g') \neq rel(h, h')$

hold. In order to construct the magic unitary u, we now identify each vertex in V with the index of the row respectively column that is associated to it in the corresponding adjacency matrix and we put

$$u := (A_{gh})_{g \in V(G), h \in V(H)},$$

that is each row of u has entries A_{gh} with g fixed and h running through $V(H)$ and similar for the columns. Then u is in fact a magic unitary, since its rows and columns sum up to one and the entries are projection matrices. Moreover, by Lemma 2.2.4, since u fulfills property (iii), we get that $M_G u = u M_H$ as desired.

For the other direction, we are given a magic unitary u, that fulfills $M_G u = u M_H$. Again, we identify the rows and columns of u with vertices from G and H and therefore write $u = (A_{gh})_{g \in V(G), h \in V(H)}$. We get by Lemma 2.2.4, that for all $g, g' \in V(G)$ and $h, h' \in V(H)$ the orthogonality property $A_{gh}A_{g'h'} = 0$ if $rel(g, g') \neq rel(h, h')$ holds. Since u is a magic unitary, its rows and columns sum up to one, and therefore we can use Lemma 2.2.3 to get that G and H are quantum isomorphic. □

3. Arrangements

Recall from Lemma 2.1.5 that all isomorphic graphs are quantum isomorphic, but not all quantum isomorphic graphs are isomorphic. On the way to constructing examples of graphs that are quantum isomorphic but not classically isomorphic, arrangements, and especially statements about arrangements shown by Arkhipov in [1], play an important role. In this section, we will state the basic definitions and the relevant facts from [1] about arrangements.

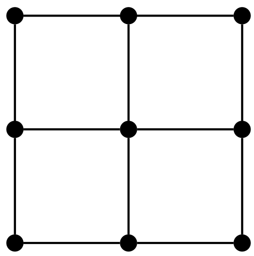
3.1. Basic Definitions.

3.1.1. **Definition.** A finite *hypergraph* $G = (V, E)$ consists of a finite set of vertices V and a finite set of *hyperedges* E, where each hyperedge $e \in E$ is a nonempty subset of V: $e \subseteq V$. A hypergraph is thus a generalization of a graph as each hyperedge might connect more than just two vertices.

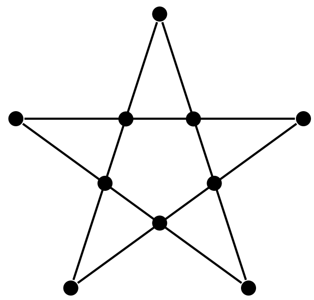
We call G a *connected* hypergraph if for every pair of vertices $u, v \in V$ there are edges $\{e_1, \ldots, e_n\}$ with $n \geq 1$ and $e_i \cap e_{i+1}$ being nonempty for $i = 1, \ldots, n-1$ such that $u \in e_1$ and $v \in e_n$.

3.1.2. **Definition.** An *(unsigned) arrangement* $A = (V, E)$ is a finite connected hypergraph with vertex set V and hyperedge set E, where each vertex lies in exactly two hyperedges. A *signed arrangement* $A = (V, E, l)$ is an arrangement with a labelling $l : E \to \{+1, -1\}$ that assigns each hyperedge in E a sign of $+1$ or -1.

3.1.3. **Example.** Two important examples of arrangements are the magic square and the magic pentagram as shown below. We will later see that in a way they form the basis of the construction of the main result of this thesis.



The magic square



The magic pentagram

3.1.4. **Definition.** A *classical realization* of a signed arrangement $A = (V, E, l)$ is a labelling $c : V \to \{+1, -1\}$ of the vertices, such that the product of the labels of all vertices that are in one hyperedge matches the label of that hyperedge:

$$\prod_{v \in e} c(v) = l(e) \text{ for each } e \in E$$

3.1.5. **Definition.** A *quantum realization* of a signed arrangement $A = (V, E, l)$ is a labelling $c : V \to B(H)$ of the vertices with operators on a fixed finite-dimensional Hilbert space H such that:

(i) The operator $c(v) \in B(H)$ assigned to a vertex $v \in V$ is a symmetry as defined in Definition 1.2.7, i. e. it is selfadjoint and squares to the identity $c(v)^2 = 1$, or equivalently, each operator diagonalizes orthogonally with eigenvalues $+1$ and -1.
(ii) For each hyperedge, the operators assigned to its vertices pairwise commute.
(iii) For each hyperedge, the product of the observables assigned to its vertices equals either the identity in $\mathcal{B}(H)$ or its negation, according to the label of that hyperedge:

$$\prod_{u \in e} c(u) = l(e)1 \text{ for each } e \in E$$

3.1.6. **Definition.** We call an arrangement *classically realizable* if it has a classical realization and *quantum realizable* if it has a quantum realization.

3.1.7. **Lemma.** *Any classically realizable arrangement is also quantum realizable.*

Proof. A classical realization is just a quantum realization where $H = \mathbb{C}$, as the symmetries in $\mathbb{C}$ are just $\{-1, 1\}$ and moreover in $\mathbb{C}$ all elements commute. ☐

3.2. **Magic Arrangements.** We next define parities of the labellings of edges, as it turns out that the realizability of a signed arrangement only depends on the labelling of the edges up to the parity of the labelling.

3.2.1. **Definition.** The *parity* $p(l)$ of a signed arrangement $A = (V, E, l)$ is

$$p(l) = \prod_{e \in E} l(e)$$

which is just the parity of the number of times -1 is a label: if there is an odd number of labels -1, $p(l)$ is -1, otherwise it is $+1$. We thus call l an *even parity labelling* if $p(l) = 1$ and an *odd parity labelling* if $p(l) = -1$.

The following two propositions and the corollary are Propositions 10 to 12 in [1].

3.2.2. **Proposition** (Prop. 12 in [1]). *The quantum realizability of a signed arrangement $A = (V, E, l)$ depends on the parity $p(l)$ of the labelling l only: if $A' = (V, E, l')$ is another arrangement with the same vertices and hyperedges but a different labelling that has the same parity, i. e. $p(l) = p(l')$, then A has the same quantum realizability as A'.*

Proof. Assuming that A has a quantum realization c, we will construct a corresponding realization c' for the new labelling l'.

For this, first look at the case, where l' flips the labels of two edges, a and b, in comparison with the labelling l:

- We choose a path of hyperedges connecting a and b, denoted by $a = e_0, e_1, \ldots, e_n = b$, such that each pair of hyperedges from this path e_i and e_{i+1} share exactly one vertex v_i for $i \in \{0, \ldots, n-1\}$. This path exists and is finite as A is a finite connected hypergraph.
- We define the label c' via

$$c'(v) := \begin{cases} c(v) \text{ if } v \notin \{v_0, \ldots, v_{n-1}\} \\ -c(v) \text{ if } v \in \{v_0, \ldots, v_{n-1}\} \,. \end{cases}$$

- Then c' is a quantum realization of A':
  - Each operator $c'(v)$ for $v \in V$ is still selfadjoint and squares to identity, as $c'(v) = c(v)$ for $v \in V \setminus \{v_0, \ldots, v_{n-1}\}$, which fulfills both properties by assumption, and for $v_i \in \{v_0, \ldots, v_{n-1}\}$ we have $c'(v_i) = -c(v_i) = -c(v_i)^* = c'(v_i)^*$ and $c'(v_i)^2 = (-c(v_i))^2 = c(v_1)^2 = 1$.
  - All operators assigned to vertices of the same hyperedge still commute, as the only change to the original labelling c is a scalar multiple.
  - For vertices lying in hyperedges $e \notin \{e_0, \ldots, e_n\}$, c' is the same as c, as none of the vertices $\{v_0, \ldots, v_{n-1}\}$ appear in e, since each vertex appears in exactly two hyperedges in an arrangement and it already holds that $v_i \in e_i$ and $v_i \in e_{i+1}$. But c was already a quantum realization and thus for $e \notin \{e_0, \ldots e_n\}$, the condition for being a quantum realization is fulfilled.
  - For $e_i \in \{e_1, \ldots, e_{n-1}\}$, we have that the label for all vertices in e_i except for v_{i-1} and v_i are the same as before:

    $$c'(v) = c(v) \text{ for } v \in e_i \setminus \{v_{i-1}, v_i\}$$

    and for v_{i-1} and v_i it is just the same but multiplied with -1. We thus have

    $$\prod_{v \in e_i} c'(v) = (-1) \cdot (-1) \cdot \prod_{v \in e_i} c(v) = \prod_{v \in e_i} c(v) = l(e_i)1 = l'(e_i)1$$

  - For $i \in \{0, n\}$, only one vertex from $\{v_0, \ldots v_{n-1}\}$ lies in e_i. In this case

    $$\prod_{v \in e_i} c'(v) = -\prod_{v \in e_i} c(v) = -l(e_i)1 = l'(e_i)1$$

    holds.

By repeatedly flipping the labels of two hyperedges, we can arrive at any labelling with the same parity, and therefore we can apply the above algorithm multiple times to arrive at the new realizations. □

3.2.3. **Corollary** (Prop. 10 in [1])**.** *The classical realizability of a signed arrangement* $A = (V, E, l)$ *depends on the parity* $p(l)$ *of the labelling* l *only.*

Proof. Since any classical realization is already a quantum realization with $H = \mathbb{C}$ by Lemma 3.1.7, this follows directly from Proposition 3.2.2. □

3.2.4. **Proposition** (Prop. 11 in [1])**.** *A signed arrangement* $A = (V, E, l)$ *is classically realizable if and only if* $p(l) = +1$*.*

Proof. For even-parity labellings, we note that the labelling that assigns +1 to each hyperedge is realizable with the vertex-labelling that also assigns +1 to each vertex. Thus, by Proposition 3.2.3, every even-parity labelling is realizable.

