UNIVERSITÄT DES SAARLANDES

EXPLORING NEW HYPERGRAPH C^*-ALGEBRAS

Bachelor Thesis

by

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March 18, 2026

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Contents

The theory of C^*-algebras is a central area of modern functional analysis and has deep connections to operator theory, topology, dynamical systems, and mathematical physics. Its origins lie in the mathematical formulation of quantum mechanics, where algebras of bounded operators on Hilbert spaces serve as models for physical observables. In this context, observables are represented by self-adjoint operators, and the algebraic relations between these operators naturally lead to the study of operator algebras. As Blackadar writes in [Blackadar2006, p. 1],

> "C^*-algebras are self-adjoint operator algebras on Hilbert space which are closed in the norm topology. Their study was begun in the work of Gelfand and Naimark who showed that such algebras can be characterized abstractly as involutive Banach algebras satisfying a relation connecting the norm and the involution."

The abstract theory of C^*-algebras was established in the 1940s through the fundamental work of Israel Gelfand and Mark Naimark. The Gelfand-Naimark Theorem motivates the interpretation of C^*-algebras as a noncommutative or "quantum version" of topology. The commutative version of the Gelfand-Naimark theorem (1.9) states that every commutative unital C^*-algebra is isometrically *-isomorphic to the algebra $C(X)$ of continuous complex-valued functions on a compact Hausdorff space X. Conversely, every such function algebra arises in this way. This correspondence, often referred to as *Gelfand duality*, establishes an equivalence between the category of compact Hausdorff spaces and the category of commutative unital C^*-algebras. From this perspective, noncommutative C^*-algebras can be interpreted as generalizations of function algebras on hypothetical "noncommutative spaces", which motivates the viewpoint of C^*-algebra theory as a form of *noncommutative topology*. The Gelfand-Naimark Segal theorem (1.11) shows that every C^*-algebra is isometrically *-isomorphic to a norm-closed *-subalgebra of the bounded operators on some Hilbert space. This provides a representation of abstract C^*-algebras as concrete operator algebras. We conclude that quantum/noncommutative mathematics follows the philosophy:

$$\text{commutative algebras} \Leftrightarrow \text{classical situation}$$
$$\text{noncommutative algebras} \Leftrightarrow \text{quantum/noncommutative situation}$$

The terminology "C^*-algebra" was introduced by Irving Segal in 1947. Originally, the letter C meant not "continuous", but referred to the fact that these algebras are closed in the norm topology as subalgebras of $B(H)$, the algebra of bounded operators on a Hilbert space.

A major step in the development of concrete and combinatorial examples of C^*-algebras occurred in the 1970s with the work of Joachim Cuntz. In 1977, Cuntz introduced a family of simple C^*-algebras $\mathcal{O}_n$ generated by n isometries satisfying certain relations. These algebras, now known as the *Cuntz algebras*, became fundamental examples in the study of simple and purely infinite C^*-algebras and have played an important role in classification theory.

Shortly thereafter, Joachim Cuntz and Wolfgang Krieger generalized this construction in their 1980 work on C^*-algebras associated to topological Markov chains. The resulting *Cuntz-Krieger algebras* are generated by partial isometries satisfying relations determined by a finite $\{0, 1\}$-matrix. These algebras revealed deep connections between operator algebras and symbolic dynamics.

They were later extended to the more general class of *graph C^*-algebras*. Given a graph, the idea is to associate a universal C^*-algebra generated by projections corresponding to the vertices and partial isometries corresponding to the edges, satisfying certain relations coming from the graph. Graph C^*-algebras form one of the most important and well understood classes of examples in the theory of operator algebras. Their structure can be analyzed using graph-theoretic properties, which allows questions about ideals, simplicity, or K-theory to be translated into combinatorial conditions. We refer to [Raeburn2005] for an overview on this topic.

During the 1990s, it became clear that many important classes of C^*-algebras could be described using directed graphs. This insight led to the systematic study of *graph C^*-algebras*. Important contributions to this theory were made by Alex Kumjian, David Pask, Iain Raeburn, and Jean Renault. In particular, they showed that C^*-algebras associated with directed graphs generalize both Cuntz algebras and Cuntz-Krieger algebras. As Raeburn writes in [Raeburn2005, p. 1],

> "A directed graph is a combinatorial object consisting of vertices and oriented edges joining pairs of vertices. We can represent such a graph by operators on a Hilbert space H: the vertices are represented by mutually orthogonal closed subspaces, or more precisely the projections onto these subspaces, and the edges by operators between the appropriate subspaces. The graph algebra is, loosely speaking, the C^*-subalgebra of $B(H)$ generated by these operators. When the graph is finite and highly connected, the graph algebras coincide with a family of C^*-algebras first studied by Cuntz and Krieger in 1980."

Another natural generalization arises from replacing ordinary graphs by *hypergraphs*. In contrast to graphs, where each edge connects exactly two vertices, a hyperedge may connect multiple vertices simultaneously. Formally, the source and range mappings of a hypergraph map into the power set of the vertices. This additional flexibility allows hypergraphs to encode more complicated combinatorial relations. Hypergraphs therefore provide a natural generalization of directed graphs. The associated *hypergraph C^*-algebras* extend the construction of graph C^*-algebras by assigning generators and relations to hyperedges and vertex sets. In fact, every graph C^*-algebra can be realized as a hypergraph C^*-algebra, so their class is strictly contained in the class of hypergraph C^*-algebras. Moreover, while they are always nuclear, there exist hypergraph C^*-algebras which are not nuclear, for instance the example $(C(S^1) * C^2)$ discussed in [TWZ24, Proposition 4.2]. This shows that hypergraph C^*-algebras form a significantly larger and more flexible class of operator algebras, which makes them interesting for further investigation.

Hypergraph C^*-algebras represent a relatively recent direction in the study of combinatorial operator algebras. They provide a framework in which various universal C^*-algebras can be described using hypergraph structures and allow new examples to be constructed by modifying or extending known graph models. In particular, they can be used to study operator-algebraic structures arising from more general combinatorial configurations than ordinary graphs.

The main goal of this thesis is to study constructions of hypergraph C^*-algebras and to investigate how modifications of the underlying hypergraphs affect the associated operator algebras. In particular, we consider examples obtained by modifying known graphs and hypergraphs, such as those related to the Toeplitz algebra or the Cuntz algebra, combining hypergraphs, or applying certain hypergraph moves. These constructions illustrate how combinatorial changes in the hypergraph influence structural properties of the corresponding C^*-algebra. One interesting example arising in this context is the realization of the extended Cuntz algebra as a hypergraph C^*-algebra (3.2).

The thesis is organized as follows. In the preliminaries we recall the basic definitions and results concerning C^*-algebras, graph and hypergraph C^*-algebras that will be needed throughout the work. In the main chapter we present new examples and investigate how various modifications of hypergraphs influence the associated C^*-algebras and in the end we study the application of moves.

1 Preliminaries

Since we want to explore new examples of hypergraph C^*-algebras, we need to understand the basic theory behind them. For this reason, we introduce, among other things, the concept of universal- and graph- C^*-algebras. We will see some interesting examples in this chapter and refer for more details to the standard Literature [Blackadar2006], [ISem24] and [Zen21].

1.1 Projections and partial isometries in C^*-algebras

We first want to provide the basic properties about projections and partial isometries, since they are the basic tools for the hypergraph C^*-algebra setting.

Definition 1.1: Let A be a C^*-algebra. We call $p \in A$ a *projection* iff the equation $p = p^* = p^2$ holds.

Definition 1.2: Let A be a C^*-algebra. We call $s \in A$ a *partial isometry* iff the equation $ss^*s = s$ holds.

Remark 1.3: Let A be a C^*-algebra and $s \in A$ a partial isometry. Then it holds the following.

$$s = ss^*s \iff s^*s \text{ is a projection} \iff ss^* \text{ is a projection}$$

Remark 1.4: Notice that for every partial isometry s in a C^*-algebra A we have that $s^* = s^*ss^*$.

Now we come the main statement of this section, namely that the sum of all projections is itself a projection. We need this later for the definition of the graph and hypergraph C^*-algebra.

Proposition 1.5: *Let $\{p_i \mid 1 \leq i \leq n\}$ be projections in a C^*-algebra A. Then we have that $p := \sum_{i=1}^{n} p_i$ is a projection iff $p_i p_j = 0$ for all $i \neq j$. In that case we say that the projections are mutually orthogonal.*

Proof: See [Zen21, Proposition 1.9.] □

1.2 C^*-algebras

In this section we introduce the theory of Banach algebras and C^*-algebras, because everything else in this thesis builds up on this. So, in order to understand what a hypergraph C^*-algebra actually is, we first need to understand what a C^*-algebra is.

Definition 1.6: (a) An *involution* on a $\mathbb{C}$-algebra A is an antilinear map $^*\colon A \to A$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$. A **-algebra* is an algebra A equipped with an involution. $B \subseteq A$ is a *-subalgebra of A, if $xy, \lambda x + \mu y, x^* \in B$ for all $x, y \in B$ and $\lambda, \mu \in \mathbb{C}$.

(b) A *Banach algebra* is a normed $\mathbb{C}$-algebra which is complete; its norm satisfies

$$\|xy\| \leq \|x\|\|y\|.$$

A *Banach *-algebra* is a Banach algebra with an involution.

(c) A *C^*-algebra* is a Banach *-algebra A satisfying the C^*-identity

$$\|x^*x\| = \|x\|^2.$$

A *-subalgebra $B \subseteq A$ is a C^*-subalgebra, if B is (topologically) closed.

(d) An algebra is *unital*, if it contains a unit with respect to multiplication.

(e) An algebra A is *commutative*, if $xy = yx$ for all $x, y \in A$.

(f) An element $x \in A$ of a C^*-algebra is *normal*, if $x^*x = xx^*$. It is *selfadjoint*, if $x^* = x$.

The only difference between C^*-algebras and Banach *-algebras is the C^*-identity. This condition turns the class of C^*-algebras into a very special subclass of Banach algebras. In a C^*-algebra we have the consequence $x^*x = 0 \Rightarrow x = 0$. As Blackadar writes in [Blackadar2006], one can easily see that the C^*-identity implies that the involution is isometric. Therefore, it is unnecessary to include isometry beside the involution as an axiom. The C*-axiom has sometimes been replaced by the apparently weaker axiom that $\|x^*x\| = \|x^*\|\|x\|$ for all x, which also implies isometry (a much harder result). Therefore, the weakened axiom is equivalent to the C^*-identity.

The following remark play a role in the theory of ideals, which we will need in a later section, e.g. where we see that the Toeplitz algebra can be seen as an extension of the continuous functions on the unit circle by the compact operators.

Remark 1.7: If A is a Banach algebra and I is a closed (two sided) ideal in A, then the quotient norm makes A/I into a Banach algebra. If A is a Banach *-algebra and I is a *-ideal (i.e. closed under *), then A/I is a Banach *-algebra. It turns out that if A is a C^*-algebra and I is a closed ideal in A, then I is automatically a *-ideal and A/I is a C^*-algebra in the quotient norm.

The following statements are are very important for the theory of C^*-algebras. They are being proven in [Raeburn2005]. Before we present the first theorem we need a further defintion.

Definition 1.8 (Gelfand transform): Let A be a commutative, unital Banach algebra. The *Gelfand transform* $\chi\colon A \to C(Spec(A))$ is defined by $\chi(x) := \hat{x}$ and $\hat{x}(\phi) := \phi(x)$ for $\phi \in Spec(A)$.

The following theorem is called the commutative Gelfand-Naimark theorem.

Theorem 1.9 (Gelfand-Naimark Theorem, 1943): *The Gelfand transform is an isometric *-isomorphism for commutative, unital C^*-algebras. Hence, we have the following equivalence given a unital C^*-algebra A:*

$$A \text{ is commutative } \Leftrightarrow \exists\, X \text{ compact} : A \cong C(X)$$

The space X is then given by $Spec(A)$. In the non-unital case, we have $A \cong C_0(X)$ for some locally compact space X.

Note, that we allow ourselves to call this commutative Gelfand-Naimark theorem the "1st fundamental Theorem of C^*-algebras" to emphasize the role of this theorem. Furthermore, we need the GNS construction for the noncommutative Gelfand-Naimark theorem, so we present it in the following.

Theorem 1.10 (GNS-construcion (Gelfand-Naimark Segal construction)): *Let A be a C^*-algebra and let $\phi\colon A \to \mathbb{C}$ be a state. There are a Hilbert space H_ϕ, a representation $\pi_\phi\colon A \to B(H_\phi)$, and a cyclic vector $x_\phi \in H_\phi$ such that $\phi(a) = \langle \pi_\phi(a)x_\phi, x_\phi\rangle$ for all $a \in A$. With these properties, the triple $(H_\phi, \pi_\phi, x_\phi)$ is unique up to equivalence.*

Now we derive the noncommutative version of Theorem 1.9 as a consequence of the GNS construction 1.10. Analogously we call it the "2nd Fundamental Theorem of C^*-algebras".

Theorem 1.11 (Gelfand-Naimark Segal Theorem): *Any C^*-algebra A possesses a faithful (i.e. injective) representation $\pi\colon A \to B(H)$ on some Hilbert space H. Thus, A is isomorphic to a C^*-subalgebra of $B(H)$.*

Remark 1.12: We say that "A is concretely represented" in the setting of Theorem 1.11. In particular the abstract definition of C^*-algebras, which means a *-Banach algebra with the C^*-identity, agrees with the concrete defintion of C^*-algebras, which means a norm-closed *-subalgebra of $B(H)$.

1.3 Examples

Now we look at a few examples of C^*-algebras.

Example 1.13: The algebra $B(H)$ of bounded linear operators on a Hilbert space H is a unital C^*-algebra. This is $M_N\mathbb{C}$ in the finite dimensional case.

Example 1.14: If H is infinite dimensional, then also the compact operators $K(H)$ form a C^*-algebra, in fact a non-unital one; if H is finite dimensional, then $K(H) = B(H)$.

The following example is closely related to the commutative Gelfand–Naimark theorem.

Example 1.15: Let X be a compact Hausdorff space and $C(X) \coloneqq \{f\colon X \to \mathbb{C} \text{ continuous}\}$ the complex valued continuous functions on X. Equip $C(X)$ with the following norm $\|f\|_\infty \coloneqq \sup_{x\in X}|f(x)|$, then it is a unital commutative C^*-algebra.

The next example is a non-unital version of the previous one and therefore, closely related to the non-unital version of the Gelfand–Naimark theorem.

Example 1.16: Let X be a locally compact Hausdorff space and $C_0(X) \coloneqq \{f\colon X \to \mathbb{C} \text{ continuous functions vanishing an infinity}\}$ the complex valued continuous functions on X vanishing an infinity. Equip $C_0(X)$ with the following norm $\|f\|_\infty \coloneqq \sup_{x\in X}|f(x)| < \infty$, then it is a non-unital C^*-algebra.

The last example corresponds to the Gelfand–Naimark Segal theorem.

Example 1.17: Any closed C^*-subalgebra of $B(H)$ is a C^*-algebra.

1.4 Universal C^*-algebras

Now we look at the construction of universal C^*-algebras, which we need to define graph C^*-algebras.

Definition 1.18: Let $E = \{x_i \mid i \in I\}$ be a set of elements indexed by a set I.

(a) *A noncommutative monomial* in E is a word $x_{i_1} \dots x_{i_m}$ with $i_1, \dots, i_m \in I$ and $m \in \mathbb{N} \setminus \{0\}$.

(b) *A noncommutative polynomial* in E is a complex linear combination of noncommutative monomials: $\sum_{k=1}^{N} \alpha_k y_k$ with $N \in \mathbb{N}, \alpha_k \in \mathbb{C}$ and $y_1, \dots, y_N$ being noncommutative monomials in E.

(c) The concatenation of two words is defined by the following

$$(x_{i_1} \cdots x_{i_m}) \cdot (x_{j_1} \cdots x_{j_n}) \coloneqq x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_n}$$

where $x_{i_1} \cdots x_{i_m}$ and $x_{j_1} \cdots x_{j_n}$ are two monomials.

(d) The *free algebra on the generator set* E is the set of all noncommutative polynomials with the canonical addition and scalar multiplication. The multiplication of two elements from the free algebra is given by the concatenation.

Note that the order of the elements plays a role for such noncommutative monomials, i.e. $x_1 x_2 \neq x_2 x_1$ in the free algebra. Moreover, the algebra is "free" in the sense that the elements x_i satisfy no relations, i.e. the only polynomial in the generators which is zero, is the zero polynomial itself. Hence, the free algebra has the following universal property: For every algebra B containing a set of elements $\{y_i \mid i \in I\}$ that is indexed by the same set I, we will find a replacement homomorphism from the free algebra to the algebra B sending x_i to y_i, for all $i \in I$. Let $E = \{x_i \mid i \in I\}$ be a set of elements. By adding another set $E^* = \{x_i^* \mid i \in I\}$ which is disjoint with E and by defining an involution on $E \cup E^*$ using the following

$$(\alpha x_{i_1}^{\varepsilon_1} \cdots x_{i_m}^{\varepsilon_m})^* \coloneqq \overline{\alpha} x_{i_m}^{\overline{\varepsilon_1}} \cdots x_{i_m}^{\overline{\varepsilon_1}}$$

where $\alpha \in \mathbb{C}, \varepsilon_k \in \{1, *\}$ and

$$\overline{\varepsilon_k} \coloneqq \begin{cases} 1 & \text{, if } \varepsilon_k = * \\ * & \text{, if } \varepsilon_k = 1 \end{cases}$$

we obtain the *free *-algebra* $P(E)$ on the generator set E.

Definition 1.19: Let $E = \{x_i \mid i \in I\}$ be a set of elements indexed by a set I.

(a) Let $R \subset P(E)$ be a set of polynomials.

(b) Let $J(R) \subset P(E)$ be a two-sided *-ideal generated by R. The *universal *-algebra* with *generator* E and *relations* R is defined as the quotient space $A(E \mid R) \coloneqq P(E)/J(R)$.

Since $J(R)$ is an ideal, the structure of an algebra is kept. By abuse of notation we will write x_i for $\dot{x}_i \in A(E \mid R)$. We still need a C^*-norm, so we first will have a look at C^*-seminorms.

Definition 1.20: Let A be a *-algebra. A *C^*-seminorm* on A is a mapping $p\colon A \to [0, \infty)$, such that

(a) $p(\lambda x) = |\lambda| p(x)$ and $p(x + y) \leq p(x) + p(y)$ for all $x, y \in A$ and $\lambda \in \mathbb{C}$,

(b) $p(xy) \leq p(x)p(y)$ for all $x, y \in A$,

(c) $p(x^*x) = p(x)^2$ for all $x \in A$

holds. We are now able to define universal C^*-algebras.

Definition 1.21: Let E be a set of generators and $R \subset P(E)$ relations. Define

$$\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E \mid R)\}.$$

If $\|x\| < \infty$ for all $x \in A(E \mid R)$ holds, then $\|\cdot\|$ is a C^*-seminorm. To convince oneself, one can easily check (a)-(c) from Definition 1.20. Furthermore, note that $\{x \in A(E \mid R) \mid \|x\| = 0\}$ is a two-sided *-ideal: let $z \in \{x \in A(E \mid R) \mid \|x\| = 0\}$ and $y \in A(E \mid R)$. By using Definition 1.20 (b) we see that

$$0 \leq \|yz\| \leq \|y\|\|z\| = 0$$
$$0 \leq \|zy\| \leq \|z\|\|y\| = 0$$

applies. So if $\|x\| < \infty$ holds for all $x \in A(E \mid R)$, we define the *universal C^*-algebra* $C^*(E \mid R)$ as the completion with respect to the norm $\|\dot{x}\| := \|x\|$:

$$C^*(E \mid R) := \overline{A(E \mid R)/\{x \in A(E \mid R) \mid \|x\| = 0\}}^{\|\cdot\|}$$

where $\dot{x} \in A(E \mid R)/\{x \in A(E \mid R) \mid \|x\| = 0\}$ is the equivalence class of x. Observe that by taking the quotient space the C^*-seminorm becomes a C^*-norm. The completion yields a C^*-algebra.

Let us take a look on a helpful statement to show the existence of a C^*-algebra.

Lemma 1.22: *Let $E = \{x_i \mid i \in I\}$ be a set of generators and $R \subset P(E)$ relations. If a constant C exists such that $p(x_i) < C$ for all $i \in I$ and all C^*-seminorms p on $A(E \mid R)$, then it follows that $\|x\| < \infty$ holds for all $x \in A(E \mid R)$. In that case, we say that the universal C^*-algebra exists in the sense as described above.*

Proof: The norm of a monomial of length N is bounded by C^N and hence every polynomial in $A(E \mid R)$ is bounded. □

Therefore, we have some criterion for the existence of $C^*(E \mid R)$, but it could be possible, that the construction yields the trivial C^*-algebra: $C^*(E \mid R) = 0$. To avoid this triviality, we need to find a non-trivial *-homomorphism from our universal C^*-algebra to another (non-trivial) C^*-algebra. For this the following property is very useful, ensuring the existence of many *-homomorphisms.

Proposition 1.23 (universal property): *Let $E = \{x_i \mid i \in I\}$ be a generator set and $R \subset P(E)$ relations, such that the universal C^*-algebra $C^*(E \mid R)$ exists. Let $E' = \{y_i \mid i \in I\}$ be a subset of some C^*-algebra B. If the elements in E' satisfy the relations R, then there exists a unique *-homomorphism $\phi\colon C^*(E \mid R) \to B$, sending x_i to y_i for all $i \in I$.*

Proof: Recall the replacement *-homomorphism $\phi\colon P(E) \to B$, sending x_i to y_i. Since the elements y_i satisfy the relations R and hence the two-sided *-ideal $P(R)$, generated by R, vanishes in B, the *-homomorphism ϕ induces another *-homomorphism $\phi_0\colon A(E \mid R) \to B$. To prove this, one can consider the following definition $\phi_0\colon A(E|R) \to B$, $\phi_0(\dot{x}) \coloneqq \phi(x)$, where $\dot{x} \in A(E \mid R)$. It is well defined. Let $\dot{x}, \dot{z} \in A(E|R)$ with $\dot{x} = \dot{z}$ and therefore, $x - z \in R$. Then we have

$$\phi_0(\dot{x}) - \phi_0(\dot{z}) = \phi(x - z) = 0.$$

Keep in mind that ϕ_0 is sending $\dot{x_i}$ respectively x_i to y_i. Define $p(\dot{x}) \coloneqq \|\phi_0(\dot{x})\|_B$ for all $\dot{x} \in A(E \mid R)$. One can show that this a C^*-seminorm and hence it follows that $\|\phi_0(\dot{x})\|_B \leq \|\dot{x}\|$ holds. Therefore, ϕ_0 is continuous. We may extend it to a *-homomorphism $\phi\colon C^*(E \mid R) \to B$, sending x_i to y_i for all $i \in I$. Uniqueness is by [ISem24, Lemma 3.26.]. □

The following statement gives a criterion for the existence of universal- and later on for graph and hypergraph C^*-algebras.

Lemma 1.24: *Let A be a universal C^*-algebra that is generated by a partial isometry x and/or projection y. Then the C^*-algebra A exists.*

Proof: Let p be a C^*-seminorm on A. We have

$$\begin{aligned} &p(y)^2 = p(y^*y) = p(y^2) = p(y) \in \{0,1\} \text{ and} \\ &p(x)^4 = p(x^*x)^2 = p(x^*xx^*x) = p(x^*x) = p(x)^2 \in \{0,1\}. \end{aligned}$$

By Lemma 1.22 the C^*-algebra exists. □

We define in the next step a product of two C^*-algebras. We will see what that looks like in an example later on.

Definition 1.25: Let $A = C^*(E_1 \mid R_1)$ and $B = C^*(E_2 \mid R_2)$ be unital universal C^*-algebras. We call

$$A *_{\mathbb{C}} B \coloneqq C^*(E_1, E_2 \mid R_1, R_2 \text{ and } 1_A = 1_B)$$

the *free product* of A and B.

1.5 Examples

In this section we take a look on a few examples, that will come up several times.

Definition 1.26: Let $N \in \mathbb{N}$ with $N \geq 2$. The following C^*-algebras are isomorphic:

(a) $M_N(\mathbb{C})$,

(b) $C^*(e_{ij}; i,j = 1,\dots,N \mid e_{ij}^* = e_{ji}; e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i,j,k,l)$.

The next definition is a infinte version of Definition 1.26. By $\mathcal{K}(H)$ we mean the space of all compact operators on a separable Hilbert space H.

Definition 1.27: The following C^*-algebras are isomorphic:

(a) $\mathcal{K}(H)$,

(b) $C^*(e_{ij}; i,j \in \mathbb{N} \mid e_{ij}^* = e_{ji}; e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i,j,k,l)$.

The following example describes the functions on the unit circle.

Definition 1.28: Let A be a unital C^*-algebra and $z \in A$ a unitary element with $sp(z) = S^1$, where $S^1 := \{x \in \mathbb{C} \mid |x| = 1\}$. Then we have $C^*(u,1 \mid u^*u = 1uu^*) \cong C^*(z) \subset A$. Notice that we write $C^*(z)$ instead of $C^*(z,1)$ because z is a unitary. Furthermore, we have $C^*(u,1 \mid u^*u = 1 = uu^*) \cong C(S^1)$.

This algebra will be one of the most interesting examples in the thesis.

Definition 1.29 (Toeplitz algebra): The universal C^*-algebra

$$\mathcal{T} := C^*(u,1 \mid u^*u = 1)$$

generated by an isometry u, meaning $u^*u = 1$, is the so called *Toeplitz algebra*.

Remark 1.30: The Toeplitz algebra $\mathcal{T}$ can be viewed as an extension of the continuous functions on the unit circle $C(S^1)$ by the compact operators $\mathcal{K}(H)$. Thus $C(S^1)$ is obtained as the quotient $\mathcal{T}/\mathcal{K}$.

More precisely, there is the following short exact sequence.

Proposition 1.31: *The ideal* $(1 - vv^*) \triangleleft \mathcal{T}$ *is isomorphic to* $\mathcal{K}(H)$, *where* H *is a separable Hilbert space. The quotient by this ideal is isomorphic to* $C(S^1)$. *Hence, we have the following short exact sequence:*

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

Proof: See [ISem24, Proposition 6.23]. □

Remark 1.32: Let $\mathcal{T}$ be the Toeplitz algebra and $C(S^1)$ be the continuous functions on the unit circle. Then we get with the universal property 1.23 a *-homomorphism $\phi : \mathcal{T} \coloneqq C^*(v, 1 \mid v^*v = 1) \to C(S^1) \cong C^*(u, 1 \mid u^*u = 1 = uu^*)$, sending v to u, since any unitary is already an isometry. The reverse direction does not exist, because of 1.31.

The following two examples will be very interesting through this work, too.

Definition 1.33 (Cuntz algebra): Let $n \in \mathbb{N}$ and $n \geq 2$. We call the universal C^*-algebra

$$\mathcal{O}_n \coloneqq C^*(S_1, \ldots, S_n \mid S_i^* S_i = 1 \text{ for all } i = 1, \ldots, n; \sum_{i=1}^{n} S_i S_i^* = 1)$$

the *Cuntz algebra*.

Definition 1.34 (Extended Cuntz algebra): Let $n \in \mathbb{N}$ and $n \geq 2$. We call the universal C^*-algebra

$$\mathcal{E}_n \coloneqq C^*(T_1, \ldots, T_n \mid T_i \text{ isometries, for all } i = 1, \ldots, n, T_i^* T_j = \delta_{ij})$$

the *extended Cuntz algebra*.

We now give an analogous remark to 1.30 for the extended Cuntz algebra.

Remark 1.35: The extended Cuntz algebra $\mathcal{E}_n$ can be viewed as an extension of the Cuntz algebra $\mathcal{O}_n$ by the compact operators $\mathcal{K}(H)$. Thus $\mathcal{O}_n$ is obtained as the quotient $\mathcal{E}_n/\mathcal{K}$.

More precisely, there is the following short exact sequence.

Remark 1.36: The ideal $(1 - \sum T_i T_i^*) \lhd \mathcal{E}_n$ is isomorphic to $\mathcal{K}(H)$, where H is a separable Hilbert space. The quotient by this ideal is isomorphic to $\mathcal{O}_n$. Hence, we have the following short exact sequence:

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{O}_n \longrightarrow 0$$

Remark 1.37: Let $n \in \mathbb{N}$, $n \geq 2$ and $\mathcal{E}_n$ be the extended Cuntz algebra. It holds the following.

$$\begin{aligned} \mathcal{E}_n &\cong C^*(T_1, \ldots, T_n \mid T_i \text{ isometries, for all } i = 1, \ldots, n, T_i^* T_j = \delta_{ij}) \\ &\cong C^*(T_1, \ldots, T_n \mid T_i \text{ isometries, for all } i = 1, \ldots, n, \sum_{i=1}^{n} T_i T_i^* \leq 1). \end{aligned}$$

We get with the universal property 1.23 a *-homomorphism $\phi\colon \mathcal{E}_n \to \mathcal{O}_n$, sending T_i to S_i since the generators of the Cuntz-algebra satisfy the relations of the extended Cuntz algebra:

$$\sum_{i=1}^{n} T_i T_i^* = 1 \Rightarrow T_i T_i^* \perp T_j T_j^*, \text{ i.e. } T_i T_i^* T_j T_j^* = 0 \Rightarrow T_I^* T_j = \delta_{ij}.$$

We don't get a *-homomorphism the other direction, because of 1.35.

Definition 1.38: Let $N \in \mathbb{N}$. The following C^*-algebras are isomorphic:

(a) $\mathbb{C}^N$ as a C^*-algebra with pointwise operations,

(b) $C^*(p_1, \ldots, p_N, 1 \mid p_i = p_i^* = p_i^2,\ i = 1, \ldots, N,\ \sum_{i=1}^{N} p_i = 1)$.

Definition 1.39: Let $N \in \mathbb{N}$. The following C^*-algebras are isomorphic:

(a) $M_N(\mathbb{C}) * (C(S^1))$

(b) $C^*(e_{ij}, i, j = 1, \ldots, N, u \mid u \text{ is a unitary}, e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il})$.

1.6 Graph C^*-algebras

We are slowly approaching the main topic of this paper, but first we have introduce the concept of graph C^*-algebras. We define them based on our knowledge about universal C^*-algebras, and construct them from graphs.

Definition 1.40: A *directed finite graph* $\Gamma = (V, E, r, s)$ consists of two finite sets V, E and functions $r\colon E \to V$ for the *range map* and $s\colon E \to V$ for the *source map*. The elements of V are called *vertices* and the elements of E are called *edges*. Throughout this thesis we will only look at directed finite graphs. Therefore, we will just say graphs. We say that $v \in V$ is a *sink* iff the set $s^{-1}(v)$ is empty and we call v a *source* iff $r^{-1}(v)$ is empty.

For a better understanding, we take a look at an example of a graph.

Example 1.41: Consider the following graph Γ with vertices $V = \{v_1, v_2, v_3\}$ and edges $E = \{e_1, e_2, e_3, e_4\}$. For the image of our range and source map we have

$$\begin{aligned} r(e_1) &= \{v_1\}, & s(e_1) &= \{v_1\}, \\ r(e_2) &= \{v_1\}, & s(e_2) &= \{v_2\}, \\ r(e_3) &= \{v_1\}, & s(e_3) &= \{v_3\}, \\ r(e_4) &= \{v_2\}, & s(e_4) &= \{v_3\}. \end{aligned}$$

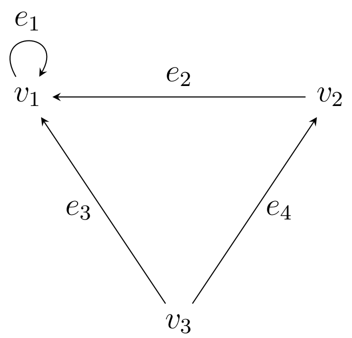

Figure 1.1: Example of a graph

In the following we define graph C^*-algebras based on our definitions of graphs and universal C^*-algebras.

Definition 1.42 (graph C^*-algebras): Let $\Gamma = (V, E, r, s)$ be a graph. The graph C^*-algebra $C^*(\Gamma)$ of the graph Γ is the universal C^*-algebra generated by mutually orthogonal projections p_v for all $v \in V$ and partial isometries s_e for all $e \in E$ such that the following relations hold

$$\text{(R1)} \quad s_e^* s_f = \delta_{ef} p_r(e) \qquad \text{for all } e, f \in E,$$
$$\text{(R2)} \quad p_v = \sum_{\substack{e \in E \\ s(e)=v}} s_e s_e^* \qquad \text{for all } v \in V, \text{ in case that v is not a sink.}$$

It is to mention that every graph C^*-algebra exists by Lemma 1.24.

Remark 1.43: Notice that $s_e^* s_f = 0$ iff $e \neq f$ and hence $s_e s_e^* s_f s_f^* = 0$ for all $e, f \in E$ with $e \neq f$. So the projections $\{s_e s_e^* \mid e \in E\}$ are mutually orthogonal.

Remark 1.44: If the graph is infinite, meaning that V and E are infinite but countable sets, we only consider vertices $v \in V$ for (R2) where $s^{-1}(v)$ is non-empty and finite. Because otherwise we need to consider the infinite sum $\sum_{\substack{e \in E \\ s(e)=v}} s_e s_e^*$ of mutually orthogonal projections which does not converge in norm. To see this let $0 < \varepsilon < 1$ and assume that the sum converges. Notice that we replace E with $\mathbb{N}$ since E is countable. Since it converges it is also a Cauchy sequence. Hence there exists $N \in \mathbb{N}$ such that $\|\sum_{i=n}^{m} s_i s_i^*\| < \varepsilon$ for $n, m \geq N$ which implies that $\|s_n s_n^*\| < \varepsilon$ for $n \geq N$. Since ε was arbitrary we conclude $\|s_n s_n^*\| = 0$ for $n \geq N$. This would imply that $s_n s_n^* = 0$ for all $n \geq N$ which would be a contradiction. We also add Relation (R3) to our relations:

$$\text{(R3)} \quad s_e s_e^* \leq p_{s(e)} \text{ for all } e \in E.$$

We need this relation (R3) for vertices where $s^{-1}(v)$ is infinite. If for all $v \in V$ the set $s^{-1}(v)$ is finite or empty one can show that (R2) implies (R3). In that case we call the graph row-finite. In this thesis we only consider row-finite graphs.