On the other hand, no odd-parity labelling is realizable, as every vertex lies in exactly two hyperedges, and therefore in the product of the vertex labels, each vertex label will appear twice. Put differently, if A is realizable, i. e. $\prod_{v \in e} c(v) = l(e)$, we have:

$$p(l) = \prod_{e \in E} l(e) = \prod_{e \in E} \prod_{v \in e} c(v) = \prod_{v \in V} c(v)^2 = 1$$

□

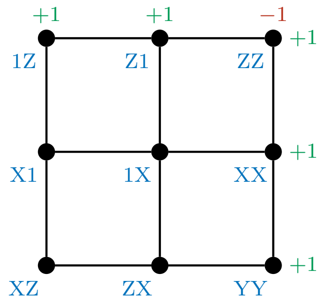
We cannot state a quantum analogue of Proposition 3.2.4, since in general the vertex labels do not have to commute and therefore it can happen that in $\prod_{e \in E} \prod_{v \in e} c(v)$ a term $c(v_1)c(v_2)c(v_1)$ appears that can not be simplified to $c(v_2)c(v_1)^2$. One can ask however, when is an arrangement quantum realizable but not classically realizable? In light of Proposition 3.2.3 and Proposition 3.2.2 one should think of this as a question about unsigned arrangements.

3.2.5. **Definition.** An unsigned arrangement is *magic*, if it has an odd-parity labeling that is quantum realizable. Moreover we call a signed arrangement magic, if its underlying arrangement is magic and the labelling has odd parity.

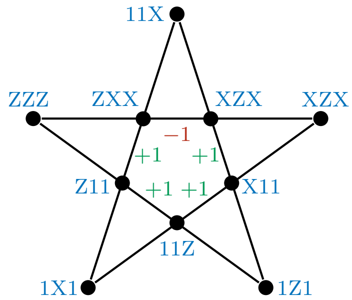
3.2.6. **Lemma.** *Let* $A = (V, E, l)$ *be a signed arrangement. If* A *is magic, then* A *is quantum realizable but not classically realizable.*

Proof. That A is quantum realizable follows from the assumption that it is magic. Since the labelling l has odd parity $p(l) = -1$, by Proposition 3.2.4 it is not classically realizable. □

3.2.7. **Example.** The magic square and the magic pentagram from Example 3.1.3 are magic arrangements. Below, they are shown again, this time with an odd parity labelling and a quantum realization. Here, X, Y and Z are the Pauli matrices, 1 is the identity matrix and juxtaposition of two operators stands for the tensor product of the two matrices.



The magic square



The magic pentagram

3.3. **Classification of Magic Arrangements.** Magic arrangements can be completely classified by their intersection graphs.

3.3.1. **Definition.** The *intersection graph* of an arrangement $A = (V, E)$ is the undirected graph $G = (V', E')$, where $V' = E$ and there is an edge between $e_1, e_2 \in V'$ for each vertex in the intersection $e_1 \cap e_2$.

A labelling l of the hyperedges of A is transferred to a labelling of the vertices of G. Similarly, a vertex labelling c of A is transferred to an edge labelling of G. Thus also the notion of (quantum) realizations can be transferred in a natural way to the intersection graph.

We call an intersection graph *magic* if the corresponding arrangement is magic.

3.3.2. **Example.** The intersection graph of the magic square is the complete bipartite graph on 6 vertices $K_{3,3}$, and the intersection graph of the magic pentagram is the complete graph on 5 vertices K_5 as defined in Example 1.1.4.

3.3.3. **Theorem** (Theorem 21 in [1]). *An arrangement is magic if and only if its intersection graph is not planar.*

In [1], Arkhipov gives a proof for both directions of this theorem. However, since it is a rather long proof and we only need one direction for our main result, we will only prove one direction, namely that a nonplanar intersection graph implies that the arrangement is magic. First however, we show a useful lemma.

3.3.4. **Lemma.** *Let G and H be two intersection graphs of arrangements. If H is a topological minor of G, then H being magic implies that G is also magic.*

Proof. We give a construction for turning a quantum realization of H into one of G. So, let some odd-parity labelling for H be given and let $c : V(H) \to B(K)$ be a quantum realization of H on some Hilbert space K. We will use the map $\Phi : H \to G$ given by the topological minor inclusion to assign a labelling and a realization of G as follows:

- Label each vertex in G that is not in the image of Φ by $+1$ and label any vertex $\Phi(v)$ where $v \in V(H)$ with $l(v)$.
- For each edge in H, label each edge of the corresponding path in G with the same quantum operator. Label all the other edges in G with 1.

Now we need to show that this actually is a quantum realization:

- Since any operator assigned to an edge of G is either an operator coming from a quantum realization or 1, they are all selfadjoint and square to identity.
- Each vertex of G that corresponds to a vertex of H touches only edges labelled with the operators that came from H and copies of identity. Since

the operators coming from H have to commute and only 1 is added, they still commute and their product is still the same, which is the label of the vertex.

- Any vertex in G that lies in a path corresponding to an edge in H touches only the operator assigned to this edge in H twice and again possibly copies of identity. Therefore they commute again and they square to identity, which is the label of the vertex.
- Every other vertex in G only touches edges labelled with 1 and the label of the vertex is also 1.

We thus have a labelling of G with odd parity and a quantum realization of the corresponding arrangement. Therefore, G is magic. □

3.3.5. **Theorem.** *If the intersection graph of an arrangement is nonplanar, then the arrangement is magic.*

Proof. Since we know by the Theorem of Kuratowski (1.1.6) that any nonplanar graph contains either $K_{3,3}$ or K_5 as a topological minor and we moreover know by Example 3.3.2 that both of these graphs belong to magic arrangements, we see by Lemma 3.3.4 that any nonplanar intersection graph is magic. □

4. Binary Constraint Systems

We now look at binary constraint systems (BCS), the last step before we get to the main result. We will introduce binary constraint system games and see how to get perfect quantum strategies for certain BCSs coming from quantum realizations of magic arrangements.

4.1. Basic Definitions.

4.1.1. **Definition.** A *linear binary constraint system* (BCS) $\mathcal{F}$ consists of a family of variables $x_1, \ldots, x_n$ and constraints $C_1, \ldots, C_m$. Each constraint is of the form $C_l : \prod_{x \in S_l} x = b_l$ where $S_l \subseteq \{x_1, \ldots, x_n\}$ is the set of variables appearing in C_l and b_l is in $\{-1, +1\}$. We say that $\mathcal{F}$ is *satisfiable* if there is an assignment from $\{-1, +1\}$ to the variables x_i such that every constraint C_l is satisfied. We call such an assignment a *satisfying assignment*.

4.1.2. **Remark.** In the definition above, we used the multiplicative notation for binary constraint systems. The additive notation, where the constraints are of the form $C_l : \sum_{x \in S_l} x = b_l$ with $b_l \in \{0, 1\}$ and the assignments of values to the x_i are coming from $\mathbb{F}_2$, is also widely used. However the multiplicative notation is more natural in our setting, as we will see in the next definition. The notion of *linear* BCS comes from the fact that in the additive notation, each of the constraints is a linear equation.

4.1.3. **Lemma.** *Linear binary constraint systems generalize signed arrangements, since any signed arrangement is just a linear BCS where each variable appears in exactly two constraints.*

4.1.4. **Example.** An example for a BCS with 9 variables and 6 constraints is given by the equations

$$x_1 \cdot x_2 \cdot x_3 = 1 \qquad x_1 \cdot x_4 \cdot x_7 = 1$$
$$x_4 \cdot x_5 \cdot x_6 = 1 \qquad x_2 \cdot x_5 \cdot x_8 = 1$$
$$x_7 \cdot x_8 \cdot x_9 = 1 \qquad x_3 \cdot x_6 \cdot x_9 = -1$$

This BCS corresponds to the magic square introduced in Example 3.1.3.

4.1.5. **Definition.** Given a linear BCS $\mathcal{F}$ with variables $x_1, \dots, x_n$ and constraints $C_1, \dots, C_m$, we call $\mathcal{F}$ *quantum satisfiable* if there exists a *quantum satisfying assignment* which is an assignment of finite-dimensional operators $A_1, \dots, A_n$ to the variables $x_1, \dots, x_n$, such that:

(i) Every A_i is a symmetry in the sense of Definition 1.2.7, i. e. every A_i is selfadjoint and squares to the identity, $A_i^2 = 1$;
(ii) All observables that appear in the same constraint mutually commute;
(iii) The A_i satisfy the constraints in the sense that $\prod_{A_i : x_i \in S_l} A_i = b_l 1$ for each constraint C_l.

4.1.6. **Lemma.** *If the operators from a quantum satisfying assignment are one-dimensional, then the quantum satisfying assignment is the same as a classical satisfying assignment.*

We now want to define a BCS game which is similar to the one discussed in [5] by Richard Cleve and Rajat Mittal. However we will use a synchronous version of the game they investigated, as was introduced in [2], since we want to use facts that hold for strategies for synchronous games.

4.1.7. **Definition.** To a linear BCS $\mathcal{F}$, we associate the *BCS game*. In this game, the verifier gives Alice a constraint C_l and Bob a constraint C_k. They both answer with assignments of values to the variables appearing in their respective constraint. The winning conditions are that:

(i) Each player satisfies their own constraint with the value assignment they give.
(ii) For all variables in $S_l \cap S_k$ Alices and Bobs assignments of values match.

Note that this game is again synchronous, since both Alice and Bob can receive any constraint and they are both posed the same question. Moreover, if $C_l = C_k$ we have that $S_l = S_k$ and therefore Alice and Bob need to return the same partial assignment in order to win.

4.1.8. **Remark.** It is not difficult to see that the above game can be won classically with probability 1 if and only if the underlying BCS is satisfiable.

We will see that the game can be won perfectly using quantum strategies if and only if the BCS is quantum satisfiable.

4.2. **Perfect Quantum Strategies for the BCS Game.** We will now show how to get a perfect quantum strategy for the BCS game given a quantum satisfying assignment. However, we first need a little lemma.