Now we are going to prove a few properties of graph C^*-algebras, which we will need for the characterization of our examples.

Proposition 1.45: *Let $\Gamma = (V, E, r, s)$ be a graph and $C^*(\Gamma)$ the corresponding graph C^*-algebra. Then the following equations hold*

$$s_e = s_e p_{r(e)} \tag{1.1}$$
$$s_e = p_{s(e)} s_e \tag{1.2}$$

for all $e \in E$.

Proof: Let $e \in E$. By (R1) and the fact that s_e is a partial isometry we have $s_e = s_e s_e^* s_e = s_e p_{r(e)}$, which shows (1.1). We are going to prove Equation (1.2). Observe that we can write $p_{s(e)} = \sum_{s(f)=s(e)} s_f s_f^*$. Consequently it is

$$p_{s(e)} s_e = \sum_{s(f)=s(e)} s_f s_f^* s_e \overset{(R1)}{=} s_e s_e^* s_e = s_e.$$

□

Proposition 1.46: *Let $\Gamma = (V, E, r, s)$ be a graph and $C^*(\Gamma)$ the corresponding graph C^*-algebra. Then we have that $\sum_{v \in V} p_v$ is the identity in $C^*(\Gamma)$ and hence $\sum_{v \in V} p_v = 1$.*

Proof: Let $w \in V$. Since the projections are mutually orthogonal we have

$$p_w \sum_{v \in V} p_v = \sum_{v \in V} p_w p_v = p_w^2 = p_w = (\sum_{v \in V} p_v) p_w.$$

Let $e \in E$. Since s_e is a partial isometry we have by Relation (R1)

$$s_e \sum_{v \in V} p_v = s_e s_e^* s_e \sum_{v \in V} p_v = s_e p_{r(e)} \sum_{v \in V} p_v = s_e p_{r(e)} = s_e.$$

Using Equation (1.2) it follows

$$(\sum_{v \in V} p_v) s_e = (\sum_{v \in V} p_v) p_{s(e)} s_e = p_{s(e)} s_e = s_e$$

and hence $\sum_{v \in V} p_v = 1$. □

Proposition 1.47: *Let $\Gamma = (V, E, r, s)$ be a graph and $C^*(\Gamma)$ the corresponding graph C^*-algebra. Then we have that* *(a) the projections* $\{s_e s_e^* \mid e \in E\}$ *are mutually orthogonal*

(b) $s_e s_f^* \neq 0 \Rightarrow r(e) = r(f)$.

Proof: (a) We have $(s_e s_e^*)^* = s_e s_e^*$ and $s_e s_e^* s_e s_e^* = s_e s_e^*$. Therefore, $s_e s_e^*$ is a projection. To show orthogonality see Remark 1.43.

(b) It apllies that $s_e s_f^* = s_e p_{r(e)} p_{r(f)} s_f^* = 0$, if $r(e) \neq r(f)$. □

The following proposition is adapted from [Raeburn2005, Proposition 1.18]. Since Raeburn formulates the result using the Australian convention, while we work with the European convention, the roles of source and range are then reversed. We therefore, restate the proposition in our notation. We need this statement for some example.

Proposition 1.48: *Suppose* $\Gamma = (V, E, r, s)$ *is a finite directed graph with no cycles, and* $w_1, \ldots, w_n$ *are the sinks in* Γ. *Then we have*

$$C^*(\Gamma) \cong \bigoplus_{i=1}^{n} M_{n_i}(\mathbb{C}),$$

where $n_i = |r^{-1}(w_i)| = |\{\mu \in E^* \mid r(\mu) = w_i\}|$ *denotes the number of paths, which ends at the sink* w_i, $E^* \coloneqq \bigcup_{n \geq 1} E^n$.

1.7 Examples

We will now revisit some familiar examples from previous sections as graph C^*-algebras.

Example 1.49 (Toeplitz algebra): Consider the following graph Γ with vertices $\{v, w\}$, edges e, f and the mappings defined as $r(e) = \{v\}$, $s(e) = \{v\}$ and $r(f) = \{w\}$, $s(f) = \{v\}$ and the associated graph C^*-algebra $C^*(\Gamma)$. We have $C^*(\Gamma) \cong \mathcal{T}$ where $\mathcal{T}$ is the Toeplitz algebra (see [Zen21, Proposition 2.10]).

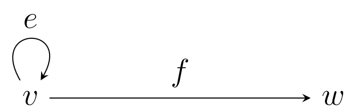

Figure 1.2: Toeplitz graph

Example 1.50 (Cuntz algebra): Let $n \in \mathbb{N}$ and $n \geq 2$. Consider to the next graph Γ with vertices $\{v\}$, edges $\{e_1, \ldots, e_n\}$ and the mappings defined as $r(e_i) = \{v\}$, $s(e_i) = \{v\}$ for all $i = 1, \ldots, n$ the associated graph C^*-algebra $C^*(\Gamma)$. It follows $C^*(\Gamma) \cong \mathcal{O}_n$ where $\mathcal{O}_n$ is the Cuntz algebra (see [Zen21, Proposition 2.13]).

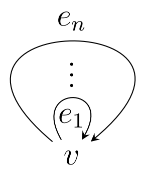

Figure 1.3: Cuntz graph

The graph associated with the C^*-algebra $C(S^1)$ arises as the special case $n = 1$ of the Cuntz graph construction.

Example 1.51: Consider to the next graph Γ with vertices $\{v\}$, edges $\{e\}$ and the mappings defined as $r(e) = \{v\}, \quad s(e) = \{v\}$ the associated graph C^*-algebra $C^*(\Gamma)$. It follows $C^*(\Gamma) \cong C^*(u, 1 \mid u^*u = 1 = uu^*) \cong C(S^1)$ where $C(S^1)$ are the continuous functions on the unit circle (see [Zen21, Proposition 2.9]).

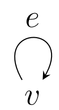

Figure 1.4: $C(S^1)$ graph

Example 1.52: Let $n \in \mathbb{N}$ and let $C^*(\Gamma)$ be the graph C^*-algebra of the following graph Γ with vertices $\{v_1, \dots, v_n\}$, edges $\{e_1, \dots, e_{n-1}\}$ and the mappings defined as $r(e_i) = \{v_i\}, \quad s(e_i) = \{v_{i+1}\}$ for all $i = 1, \dots, n-1$. It follows $C^*(\Gamma) \cong M_N\mathbb{C}$ where $M_N\mathbb{C}$ is the set of n×n-matrices (see [Zen21, Proposition 2.11]).

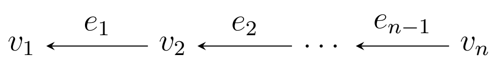

Figure 1.5: Matrix graph

One can check with Proposition 1.48 that the associated graph C^*-algebra is $M_N\mathbb{C}$.

2 Hypergraph C^*-algebras

We now come to the main theory of the work, where we introduce the so called *hypergraph C^*-algebras*. Since this topic is still relatively new and has not been studied extensively yet, we will try to find some new examples. But first, we need to define hypergraphs.

Definition 2.1: A *directed finite hypergraph* $H\Gamma = (V, E, r, s)$ consists of two finite sets V, E and two mappings $r, s \colon E \to \mathcal{P}(V) \backslash \{\varnothing\}$. The set V contains *vertices*, while the set E contains *hyperedges*. The difference to a directed finite graph, as in Definition 1.40, is that the hyperedges can join any number of vertices whereas for graphs, we always have the restriction $|r(e)| = 1 = |s(e)|$ for all $e \in E$. Therefore, the range map r and the source map s map to the power set $\mathcal{P}(V)$ of V rather than to V. We only study directed finite hypergraphs. We write hypergraph instead of directed finite hypergraph.

For a better understanding, we take a look at an example of a hypergraph.

Example 2.2: Consider the following hypergraph $H\Gamma$ with vertices $V = \{v_1, v_2, v_3, v_4\}$ and edges $E = \{e_1, e_2, e_3\}$. For the image of our range and source map we have

$$\begin{aligned} r(e_1) &= \{v_1, v_2, v_4\}, & s(e_1) &= \{v_1\}, \\ r(e_2) &= \{v_2, v_4\}, & s(e_2) &= \{v_3\}, \\ r(e_3) &= \{v_3\}, & s(e_3) &= \{v_2, v_4\}. \end{aligned}$$

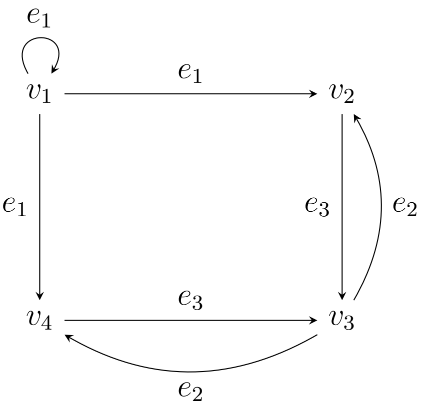

Figure 2.1: Example of a hypergraph

In this thesis we represent hypergraphs as shown in the example above. There are different ways to represent hypergraphs, but for most of the following examples, this method of representation is entirely sufficient. We will also look at an example 3.13, where we represent the hypergraph in a different way. In this case, it is a hyperbranch. We will take a closer look at what that is at the end of this section.

Remark 2.3: Notice that every graph $\Gamma = (V, E, r, s)$ is also a hypergraph $H\Gamma = (V, E, r', s')$ by defining $r' : E \to \mathcal{P}(V)$, $e \to \{r(e)\}$ and $s' : E \to \mathcal{P}(V)$, $e \to \{s(e)\}$.

Similar as in Definition 1.42 for constructing graph C^*-algebras, we will define hypergraph C^*-algebras based on our knowledge about hypergraphs and universal C^*-algebras.

Definition 2.4 (hypergraph C^*-algebras): Let $H\Gamma = (V, E, r, s)$ be a hypergraph. The hypergraph C^*-algebra $C^*(H\Gamma)$ of the hypergraph $H\Gamma$ is the universal C^*-algebra generated by mutually orthogonal projections p_v for all $v \in V$ and partial isometries s_e for all $e \in E$ such that the following relations hold

$$\text{(HR1)} \quad s_e^* s_f = \delta_{ef} \sum_{v \in r(e)} p_v \qquad \text{for all } e, f \in E$$

$$\text{(HR2)} \quad s_e s_e^* \leq \sum_{v \in s(e)} p_v \qquad \text{for all } e \in E$$

$$\text{(HR3)} \quad p_w \leq \sum_{\substack{e \in E \\ w \in s(e)}} s_e s_e^* \qquad \text{if } s^{-1}(w) \neq \varnothing \text{ for } w \in V.$$

It is to mention that every hypergraph C^*-algebra exists by Lemma 1.24.

In the following, we present several remarks on projections that will be useful in subsequent arguments.

Remark 2.5: We have for every projection p and q in some C^*-algebra A the equivalence $p \leq q \iff pq = p = qp$.

Lemma 2.6: *Let p and q be projections in some C^*-algebra A. If $p \leq q$ and $q \leq p$ applies, then we have $p = q$.*

Proof: It follows
$$p = pq = q.$$
□

Lemma 2.7: *Let p_1, p_2, q be projections in some C^*-algebra A. If $p_i \leq q$ for $i = 1, 2$ and $q \leq p_1 + p_2$ applies, then we have $q = p_1 + p_2$.*

Proof: It follows
$$q = q(p_1 + p_2) = qp_1 + qp_2 = p_1 + p_2.$$
□

The question remains whether the hypergraph C^*-algebra $C^*(H\Gamma)$ of an arbitrary hypergraph $H\Gamma$ is trivial or not. The following statement shows that the class of graph C^*-algebras is contained in the class of hypergraph C^*-algebras and thus our definition of hypergraph C^*-algebras represents the desired generalization. We know that every graph C^*-algebra is non-trivial (see [Raeburn2005]) and have thus found a class of non-trivial hypergraph C^*-algebras. The proposition also shows that the Relations (HR2) and (HR3) are the corresponding relations to Relation (R2).

Proposition 2.8: *Consider a graph $\Gamma = (V, E, r, s)$ and interpret it as a hypergraph $H\Gamma = (V, E, r', s')$ in the sense of Remark 2.3. For our graph C^*-algebra we write*

$$C^*(\Gamma) = C^*(\tilde{s}_e, e \in E; \tilde{p}_v, v \in V \mid \tilde{p}_v\tilde{p}_w = 0, v \neq w; \tilde{s}_e^*\tilde{s}_f = \delta_{ef}\tilde{p}_{r(e)}; \sum_{\substack{e \in E \\ s(e)=w}} \tilde{s}_e\tilde{s}_e^* = \tilde{p}_w)$$

where $\tilde{s}_e$ is a partial isometry for all $e \in E$ and $\tilde{p}_v$ is a projection for all $v \in V$. Then we have $C^*(\Gamma) \cong C^*(H\Gamma)$.

Proof: See [Zen21, Proof of Proposition 3.8.] □

In the following, we introduce a useful result for the remaining parts of this thesis. It says that the projections sum up to the identity. We know that it holds for every graph C^*-algebra (see Proposition 1.46). It is nice to see that, even though we are generalizing graph C^*-algebras, we can show that this statement still holds.

Theorem 2.9: *For every hypergraph $H\Gamma = (V, E, r, s)$ and hypergraph C^*-algebra $C^*(H\Gamma)$ we have that $\sum_{v\in V} p_v$ is the unit element in $C^*(H\Gamma)$ and therefore, $\sum_{v\in V} p_v = 1$.*

Proof: Using Relation (HR1) and 1.1, we have

$$\begin{aligned} s_e \sum_{v\in V} p_v &= s_e s_e^* s_e \sum_{v\in V} p_v \\ &= s_e \sum_{v\in r(e)} p_v \sum_{v\in V} p_v \\ &= s_e \sum_{v\in r(e)} p_v \\ &= s_e. \end{aligned}$$

It follows with 1.2 that

$$\begin{aligned} (\sum_{v\in V} p_v)s_e &= (\sum_{v\in V} p_v)(\sum_{v\in s(e)} p_v)s_e \\ &= (\sum_{v\in s(e)} p_v)s_e \\ &= s_e. \end{aligned}$$

Notice that we have

$$\sum_{v\in V} p_v p_w = p_w = p_w \sum_{v\in V} p_v \quad \text{for all } w \in V,$$

and

$$(\sum_{v\in V} p_v)^2 = \sum_{v\in V} p_v = (\sum_{v\in V} p_v)^*.$$

We conclude that $\sum_{v\in V} p_v$ is the unit element in $C^*(H\Gamma)$ (see [Zen21, Proof of Theorem 3.9.]). □

2.1 Further interesting definitions

In this section, we will briefly mention some other interesting topics that we will need for the upcoming examples.

Definition 2.10 (full corner): Let A be a C^*-algebra and p be a projection. Then the set

$$pAp := \{pap \mid a \in A\}$$

is a C^*-subalgebra of A, which is called a *corner* of A. The corner is *full* if it is not contained in any proper ideal of A, or in other words if the set ApA spans a dense subspace of A.

We now give the defintion of the so called *hyperbranches* (see [AK26]), which we represent in a different way to the normal hypergraphs.

Definition 2.11 (Hyperbranch): Let $H\Gamma = (V, E, r, s)$ be a hypergraph. We call $H\Gamma$ a *hyperbranch of length* k if the vertex set V can be partitioned into k disjoint sets $V_1, \ldots, V_k$ such that:

(i) The vertex set is the disjoint union
$$V = \bigsqcup_{i=1}^{k} V_i.$$

(ii) The edge set consists of $k-1$ edges, denoted by
$$E = \{e_1, \ldots, e_{k-1}\}.$$

(iii) For each edge e_i, the source is the set V_i and the range is the set V_{i+1}. That is,
$$s(e_i) = V_i \qquad \text{and} \qquad r(e_i) = V_{i+1}$$
for all $1 \leq i \leq k-1$.

Hyperbranches can be visualized as a sequence of vertex sets connected by hyperedges, where each hyperedge connects all vertices in one set to all vertices in the next set.

Example 2.12: Consider the hypergraph $H\Gamma = (V, E, r, s)$ with vertex set $V = \{V_1, V_2, V_3\}$, where each V_i consists of 2 vertices for all $i = 1, \ldots, 3$ and edge set $E = \{e_1, e_2, e_3\}$. The range and source map are defined as $r(e_i) = V_{i+1}$ and $s(e_i) = V_i$ for all $i = 1, 2$.

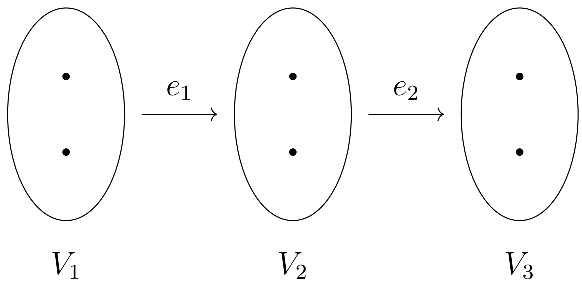

Figure 2.2: Hyperbranch of length 3

The next proposition is due to [AK26, Theorem 3.11], which is a version of Raeburn's Proposition 1.48 for hypergraphs respectively hyperbranches.