4.2.1. **Lemma.** *Let A and B be finite dimensional selfadjoint operators on a Hilbert space H with eigenvalues $\{-1, +1\}$. If A and B commute, then so do the projections onto their eigenspaces.*

Proof. We know by Lemma 1.2.10 that both A and B are symmetries, as they are selfadjoint and have eigenvalues $\{-1, +1\}$. Moreover, again by Lemma 1.2.10, we know that the projection P_j^v on the eigenspace belonging to eigenvalue j of the symmetry v can be written as $P^v_j = \frac{1}{2}(v + j1)$. We thus have that

$$\begin{aligned} P^A_{j_1} P^B_{j_2} &= \frac{1}{2}(A + j_1 1)\frac{1}{2}(B + j_2 1) \\ &= \frac{1}{4}(AB + j_2 A + j_1 B + j_1 j_2 1) \\ &= \frac{1}{4}(BA + j_2 A + j_1 B + j_2 j_1 1) \\ &= \frac{1}{2}(B + j_2 1)\frac{1}{2}(A + j_1 1) \\ &= P^B_{j_2} P^A_{j_1}. \end{aligned}$$

□

The following construction is similar to the one presented by Cleve and Mittal in [5], but since the game is a bit different, it is adapted for the new game.

4.2.2. **Construction.** Given a linear BCS $\mathcal{F}$ with variables $x_1, \ldots, x_n$ and constraints $C_1, \ldots, C_m$ and a quantum satisfying assignment of $\mathcal{F}$ mapping a finite dimensional selfadjoint operator A_i on the Hilbert space $H = \mathbb{C}^d$ to each variable x_i, we construct a quantum strategy for the BCS game for $\mathcal{F}$ as follows:

- We define the shared entangled state as the canonical maximally entangled state $\psi := \frac{1}{\sqrt{d}} \sum_{i=1}^d e_i \otimes e_i$.
- Alice associates the observables A_i to each variable x_i while Bob associates their transpose A_i^T to x_i.
- On input s, Alice conducts the measurements described by the observables she associated to the variables appearing in C_s and similarly Bob conducts the measurements described by the observables he associated to the variables appearing in C_t given input t.

Recall from Definition 1.3.5 what measurements described by an observable M look like: The observable has a spectral decomposition, which we write as $M = \sum_m m P_m$. Here the m run over the eigenvalues of M and the P_m are the corresponding projections onto eigenspaces. The measurement described by M is then the same as the one described by the POVM $\{P_m \mid m \text{ is an eigenvalue of } M\}$. Note moreover, that since all observables appearing in the same constraint commute, so do the projections on their eigenspaces by Lemma 4.2.1 and therefore it does not matter in which order Alice and Bob perform their measurements.

4.2.3. **Proposition.** *Given a binary constraint system $\mathcal{F}$ and a quantum satisfying assignment for $\mathcal{F}$ the quantum strategy for the BCS game for $\mathcal{F}$ described in Construction 4.2.2 is perfect.*

Proof. We want to show that the quantum strategy given above is in fact a perfect strategy. Recall from Lemma 1.4.3, that for this it suffices to show that $p(f_A, f_B \,|s, t) = 0$ for partial assignments f_A and f_B that do not fulfill condition (i) or (ii) from the BCS game of Definition 4.1.7. We will first show that condition (i) is always fulfilled. Let us therefore take a closer look at the quantum strategy we defined above. As was mentioned above, the measurements of observables that Alice and Bob conduct can be seen as POVMs. The POVM belonging to Alices operator A_i is of the form

$$\left\{P^i_1, P^i_{-1} \mid P^i_j \text{ is the projection on the eigenspace of } A_i \text{ belonging to eigenvalue } j\right\}$$

since A_i comes from a quantum satisfying assignment and therefore has eigenvalues $\{+1, -1\}$. The same holds true for Bob, only considering A_i^T:

$$\left\{Q_1^i, Q_{-1}^i \mid Q_j^i \text{ is the projection on the eigenspace of } A_i^T \text{ belonging to eigenvalue } j\right\}$$

Since cascaded measurements can be seen as a single measurement, as was noted in Lemma 1.3.4, we see that upon receiving input s, Alice measures the POVM

$$\mathcal{A}_s = \left\{P_{j_1}^{i_1} \cdots P_{j_k}^{i_k} \mid x_{i_1}, \ldots x_{i_k} \in S_s\right\}$$

while Bob measures

$$\mathcal{B}_t = \left\{Q_{j_1}^{i_1} \cdots Q_{j_l}^{i_l} \mid x_{i_1}, \ldots x_{i_l} \in S_t\right\}$$

upon input t, where the order of the projections does not matter, since they commute by Lemma 4.2.1. If Alice now receives the result $(i_1, j_1), \ldots, (i_k, j_k)$ upon measuring her POVM $\mathcal{A}_s$, where the (i_k, j_k) come from the projections $P_{j_k}^{i_k}$, the partial assignment she returns will be $f_A : S_s \to \{-1, +1\}$ defined by $f_A(x_{i_l}) = j_l$ and similar for Bob.

Because of this, we can already see, that Alice and Bob will always return a partial assignment that assigns values to the correct variables. Next, we want to show that these assignments do in fact satisfy the corresponding constraint. We will show it for Alice, because in Bob's case it is very similar. For this, let the input for Alice be s and let $P_{j_1'}^{i_1}, \ldots, P_{j_k'}^{i_k}$ be the projections associated to Alice's result $(i_1, j_1'), \ldots, (i_k, j_k')$ of the measurement of $\mathcal{A}_s$. We have that

$$\prod_{l=1}^{k} A_{i_l} = b_s 1$$

since the A_i come from a quantum satisfying assignment of $\mathcal{F}$. As P_j^i is the projection on the eigenraum to the eigenvalue j of the operator A_i and as all A_i have eigenvalues $\{1, -1\}$ and are selfadjoint, each A_i can be written as

$$A_i = P_1^i - P_{-1}^i.$$

We thus can rewrite

$$\prod_{l=1}^{k} A_{i_l} = \prod_{l=1}^{k} (P_1^{i_l} - P_{-1}^{i_l})$$

and by expanding we get

$$\prod_{l=1}^{k} (P_1^{i_l} - P_{-1}^{i_l}) = \sum_{\overline{j} \in \{1,-1\}^k} sgn(\overline{j}) \prod_{l=1}^{k} P_{j_l}^{i_l} = b_s 1$$

for multiindices $\overline{j} = (j_1, \ldots, j_k) \in \{1, -1\}^k$ where we put $sgn(\overline{j}) := \prod_{l=1}^{k} j_l$. By multiplying both sides of the above equation with $\prod_{l=1}^{k} P_{j_l'}^{i_l}$ we get

$$\sum_{\overline{j} \in \{1,-1\}^k} sgn(\overline{j}) \prod_{l=1}^{k} P_{j_l}^{i_l} \prod_{l=1}^{k} P_{j_l'}^{i_l} = b_s \prod_{l=1}^{k} P_{j_l'}^{i_l}.$$

We know by Lemma 1.2.10 that $P_{-1}^i = 1 - P_1^i$ and thus P_1^i and P_{-1}^i are orthogonal for the same i. Since however all of the projections appearing in the equation commute by Lemma 4.2.1 and since for every $\overline{j} \neq (j'_1, \ldots, j'_k) = \overline{j}'$ there is an i such that both P^i_1 and P^i_{-1} appear in $\prod_{l=1}^k P^{i_l}_{j_l} \prod_{l=1}^k P^{i_l}_{j'_l}$, the above equation can be transformed into

$$sgn(\overline{j}') \prod_{l=1}^{k} P^{i_l}_{j'_l} = b_s \prod_{l=1}^{k} P^{i_l}_{j'_l}.$$

If now $sgn(\overline{j}') \neq b_s$ we get that $\prod_{l=1}^k P^{i_l}_{j'_l} = -\prod_{l=1}^k P^{i_l}_{j'_l}$ and thus the product has to be 0. Therefore the probability for Alice to receive $(i_1, j'_1), \ldots, (i_k, j'_k)$ as a result of her measurement is also 0. If on the other hand $sgn(\overline{j}') = b_s$ then the corresponding partial assignment f_A will fulfill the constraint s, since

$$\prod_{l=1}^{k} f_A(x_{i_l}) = \prod_{l=1}^{k} j'_l = sgn(\overline{j}') = b_s.$$

Therefore, condition (i) of Definition 4.1.7 is fulfilled.

We now take a closer look at condition (ii) of Definition 4.1.7. Let f_A and f_B be two inconsistent partial assignments, assigning values to S_s and S_t respectively. Being inconsistent means, that there is a variable $x_i \in S_s \cap S_t$ such that $f_A(x_i) \neq f_B(x_i)$. In order to return f_A and f_B, Alice and Bob would have to measure $P^{i_1}_{j_1} \cdots P^{i_k}_{j_k} P^i_{f_A(x_i)}$ and $Q^{i_1}_{j_1} \cdots Q^{i_l}_{j_l} Q^i_{f_B(x_i)}$ respectively. However, we have that

$$\begin{aligned} p(f_A, f_B \mid s, t) &= \psi^*(P^{i_1}_{j_1} \cdots P^{i_k}_{j_k} P^i_{f_A(x_i)} \otimes Q^{i_1}_{j_1} \cdots Q^{i_l}_{j_l} Q^i_{f_B(x_i)})\psi \\ &= Tr(P^{i_1}_{j_1} \cdots P^{i_k}_{j_k} P^i_{f_A(x_i)} (Q^{i_1}_{j_1} \cdots Q^{i_l}_{j_l} Q^i_{f_B(x_i)})^T) \\ &= Tr(P^{i_1}_{j_1} \cdots P^{i_k}_{j_k} P^i_{f_A(x_i)} Q^{i^T}_{f_B(x_i)} Q^{i_l^T}_{j_l} \cdots Q^{i_1^T}_{j_1}) \end{aligned}$$

by Lemma 1.3.12. But the Q^i_j are just the projections on eigenspaces of A_i^T, while the P^i_j project on the eigenspaces of A_i. Therefore, $P^i_j = Q^{i^T}_j$ and thus also $Q^{i^T}_{f_B(x_i)} = P^i_{f_B(x_i)}$. But since $f_A(x_i) \neq f_B(x_i)$, we have that $P^i_{f_A(x_i)}$ and $P^i_{f_B(x_i)}$ are distinct projections onto eigenspaces of the same operator, and therefore orthogonal to each other and we have

$$p(f_A, f_B \mid s, t) = Tr(P^{i_1}_{j_1} \cdots P^{i_k}_{j_k} P^i_{f_A(x_i)} Q^{i^T}_{f_B(x_i)} Q^{i_l^T}_{j_l} \cdots Q^{i_1^T}_{j_1}) = 0.$$

This means that the probability of returning inconsistent responses is 0 and therefore the given quantum strategy is perfect. □

4.2.4. **Proposition.** *If $\mathcal{F}$ is a binary constraint system and there is a perfect quantum strategy for the corresponding BCS game, then a quantum satisfying assignment of $\mathcal{F}$ exists.*

Proof. In [5], Cleve and Mittal show the proposition for a BCS game that is similar to our version, but not quite the same. To be precise, in their version of the game, Alice still gets a constraint C_l and has to answer with a partial assignment for this constraint, however Bob gets a variable that appears in Alice's constraint $x \in S_l$ and has to answer with an assignment to x that matches Alice's partial assignment of C_l. Given a perfect quantum strategy for our version of the game, one can use it as a subroutine for a perfect quantum strategy for Cleve and Mittal's game: If Alice gets constraint C_l and Bob gets variable $x \in C_l$, Alice and Bob use the perfect quantum strategy for the game from Definition 4.1.7 to both get assignments for C_l. Alice then returns her assignment, while Bob only returns the value for x. Then we have a perfect quantum strategy and thus, by [5], there is a quantum satisfying assignment for $\mathcal{F}$. □

5. Getting Examples for Quantum Isomorphisms of Graphs via BCS Strategies

We now present the main result of this thesis by combining the results of the previous sections and by introducing the construction of quantum isomorphic but not isomorphic graphs from [2].

5.1. **Graphs Associated to Binary Constraint Systems.** In this section, we want to present the construction of two quantum isomorphic graphs from a binary constraint system with a perfect quantum strategy. The construction associates a graph to a given binary constraint system. It was first introduced in Section 6 in [2].

5.1.1. **Construction.** Given a binary constraint system $\mathcal{F}$, we associate to it a graph $G_{\mathcal{F}}$ defined as follows. For each constraint C_l of $\mathcal{F}$ and each possible partial assignment $f : S_l \to \{-1, 1\}$ that satisfies C_l, we add a vertex (l, f) to $G_{\mathcal{F}}$. Next, we add an edge between two vertices (l, f) and (k, g), if f and g are inconsistent, i. e. if there is an $x \in S_l \cap S_k$ such that $f(x) \neq g(x)$.

Note that with this construction, all vertices belonging to the same constraint of $\mathcal{F}$ are connected.

5.1.2. **Definition.** For any linear BCS $\mathcal{F}$, we define its *homogenization* $\mathcal{F}_1$ as the binary constraint system with the constraints of $\mathcal{F}$, but with the right hand side of all equations set to 1.

Note that the homogenization of a BCS always has a solution, namely the all 1 assignment. Note moreover, that $G_{\mathcal{F}_1}$ and $G_{\mathcal{F}}$ always have the same number of vertices. This is important, since in the next sections, we want to take a closer look at the relation between (quantum) satisfiability of $\mathcal{F}$ and the (quantum) isomorphisms between $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$.

In order to do so, we first consider the next little lemma. Recall for this from Definition 1.1.7 that $\alpha(G)$ denotes the size of the largest independent set in the graph G.

5.1.3. **Lemma.** *If $\mathcal{F}$ is a BCS with m constraints, then $\alpha(G_{\mathcal{F}_1}) = m$ holds for the graph $G_{\mathcal{F}_1}$ associated to the homogenization of $\mathcal{F}$.*

Proof. First note that it is clear that $\alpha(G_{\mathcal{F}_1}) \leq m$, since all vertices belonging to the same constraint are connected, as was remarked above. Therefore, as soon as two vertices belonging to the same constraint are in a set, the set is no longer independent. Thus maximally one vertex of each constraint can belong to an independent set.

Next, we consider the vertices (l, f_l^1) for each constraint C_l, where f_l^1 is the all one assignment. These vertices form an independent set of size m, since for two distinct vertices (l, f_l^1) and (k, f_k^1), f_l^1 and f_k^1 agree on all shared variables, as they assign 1 to each of them. Thus, by the construction of $G_{\mathcal{F}_1}$, they are independent. □

5.2. **Characterization of Isomorphic Graphs Arising From Binary Constraint Systems.** We now present the proof shown in [2] that classical satisfiability of a linear BCS $\mathcal{F}$ is the same as the graphs $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ being isomorphic.

5.2.1. **Lemma.** *If $\mathcal{F}$ is a linear BCS that is classically satisfiable, then $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are isomorphic.*

Proof. Suppose we have a satisfying assignment of $\mathcal{F}$ given by $F : \{x_1, \ldots, x_n\} \to \{-1, 1\}$. We now construct a graph isomorphism $\varphi : V(G_{\mathcal{F}}) \to V(G_{\mathcal{F}_1})$ by mapping the vertex (l, f_l) to the vertex $(l, f_l \cdot F_{|S_l})$.

Let us first show, that $(l, f_l \cdot F_{|S_l})$ is in fact a vertex of $G_{\mathcal{F}_1}$. For this, we need to show that the function $f_l \cdot F_{|S_l}$ satisfies the constraint C_l^1, i. e. that $\prod_{x\in S_l} f(x)F_{|S_l}(x) = 1$. But we know, that both f_l and $F_{|S_l}$ satisfy C_l, i. e.

$$\prod_{x\in S_l} f_l(x) = b_l = \prod_{x\in S_l} F_{|S_l}(x)$$

and therefore

$$\prod_{x\in S_l} f(x)F_{|S_l}(x) = b_l^2 = 1.$$

It is also easy to see that φ is an injection and therefore also a bijection, as $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ have the same number of vertices: for two distinct vertices (l, f_1) and (k, f_2) it is immediately obvious, that φ maps them to distinct vertices if $l \neq k$. If $l = k$ we have that $f_1 \neq f_2$ and therefore $f_1 \cdot F_{|S_l} \neq f_2 \cdot F_{|S_l}$, which shows injectivity.

Lastly we check that φ is indeed an isomorphism, i. e. that it preserves adjacency. Let (l, f_1) and (k, f_2) be two adjacent vertices in $G_{\mathcal{F}}$, which means that there is an $x \in S_l \cap S_k$ such that $f_1(x) \neq f_2(x)$. But then it follows that $f_1(x)F_{|S_l}(x) \neq f_2(x)F_{|S_k}(x)$, since $F_{|S_l}(x) = F_{|S_k}(x) = F(x)$ and therefore the vertices $(l, f_1 \cdot F_{|S_l})$ and $(k, f_2 \cdot F_{|S_k})$ are also adjacent. Showing that φ also preserves non-adjacency is very similar. We thus have that $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are isomorphic. □

5.2.2. **Lemma.** *If $\mathcal{F}$ is a linear BCS with m constraints, then $\alpha(G_{\mathcal{F}}) = m$ implies that $\mathcal{F}$ is satisfiable.*

Proof. We assume that $\alpha(G_{\mathcal{F}}) = m$ and let T be an independent set of $G_{\mathcal{F}}$ of size m. Since all vertices belonging to the same constraint are connected, we infer that T must contain exactly one vertex coming from each of the m constraints. Because of this we can define a function $F : \{x_1, \dots x_n\} \to \{-1, +1\}$ by putting $F(x) := f_l(x)$ if $x \in S_l$ and $(l, f_l) \in T$. To see that F is well-defined, consider two distinct vertices $(l, f_l), (k, f_k) \in T$ such that there is an $x \in S_l \cap S_k$. Since the two vertices are not connected, they must agree on all such variables x by construction of $G_{\mathcal{F}}$. Therefore it does not matter, which function from f_l and f_k is chosen to define the value of F at the variable x.

It thus only remains to show that F satisfies the BCS $\mathcal{F}$. But for any constraint C_l of $\mathcal{F}$ we have that $F(x) = f_l(x)$ for all $x \in S_l$, where f_l is a partial assignment satisfying C_l. Therefore, F satisfies all constraints, and thus also $\mathcal{F}$. □

5.2.3. **Proposition** (Theorem 6.2 in [2]). *Given a linear BCS $\mathcal{F}$ with m constraints, the following are equivalent:*

(i) $\mathcal{F}$ is satisfiable;
(ii) $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are isomorphic;
(iii) $\alpha(G_{\mathcal{F}}) = m$.

Proof. $(i) \implies (ii)$. This holds by Lemma 5.2.1.

$(ii) \implies (iii)$. In Lemma 5.1.3 we already saw that $\alpha(G_{\mathcal{F}_1}) = m$. Since by assumption $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are isomorphic, we get that also $\alpha(G_{\mathcal{F}}) = m$ as desired.

$(iii) \implies (i)$. This holds by Lemma 5.2.2. □

5.3. **Characterization of Quantum Isomorphic Graphs Arising From Binary Constraint Systems.** We now want to show the quantum equivalent of Proposition 5.2.3. However, before we can do that, we need to introduce a new concept, namely that of a projective packing of a graph.

5.3.1. **Definition.** Let G be a graph. A *projective packing* of G is an assignment $g \mapsto E_g \in \mathbb{C}^{d\times d}$ of $d \times d$ projections for some dimension $d \in \mathbb{N}$ such that for adjacent vertices g and g' the assigned projections E_g and $E_{g'}$ are orthogonal.

The *value* for such a projective packing is defined as $\frac{1}{d}\sum_{g\in V(G)} rk(E_g)$, where $rk(E)$ denotes the rank of the matrix E. Moreover we define the *projective packing number* of G, $\alpha_p(G)$, as the supremum of all values of projective packings of G.

In [2], the following lemma is presented. Recall from Definition 1.1.1 that $\overline{G}$ denotes the complement of the graph G.

5.3.2. **Lemma.** *Let G and H be finite graphs. If G and H are quantum isomorphic, it holds that $\alpha_p(G) = \alpha_p(H)$ and $\alpha_p(\overline{G}) = \alpha_p(\overline{H})$. Moreover, if G has a projective packing of value γ then H also has a projective packing of value γ.*

It also holds that the projective packing value of a graph is always greater or equal than the size of its largest independent set.