Proposition 2.13 (Hyperbranch Structure Theorem): *Let $H\Gamma$ be a hyperbranch of length k with vertex sets $V_1, \ldots, V_k$ and edges $e_1, \ldots, e_{k-1}$. Let $n_j = |V_j|$ denote the number of vertices in the set V_j. Then there is an isomorphism:*
$$C^*(H\Gamma) \cong M_k(\mathbb{C}^{n_1} *_{\mathbb{C}} \mathbb{C}^{n_2} *_{\mathbb{C}} \cdots *_{\mathbb{C}} \mathbb{C}^{n_k}),$$
where $*_{\mathbb{C}}$ denotes the unital free product of C^*-algebras.

2.2 Examples

To conclude this chapter we will look at some interesting examples of hypergraph C^*-algebras. It is notable that the Toeplitz algebra have been well-studied to this point and yield promising results in this theory of operator algebras. The graph and hypergraph examples of the Toeplitz algebra are therefore very promising and could be well-suited for further investigation. Hence, the first example, in particular, will come up frequently.

Example 2.14 (Toeplitz algebra): Consider the two following hypergraphs $H\Gamma_1$ with vertices $\{v, w\}$, edges $\{e\}$ and the mappings defined as $r(e) = \{v, w\}, \quad s(e) = \{v\}$ and $H\Gamma_2$ with vertices $\{v, w\}$, edges $\{e\}$ and the mappings defined as $r(e) = \{v\}, \quad s(e) = \{v, w\}$ and their associated hypergraph C^*-algebras $C^*(H\Gamma_1)$ and $C^*(H\Gamma_2)$. We have $C^*(H\Gamma_i) \cong \mathcal{T}$ for all $i = 1, 2$ where $\mathcal{T}$ is the Toeplitz algebra. (see [TWZ24, Appendix A])

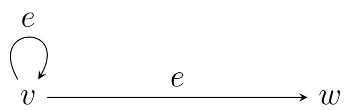

Figure 2.3: Toeplitz hypergraph 1

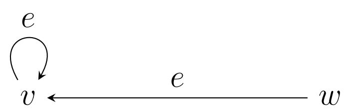

Figure 2.4: Toeplitz hypergraph 2

Example 2.15: Let $n \in \mathbb{N}$ and consider the hypergraph $H\Gamma$ with vertices $\{v_1, \dots, v_n\}$, edges $\{e_1\}$ and the mappings defined as $r(e_1) = \{v_1, \dots, v_n\}$ and $s(e_1) = \{v_1, \dots, v_n\}$. We have $C^*(H\Gamma) \cong C(S^1) * \mathbb{C}^n$. (see [Zen21, Proposition 3.12])

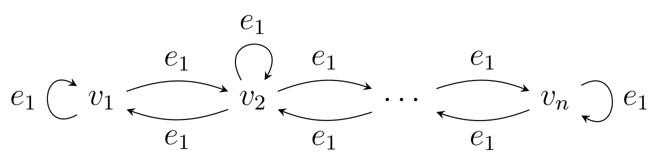

Figure 2.5: $C(S^1) * \mathbb{C}^n$ hypergraph

3 Construction of new hypergraph C^*-algebras

In the previous chapter we introduced the basic definitions of hypergraph C^*-algebras. We now turn to the central part of this thesis, namely the construction of new examples of hypergraph C^*-algebras. The main idea underlying these constructions is to start from well-known graph and hypergraph C^*-algebras and to systematically extend them to the hypergraph setting. More precisely, we will repeatedly consider a specific graph example and then apply certain structural modifications to it. Typical modifications include renaming edges, introducing additional loops, or duplicating structural features of the graph. By carrying out these changes step by step, we obtain new hypergraph constructions whose associated C^*-algebras can then be analysed. This approach leads to a number of interesting examples and structural results. In particular, one of the main results of this thesis is the realization of the extended Cuntz algebra $\mathcal{E}_n$ (3.2) as a hypergraph C^*-algebra, which arises from a modification of the Toeplitz graph (1.49). Another method for constructing new examples consists of combining different hypergraphs to form larger ones. This construction will be studied in detail in a later section. Throughout this chapter, the Toeplitz algebra (1.29) and the Cuntz algebra (1.33) will serve as recurring reference examples. Various modifications of these hypergraphs will illustrate how the described procedures lead to new hypergraph C^*-algebras and how the corresponding algebraic structures behave.

3.1 Modifications to well-known isomorphic C^*-algebras

We present in this section the most interesting examples of this chapter, since we found in each case a *-isomorphism between the associated hypergraph C^*-algebra and a well-known C^*-algebra. We begin with a modification of the Toeplitz hypergraph (2.14). This construction also appears in the work of Dean Zenner [Zen21]. The following hypergraph is obtained by adding an additional loop to the Toeplitz hypergraph.

3.1.1 $C(S^1) * \mathbb{C}^2$

Proposition 3.1: *Consider the hypergraph $H\Gamma = (V, E, r, s)$ with vertices $V = \{v, w\}$ and edges $E = \{e\}$. The range and source map are defined as follows $r(e) = V = s(e)$. It holds that $C^*(H\Gamma) \cong C(S^1) * \mathbb{C}^2$ where $C(S^1) * \mathbb{C}^2$ is the case with $n = 2$ from 2.15.*

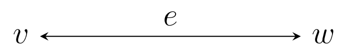

Figure 3.1: Toeplitz with extra loop

Proof: We have:

$$\begin{aligned} s_e^* s_e &= p_v + p_w = 1, \\ s_e s_e^* &\leq p_v + p_w = 1, \\ p_v &\leq s_e s_e^*, \\ p_w &\leq s_e^* s_e. \end{aligned}$$

Hence

$$1 = p_v + p_w = s_e s_e^* = s_e^* s_e.$$

Therefore, s_e is a unitary. For more details of the proof see [Zen21, Proposition 3.12.]. □

3.1.2 Extended Cuntz-algebra

We now consider an n-fold version of the Toeplitz hypergraph (2.14). We will show that the associated hypergraph C^*-algebra is the extended Cuntz algebra $\mathcal{E}_n$.

Proposition 3.2: *Let $n \in \mathbb{N}$. Consider to the next hypergraph $H\Gamma$ with vertices $V = \{v, w\}$ and edges $E = \{e_1, \ldots, e_n\}$ the associated C^*-algebra $C^*(H\Gamma)$. The image of the range and source map looks as follows $r(e_i) = V$ and $s(e_i) = \{v\}$ for all $i = 1, \ldots, n$. It follows $C^*(H\Gamma) \cong \mathcal{E}_n$ where $\mathcal{E}_n$ is the extended Cuntz-algebra (1.34).*

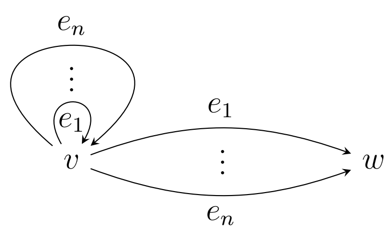

Figure 3.2: Extended Cuntz-algebra hypergraph

Proof: Step 1: There exists a *-homomorphism $\phi_1 \colon \mathcal{E}_n \to C^*(H\Gamma)$, sending T_i to s_i and 1 to $p_v + p_w = s_i^* s_i$. We know with the Remark 1.37 that

$$\begin{aligned}\mathcal{E}_n &\cong C^*(T_1, \dots, T_n \mid T_i \text{ isometries, for all } i = 1, \dots, n, T_i^* T_j = \delta_{ij}) \\ &\cong C^*(T_1, \dots, T_n \mid T_i \text{ isometries, for all } i = 1, \dots, n, \sum_{i=1}^n T_i T_i^* \leq 1).\end{aligned}$$

For simplicity, write $s_i := s_{e_i}$. We know from the Proposition 1.46, that $p_v + p_w$ is the identity in $C^*(H\Gamma)$. Hence we get with the hypergraph relation (HR1) $s_i^* s_i = p_v + p_w = 1$ for all $i = 1, \dots, n$. Using the relations (HR2) and (HR3) imply $s_i s_i^* \leq p_v$ for all $i = 1, \dots, n$ and $p_v \leq \sum_{i=1}^n s_i s_i^*$. Notice that w is a sink and so the third hypergraph relation (HR3) does not apply for this vertex, just for v. Hence each s_i is an isometry for all $i = 1, \dots, n$ and with the Lemma 2.7 it holds

$$\sum_{i=1}^n s_i s_i^* = p_v \leq p_v + p_w = 1.$$

Also $\{s_i \mid i = 1, \dots, n\}$ is the generator of $C^*(H\Gamma)$. We obtain with the universal property 1.23 a *-homomorphism $\phi_1 \colon \mathcal{E}_n \to C^*(H\Gamma)$, sending T_i to s_i and 1 to $p_v + p_w$.

Step 2: Consider now the definition $\tilde{p}_v := \sum_{i=1}^n t_i t_i^*, \tilde{p}_w := 1 - \tilde{p}_v$ and $\tilde{s}_i := t_i$. There exists a *-homomorphism $\phi_2 \colon C^*(H\Gamma) \to \mathcal{E}_n$, sending p_v to $\tilde{p}_v$, p_w to $\tilde{p}_w$ and s_i to $\tilde{s}_i$. We check that $\tilde{s}_i$ are partial isometries for all $i = 1, \dots, n$ and $\tilde{p}_v, \tilde{p}_w$ are projections.

$$\begin{aligned}\tilde{s}_i \tilde{s}_i^* \tilde{s}_i &= t_i t_i^* t_i = t_i 1 = t_i = \tilde{s}_i \\ \tilde{p}_v^2 &= (\sum_{i=1}^n t_i t_i^*)(\sum_{i=1}^n t_i t_i^*) = (\sum_{i=1}^n q_i)(\sum_{j=1}^n q_j) = \sum_{i=1}^n q_i = \sum_{i=1}^n t_i t_i^* = \tilde{p}_v \\ \tilde{p}_v^* &= (\sum_{i=1}^n t_i t_i^*)^* = \sum_{i=1}^n (t_i t_i^*)^* = \sum_{i=1}^n t_i t_i^* = \tilde{p}_v \\ \tilde{p}_w^2 &= (1 - \tilde{p}_v)^2 = (1 - \sum_{i=1}^n t_i t_i^*)(1 - \sum_{i=1}^n t_i t_i^*) \\ &= 1 - \sum_{i=1}^n t_i t_i^* - \sum_{i=1}^n t_i t_i^* + (\sum_{i=1}^n t_i t_i^*)(\sum_{i=1}^n t_i t_i^*) \\ &= 1 - \tilde{p}_v - \tilde{p}_v + \tilde{p}_v^2 = 1 - \tilde{p}_v = \tilde{p}_w \\ \tilde{p}_w^* &= (1 - \tilde{p}_v)^* = 1 - \tilde{p}_v = \tilde{p}_w \\ \tilde{p}_v \tilde{p}_w &= \tilde{p}_v (1 - \tilde{p}_v) = \tilde{p}_v - \tilde{p}_v^2 = \tilde{p}_v - \tilde{p}_v = 0\end{aligned}$$

We used Proposition 1.5, such that $q_j := t_i t_i^*$ for all $i = 1, \dots, n$ are mutually orthogonal projections. Now, we have to check the relations of $C^*(H\Gamma)$. We start with (HR1). We have for all $i = 1, \dots, n$

$$\tilde{s}_i^* \tilde{s}_i = t_i^* t_i = 1 = \tilde{p}_v + \tilde{p}_w.$$

Therefore, (HR1) is fulfilled. Furthermore, it is

$$\tilde{s}_i \tilde{s}_i^* = t_i t_i^* \leq \sum_{i=1}^{n} t_i t_i^* = \tilde{p}_v$$

and we see that (HR2) and (HR3) are satisfied. Hence, $\tilde{s}_i, \tilde{p}_v$ and $\tilde{p}_w$ fulfill the relations of $C^*(H\Gamma)$. Hence, there exists a *-homomorphism $\phi_2 \colon C^*(H\Gamma) \to \mathcal{E}_n$.

Step 3: To the end we have to check, that the *-homomorphisms are inverse to each other, which means $\phi_2 \circ \phi_1 = id_{\mathcal{E}_n}$ and $\phi_1 \circ \phi_2 = id_{C^*(H\Gamma)}$. We get

$$\begin{aligned}
&\phi_1(\phi_2(p_v)) = \phi_1(\tilde{p}_v) = \phi_1(\sum_{i=1}^{n} t_i t_i^*) = \sum_{i=1}^{n} s_i s_i^* = p_v \\
&\phi_1(\phi_2(p_w)) = \phi_1(\phi_2(1 - p_v)) = \phi_1(1 - \tilde{p}_v) = (p_v + p_w) - p_v = p_w \\
&\phi_1(\phi_2(s_i)) = \phi_1(\tilde{s}_i) = \phi_1(t_i) = s_i \\
&\phi_2(\phi_1(t_i)) = \phi_2(s_i) = \tilde{s}_i = t_i \\
&\phi_2(\phi_1(1)) = \phi_2(p_v + p_w) = \tilde{p}_v + \tilde{p}_w = \tilde{p}_v + 1 - \tilde{p}_v = 1.
\end{aligned}$$

So we checked that ϕ_1 and ϕ_2 are inverse to each other and so we have $C^*(H\Gamma) \cong \mathcal{E}_n \cong C^*(T_1, \dots, T_n \mid T_i \text{ isometries, for all } i = 1, \dots, n, \sum_{i=1}^{n} T_i T_i^* \leq 1)$. □

3.2 Toeplitz and its modifications

In this section we investigate several constructions that arise from the Toeplitz algebra and its associated graph and hypergraphs. The main idea is to start with a graph example and then systematically extend it to the hypergraph setting in various ways. This section helps to understand how combinatorial changes in the underlying graph influence the structure of the resulting hypergraph C^*-algebra. Several of the central examples of this thesis arise from such modifications of Toeplitz-type graphs, just like the first two examples in 3.1.

Remark 3.3: Let $n \in \mathbb{N}$. Then it holds

$$\mathcal{T} * \mathbb{C}^n \cong C^*(u, p_1, \dots, p_n \mid u \text{ is an isometry}, \sum_{i=1}^{n} p_i = 1).$$

Let us now consider a multiple of the Toeplitz graph.

Proposition 3.4: *Let $n \in \mathbb{N}$ and consider the graph Γ with vertices $V = \{v_1, \dots, v_n\}$ and edges $E = \{e_1, \dots, e_n\}$, where the range and source map are defined as follows $r(e_i) = \{v_i\}$ and $s(e_i) = \{v_n\}$ for all $i = 1, \dots, n$. There exists a *-homomorphism $\phi\colon \mathcal{T} * \mathbb{C}^n \to C^*(\Gamma)$.*

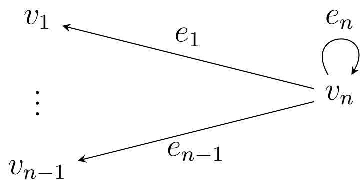

Figure 3.3: Multiple of the Toeplitz graph

Proof: We have with the Proposition 1.46, that $\sum_{i=1}^n p_{v_i}$ is the identity in $C^*(\Gamma)$ and by the relation (R1) of the graph C^*-algebra $C^*(\Gamma)$ we get $s_{e_i}^* s_{e_i} = p_{v_i}$ for all $i = 1, \dots, n$. Notice that p_{v_i} are sinks for all $i = 1, \dots, n-1$, so the second relation (R2) only applies for p_{v_n}. Hence we have $p_{v_n} = \sum_{i=1}^n s_{e_i} s_{e_i}^*$ and it holds $\sum_{i=1}^n s_{e_i}^* s_{e_i} = \sum_{i=1}^n p_{v_i} = 1$. We obtain with the universal property 1.23 a *-homomorphism $\phi\colon \mathcal{T} * \mathbb{C}^n \to C^*(\Gamma)$, sending u to $\sum_{i=1}^n s_{e_i}$, 1 to $\sum_{i=1}^n p_{v_i}$ and p_i to p_{v_i} for all $i = 1, \dots, n$. Therefore, we have to check that u is an isometry. Note that (R1) is $s_e^* s_f = \delta_{ef} p_{r(e)}$ for all $e, f \in E$ and therefore, $s_{e_i}^* s_{e_j} = 0$ for $i \neq j$. We have

$$u^* u = (\sum_{i=1}^n s_{e_i})^* (\sum_{j=1}^n s_{e_j}) = \sum_{i=1}^n \sum_{j=1}^n s_{e_i}^* s_{e_j} = \sum_{i=1}^n s_{e_i}^* s_{e_i} = \sum_{i=1}^n p_{v_i} = 1,$$

hence u is an isometry. □

We now pass from this graph to a hypergraph by replacing one of its edges with a hyperedge.