5.3.3. **Lemma.** *Let G be a graph. Then $\alpha(G) \le \alpha_p(G)$.*

Proof. Let an independent set T of size m of G be given. We get a projective packing of value m by assigning the identity matrix to each vertex in T and the zero matrix to all other vertices of G. □

We now start showing the quantum analogue of Proposition 5.2.3, which was first presented in [2]. First, we look at how to get a perfect quantum strategy for the $(G_{\mathcal{F}}, G_{\mathcal{F}_1})$-isomorphism game given a quantum satisfying assignment of a linear BCS $\mathcal{F}$.

5.3.4. **Lemma.** *If $\mathcal{F}$ is a linear BCS that is quantum satisfiable, then the graphs $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are quantum isomorphic.*

Proof. Since $\mathcal{F}$ is quantum satisfiable, there is a quantum strategy for the BCS game of $\mathcal{F}$ as given in Construction 4.2.2 that is perfect by Proposition 4.2.3. Using this strategy, we will present a perfect quantum strategy for the $(G_{\mathcal{F}}, G_{\mathcal{F}_1})$ isomorphism game. If we have this strategy, by definition, $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are isomorphic.

In the isomorphism game, Alice gets a vertex (l_A, f_A) from one of the two graphs. Using the perfect quantum strategy for the BCS game, Alice now obtains an assignment $f'_A : S_{l_A} \to \{-1, +1\}$ that fulfills the constraint C_{l_A} in $\mathcal{F}$ and she returns the vertex $(l_A, f_A \cdot f'_A)$. Similarly, Bob receives the vertex (l_B, f_B) and returns the vertex $(l_B, f_B \cdot f'_B)$, where he got the assignment $f'_B : S_{l_B} \to \{-1, +1\}$ satisfying C_{l_B} again by using the strategy for the BCS game.

We now need to show that this strategy for the graph isomorphism game is perfect. First note, that Alice and Bob return vertices of the correct graph, i. e. that condition (1) of Definition 2.1.1 is fulfilled. We know, that the assignment f'_A fulfills the constraint C_{l_A} in $\mathcal{F}$. If her given vertex comes from the graph $G_{\mathcal{F}}$, we therefore get that f_A also fulfills the constraint C_{l_A} in $\mathcal{F}$ and thus we have $\prod_{x\in S_{l_A}} f_A(x) = b_l = \prod_{x\in S_{l_A}} f'_A(x)$ and therefore $\prod_{x\in S_{l_A}} f_A(x)f'_A(x) = b_l^2 = 1$, which fulfills the constraint $C^1_{l_A}$ in $\mathcal{F}_1$, that is she returns a vertex of $G_{\mathcal{F}_1}$. If however Alice received a vertex from $G_{\mathcal{F}_1}$, we get that $\prod_{x\in S_{l_A}} f_A(x) = 1$ and thus $\prod_{x\in S_{l_A}} f_A(x)f'_A(x) = \prod_{x\in S_{l_A}} f'_A(x) = b_l$ and therefore $f_A \cdot f'_A$ fulfills constraint C_{l_A} in $\mathcal{F}$ and thus $(l_A, f_A \cdot f'_A)$ is a vertex of $G_{\mathcal{F}}$. That Bob returns vertices of the correct graphs can be shown similarly.

Now we will show, that condition (2) of Definition 2.1.1 is fulfilled. There are two cases two consider, namely the one where Alice and Bob receive vertices from the same graph and the one where they receive vertices from different graphs.

Case 1: Suppose, Alice and Bob receive vertices from the same graph. We will call again Alice's vertex (l_A, f_A) and Bob's vertex (l_B, f_B). Now, there are again three different cases two consider: the vertices Alice and Bob receive could either be equal, adjacent or distinct and non-adjacent.

Case 1.1: We suppose first that Alice and Bob receive the same vertex. Then f_A and f_B are equal. Since f'_A and f'_B come from a perfect quantum strategy and are assignments for the same constraint, they are also equal, and thus Alice and Bobs outputs $(l_A, f_A \cdot f'_A)$ and $(l_B, f_B \cdot f'_B)$ are also equal.

Case 1.2: Second, suppose Alice and Bob are given adjacent vertices. That means that there is an $x_0 \in S_{l_A} \cap S_{l_B}$ such that $f_A(x_0) \neq f_B(x_0)$. But since f'_A and f'_B come from a perfect strategy, they agree on all shared vertices $x \in S_{l_A} \cap S_{l_B}$ and thus we have that $f_A \cdot f'_A(x_0) \neq f_B \cdot f'_B(x_0)$ and therefore Alice and Bob return again adjacent vertices.

Case 1.3: Lastly, suppose that Alice and Bob receive distinct non-adjacent vertices. In this case, f_A and f_B will agree on all variables $x \in S_{l_A} \cap S_{l_B}$. But since f'_A and f'_B come from a perfect strategy, they will also agree on all these variables and thus so will $f_A \cdot f'_A$ and $f_B \cdot f'_B$. Therefore, $(l_A, f_A \cdot f'_A)$ and $(l_B, f_B \cdot f'_B)$ will again be distinct and non-adjacent.

Case 2: We now consider the case where Alice and Bob receive vertices from different graphs. If Alice receives the vertex (l_A, f_A) from one graph and Bob receives the vertex (l_B, f_B) from the other, then the vertex $(l_B, f_B \cdot f'_B)$ will be from the same graph as Alice's vertex. Now we can use the reasoning from case 1 on the vertices (l_A, f_A) and $(l_B, f_B \cdot f'_B)$, which are from the same graph, to see that (l_A, f_A), $(l_B, f_B \cdot f'_B)$, $(l_A, f_A \cdot f'_A)$ and $(l_B, f_B \cdot f'_B \cdot f'_B)$ fulfill the corresponding conditions from the graph isomorphism game. However, since $f'_B \cdot f'_B = 1$, the vertex $(l_B, f_B \cdot f'_B \cdot f'_B)$ is the same as Bob's original vertex (l_B, f_B) and thus condition (2) of Definition 2.1.1 is fulfilled.

We thus have a perfect quantum strategy for the isomorphism game and $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are quantum isomorphic. □

We next show a lemma about the connection between the projective packing value of $G_{\mathcal{F}}$ and the quantum satisfiability of $\mathcal{F}$.

5.3.5. **Lemma.** *Let $\mathcal{F}$ be a linear BCS with m constraints. If there is a projective packing of the graph $G_{\mathcal{F}}$ of value m, then $\mathcal{F}$ is quantum satisfiable.*

Proof. Suppose there is a projective packing of $G_{\mathcal{F}}$ of value m given by $(l, f) \mapsto E_{(l,f)} \in \mathbb{C}^{d \times d}$. We will use this to construct a quantum strategy for the BCS game of $\mathcal{F}$. If that is the case, by Proposition 4.2.4 there is a quantum satisfying assignment for $\mathcal{F}$.

Step 1: We first will show that for each $l \in [m]$, the set $\{E_{l,f} \mid (l, f) \in V(G_{\mathcal{F}})\}$ is a POVM, i. e. that for each $l \in [m]$ it holds that

$$\sum_{f:(l,f) \in V(G_{\mathcal{F}})} E_{(l,f)} = 1.$$

It holds that all the projections assigned to the vertices belonging to the same constraint l are pairwise orthogonal. This is the case, since in $V_{G_{\mathcal{F}}}$ all vertices belonging to the same constraint form a clique, i. e. they are all connected, and the condition for a projective packing asserts that adjacent vertices are assigned orthogonal projections. Since it holds for orthogonal matrices A and B that $rk(A+B) = rk(A) + rk(B)$ and since the dimension of the entire space is d, we have

$$\sum_{f:(l,f)\in V(G_{\mathcal{F}})} rk(E_{(l,f)}) = rk\left(\sum_{f:(l,f)\in V(G_{\mathcal{F}})} E_{(l,f)}\right) \leq d, \text{ for all l} \in [m]\,.$$

Since the value of the projective packing is m, we have that

$$m = \frac{1}{d}\sum_{(l,f)\in V(G_{\mathcal{F}})} rk(E_{(l,f)}).$$

Putting this together with the inequality above, we get

$$m = \frac{1}{d}\sum_{(l,f)\in V(G_{\mathcal{F}})} rk(E_{(l,f)}) = \frac{1}{d}\sum_{l=1}^{m}\sum_{f:(l,f)\in V(G_{\mathcal{F}})} rk(E_{(l,f)}) \leq \frac{1}{d}\, m\, d = m$$

and from this it follows that

$$\sum_{f:(l,f)\in V(G_{\mathcal{F}})} rk\left(E_{(l,f)}\right) = d, \text{ for all } l \in [m]\,.$$

But having mutually orthogonal projections whose rank sums up to d, means that the projections themselves must sum up to identity:

$$\sum_{f:(l,f)\in V(G_{\mathcal{F}})} E_{(l,f)} = 1.$$

This means, that the projections $\{E_{l,f} \mid (l,f) \in V(G_{\mathcal{F}})\}$ form a POVM and in particular a projective measurement for each $l \in [m]$, as each $E_{(l,f)}$ is a projection.

Step 2: We will use these measurements $\left\{E_{(l,f)} \mid (l,f) \in V(G_{\mathcal{F}})\right\}_{l\in[m]}$ to build our quantum strategy for the BCS game. The strategy is as follows: first, the shared entangled state is the canonical maximally entangled state $\psi_d = \frac{1}{d}\sum_{i=1}^{d} e_i \otimes e_i$. When Alice receives constraint C_l, she will measure the POVM $\left\{E_{(l,f)} \mid (l,f) \in V(G_{\mathcal{F}})\right\}$ on her half of ψ_d. Upon receiving (l,f) as the output of the measurement, she will return the partial assignment $f : S_l \to \{-1,+1\}$, which by construction of $G_{\mathcal{F}}$ satisfies her constraint C_l. Bob will proceed similarly, only measuring $\left\{E^T_{(k,f')} \mid (k,f') \in V(G_{\mathcal{F}})\right\}$ upon input C_k to receive a partial assignment $f' : S_k \to \{-1,+1\}$ that satisfies C_k. The probability of Alice and Bob returning f and f' upon input C_l and C_k is thus given by

$$p(f,f' \mid C_l, C_k) = \psi_d^*(E_{(l,f)} \otimes E^T_{(k,f')})\psi = \frac{1}{d} Tr\left(E_{(l,f)}E_{(k,f')}\right),$$

where the last equality is by Lemma 1.3.12.