Proposition 3.5: *Let $n \in \mathbb{N}$ and consider the hypergraph $H\Gamma$ with vertices $V = \{v_1, \dots, v_n\}$, edges $E = \{e_1, \dots, e_{n-1}\}$ and the mappings are defined as $r(e_1) = \{v_1, v_n\}, s(e_1) = \{v_n\}$, and $r(e_i) = \{v_i\}, s(e_i) = \{v_n\}$ for all $i = 2, \dots, n-1$. There exists a *-homomorphism $\phi\colon \mathcal{T} * \mathbb{C}^n \to C^*(H\Gamma)$.*

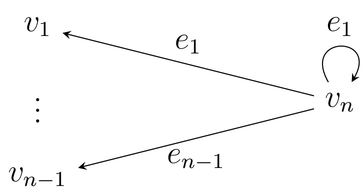

Figure 3.4: Multiple of the Toeplitz hypergraph 1

Proof: We have with the Proposition 1.46, that $\sum_{i=1}^n p_{v_i}$ is the identity in $C^*(H\Gamma)$ and by the relations of the hypergraph C^*-algebra $C^*(H\Gamma)$ we get

$$\begin{aligned} s_{e_1}^* s_{e_1} &= p_{v_1} + p_{v_n}, \\ s_{e_i}^* s_{e_i} &= p_{v_i}, \quad \text{for all } i = 2, \dots, n-1, \\ s_{e_i} s_{e_i}^* &\leq p_{v_n} \quad \text{for all } i = 1, \dots, n-1, \\ p_{v_n} &\leq \sum_{i=1}^{n-1} s_{e_i} s_{e_i}^*. \end{aligned}$$

Notice that p_{v_i} are sinks for all $i = 1, \dots, n-1$ and so the third hypergraph relation (HR3) only applies for p_{v_n}. Hence it holds that $p_{v_n} = \sum_{i=1}^{n-1} s_{e_i} s_{e_i}^*$ and it is

$$\sum_{i=1}^{n-1} s_{e_i}^* s_{e_i} = \sum_{i=1}^{n} p_{v_i} = 1.$$

We obtain with the universal property 1.23 a *-homomorphism $\phi\colon \mathcal{T} * \mathbb{C}^n \to C^*(H\Gamma)$, sending u to $\sum_{i=1}^{n-1} s_{e_i}$, 1 to $\sum_{i=1}^{n} p_{v_i}$ and p_i to p_{v_i} for all $i = 1, \dots, n$. Therefore, we have to check that u is an isometry. Note that (HR1) is $s_{e_i}^* s_{e_j} = \delta_{ij} \sum_{v \in r(e)} p_v$ and therefore, $s_{e_i}^* s_{e_j} = 0$ for $i \neq j$. We have

$$u^*u = (\sum_{i=1}^{n-1} s_{e_i})^* (\sum_{j=1}^{n-1} s_{e_j}) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} s_{e_i}^* s_{e_j} = \sum_{i=1}^{n-1} s_{e_i}^* s_{e_i} = \sum_{i=1}^{n} p_{v_i} = 1,$$

hence u is an isometry. □

The next example is derived from the previous ones by replacing all edges with one hyperedge.

Proposition 3.6: *Let $n \in \mathbb{N}$ and consider the hypergraph $H\Gamma = (V, E, r, s)$ with vertices $V = \{v_1, \dots, v_n\}$ and edges $E = \{e\}$. The following holds for the range and source map $r(e) = V$ and $s(e) = \{v_n\}$. There exists a *-homomorphism $\phi\colon \mathcal{T} * \mathbb{C}^n \to C^*(H\Gamma)$.*

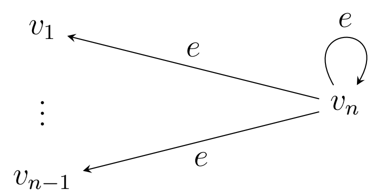

Figure 3.5: Multiple of the Toeplitz hypergraph 2

Proof: By the relations of the hypergraph C^*-algebra we have

$$s_e^* s_e = \sum_{i=1}^{n} p_{v_i} = 1,$$
$$s_e s_e^* \leq p_{v_n},$$
$$p_{v_n} \leq s_e s_e^*.$$

So it holds that s_e is an isometry and $s_e s_e^* = p_{v_n}$. We obtain with the universal property 1.23 a *-homomorphism $\phi\colon \mathcal{T} * \mathbb{C}^n \to C^*(H\Gamma)$, sending u to s_e, 1 to $\sum_{i=1}^n p_{v_i}$ and p_i to p_{v_i} for all $i = 1, \dots, n$. Therefore, we have to check that u is an isometry. We have

$$u^*u = (s_e)^*(s_e) = s_e^* s_e = \sum_{i=1}^{n} p_{v_i} = 1,$$

hence u is an isometry. □

We observe that the previous graph example extends to different hypergraphs, and that a *-homomorphism from the same algebra into the associated hypergraph C^*-algebras still exists. We thus see that we can transfer certain properties from the graph setting to the hypergraph setting. The next example arises from the previous one by taking a mulitple of the hyperedges. This construction can be seen as a multiple version of the hypergraph of the extended Cuntz-algebra 3.2.

Proposition 3.7: *Let $n, m \in \mathbb{N}$ and consider the hypergraph $H\Gamma = (V, E, r, s)$ with vertices $V = \{v_1, \dots, v_n\}$ and edges $E = \{e_1, \dots, e_m\}$. The range and source map are defined as follows $r(e_i) = V$ and $s(e_i) = \{v_n\}$ for all $i = 1, \dots, m$. There exists a *-homomorphism $\phi\colon \mathcal{E}_n \to C^*(H\Gamma)$.*

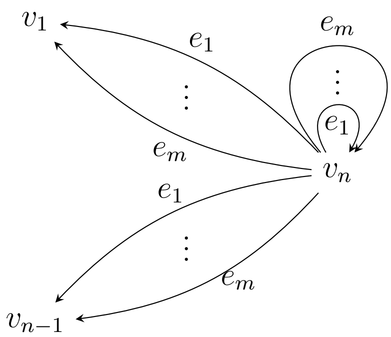

Figure 3.6: Multiple of the Toeplitz hypergraph 3

Proof: For the hypergraph C^*-algebra we obtain $s_{e_i}^* s_{e_i} = \sum_{i=1}^n p_{v_i} = 1$ by relation (HR1). Further by relations (HR2) and (HR3) we have $s_{e_i} s_{e_i}^* \leq p_{v_n}$ and $p_{v_n} \leq$

$s_{e_i}s_{e_i}^*$ for all $i = 1, \ldots, m$. So it follows that $p_{v_n} = \sum_{i=1}^{m} s_{e_i}s_{e_i}^*$. So it holds that s_{e_i} are isometries for all $i = 1, \ldots, m$ and

$$\sum_{i=1}^{m} s_{e_i}s_{e_i}^* = p_{v_n} \leq \sum_{i=1}^{n} p_{v_i} = 1.$$

We obtain with the universal property 1.23 a *-homomorphism $\phi\colon \mathcal{E}_n \to C^*(H\Gamma)$, sending T_i to s_{e_i} and 1 to $\sum_{i=1}^{n} p_{v_i}$. □

We considered a multiple of the hypergraph associated with the extended Cuntz algebra and observe that a *-homomorphism from $\mathcal{E}_n$ into the corresponding hypergraph C^*-algebra still exists. This demonstrates that certain structural properties can be preserved when passing to multiples of hypergraphs.

Remark 3.8: Let $\mathcal{T}$ be the Toeplitz algebra and u, v are ismoetries. Then it holds that

$$C^*(u, v, 1 \mid u, v \text{ are isometries }) \cong \mathcal{T} * \mathcal{T}.$$

The following hypergraph is constructed from a multiple of the Toeplitz hypergraph 3.6 by duplicating the hyperedges reversed and introducing additional loops.

Proposition 3.9: *Let $n \in \mathbb{N}$ and consider the hypergraph $H\Gamma = (V, E, r, s)$ with vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{e, f\}$. The range and source map are defined as follows $r(e) = V$, $s(e) = \{v_n\}$ and $r(f) = V$, $s(f) = \{v_1, \ldots, v_{n-1}\}$. There exists a *-homomorphism $\phi\colon \mathcal{T} * \mathcal{T} \to C^*(H\Gamma)$.*

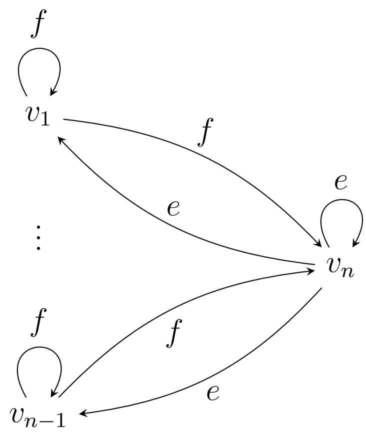

Figure 3.7: Multiple of the Toeplitz hypergraph 4

Proof: By the relations of the hypergraph C^*-algebra we have for the hyperedge e

$$\begin{aligned} s_e^* s_e &= \sum_{i=1}^{n} p_{v_i} = 1, \\ s_e s_e^* &\leq p_{v_n}, \\ p_{v_n} &\leq s_e s_e^*. \end{aligned}$$

So it follows that $p_{v_n} = s_e s_e^*$ and for the other hyperedge f we have

$$\begin{aligned} s_f^* s_f &= \sum_{i=1}^{n} p_{v_i} = 1, \\ s_f s_f^* &\leq \sum_{i=1}^{n-1} p_{v_i}, \\ p_{v_i} &\leq s_f s_f^* \text{ for all } i = 1, \dots, n-1. \end{aligned}$$

It follows that $\sum_{i=1}^{n-1} p_{v_i} = s_f s_f^*$ for all $i = 1, \dots, n-1$. If we combine all this together we have

$$1 = \sum_{i=1}^{n} p_{v_i} = s_e^* s_e = s_f^* s_f = s_e s_e^* + s_f s_f^*.$$

So it holds that s_e and s_f are isometries. We obtain with the universal property 1.23 a *-homomorphism $\phi\colon \mathcal{T} * \mathcal{T} \to C^*(H\Gamma)$, sending u to s_e, v to s_f, 1 to $\sum_{i=1}^{n} p_{v_i}$ and p_i to p_{v_i} for all $i = 1, \dots, n$. □

If we remove all loops from the previous example, the defining relations reduce in such a way that a unitary can be obtained from two isometries.

Proposition 3.10: *Let $n \in \mathbb{N}$ and $H\Gamma = (V, E, r, s)$ be the hypergraph with vertices $V = \{v_1, \dots, v_n\}$ and edges $E = \{e, f\}$. The following holds for the range and source map $r(e) = \{v_1, \dots, v_{n-1}\}$, $s(e) = \{v_n\}$ and $r(f) = \{v_n\}$, $s(f) = \{v_1, \dots, v_{n-1}\}$. There exists a *-homomorphism $\phi\colon C(S^1) \cong C^*(u, 1 \mid u^* u = 1 = u u^*) \to C^*(H\Gamma)$, meaning we find a *-homomorphism from the set of continuous functions on the unit circle into our hypergraph C^*-algebra.*

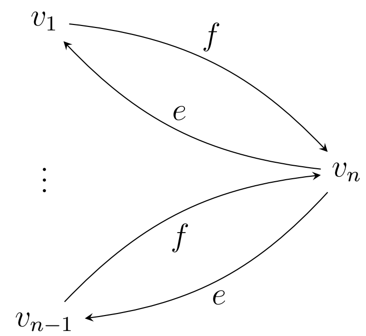

Figure 3.8: Multiple of the Toeplitz hypergraph 5

Proof: By the relations of the hypergraph C^*-algebra we have for the hyperedge e

$$\begin{aligned} s_e^* s_e &= \sum_{i=1}^{n-1} p_{v_i}, \\ s_e s_e^* &\leq p_{v_n}, \\ p_{v_n} &\leq s_e s_e^*. \end{aligned}$$

So it follows $p_{v_n} = s_e s_e^*$ and for the other hyperedge f we have

$$\begin{aligned} s_f^* s_f &= p_{v_n}, \\ s_f s_f^* &\leq \sum_{i=1}^{n-1} p_{v_i}, \\ p_{v_i} &\leq s_f s_f^* \text{ for all } i = 1, \ldots, n-1. \end{aligned}$$

It follows that $\sum_{i=1}^{n-1} p_{v_i} = s_f s_f^*$ for all $i = 1, \ldots, n-1$. If we combine all this together we have

$$1 = \sum_{i=1}^{n} p_{v_i} = s_e^* s_e + s_f^* s_f = s_e s_e^* + s_f s_f^*.$$

So it holds that $s_e + s_f$ is an unitary. We obtain with the universal property 1.23 a *-homomorphism $\phi\colon C^*(u, 1 \mid u^*u = 1 = uu^*) \to C^*(H\Gamma)$, sending u to $s_e + s_f$, 1 to $\sum_{i=1}^{n} p_{v_i}$ and p_i to p_{v_i} for all $i = 1, \ldots, n$. □

Now let's consider another example that combines the Toeplitz graph (1.49) with the Toeplitz hypergraph (2.14).

Proposition 3.11: *Consider the hypergraph $H\Gamma$ with vertices $V = \{v_1, v_2\}$ and edges $E = \{e, f\}$. The range and source map are defined as $r(e) = V$, $s(e) = \{v_2\}$ and $r(f) = \{v_2\}$, $s(f) = \{v_1\}$. We find*

(i) *a *-homomorphism* $\phi\colon \mathcal{T} \to C^*(H\Gamma)$*, sending* u *to* s_e *and* 1 *to* $p_{v_1} + p_{v_2}$.

(ii) *a *-homomorphism* $\psi\colon \mathcal{E}_1 \to C^*(H\Gamma)$*, sending* T *to* s_e *and* 1 *to* $p_{v_1} + p_{v_2}$.

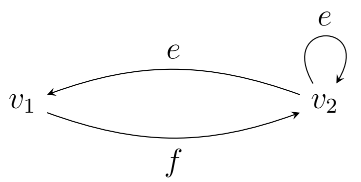

Figure 3.9: Multiple of the Toeplitz hypergraph 6

Proof: We have by the relations of the hypergraph C^*-algebra

$$s_e^* s_e = p_{v_1} + p_{v_2} = 1 \quad \text{and} \quad s_e s_e^* = p_{v_2},$$
$$s_f^* s_f = p_{v_2}, \quad \text{and} \quad s_f s_f^* = p_{v_1}.$$

Hence, s_e is an isometry and it holds that $s_e s_e^* \leq 1$. Therefore, all relations of $\mathcal{T}$ are fulfilled and we obtain the *-homomorphism $\phi\colon \mathcal{T} \to C^*(H\Gamma)$, sending u to s_e and 1 to $p_{v_1} + p_{v_2}$. It holds the same for $\mathcal{E}_1$ since we do not map to the edge f and so we have the *-homomorphism $\psi\colon \mathcal{E}_1 \to C^*(H\Gamma)$, sending T to s_e and 1 to $p_{v_1} + p_{v_2}$. ☐

3.3 Combination of known hypergraphs

In this section we investigate another method for constructing new examples of hypergraph C^*-algebras, namely by combining already known graphs and hypergraphs. The idea is to take hypergraphs whose associated C^*-algebras are well understood and to merge their structures in a controlled way. The resulting hypergraphs often lead to new C^*-algebras whose properties can be related to those of the original examples. As a first example, we combine the hypergraph of the C^*-algebra "$C(S^1) * \mathbb{C}^2$" (3.1) with the graph of the matrix algebra "$M_n\mathbb{C}$" (1.52). Further constructions will consider hypergraphs related to the matrix algebra and the Cuntz algebra, illustrating how these well-known examples can be used as building blocks for more complicated hypergraph structures.

Proposition 3.12: *Let* $n \in \mathbb{N}$ *and consider the hypergraph with vertices* $V = \{v_0, v_1, \ldots, v_n\}$ *and edges* $E = \{e_1, \ldots, e_n\}$*. The range and source map are defined as follows* $r(e_1) = \{v_0, v_1\} = s(e_1)$, $r(e_i) = \{v_{i-1}\}$ *and* $s(e_i) = \{v_i\}$ *for all* $i = 2, \ldots, n$.

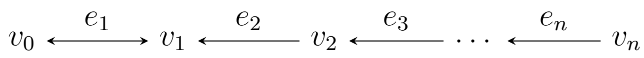

Figure 3.10: Combined hypergraphs

We have

(i) *a *-homomorphism* $\phi\colon \mathcal{E}_n \to C^*(H\Gamma)$.

(ii) *a *-homomorphism* $\psi\colon M_n\mathbb{C} \to C^*(H\Gamma)$.

(iii) *a *-homomorphism* $\alpha\colon \mathcal{T} \to C^*(H\Gamma)$.

Proof: By the relations of the hypergraph C^*-algebra we have

$$\begin{aligned} s_{e_1}^* s_{e_1} &= p_{v_0} + p_{v_1}, \\ s_{e_1} s_{e_1}^* &\leq p_{v_0} + Pv_1, \\ p_{v_0}, p_{v_1} &\leq s_{e_1} s_{e_1}^* \\ s_{e_i}^* s_{e_i} &= p_{v_{i-1}} \quad \text{for all } i = 2, \dots, n, \\ s_{e_i} s_{e_i}^* &\leq p_{v_i} \quad \text{for all } i = 2, \dots, n, \\ p_{v_i} &\leq s_{e_i} s_{e_i}^* \quad \text{for all } i = 2, \dots, n. \end{aligned}$$

So it holds, that $\sum_{i=1}^n s_{e_i} s_{e_i}^* = 1$.