Step 3: Now it only remains to check that the given strategy is also perfect. Note that by construction of the strategy, both Alice and Bob will always return a partial assignment of the correct constraint that will also satisfy this given constraint. We thus only need to verify that for inconsistent partial assignments f and f' of constraints C_l and C_k respectively it holds that $p(f,f' \mid C_l, C_k) = 0$. But if they are inconsistent, i.e. there is an $x \in S_l \cap S_k$ such that $f(x) \neq f'(x)$, then the corresponding vertices (l,f) and (k,f') are adjacent. In this case however, the projections assigned to these vertices $E_{(l,f)}$ and $E_{(k,f')}$ are orthogonal, since they come from a projective packing of $G_{\mathcal{F}}$, and therefore we have that

$$p(f,f' \mid C_l, C_k) = \frac{1}{d} Tr\left(E_{(l,f)}E_{(k,f')}\right) = 0.$$

Thus the given quantum strategy is perfect. □

Now we can state the quantum analogue of Proposition 5.2.3, which was also shown in [2].

5.3.6. **Theorem** (Theorem 6.3 in [2])**.** *Given a linear BCS $\mathcal{F}$ with m constraints, the following are equivalent:*

(i) There is a perfect quantum strategy for the BCS game of $\mathcal{F}$;
(ii) The graphs $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are quantum isomorphic;
(iii) There exists a projective packing of $G_{\mathcal{F}}$ of value m.

Proof. $(i) \implies (ii)$. This holds by Lemma 5.3.4.

$(ii) \implies (iii)$. We know by Lemma 5.1.3 that $G_{\mathcal{F}_1}$ has an independent set of size m and therefore, by Lemma 5.3.3, it also admits a projective packing of value m. By Lemma 5.3.2 we therefore also get a projective packing of $G_{\mathcal{F}}$ of value m, since $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are quantum isomorphic by assumption.

$(iii) \implies (i)$. This holds by Lemma 5.3.5. □

5.3.7. **Corollary.** *We thus see that as soon as we have a linear binary constraint system $\mathcal{F}$ that is quantum satisfiable but not classically satisfiable, we have an example of two quantum isomorphic graphs that are not classically isomorphic, namely $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$.*

5.4. **A Closer Look at the Resulting Quantum Strategy.** By Corollary 5.3.7, we now know that for a quantum satisfiable but not satisfiable BCS $\mathcal{F}$, we get two graphs that are quantum isomorphic but not isomorphic, and thus, by Theorem 2.2.1, there is also a magic unitary u that fulfills

$$M_{G_{\mathcal{F}}} u = u M_{G_{\mathcal{F}_1}}.$$

However, we do not know, what this magic unitary looks like, since in Lemma 5.3.4 the strategy for the graph isomorphism game is not explicitly stated. In this section, we will explicitly state the strategy for the $(G_{\mathcal{F}}, G_{\mathcal{F}_1})$-isomorphism game given a BCS $\mathcal{F}$ with a quantum satisfying assignment. First, let us recall the perfect quantum strategy for the BCS game coming from a quantum satisfying assignment as given in Construction 4.2.2.

5.4.1. **Proposition.** *Let $\mathcal{F}$ be a linear BCS and let a quantum satisfying assignment for $\mathcal{F}$ be given by $x_i \mapsto A_i$ for all variables x_i of $\mathcal{F}$ and some finite dimensional symmetries A_i. Let P^i_j be the projection on the eigenspace of A_i belonging to the eigenvalue j. The POVM Alice will measure upon receiving a constraint C_s as input is of the form*

$$\mathcal{A}_s = \left\{ P^{i_1}_{j_1} \cdot P^{i_k}_{j_k} \mid x_{i_1}, \dots, x_{i_k} \in S_s \right\}.$$

Each of the products $P^{i_1}_{j_1} \cdot P^{i_k}_{j_k}$ for $x_{i_1}, \dots, x_{i_k} \in S_s$ defines a partial assignment $f : S_s \longrightarrow \{-1, 1\}$ with $f(x_{i_l}) = j_l$. For easier notation, we denote such a product defining the partial assignment f as

$$M_f := P^{i_1}_{j_1} \cdot P^{i_k}_{j_k}.$$

The POVM Bob measures if he receives the constraint C_t as input is then of the form

$$\mathcal{B}_t = \left\{ M_f^T \mid x_{i_1}, \dots, x_{i_k} \in S_t \right\}.$$

Next, we need to consider how this strategy for the BCS game translates into a strategy for the graph isomorphism game.

5.4.2. **Proposition.** *Let $\mathcal{F}$ be a linear BCS and let a perfect quantum strategy for the BCS game of $\mathcal{F}$ be given by a shared entangled state ψ and POVMs*

$$\{\mathcal{A}_s \,|\, C_s \text{ is a constraint in } \mathcal{F}\} \text{ and } \{\mathcal{B}_s \,|\, C_s \text{ is a constraint in } \mathcal{F}\}$$

as described in Proposition 5.4.1. A perfect quantum strategy for the $(G_{\mathcal{F}}, G_{\mathcal{F}_1})$-isomorphism game is given by the same entangled state ψ and POVMs

$$\{\mathcal{E}_x | x \in V(G_{\mathcal{F}}) \cup V(G_{\mathcal{F}_1})\}$$

for Alice and

$$\{\mathcal{F}_x | x \in V(G_{\mathcal{F}}) \cup V(G_{\mathcal{F}_1})\}$$

for Bob given as follows:

$$\mathcal{E}_{(l,f)} = \{M_{f'} \,|\, (l, f \cdot f') \in V(G_{\mathcal{F}_1})\} \text{ for } (l,f) \in V(G_{\mathcal{F}})$$
$$\mathcal{E}_{(l,f)} = \{M_{f'} \,|\, (l, f \cdot f') \in V(G_{\mathcal{F}})\} \text{ for } (l,f) \in V(G_{\mathcal{F}_1})$$

and

$$\mathcal{F}_{(l,f)} = \left\{M_{f'}^T \,|\, (l, f \cdot f') \in V(G_{\mathcal{F}_1})\right\} \text{ for } (l,f) \in V(G_{\mathcal{F}})$$
$$\mathcal{F}_{(l,f)} = \left\{M_{f'}^T \,|\, (l, f \cdot f') \in V(G_{\mathcal{F}})\right\} \text{ for } (l,f) \in V(G_{\mathcal{F}_1})$$

If Alice gets the vertex (l,f) as input, she will measure the POVM $\mathcal{E}_{(l,f)}$. If the result of this measurement is f' (associated to the operator $M_{f'} \in \mathcal{E}_{(l,f)}$), Alice will return the vertex $(l, f \cdot f')$. The same holds for Bob only with POVMs $\mathcal{F}_{(l,f)}$.

Proof. Recall from the proof of Lemma 5.3.4 that given a vertex (l,f) from one of the two graphs, Alice will measure her POVM from the BCS strategy $\mathcal{A}_l = \{M_{f'}\}$ to receive a function f' and return the vertex $(l, f \cdot f')$. That means that the operators in the POVM she measures for the graph isomorphism game will be the same, just the interpretation will be different. Therefore, in the strategy for the graph isomorphism game, the operator $M_{f'}$ is associated to the vertex $(l, f \cdot f')$, which will be her return vertex. This is again similar for Bob.

Since in some cases the labels of some vertices in $V(G_{\mathcal{F}})$ and $V(G_{\mathcal{F}_1})$ might be the same, we made a distinction between the POVMs for the vertices from $V(G_{\mathcal{F}})$ and $V(G_{\mathcal{F}_1})$ in order to make sure that Alice and Bob always return the vertices of the correct graph. □

To summarize, we will now write down constructively what the magic unitary witnessing the quantum isomorphism looks like.

5.4.3. **Proposition.** *Let $\mathcal{F}$ be a linear BCS with a quantum satisfying assignment A_i for each variable x_i of $\mathcal{F}$. Denote by M_f again the products of projections onto eigenspaces of A_i as defined in Proposition 5.4.1. The magic unitary $u = (u_{xy})_{x \in V(G_{\mathcal{F}}), y \in V(G_{\mathcal{F}_1})}$ fulfilling $M_{G_{\mathcal{F}}} u = u M_{G_{\mathcal{F}_1}}$ has the entries:*

$$u_{xy} = \begin{cases} M_{f'} & \text{if } x = (l,f) \text{ and } y = (l, f \cdot f') \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We put $V := V(G_{\mathcal{F}} \cup V(G_{\mathcal{F}_1}))$. If Alice's POVMs in the perfect quantum strategy are of the form $\mathcal{E}_x = \{E_{xy} | y \in V\}$ for all $x \in V$, we know by Theorem 2.2.1 that the magic unitary u has the (x,y)-entry E_{xy} for $x \in V(G_{\mathcal{F}})$ and $y \in V(G_{\mathcal{F}_1})$. Moreover, we know by Proposition 5.4.2 that

$$E_{xy} = \begin{cases} M_{f'} & \text{if } x = (l,f) \text{ and } y = (l, f \cdot f') \\ 0 & \text{otherwise.} \end{cases}$$

□

5.5. **Application.** In the previous section, we saw how to construct the magic unitary given a linear binary constraint system with a quantum satisfying assignment. In this section, we will see how to get these BCSs and their quantum satisfying assignments.

Recall from Theorem 3.3.5, that for any nonplanar graph, we get a magic arrangement.