Step 1: By the hypergraph relations it follows that $s_{e_i}^* s_{e_j} = 0$ for $i \neq j$. So we have $s_{e_i}^* s_{e_j} = \delta_{ij} p_{v_{i-1}}$. Furthermore, it holds $\sum_{i=1}^n s_{e_i} s_{e_i}^* \leq 1$. Hence, the relations of the extended Cuntz algebra are fulfilled and we have a *-homomorphism $\phi\colon \mathcal{E}_n \to C^*(H\Gamma)$, sending T_i to s_{e_i}.

Step 2: Analogously as in the proof of [Zen21, Proposition 2.11.] we define $\tilde{E}_{i,i} \coloneqq p_{v_i}$ for $i = 1, \dots, n$ and for $i, j = 1 \dots, n$ with $i > j$ we map $\tilde{E}_{i,j} \coloneqq s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{j+1}} s_{e_j}$ and else $\tilde{E}_{i,j} = \tilde{E}_{j,i}^*$, where $\tilde{E}_{(j+1),j} = s_{e_j}$. We have to prove, that $\tilde{E}_{ij}$ satisfy the relations of the matrix algebra, $\tilde{E}_{ii}$ are obviously matrix units. For the remaining relations let $i > j$ and $k > l$, then we have

$$\begin{aligned} \tilde{E}_{i,j} \tilde{E}_{k,l} &= s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{j+1}} s_{e_j} s_{e_{k-1}} s_{e_{k-2}} \dots s_{e_{l+1}} s_{e_l} \\ &= \delta_{jk} s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{j+1}} s_{e_j} s_{e_{i-j}} s_{e_{j-2}} \dots s_{e_{l+1}} s_{e_l} = \delta_{jk} \tilde{E}_{i,l}. \end{aligned}$$

For the next case let $j > i$, $k > l$ and with no loss of generality $i < j$. The remaining cases can be treated analogously. Then by using Equation 1.2

$$\begin{aligned}
\tilde{E}_{i,j}\tilde{E}_{k,l} = \tilde{E}^*_{j,i}\tilde{E}_{k,l} &= s^*_{e_i}s^*_{e_{i+1}} \dots s^*_{e_{j-2}}s^*_{e_{j-1}}s_{e_{k-1}}s_{e_{k-2}} \dots s_{e_{l+1}}s_{e_l} \\
&= \delta_{jk}s^*_{e_i}s^*_{e_{i+1}} \dots s^*_{e_{j-2}}s^*_{e_{j-1}}s_{e_{j-1}}s_{e_{j-2}} \dots s_{e_{l+1}}s_{e_l} \\
&= \delta_{jk}s^*_{e_i}s^*_{e_{i+1}} \dots s^*_{e_{j-2}}p_{v_{j-1}}s_{e_{j-2}} \dots s_{e_{l+1}}s_{e_l} \\
&= \delta_{jk}s^*_{e_i}s^*_{e_{i+1}} \dots s^*_{e_{j-2}}s_{e_{j-2}} \dots s_{e_{l+1}}s_{e_l} \\
&= \dots \\
&= \delta_{jk}s_{e_{i-1}}s_{e_{i-2}} \dots s_{e_{l+1}}s_{e_l} \\
&= \delta_{jk}\tilde{E}_{il}
\end{aligned}$$

holds. In addition to that we have

$$\begin{aligned}
\tilde{E}^*_{il} &= \tilde{E}_{ji} \text{ for } i > j \\
\tilde{E}^*_{il} &= (\tilde{E}^*_{ji})^* else.
\end{aligned}$$

Hence there exists a *-homomorphism $\psi\colon M_n\mathbb{C} \to C^*(H\Gamma)$, sending E_{ij} to $\tilde{E}_{il}$ and E_{ii} to $\tilde{E}_{ii} = p_{v_i}$.

Step 3: The next *-homomorphism maps the unit 1 of the Toeplitz algebra to $p_{v_0} + p_{v_1}$, the other projections are not displayed. Therefore, we are locally in a Toeplitz algebra, so we map the isometry u to s_{e_1} and hence it holds that $s^*_{e_1}s_{e_1} = p_{v_0} + p_{v_1} = \alpha(1)$ is an isometry. Hence, there exists a *-homomorphism $\alpha\colon \mathcal{T} \to C^*(H\Gamma)$, sending 1 to $p_{v_0} + p_{v_1}$ and u to s_{e_1}. □

The following proposition considers an m-fold combination of the graph associated with the matrix algebra. More precisely, each vertex v_i is replaced by a set $V_i = \{a_i, \dots, m_i\}$ consisting of m vertices for $i = 1, \dots, n$. The original edges $\{e_1, \dots, e_{n-1}\}$ are then regarded as hyperedges with range V_{i+1} and source V_i for all $i = 1, \dots, n-1$.

Proposition 3.13: *Let $n, m \in \mathbb{N}$. Consider the hypergraph $H\Gamma = (V, E, r, s)$ with vertex set $V = \{V_1, \dots, V_n\}$, where each $V_i = \{a_i, \dots, m_i\}$ consists of m vertices for all $i = 1, \dots, n$ and edge set $E = \{e_1, \dots, e_{n-1}\}$. The range and source map are defined as $r(e_i) = V_{i+1}$ and $s(e_i) = V_i$ for all $i = 1, \dots, n-1$. Then we have that*

$$C^*(H\Gamma) \cong M_n(C^m *_{\mathbb{C}} C^m *_{\mathbb{C}} \dots *_{\mathbb{C}} C^m),$$

where $*_{\mathbb{C}}$ denotes the unital free product of C^*-algebras.

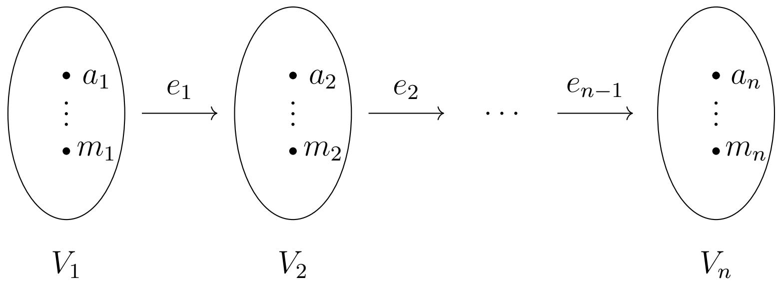

Figure 3.11: Hyperbranch of length n

Proof: It suffices to verify that this hypergraph is a hyperbranch of length n. By construction, this condition is satisfied. Hence, the Hyperbranch Structure Theorem 2.13 yields

$$C^*(H\Gamma) \cong M_n(C^m *_{\mathbb{C}} C^m *_{\mathbb{C}} \cdots *_{\mathbb{C}} C^m),$$

where $*_{\mathbb{C}}$ denotes the unital free product of C^*-algebras. $\square$

As noted earlier, it is possible to transfer certain properties when duplicating graphs and hypergraphs. Next we look at an example, where we combined the graph of the Cuntz algebra (1.50) and the hypergraph of the Toeplitz algebra (2.14).

Proposition 3.14: *Let $n \in \mathbb{N}$. Consider the hypergraph $H\Gamma$ with the following properties: vertices $V = \{v, w\}$, edges $E = \{e_1, \dots, e_n\}$, mappings $r(e_1) = V$, $s(e_1) = \{v\}$, and $r(e_i) = \{v\} = s(e_i)$ for $i = 2, \dots, n$. We have*

*(a) a *-homomorphism $\phi\colon \mathcal{T} \to C^*(H\Gamma)$, sending u to s_{e_1} and 1 to $p_v + p_w$.*

*(b) a *-homomorphism $\psi\colon \mathcal{E}_{n-1} \to C^*(H\Gamma)$, sending T_i to s_{e_i} for all $i = 2, \dots, n$ and 1 to p_v.*

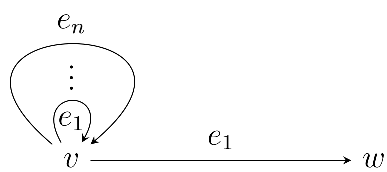

Figure 3.12: Combination of Cuntz and Toeplitz

Proof: Step 1: We have with relation (HR1) $s_{e_1}^* s_{e_1} = p_v + p_w = 1$. Hence, s_{e_1} is an isometry. Therefore, we find a *-homomorphism $\phi\colon \mathcal{T} \to C^*(H\Gamma)$, sending u to s_{e_1} and 1 to $p_v + p_w$.

Step 2: Moreover it follows with the relations (HR1)-(HR3) $s_{e_i}^* s_{e_i} = p_v$, for all $i = 2, \dots, n$ and $\sum_{i=1}^n s_{e_i} s_{e_i}^* = p_v$. In the situation of the *-homomorphism ψ we map p_v to 1, such that the relations of the extended Cuntz-algebra are fulfilled. So we have the *-homomorphism $\psi\colon \mathcal{E}_{n-1} \to C^*(H\Gamma)$, sending $T_i \to s_{e_i}$ for all $i = 2, \dots, n$ and $1 \to p_v$. □

The following proposition considers a modification obtained by combining two copies of the Cuntz graph. We first describe the corresponding graph construction and then extend it, analogously to the Toeplitz examples, to a hypergraph obtained by combining m copies of the Cuntz graph.

Proposition 3.15: *Let $n \in \mathbb{N}$ and consider the graph Γ with vertices $V = \{v, w\}$ and edges $E = \{e_{-1}, \dots, e_{2n}\}$. The image of the range and source map is defined like this for all $i = 1, \dots, n$*

$$
\begin{aligned}
r(e_0) &= \{v\}, & s(e_0) &= \{w\},\\
r(e_{-1}) &= \{w\}, & s(e_{-1}) &= \{v\},\\
r(e_{2i-1}) &= \{v\}, & s(e_{2i-1}) &= \{v\},\\
r(e_{2i}) &= \{w\}, & s(e_{2i}) &= \{w\}.
\end{aligned}
$$

*There exists a *-homomorphism $\phi\colon O_{n+1} \to C^*(\Gamma)$.*

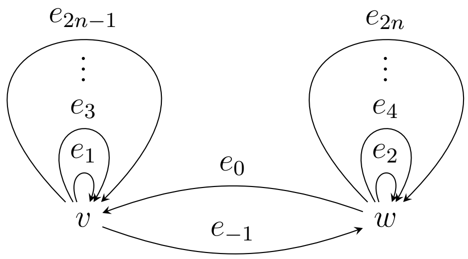

Figure 3.13: Combination of Cuntz graphs

Proof: We have by the relations (R1) and (R2) of the graph C^*-algebra

$$
\begin{aligned}
s_{e_0}^* s_{e_0} &= p_v,\\
s_{e_{-1}}^* s_{e_{-1}} &= p_w,\\
s_{e_{2i}}^* s_{e_{2i}} &= p_w, \quad \text{for all } i = 1, \dots, n,\\
s_{e_{2i-1}}^* s_{e_{2i-1}} &= p_v, \quad \text{for all } i = 1, \dots, n,\\
p_v &= \sum_{i=0}^{n} s_{e_{2i-1}} s_{e_{2i-1}}^*,\\
p_w &= \sum_{i=0}^{n} s_{e_{2i}} s_{e_{2i}}^*.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned} 1 = p_v + p_w &= s^*_{e_{2i-1}} s_{e_{2i-1}} + s^*_{2e_i} s_{2e_i} \text{ for all } i = 0, \dots, n \\ &= \sum_{i=0}^{n} s_{e_{2i-1}} s^*_{e_{2i-1}} + \sum_{j=0}^{n} s_{e_{2j}} s^*_{e_{2j}} \\ &= \sum_{i=-1}^{2n} s_{e_i} s^*_{e_i}. \end{aligned}$$

If we combine every two consecutive edges into a new edge, then these $n+1$ "edges" satisfy the Cuntz relations.

$$s_{g_k} \coloneqq s_{e_{2k-1}} + s_{e_{2k}} \text{ for all } k = 0, \dots, n$$

Note that (R1) is $s^*_e s_f = \delta_{ef} p_{r(e)}$ for all $e, f \in E$ and therefore, $s^*_{e_i} s_{e_j} = 0$ for $i \neq j$. We have

$$\begin{aligned} 1 &= \sum_{k=0}^{n} s_{g_k} s^*_{g_k} \\ &= s^*_{g_k} s_{g_k} \text{ for all } k = 0, \dots, n. \end{aligned}$$

Hence, we get a *-homomorphism $\phi : O_{n+1} \to C^*(\Gamma)$, sending s_{g_k} to $s_{e_{2k-1}} + s_{e_{2k}}$ and 1 to $p_v + p_w$. □

The following statement extends the previous graph example, as described above, to a more general hypergraph setting.

Proposition 3.16: *Let $n, m \in \mathbb{N}$. Consider the hypergraph $H\Gamma$ with the following properties: vertices $V = \{a, b, c, \dots, m\}$, edges $E = \{a_0, a_i, b_i, c_i, \dots, m_i\}$ for all $i = 1, \dots, n$ and the range and source map $r(a_0) = \{a, b, c, \dots, m\}$, $s(a_0) = \{a\}$ and $r(j_i) = \{j\} = s(j_i)$ for all $i = 1, \dots, n$ and $j = a, \dots, m$. There exists a *-homomorphism $\phi \colon O_{n+1} \to C^*(H\Gamma)$.*

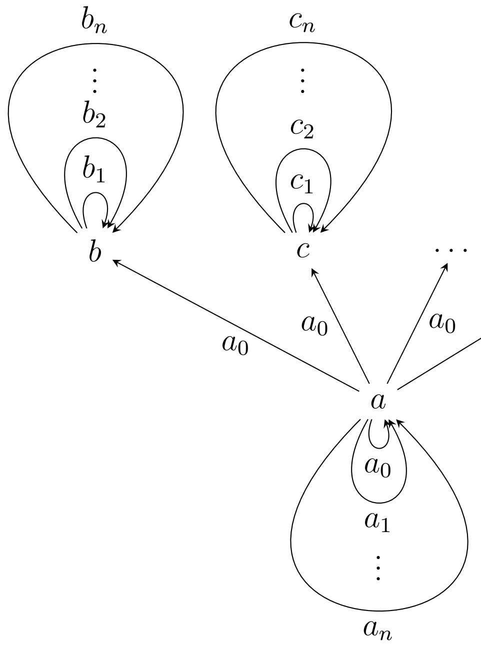

Figure 3.14: Combination of Cuntz hypergraphs

Proof: Using the relations of the associated hypergraph C^*-algebra $C^*(H\Gamma)$, we obtain for the hyperedge a_0

$$\begin{aligned} s_{a_0}^* s_{a_0} &= \sum_{k=a}^{m} p_k = 1, \\ s_{a_0} s_{a_0}^* &\leq p_a, \\ p_a &\leq \sum_{i=0}^{n} s_{a_i} s_{a_i}^*. \end{aligned}$$

For the other edges it holds that

$$\begin{aligned} s_{j_i}^* s_{j_i} &= p_j \quad \text{for all } i = 1, \dots, n \text{ and } j = a, \dots, m, \\ s_{j_i} s_{j_i}^* &\leq p_j \quad \text{for all } i = 1, \dots, n \text{ and } j = a, \dots, m, \\ p_j &\leq \sum_{i=1}^{n} s_{j_i} s_{j_i}^* \quad \text{for all } j = b, \dots, m. \end{aligned}$$

Hence, it follows that

$$p_a = \sum_{i=0}^{n} s_{a_i} s_{a_i}^* \quad \text{and} \quad p_j = \sum_{i=1}^{n} s_{j_i} s_{j_i}^* \quad \text{for all } j = b, \dots, m.$$

Therefore,

$$\begin{aligned}1 = \sum_{k=a}^{m} p_k &= \sum_{i=0}^{n} s_{a_i} s_{a_i}^* + \sum_{j=b}^{m}\sum_{i=1}^{n} s_{j_i} s_{j_i}^* \\ &= \sum_{a_l, b_i, c_i, \dots, m_i} s_k s_k^* \text{ for all } i = 1, \dots, n; l = 0, \dots, n \\ &= s_{a_0}^* s_{a_0} \\ &= \sum_{j=b}^{m} s_{j_i}^* s_{j_i} \text{ for all } i = 1, \dots, n.\end{aligned}$$

If we combine every edge with the same indexnumber $K = 0, \dots, n$ from all vertices into a new edge, then these $n+1$ "edges" satisfy the Cuntz relations.

$$s_K := s_{a_K} + s_{b_K} + s_{c_K} + \dots + s_{m_K} \text{ for all } K = 1, \dots, n$$

and $s_0 := s_{a_0}$. So it holds

$$\begin{aligned}1 &= \sum_{K=0}^{n} s_K s_K^* \\ &= s_K^* s_K \text{ for all } K = 0, \dots, n.\end{aligned}$$

Hence, we get a *-homomorphism $\phi : O_{n+1} \to C^*(H\Gamma)$, sending s_0 to s_{a_0}, s_K to $\sum_{i=a}^{m} s_{i_K}$ for all $K = 1, \dots, n$ and 1 to $\sum_{i=a}^{m} p_i$. □

Once again, we have found an example where we were able to map the structure of a graph onto a hypergraph.

3.4 A mysterious example from Dean Zenner

At the end of this chapter we look at an example taken from the bachelor thesis of Dean Zenner ([Zen21, Proposition 3.17]). He refers to this example as "mysterious" and points out that it may deserve further study. We therefore include it here and form our own conclusions.