5.5.1. **Construction.** Let $G = (V, E)$ be a nonplanar graph. G is the intersection graph of the arrangement $A = (V', E')$, where each vertex of A, $v'_i \in V'$, corresponds to an edge of G, $e_i \in E$. Moreover, each hyperedge e'_i of A corresponds to a vertex v_i of G and the hyperedges are of the form:

$$e'_i = \left\{ v'_j \,|\, r(e_j) = v_i \text{ or } s(e_j) = v_i \right\}$$

where e_j is again the edge corresponding to v'_j and r and s are the range and source maps of G. Then A is a magic arrangement. If the graph G was either the complete bipartite graph on 6 vertices $K_{3,3}$ or the complete graph on 5 vertices K_5 we get a quantum realization of A from Example 3.2.7. Otherwise, one of these two graphs is a topological minor of G and we get the quantum realization for A from the proof of Theorem 3.3.4: let $H \in \{K_{3,3}, K_5\}$ be a topological minor of G, let $\Phi : H \to G$ be the topological minor inclusion map and let l be a labelling with odd parity of H. The labelling l' for G is given by

$$l'(v) = \begin{cases} 1 & \text{if } v \notin Im(\Phi_V) \\ l(u) & \text{if } u \in V(H) \text{ and } \Phi_V(u) = v \end{cases}$$

and it translates to a labelling of A by assigning the label $l(v_i)$ to the hyperedge e'_i corresponding to v_i.

This magic arrangement corresponds to a binary constraint system.

5.5.2. **Construction.** Let $A = (V, E, l)$ be a magic arrangement with an odd-parity labelling l. A induces a BCS $\mathcal{F}$ as follows:

- for each vertex $v_i \in V$ there is a variable x_i of $\mathcal{F}$;
- for each hyperedge $e_j \in E$ there is a constraint C_j of $\mathcal{F}$ of the form

$$C_j : \prod_{v_i \in e_j} x_i = l(e_j).$$

5.5.3. **Theorem.** *Let H be a nonplanar graph. Applying Constructions 5.5.1 and 5.5.2, we get a quantum satisfiable BCS $\mathcal{F}$. Using this BCS and its homogenization $\mathcal{F}_1$ in Construction 5.1.1 yields two graphs $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$. These two graphs are quantum isomorphic but not isomorphic and by following the constructions in section 5.4 we get a magic unitary u that fulfills*

$$M_{G_{\mathcal{F}}} u = u M_{G_{\mathcal{F}_1}}$$

for the adjacency matrices of $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$.

Proof. The arrangement associated to the nonplanar graph H is magic by Theorem 3.3.5. That the corresponding BCS $\mathcal{F}$ is then quantum satisfiable is seen easily by comparing the definitions of a quantum satisfiable assignment and a quantum realization. The rest is shown in section 5.4. □

The smallest graphs one can get using Theorem 5.5 are obtained when applying it to the complete bipartite graph on 6 vertices $K_{3,3}$ from Example 1.1.4 and they have 24 vertices. Since that means that the magic unitary has 576 entries, it is not an easy task to compute it by hand. Therefore we implemented the algorithms from section 5.4 in Singular. The resulting magic unitary is presented in appendix A.

6. Summary

In this section, we give a structured summary of the construction of graphs that are quantum isomorphic but not classically isomorphic and of the magic unitary witnessing the quantum isomorphism.

Step 1: We start with any nonplanar graph Γ.

Step 2: By using Arkhipov's construction, which is reiterated in Construction 5.5.1, we get a magic arrangement A_Γ. It is the arrangement that has Γ as an intersection graph. We get a quantum realization of A_Γ by lifting the quantum realization of either the magic square or the magic pentagram from Example 3.2.7. The magic arrangement A_Γ corresponds to a linear binary constraint system $\mathcal{F}(\Gamma)$ as detailed in Construction 5.5.2 with a quantum satisfying assignment that is just the quantum realization of A_Γ.

Step 3: Using the quantum satisfying assignment of $\mathcal{F}(\Gamma)$ and Construction 4.2.2, we get a perfect quantum strategy for the BCS game associated to $\mathcal{F}(\Gamma)$. In this strategy, Alice associates the observables A_i to each variable x_i of $\mathcal{F}(\Gamma)$ and Bob associates A_i^T to x_i. Given a constraint as input, they each measure all observables belonging to variables appearing in this constraint on the canonical maximally entangled state ψ_d, where d is the dimension of the observables.

Step 4: Using the construction from [2], given again in Construction 5.1.1, on $\mathcal{F}(\Gamma)$ and its homogenization $\mathcal{F}_1(\Gamma)$, we get two graphs $G_{\mathcal{F}(\Gamma)}$ and $G_{\mathcal{F}_1(\Gamma)}$. These graphs have vertices (l, f) for each constraint C_l of the BCS and each partial assignment f satisfying C_l. The edges in the graphs are between vertices where the partial assignments have a conflict.

Step 5: Using the perfect quantum strategy for the BCS game of $\mathcal{F}(\Gamma)$, one can get a perfect quantum strategy for the $(G_{\mathcal{F}(\Gamma)}, G_{\mathcal{F}_1(\Gamma)})$-isomorphism game by following the construction in Proposition 5.4.2. In this strategy, upon receiving a vertex (l, f) from one graph, Alice and Bob measure POVMs consisting of the operators $M_{f'}$ and $M_{f'}^T$ respectively for all f' such that $(l, f \cdot f')$ is a vertex of the other graph. The operators M_f were defined in Proposition 5.4.1 for ease of notation and are products of projections onto eigenspaces of the observables from the BCS strategy.

Step 6: By using Theorem 2.2.1, we get a magic unitary u from the perfect quantum strategy of the isomorphism game such that $uM_{G_{\mathcal{F}(\Gamma)}} = M_{G_{\mathcal{F}_1(\Gamma)}}u$. This magic unitary has the entries $u_{xy} = M_{f'}$ if $x = (l, f)$ and $y = (l, f \cdot f')$ and $u_{xy} = 0$ otherwise.

References

1. Arkhipov, A. Extending and Characterizing Quantum Magic Games. *arXiv:1209.3819 [quant-ph].* arXiv: `1209.3819`. `http://arxiv.org/abs/1209.3819` (2012).
2. Atserias, A., Mančinska, L., Roberson, D. E., Šámal, R., Severini, S. & Varvitsiotis, A. Quantum and non-signalling graph isomorphisms. *Journal of Combinatorial Theory, Series B* **136,** 289–328 (2019).
3. Banica, T. Quantum automorphism groups of homogeneous graphs. *Journal of Functional Analysis* **224,** 243–280 (2005).
4. Cameron, P. J., Montanaro, A., Newman, M. W., Severini, S. & Winter, A. On the quantum chromatic number of a graph. *The Electronic Journal of Combinatorics* (2007).
5. Cleve, R. & Mittal, R. Characterization of Binary Constraint System Games. *Automata, Languages, and Programming Lecture Notes in Computer Science,* 320–331 (2014).
6. Eder, C., Levandovskyy, V., Schanz, J., Schmidt, S., Steenpass, A. & Weber, M. Existence of quantum symmetries for graphs on up to seven vertices: a computer based approach. *arXiv:1906.12097 [math].* arXiv: `1906.12097`. `http://arxiv.org/abs/1906.12097` (2019).
7. Helton, W., Meyer, K. P., Paulsen, V. I. & Satriano, M. Algebras, Synchronous Games and Chromatic Numbers of Graphs. *New York Journal of Mathematics,* 328–361 (2019).
8. Lupini, M., Mančinska, L. & Roberson, D. E. Nonlocal games and quantum permutation groups. *Journal of Functional Analysis* **279,** 108592 (2020).
9. Mančinska, L. & Roberson, D. E. Graph Homomorphisms for Quantum Players. *Journal of Combinatorial Theory, Series B* **118,** 228–267. ISSN: 00958956 (2016).
10. Mančinska, L., Roberson, D. E. & Varvitsiotis, A. On deciding the existence of perfect entangled strategies for nonlocal games. *Chicago Journal of Theoretical Computer Science,* 1–16 (2016).
11. Nielsen, M. A. & Chuang, I. L. *Quantum Computation and Quantum Information* (ed Capelin, S.) (Cambridge University Press, 2010).
12. Roberson, D. E. *Variations on a Theme: Graph Homomorphisms* PhD Thesis (University of Waterloo, 2013).
13. Schmidt, S. & Weber, M. Quantum symmetries of graph C*-algebras. *Can. Math. Bull.* **61,** 848–864 (2018).
14. Speicher, R. & Weber, M. Quantum groups with partial commutation relations. *Indiana University Mathematics Journal* **68,** 1849–1883 (2019).
15. Woronowicz, S. L. Compact matrix pseudogroups. *Communications in Mathematical Physics* **111,** 613–665 (1987).
16. Woronowicz, S. L. A remark on compact matrix quantum groups. *Letters in Mathematical Physics* **21,** 35–39 (1991).

Appendices

A. Results of the Algorithm

We present the entries of the magic unitary u satisfying $M_{G_{\mathcal{F}}}u = uM_{G_{\mathcal{F}_1}}$, where M_G is the adjacency matrix of the graph G and $G_{\mathcal{F}}$ and $G_{\mathcal{F}_1}$ are the graphs associated to the Mermin-Peres magic square BCS and its homogenization as in 5.1.1. We assume that the vertex in $G_{\mathcal{F}}$ belonging to the first constraint and the partial assignment $(1, 1, 1)$ is associated to the first row and column respectively in $M_{G_{\mathcal{F}}}$, that the assignment $(1, -1, -1)$ is associated to the index 2, $(-1, 1, -1)$ to 3 and so forth. The last row and column are therefore associated to the vertex coming from the sixth constraint and the assignment $(-1, -1, -1)$. For $G_{\mathcal{F}_1}$, we assume a similar association between the adjacency matrix and the graph.

The final result u is of the form

$$u = \begin{bmatrix} u^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & u^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & u^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & u^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & u^6 \end{bmatrix}$$

and the entries of the u^i are:

$$u^1_{1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^1_{1,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^1_{1,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^1_{1,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$u^1_{2,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^1_{2,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^1_{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad u^1_{2,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^1_{3,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^1_{3,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$u^1_{3,3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^1_{3,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^1_{4,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad u^1_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^1_{4,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^1_{4,4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^2_{1,1} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \qquad u^2_{1,2} = \begin{bmatrix} 0.25 & 0.25 & -0.25 & -0.25 \\ 0.25 & 0.25 & -0.25 & -0.25 \\ -0.25 & -0.25 & 0.25 & 0.25 \\ -0.25 & -0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$u^2_{1,3} = \begin{bmatrix} 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \\ 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \end{bmatrix} \qquad u^2_{1,4} = \begin{bmatrix} 0.25 & -0.25 & -0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & -0.25 & -0.25 & 0.25 \end{bmatrix}$$

$$u^2_{2,1} = \begin{bmatrix} 0.25 & 0.25 & -0.25 & -0.25 \\ 0.25 & 0.25 & -0.25 & -0.25 \\ -0.25 & -0.25 & 0.25 & 0.25 \\ -0.25 & -0.25 & 0.25 & 0.25 \end{bmatrix} \qquad u^2_{2,2} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$u^2_{2,3} = \begin{bmatrix} 0.25 & -0.25 & -0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & -0.25 & -0.25 & 0.25 \end{bmatrix} \qquad u^2_{2,4} = \begin{bmatrix} 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \\ 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \end{bmatrix}$$

$$u^2_{3,1} = \begin{bmatrix} 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \\ 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \end{bmatrix} \qquad u^2_{3,2} = \begin{bmatrix} 0.25 & -0.25 & -0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & -0.25 & -0.25 & 0.25 \end{bmatrix}$$

$$u^2_{3,3} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \qquad u^2_{3,4} = \begin{bmatrix} 0.25 & 0.25 & -0.25 & -0.25 \\ 0.25 & 0.25 & -0.25 & -0.25 \\ -0.25 & -0.25 & 0.25 & 0.25 \\ -0.25 & -0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$u^2_{4,1} = \begin{bmatrix} 0.25 & -0.25 & -0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & -0.25 & -0.25 & 0.25 \end{bmatrix} \quad u^2_{4,2} = \begin{bmatrix} 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \\ 0.25 & -0.25 & 0.25 & -0.25 \\ -0.25 & 0.25 & -0.25 & 0.25 \end{bmatrix}$$

$$u^2_{4,3} = \begin{bmatrix} 0.25 & 0.25 & -0.25 & -0.25 \\ 0.25 & 0.25 & -0.25 & -0.25 \\ -0.25 & -0.25 & 0.25 & 0.25 \\ -0.25 & -0.25 & 0.25 & 0.25 \end{bmatrix} \quad u^2_{4,4} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$u^3_{1,1} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & 0.25 & -0.25 \\ -0.25 & -0.25 & -0.25 & 0.25 \end{bmatrix} \quad u^3_{1,2} = \begin{bmatrix} 0.25 & 0.25 & -0.25 & 0.25 \\ 0.25 & 0.25 & -0.25 & 0.25 \\ -0.25 & -0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & -0.25 & 0.25 \end{bmatrix}$$

$$u^3_{1,3} = \begin{bmatrix} 0.25 & -0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & -0.25 & -0.25 \\ 0.25 & -0.25 & 0.25 & 0.25 \\ 0.25 & -0.25 & 0.25 & 0.25 \end{bmatrix} \quad u^3_{1,4} = \begin{bmatrix} 0.25 & -0.25 & -0.25 & -0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$u^3_{2,1} = \begin{bmatrix} 0.25 & 0.25 & -0.25 & 0.25 \\ 0.25 & 0.25 & -0.25 & 0.25 \\ -0.25 & -0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & -0.25 & 0.25 \end{bmatrix} \quad u^3_{2,2} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & 0.25 & -0.25 \\ -0.25 & -0.25 & -0.25 & 0.25 \end{bmatrix}$$

$$u^3_{2,3} = \begin{bmatrix} 0.25 & -0.25 & -0.25 & -0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \quad u^3_{2,4} = \begin{bmatrix} 0.25 & -0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & -0.25 & -0.25 \\ 0.25 & -0.25 & 0.25 & 0.25 \\ 0.25 & -0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$u_{3,1}^3 = \begin{bmatrix} 0.25 & -0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & -0.25 & -0.25 \\ 0.25 & -0.25 & 0.25 & 0.25 \\ 0.25 & -0.25 & 0.25 & 0.25 \end{bmatrix} \quad u_{3,2}^3 = \begin{bmatrix} 0.25 & -0.25 & -0.25 & -0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$u_{3,3}^3 = \begin{bmatrix} 0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & 0.25 & -0.25 \\ -0.25 & -0.25 & -0.25 & 0.25 \end{bmatrix} \quad u_{3,4}^3 = \begin{bmatrix} 0.25 & 0.25 & -0.25 & 0.25 \\ 0.25 & 0.25 & -0.25 & 0.25 \\ -0.25 & -0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & -0.25 & 0.25 \end{bmatrix}$$

$$u_{4,1}^3 = \begin{bmatrix} 0.25 & -0.25 & -0.25 & -0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \quad u_{4,2}^3 = \begin{bmatrix} 0.25 & -0.25 & 0.25 & 0.25 \\ -0.25 & 0.25 & -0.25 & -0.25 \\ 0.25 & -0.25 & 0.25 & 0.25 \\ 0.25 & -0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$u_{4,3}^3 = \begin{bmatrix} 0.25 & 0.25 & -0.25 & 0.25 \\ 0.25 & 0.25 & -0.25 & 0.25 \\ -0.25 & -0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & -0.25 & 0.25 \end{bmatrix} \quad u_{4,4}^3 = \begin{bmatrix} 0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.25 & 0.25 & -0.25 \\ -0.25 & -0.25 & -0.25 & 0.25 \end{bmatrix}$$

$$u_{1,1}^4 = \begin{bmatrix} 0.50 & 0.50 & 0 & 0 \\ 0.50 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad u_{1,2}^4 = \begin{bmatrix} 0.50 & -0.50 & 0 & 0 \\ -0.50 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u_{1,3}^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0.50 & 0.50 \end{bmatrix} \quad u_{1,4}^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & -0.50 \\ 0 & 0 & -0.50 & 0.50 \end{bmatrix}$$

$$u^4_{2,1} = \begin{bmatrix} 0.50 & -0.50 & 0 & 0 \\ -0.50 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^4_{2,2} = \begin{bmatrix} 0.50 & 0.50 & 0 & 0 \\ 0.50 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^4_{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & -0.50 \\ 0 & 0 & -0.50 & 0.50 \end{bmatrix} \qquad u^4_{2,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0.50 & 0.50 \end{bmatrix}$$

$$u^4_{3,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0.50 & 0.50 \end{bmatrix} \qquad u^4_{3,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & -0.50 \\ 0 & 0 & -0.50 & 0.50 \end{bmatrix}$$

$$u^4_{3,3} = \begin{bmatrix} 0.50 & 0.50 & 0 & 0 \\ 0.50 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^4_{3,4} = \begin{bmatrix} 0.50 & -0.50 & 0 & 0 \\ -0.50 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^4_{4,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & -0.50 \\ 0 & 0 & -0.50 & 0.50 \end{bmatrix} \qquad u^4_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0.50 & 0.50 \end{bmatrix}$$

$$u^4_{4,3} = \begin{bmatrix} 0.50 & -0.50 & 0 & 0 \\ -0.50 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^4_{4,4} = \begin{bmatrix} 0.50 & 0.50 & 0 & 0 \\ 0.50 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^5_{1,1} = \begin{bmatrix} 0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^5_{1,2} = \begin{bmatrix} 0.50 & 0 & -0.50 & 0 \\ 0 & 0 & 0 & 0 \\ -0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^5_{1,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0.50 \end{bmatrix} \qquad u^5_{1,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & -0.50 \\ 0 & 0 & 0 & 0 \\ 0 & -0.50 & 0 & 0.50 \end{bmatrix}$$

$$u^5_{2,1} = \begin{bmatrix} 0.50 & 0 & -0.50 & 0 \\ 0 & 0 & 0 & 0 \\ -0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^5_{2,2} = \begin{bmatrix} 0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^5_{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & -0.50 \\ 0 & 0 & 0 & 0 \\ 0 & -0.50 & 0 & 0.50 \end{bmatrix} \qquad u^5_{2,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0.50 \end{bmatrix}$$

$$u^5_{3,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0.50 \end{bmatrix} \qquad u^5_{3,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & -0.50 \\ 0 & 0 & 0 & 0 \\ 0 & -0.50 & 0 & 0.50 \end{bmatrix}$$

$$u^5_{3,3} = \begin{bmatrix} 0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^5_{3,4} = \begin{bmatrix} 0.50 & 0 & -0.50 & 0 \\ 0 & 0 & 0 & 0 \\ -0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^5_{4,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & -0.50 \\ 0 & 0 & 0 & 0 \\ 0 & -0.50 & 0 & 0.50 \end{bmatrix} \qquad u^5_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0.50 \end{bmatrix}$$

$$u^5_{4,3} = \begin{bmatrix} 0.50 & 0 & -0.50 & 0 \\ 0 & 0 & 0 & 0 \\ -0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^5_{4,4} = \begin{bmatrix} 0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^6_{1,1} = \begin{bmatrix} 0.50 & 0 & 0 & 0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0 & 0.50 \end{bmatrix} \qquad u^6_{1,2} = \begin{bmatrix} 0.50 & 0 & 0 & -0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.50 & 0 & 0 & 0.50 \end{bmatrix}$$

$$u^6_{1,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^6_{1,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & -0.50 & 0 \\ 0 & -0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^6_{2,1} = \begin{bmatrix} 0.50 & 0 & 0 & -0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.50 & 0 & 0 & 0.50 \end{bmatrix} \qquad u^6_{2,2} = \begin{bmatrix} 0.50 & 0 & 0 & 0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0 & 0.50 \end{bmatrix}$$

$$u^6_{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & -0.50 & 0 \\ 0 & -0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^6_{2,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^6_{3,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^6_{3,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & -0.50 & 0 \\ 0 & -0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^6_{3,3} = \begin{bmatrix} 0.50 & 0 & 0 & 0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0 & 0.50 \end{bmatrix} \qquad u^6_{3,4} = \begin{bmatrix} 0.50 & 0 & 0 & -0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.50 & 0 & 0 & 0.50 \end{bmatrix}$$

$$u^6_{4,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & -0.50 & 0 \\ 0 & -0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad u^6_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0.50 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$u^6_{4,3} = \begin{bmatrix} 0.50 & 0 & 0 & -0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.50 & 0 & 0 & 0.50 \end{bmatrix} \qquad u^6_{4,4} = \begin{bmatrix} 0.50 & 0 & 0 & 0.50 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.50 & 0 & 0 & 0.50 \end{bmatrix}$$