Example 3.17: Consider the hypergraph $H\Gamma = (V, E, r, s)$ with vertices $V = \{v_1, v_2, v_3, v_4\}$ and edges $E = \{e_1, e_2\}$. The following holds for the range and source map $r(e_2) = \{v_1, v_2\} = s(e_1)$ and $r(e_1) = \{v_3, v_4\} = s(e_2)$. Dean Zenner found a *-homomorphism from the universal C^*-algebra $B := C^*(u, p \mid up = (1-p)u)$, where u is a unitary and p a projection, to $C^*(H\Gamma)$. We have a *-homomorphism $\phi\colon C^*(H\Gamma) \to M_4(\mathbb{C})$, sending s_{e_1} to $E_{13} + E_{24}$, s_{e_2} to $E_{32} + E_{41}$, p_i to E_{ii}.

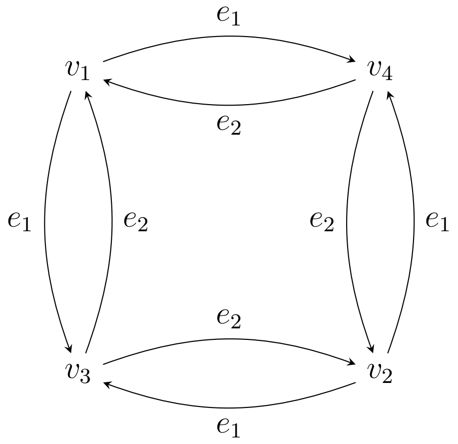

Figure 3.15: Mysterious example from Dean Zenner

Proof: By the relations of the hypergraph C^*-algebra we have

$$s_{e_1}^* s_{e_1} = p_{v_3} + p_{v_4} = p_B = s_{e_2} s_{e_2}^*$$
$$s_{e_2}^* s_{e_2} = p_{v_1} + p_{v_2} = p_A = s_{e_1} s_{e_1}^*.$$

We would like to present our interpretation of the hypergraphic characteristics, which is as follows.

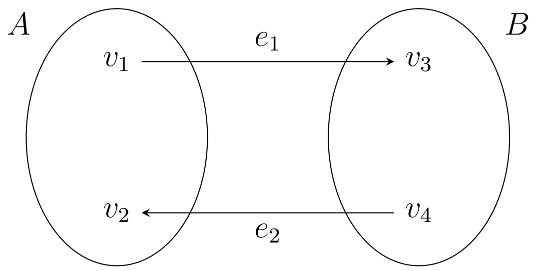

Figure 3.16: Our interpretation of Dean Zenner's example

With the matrix multiplication we get that the elementary matrices E_{ij} are partial isometries, since $E_{ij}E_{ij}^*E_{ij} = E_{ij}E_{ji}E_{ij} = E_{ij}E_{jj} = E_{ij}$ and E_{ii} are projections, because $E_{ii}^2 = E_{ii}E_{ii} = E_{ii} = E_{ii}^*$. We imagine a cyclical order such as, for example, $v_1 \to v_3 \to v_2 \to v_4 \to v_1$. Based on this idea, we choose our mapping rule. We associate the mapping of the cyclic order with the elementary matrix with indices i and j, where $i, j \in \{1, \ldots, 4\}$ as follows

$$s_{13} \to E_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, s_{32} \to E_{32}, s_{24} \to E_{24}, s_{41} \to E_{41}, p_{v_i} \to E_{ii}.$$

These elementary matrices are the generators of $M_4(\mathbb{C})$ and therefore, it remains to show that they fulfill the relations of the hypergraph C^*-algebra. We map the edges s_{e_1}, s_{e_2} to the sum of the cyclic order mappings. Hence,

$$s_{e_1} \to s_{13} + s_{24} = E_{13} + E_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, s_{e_1}^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$s_{e_2} \to s_{32} + s_{41} = E_{32} + E_{41} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, s_{e_2}^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now we check the relations of our hypergraph C^*-algebra $C^*(H\Gamma)$.

$$s_{e_1}^* s_{e_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = p_{v_3} + p_{v_4} = p_B$$

$$s_{e_1} s_{e_1}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = p_{v_1} + p_{v_2} = p_A$$

$$s_{e_2}^* s_{e_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = p_{v_1} + p_{v_2} = p_A$$

$$s_{e_2} s_{e_2}^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = p_{v_3} + p_{v_4} = p_B$$

Since they fulfill all relations of the hypergraph C^*-algebra we have a *-homomorphism $\phi\colon C^*(H\Gamma) \to M_4(\mathbb{C})$, sending s_{e_1} to $E_{13} + E_{24}$, s_{e_2} to $E_{32} + E_{41}$, p_i to E_{ii}. □

Remark 3.18: It could be possible that one can find a *-homomorphism $\psi\colon M_4(\mathbb{C}) * (C(S^1)) \to C^*(H\Gamma)$, sending E_{ij} to s_{ij}, E_{ii} to p_{v_i} and u to $s_{e_1} + s_{e_2}$, but therefore, it remains to show that s_{e_1}, s_{e_2} satisfy the matrix relations, $s_{ij}^* = s_{ji}$, $s_{ij}s_{kl} = \delta_{jk}s_{il}$. This is the part where we got stuck. As a reminder:

$M_4(\mathbb{C}) * (C(S^1)) := C^*(E_{ij}, i, j = 1, \ldots, 4, u \mid u \text{ is a unitary, } E_{ij}^* = E_{ji}, E_{ij}E_{kl} = \delta_{jk}E_{il})$.

E_{ij} are matrix units in this C^*-algebra. By the relations of the hypergraph C^*-algebra we have

$$\begin{aligned} s_{e_1}^* s_{e_1} &= p_{v_3} + p_{v_4} = p_B = s_{e_2} s_{e_2}^* \\ s_{e_2}^* s_{e_2} &= p_{v_1} + p_{v_2} = p_A = s_{e_1} s_{e_1}^*. \end{aligned}$$

We choose our mapping rule based on the same cyclical order in Proposition 3.17 as follows $E_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to s_{13}, E_{32} \to s_{32}, E_{24} \to s_{24}, E_{41} \to s_{41}, E_{ii} \to p_{v_i}$.
Therefore, it holds

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to s_{e_1} = s_{13} + s_{24},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \to s_{e_2} = s_{32} + s_{41}.$$

We map the unitary u of $M_4(\mathbb{C}) * (C(S^1))$ to $s_{e_1} + s_{e_2}$ of $C^*(H\Gamma)$ since it holds with relation (HR1) that $s_{e_i}^* s_{e_j} = 0$ for $i \neq j$. So we have

$$\begin{aligned} I_4 = 1 = p_B + p_A &= s_{e_1}^* s_{e_1} + s_{e_2}^* s_{e_2} = (s_{e_1} + s_{e_2})^* (s_{e_1} + s_{e_2}) \\ = p_A + p_B &= s_{e_1} s_{e_1}^* + s_{e_2} s_{e_2}^* = (s_{e_1} + s_{e_2})(s_{e_1} + s_{e_2})^*. \end{aligned}$$

It holds that $p_{v_i}^* = p_{v_i}$ and $p_{v_i} p_{v_j} = \delta_{ij} p_{v_i}$ since p_{v_i} are mutually orthogonal projections, so they fulfill the matrix relations.

It still remains to show that s_{e_1}, s_{e_2} satisfy the matrix relations, but we haven't been able to show that yet.

It remains an interesting open question whether this claim can be established, or whether the construction of a suitable *-homomorphism to or from another C^*-algebra could yield further insight into this example.

3.5 Conclusions

In conclusion, this work demonstrates that certain structural properties of graphs and hypergraphs can be preserved when constructing modified hypergraphs, for instance through multiplication or combination with other hypergraphs. Although the examples considered here do not establish this in full generality, they provide supporting evidence for the conjecture that such properties can be systematically transferred. These findings open up promising directions for further research in this area. Moreover, this work introduces a new hypergraph associated with the extended Cuntz algebra 3.2 and establishes several *-homomorphisms from the Cuntz algebra, the extended Cuntz algebra, the Toeplitz algebra, as well as from various free products of the Toeplitz algebra with itself or with $\mathbb{C}^n$, to different hypergraph C^*-algebras. Finally, we note that many further examples of hypergraphs can be studied using the methods developed in this work, indicating that the scope of this theory extends well beyond the cases considered here.

4 Moves of hypergraphs

In this chapter, we introduce the theory of moves and present the statements without their proofs. We follow the work of Trieb, Zenner, and Weber and refer for more details to their literature [TWZ24]. Moves provide a systematic method for modifying a hypergraph while preserving structural information at the level of the associated C^*-algebras. In particular, they induce *-homomorphisms or even isomorphisms between the corresponding C^*-algebras, provided that the moves are performed at vertices that locally behave like graphs, that is, when the hypergraph is locally ultra at the respective vertex. We will look at an example and explicitly perform various moves on it. We start with the following definition.

Definition 4.1 (locally ultra at vertex w): A hypergraph is called *locally ultra at vertex w,* if for all $e \in E$, the assertion $w \in s(e)$ implies $s(e) = \{w\}$. (see [TWZ24, Definiton 6.1.])

The following theorem summarizes the main contents of the section about moves in [TWZ24].

Theorem 4.2: *Let $H\Gamma$ be a hypergraph, and let $H\Gamma$ be locally ultra at some vertex w (possibly satisfying some further mild assumptions on w). The moves I and O produce hypergraphs $H\Gamma'$ and *-homomorphisms*

$$\pi \colon C^*(H\Gamma') \to C^*(H\Gamma),$$

*respectively, such that $Im(\pi)$ is a full corner in $C^*(H\Gamma)$. In the case of move O, π is even a *-isomorphism. (see [TWZ24, Theorem 6.2.])*

4.1 Move O - Outsplitting

Definition 4.3 (Move *O* - Outsplitting): Let $H\Gamma = (V, E, r, s)$ be a finite hypergraph and let w be a vertex that is not a sink. We partition the set of outgoing edges in finitely many nonempty sets:

$$\{e \in E \mid w \in s(e)\} = \varepsilon_1 \cup \cdots \cup \varepsilon_n.$$

The hypergraph $H\Gamma_O$ obtained by performing *move* O on $H\Gamma$ is defined as follows.

$$V_O := V \setminus \{w\} \cup \{w^1, w^2, \dots, w^n\},$$
$$E_O := \{e^1 \mid e \in E, w \notin r(e)\} \cup \{e^1, \dots, e^n \mid e \in E, w \in r(e)\},$$
$$r_O(e^i) := \begin{cases} r(e) & \text{if } i = 1 \text{ and } w \notin r(e), \\ (r(e) \setminus \{w\}) \cup \{w^1\} & \text{if } i = 1 \text{ and } w \in r(e), \\ w^i & \text{if } i > 1 \text{ and } w \in r(e), \end{cases}$$
$$s_O(e^i) := \begin{cases} s(e) & \text{if } w \notin s(e), \\ (s(e) \setminus \{w\}) \cup \{w^j\} & \text{if } w \in s(e) \text{ and } e \in \varepsilon_j. \end{cases}$$

We call $H\Gamma_O$ the hypergraph obtained by *outsplitting* $H\Gamma$ at w. (see [TWZ24, Definiton 6.5.])

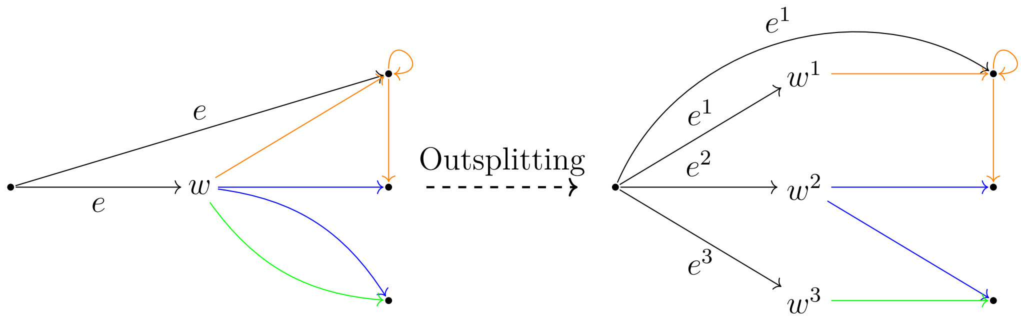

Figure 4.1: Move O - Outsplitting
This is a illustration of the outsplitting move O at a vertex w. Each color/thickness marks one edge and we partition the set of outgoing edges of w into one-point sets.

We would like to point out here that the hypergraph requires more than two different hyperedges for the move to have an effect.

Proposition 4.4 (Move O): *Let $H\Gamma = (V, E, r, s)$ be a finite hypergraph and w be a vertex that is not a sink and let $H\Gamma$ be locally ultra at w. Let $H\Gamma_O$ be the hypergraph obtained by outsplitting $H\Gamma$ at w. Then $C^*(H\Gamma) \cong C^*(H\Gamma_O)$.*

Proof: (see Proof of [TWZ24, Proposition 6.9.]) □

Now we apply the Move O on the extended Cuntz-algebra, $\mathcal{E}_n$ (Proposition 3.2).

Example 4.5: Let $n \in \mathbb{N}$ and consider the hypergraph with vertices $V = \{v, w\}$, edges $E = \{e_1, \ldots, e_n\}$ and the range and source map are defined as follows $r(e_i) = V$ and $s(e_i) = \{v\}$ for all $i = 1, \ldots, n$.

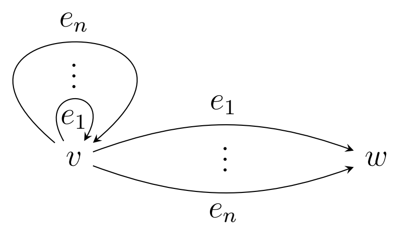

Figure 4.2: Example (extended Cuntz-algebra $\mathcal{E}_n$) before Moves

It holds

$$\begin{aligned}C^*(H\Gamma) &\cong \mathcal{E}_n \cong C^*(T_1, \ldots, T_n \mid T_i \text{ isometries, for all } i = 1, \ldots, n, T_i^* T_j = \delta_{ij}) \\ &\cong C^*(T_1, \ldots, T_n \mid T_i \text{ isometries, for all } i = 1, \ldots, n, \sum_{i=1}^{n} T_i T_i^* \leq 1).\end{aligned}$$

It holds that v is not a sink and that the hypergraph $H\Gamma$ is locally ultra in the vertex v. So we can apply the outsplitting move, which yields the following. We partition the set of outgoing edges in finitely many nonempty sets:

$$\{e \in E \mid w \in s(e)\} = \varepsilon_1 \cup \cdots \cup \varepsilon_n = \{e_1\} \cup \cdots \cup \{e_n\}.$$

The hypergraph $H\Gamma_O$ is obtained by performing move O on $H\Gamma$.

$$\begin{aligned}V_O &= \{v^1, \ldots, v^n, w\}, \\ E_O &= \{e_1^1, \ldots, e_1^n \mid e_1 \in E, v \in r(e_1)\} \cup \cdots \cup \{e_n^1, \ldots, e_n^n \mid e_n \in E, v \in r(e_n)\} \\ &= \{e_1^1, \ldots, e_1^n, \ldots, e_n^1, \ldots, e_n^n\}, \\ r_O(e_i^j) &= \{v^j\} \text{ if } j > 1, i = 1, \ldots, n, \\ r_O(e_i^1) &= (r(e_i) \setminus \{v\}) \cup \{v^1\} = \{w\} \cup \{v^1\} = \{w, v^1\}, \\ s_O(e_i^j) &= (s(e_i) \setminus \{v\}) \cup \{v^i\} = \{v^i\} \text{ for all } i, j = 1, \ldots, n.\end{aligned}$$

The isomorphic hypergraph C^*-algebra is generated by the following hypergraph $H\Gamma_O$.

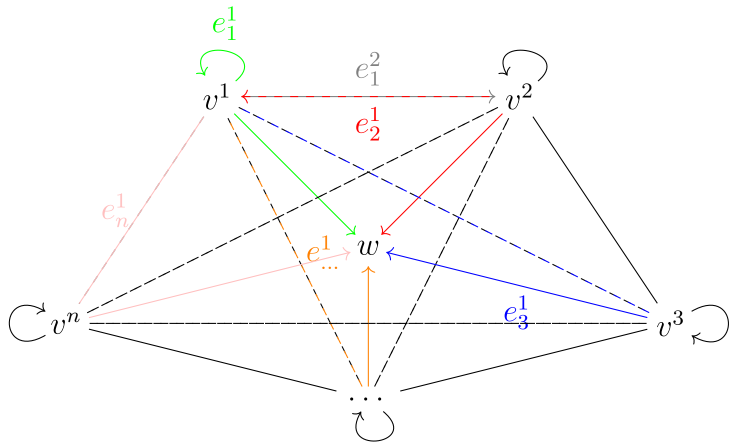

Figure 4.3: Example (extended Cuntz-algebra $\mathcal{E}_n$) after Move O

According to Proposition 4.4 one can check, that the C^*-algebra, which is generated by this hypergraph is the extended Cuntz-algebra, $\mathcal{E}_n$. We just give a proof for one direction. By the hypergraph relations it holds the following

$$\begin{aligned} s_{e_i^1}^* s_{e_i^1} &= p_{v_1} + p_w, \\ s_{e_i^j}^* s_{e_i^j} &= p_{v_j} \text{ for all } i = 1, \dots, n \text{ and } j > 1, \\ s_{e_i^j} s_{e_i^j}^* &\leq p_{v_i} \text{ for all } i, j = 1, \dots, n \\ p_{v_i} &\leq \sum_{j=1}^n s_{e_i^j} s_{e_i^j}^* \text{ for all } i = 1, \dots, n. \end{aligned}$$

So we have that $p_{v_i} = \sum_{j=1}^n s_{e_i^j} s_{e_i^j}^*$ for all $i = 1, \dots, n$. For all $i = 1, \dots, n$ we define

$$S_i := \sum_{j=1}^n s_{e_i^j}.$$

In other words we map the sum over all edges coming from one vertex to the generator of $\mathcal{E}_n$. The relation (HR1) means $s_e^* s_f = \delta_{ef} \sum_{v \in r(e)} p_v$ for all $e, f \in E$. Hence these n new edges are isometries, since

$$S_i^* S_i = (\sum_{j=1}^n s_{e_i^j})^* \sum_{k=1}^n s_{e_i^k} = \sum_{j=1}^n s_{e_i^j} = \sum_{i=1}^n p_{v^i} + p_w = 1 \text{ for all } i = 1, \dots, n$$

and it holds with (HR2) and (HR3) that:

$$\begin{aligned}\sum_{i=1}^{n} S_i S_i^* &= \sum_{i=1}^{n}(\sum_{j=1}^{n} s_{e_i^j})(\sum_{j=1}^{n} s_{e_i^j})^* \\ &= \sum_{i=1}^{n}(\sum_{j=1}^{n} s_{e_i^j} s_{e_i^j}^*) \\ &= \sum_{i=1}^{n} p_{v_i} \\ &\leq \sum_{i=1}^{n} p_{v_i} + p_w = 1\end{aligned}$$

Therefore, the generators of $C^*(H\Gamma)$ satisfy the relations of $\mathcal{E}_n$ and with the universal property 1.23 we have shown one direction of the proof.

Next, I want to introduce the next move, namely the move I for insplitting.

4.2 Move I - Insplitting

Definition 4.6 (Move *I* - Insplitting): Let $H\Gamma = (V, E, r, s)$ be a finite hypergraph and let $w \in V$ be a vertex that is not a source. We partition the set of incoming edges in finitely many nonempty sets:

$$\{e \in E \mid w \in r(e)\} = \varepsilon_1 \cup \cdots \cup \varepsilon_n.$$

The hypergraph $H\Gamma_I$ obtained by performing move I on $H\Gamma$ is defined by

$$\begin{aligned}V_I &:= (V \setminus \{w\}) \cup \{w^1, \ldots, w^n\}, \\ E_I &:= \{e^1 \mid e \in E, w \notin s(e)\} \cup \{e^1, \ldots, e^n \mid e \in E, w \in s(e)\},\end{aligned}$$

$$r_I(e^i) := \begin{cases} r(e) & \text{if } i = 1 \text{ and } w \notin r(e), \\ (r(e) \setminus \{w\}) \cup \{w^j\} & \text{if } e^i \in \varepsilon_j, \end{cases}$$

$$s_I(e^i) := \begin{cases} s(e) & \text{if } w \notin s(e), \\ (s(e) \setminus \{w\}) \cup \{w^1\} & \text{if } i = 1 \text{ and } w \in s(e), \\ \{w^i\} & \text{if } i = 2, \ldots, n. \end{cases}$$

We call $H\Gamma_I$ the hypergraph obtained by *insplitting* $H\Gamma$ at w.

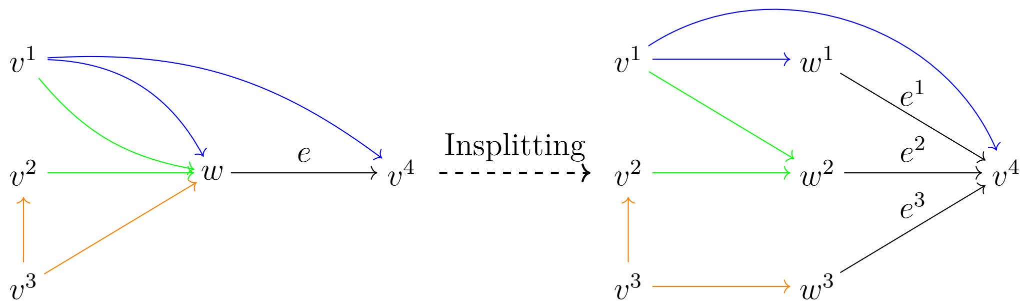

Figure 4.4: Move I - Insplitting

This is a illustration of the insplitting move I at a vertex w. Each color/ thickness marks one edge and ε_j are one-point sets. (see [TWZ24, Definiton 6.6.])

In the following we introduce the move I for indelay.

4.3 Move I - Indelay

Definition 4.7 (Move I - Indelay): Let $H\Gamma = (V, E, r, s)$ be a finite hypergraph and $w \in V$ be a vertex that is not a source. We partition the set of incoming edges in finitely many nonempty sets:

$$\{e \in E \mid w \in r(e)\} = \varepsilon_1 \cup \cdots \cup \varepsilon_n.$$

The hypergraph $H\Gamma_D$ obtained by an indelay of $H\Gamma$ at w is defined by

$$\begin{aligned}
V_D &:= V \setminus \{w\} \cup \{w^1, \dots, w^n\}, \\
E_D &:= E \cup \{f^1, \dots, f^{n-1}\}, \\
r_D(e) &:= \begin{cases} r(e) & \text{if } w \notin r(e), \\ (r(e) \setminus \{w\}) \cup \{w^j\} & \text{if } e \in \varepsilon_j, \end{cases} \\
r_D(f^j) &:= w^j, \\
s_D(e) &:= \begin{cases} s(e) & \text{if } w \notin s(e), \\ (s(e) \setminus \{w\}) \cup \{w^1\} & \text{if } w \in s(e), \end{cases} \\
s_D(f^j) &:= w^{j+1}.
\end{aligned}$$

We call $H\Gamma_D$ the hypergraph obtained by *indelay* $H\Gamma$ at w.

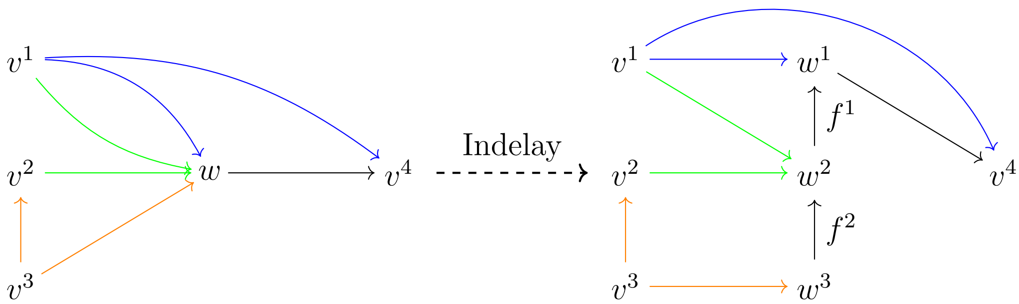

Figure 4.5: Move I - Indelay

This is a illustration of the application of the indelay move I at a vertex w. Each color/ thickness marks one edge and ε_j are one-point sets. (see [TWZ24, Definiton 6.10.])

Proposition 4.8 (Move *I* - Indelay): *Let $H\Gamma = (V, E, r, s)$ be a finite hypergraph, w be a vertex that is not a source such that $w \in r(e)$ implies $r(e) = \{w\}$. $H\Gamma_D$ be the hypergraph obtained by an indelay of $H\Gamma$ at w. Then there is a surjective *-homomorphism from $C^*(H\Gamma_D)$ onto a full corner of $C^*(H\Gamma)$.*

Proof: (see Proof of [TWZ24, Proposition 6.11.]) □

Proposition 4.9: *Let $H\Gamma = (V, E, r, s)$ be a finite hypergraph and w be a vertex that is not a source and let $H\Gamma$ be ultra locally at w. The incoming edges of w be partitioned into disjoint sets $\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n$. Let $H\Gamma_D$ and $H\Gamma_I$ be the corresponding hypergraphs formed by an indelay and an insplitting respectively. Then $C^*(H\Gamma_D) \cong C^*(H\Gamma_I)$.*

Proof: (see Proof of [TWZ24, Proposition 6.12.]) □

Combining 4.8 and 4.9 we get the desired result under the locally ultra assumption.

Proposition 4.10 (Move *I* - Insplitting): *Let $H\Gamma = (V, E, r, s)$ be a finite hypergraph and w be a vertex that is not a source and let $H\Gamma$ be locally ultra at w. Let $H\Gamma_I$ be the hypergraph obtained by insplitting $H\Gamma$ at w. Then there is a surjective *-homomorphism from $C^*(H\Gamma_I)$ onto a full corner of $C^*(H\Gamma)$. (see [TWZ24, Corollary 6.13.])*

Now we come to an interesting example for both moves I. Therefore, we consider again the extended Cuntz-algebra, $\mathcal{E}_n$ (3.2). First we calculate the indelay version.

Example 4.11: Let $\mathcal{E}_n$ denote the extended Cuntz algebra as in Example 4.5. The vertex v is not a source and the hypergraph $H\Gamma$ is locally ultra in this vertex. Hence, we can apply the move indelay. We partition the set of incoming edges in finitely many nonempty sets:

$$\{e \in E \mid v \in r(e)\} = \varepsilon_1 \cup \cdots \cup \varepsilon_n = \{e_1\} \cup \cdots \cup \{e_n\}.$$

The hypergraph $H\Gamma_D$ obtained by an indelay of $H\Gamma$ at v is defined by

$$\begin{aligned} V_D &:= (V \setminus \{v\}) \cup \{v^1, \ldots, v^n\} = \{v^1, \ldots, v^n, w\}, \\ E_D &:= E \cup \{f^1, \ldots, f^{n-1}\} = \{e_1, \ldots, e_n, f^1, \ldots, f^{n-1}\}, \\ r_D(e_i) &:= (r(e_i) \setminus \{v\}) \cup \{v^i\} = \{v^i, w\} \text{ for all } i = 1, \ldots, n, \\ r_D(f^i) &:= \{v^i\} \text{ for all } i = 1, \ldots, n-1, \\ s_D(e_i) &:= (s(e_i) \setminus \{v\}) \cup \{v^1\} = \{v^1\} \text{ for all } i = 1, \ldots, n, \\ s_D(f^i) &:= \{v^{i+1}\} \text{ for all } i = 1, \ldots, n-1. \end{aligned}$$

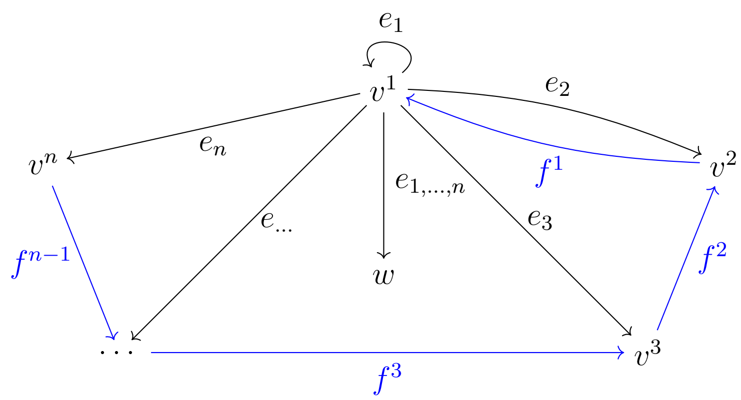

Figure 4.6: Example (extended Cuntz-algebra $\mathcal{E}_n$) after Move I - Indelay

According to Proposition 4.8, there exists a surjective *-homomorphism from the associated C^*-algebra to a full corner of $\mathcal{E}_n$. The surjective *-homomorphism is defined on the generators by $\phi(s_{e_i}) = T_i$ for all $i = 1, \ldots, n$ and $\phi(s_{f^j}) = T_{j+1}T_j^*$ for all $j = 1, \ldots, n-1$. Furthermore, $\phi(p_{v^i}) = T_iT_i^*$, and $\phi(p_w) = 1 - \sum_{i=1}^n T_iT_i^*$. This choice makes sense, because $T_i^*T_i = p_v + p_w = 1$ and $\sum_{i=1}^n T_iT_i^* \leq 1$ in the extended Cuntz-algebra. The partial isometries s_{e_i} correspond in the full corner to the isometries and the partial isometries s_{f^j} are the connections between the projections.

We also gain such a result for the insplitting move of $\mathcal{E}_n$.

Example 4.12: Let $\mathcal{E}_n$ denote the extended Cuntz algebra as in Example 4.5. Again, the requirements are satisfied. The vertex v is not a source and the hypergraph $H\Gamma$ is locally ultra in this vertex. Hence, we can apply the move insplitting. We partition the set of incoming edges in finitely many nonempty sets:

$$\{e \in E \mid v \in r(e)\} = \varepsilon_1 \cup \cdots \cup \varepsilon_n = \{e_1\} \cup \cdots \cup \{e_n\}.$$

The hypergraph $H\Gamma_I$ obtained by an insplitting of $H\Gamma$ at v is defined by

$$\begin{aligned}
V_I &\coloneqq (V \setminus \{v\}) \cup \{v^1, \dots, v^n\} = \{v^1, \dots, v^n, w\},\\
E_I &\coloneqq \{e^1 \mid e \in E, v \notin s(e)\} \cup \{e^1, \dots e^n \mid e \in E, v \in s(e)\}\\
&= \{e_1^1, \dots, e_1^n, \dots, e_n^1, \dots, e_n^n\},\\
r_I(e_j^i) &\coloneqq (r(e_j) \setminus \{v\}) \cup \{v^j\} = \{v^j, w\} \text{ for all } i, j = 1, \dots, n,\\
s_I(e_j^i) &\coloneqq \{v^i\} \text{ for all } i, j = 1, \dots, n.
\end{aligned}$$

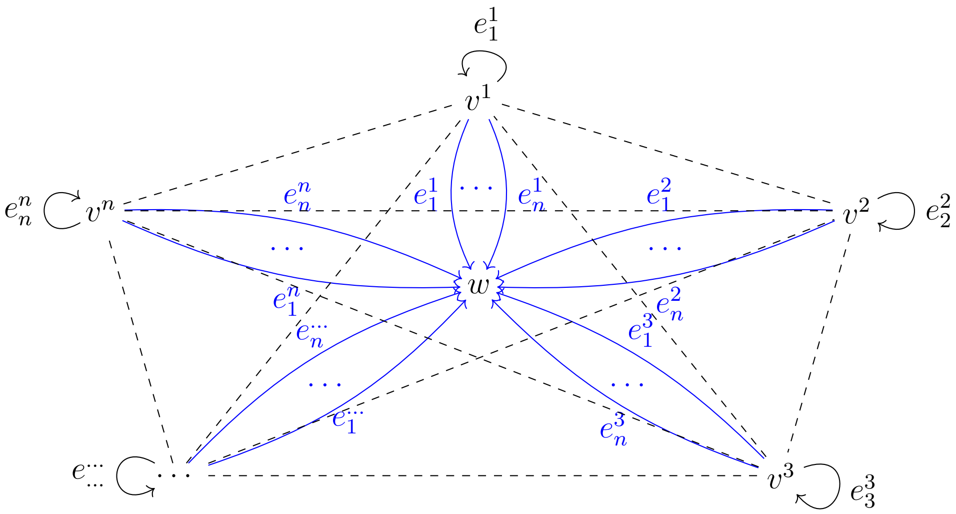

Figure 4.7: Example (extended Cuntz-algebra $\mathcal{E}_n$) after Move I - Insplitting

According to Proposition 4.10, there exists a surjective *-homomorphism from the associated C^*-algebra to a full corner of $\mathcal{E}_n$. The surjective *-homomorphism is defined on the generators by $\psi(s_{e_j^i}) = T_i T_j^*$ for all $i, j = 1, \dots, n$. Furthermore, $\psi(p_{v^i}) = T_i T_i^*$ for all $i = 1, \dots, n$ and $\psi(p_w) = 1 - \sum_{k=1}^{n} T_k T_k^*$. In the full corner it holds for the range that $\psi(s_{e_j^i})^*\psi(s_{e_j^i}) = (T_iT_j^*)^*(T_iT_j^*) = T_jT_i^*T_iT_j^* = T_jT_j^* = \psi(p_{r(s_{e_j^i})}) = \psi(p_{v^j})$ and for the source that $\psi(s_{e_j^i})\psi(s_{e_j^i})^* = (T_iT_j^*)(T_iT_j^*)^* = T_iT_j^*T_jT_i^* = T_iT_i^* = \psi(p_{s(s_{e_j^i})}) = \psi(p_{v^i})$.

Remark 4.13: Let $H\Gamma_D$ and $H\Gamma_I$ be the corresponding hypergraphs performed by indelay and insplitting $\mathcal{E}_n$ at v. According to Proposition 4.9 it holds that $C^*(H\Gamma_D) \cong C^*(H\Gamma_I)$.

List of Figures

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