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The Universal C^*-Algebra Generated By Two Projections As A Subalgebra Of Matrix Valued Continuous Functions

submitted by
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Contents

Introduction

C^*-algebras are indeed important objects as they provide a rich structure for the study of operators and abstract algebraic structures. In 1943 which is 81 years ago **I.M. Gelfand** and **M.A. Naimark** published a paper in functional analysis, *On the embedding of normed rings into the ring of operators in Hilbert space*[10] and this famous characterization later came to be known as C^*-Algebras. So, basically the study of C^*-algebras originated from this paper by Gelfand and Naimark. They accomplished in giving an abstract algebraic characterization of those Banach $*$-algebras (which they called normed $*$-rings) which are isometrically $*$-isomorphic to a norm-closed $*$-algebra of operators on a complex Hilbert space.

Universal C^*-algebras is a modern tool which is defined by sets of generators and relations and which offers a simple and natural expression for C^*-algebras but this expression is very abstract. For instance, it may very well happen that if we have a universal C^*-algebra $C^*(E|R)$ then it has no seminorm or it can even happen that it is zero. Therefore, there is a need of a concrete definition of universal C^*-algebras as well. For example we have Universal Orthogonal Projection i.e the universal C^*-algebra generated by one projection which turns out to be isomorphic to $\mathbb{C}$(details can be found in Section 5 of Chapter I). Similarly, there is another universal C^*-algebra which is unital C^*-algebra generated by one projection and it turns out to be that it is isomorphic to $\mathbb{C}^2$.

These being the two easy examples but there is another example which is a little complex but interesting, the universal C^*-algebra generated by two projections ([4], [5]) and we dig deep into this Universal C^*-Algebra in order to study it as a subalgebra of matrix valued continuous functions on [0,1] which is being the main focus of this thesis. Indeed this Universal C^*-Algebra generated by two projections is also isomorphic to a group C^*-algebra $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ (shown in Section 3 of Chapter II).

The main references of this thesis are the research papers [4] and [5]. The paper by **Iain Raeburn** and **Allan M. Sinclair** already shows that the Universal C^*-Algebra generated by two projections is isomorphic to the algebra of matrix valued continuous functions on [0,1] but the proof uses an application of a concept called "Mackey Machine" which is not used in the proof in this thesis. Rather a different approach is adopted to prove it in a more detailed and clear manner and therefore no prior knowledge of "Mackey Machine" is required to read this thesis. If you have a little background in Functional Analysis, you are good to go and read it further.

In Chapter I, we begin with an introduction of several preliminary concepts for instance Operator Algebras, C^*-Algebras, Universal C^*-Algebras, Group C^*-Algebras and Irreducible Representations which would lay the foundation for our main focus of the thesis.

In Chapter II, there are some of the own computations to show the existence, non-triviality and non-commutativity of Universal C^*-Algebra generated by two projections denoted as $C^*(p, q, 1)$. It is also shown that there also exists a commutative version of this C^*-Algebra and it is non-trivial as well. In fact there exists a surjective relationship between the non-commutative one and the commutative one. In the last section it is shown that this Universal C^*-Algebra generated by two projections is isomorphic to the group C^*-algebra $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$.

In Chapter III, we study the Universal C^*-Algebra generated by two projections as a sub-algebra of matrix valued continuous functions on [0,1] and hence prove the main theorem of the thesis stated below.

Main Theorem:

$C^*(p, q, 1)$ is isomorphic to

$$A' = \{f \in C([0,1], M_2(\mathbb{C})) : f(0) \text{ and } f(1) \text{ are diagonal matrices.}\}$$

This isomorphism carries the generating projections into the functions

$$p : x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q : x \mapsto \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}.$$

We first prove a series of lemmas which act as the ingredients of the proof and ultimately prove this theorem. We will use the Universal property of C^*-algebras to show that there exists a unique $*$-homomorphism between $C^*(p,q,1)$ and A'. The representation theory of C^*-algebras also plays a crucial role in this proof as we use the irreducible representation of $C^*(p,q,1)$ on $B(H)$ to show the one to one relation between $C^*(p,q,1)$ and A'. This approach of using the representation theory is an alternative to the "Mackey Machine" approach. One can observe that proving this theorem gives us a nice description of $C^*(p,q,1)$ and this description is indeed concrete as well because these generating projections can then be described as matrices in $M_2(\mathbb{C})$. Moreover, this proof is helpful in deducing the nuclearity of C^*-algebras. A nuclear C^*-algebra has a certain factorization property for its representations. Moreover, they have approximating properties by finite-dimensional representations. Nuclear C^*-algebras also have better behaved K-theory which further facilitates the study of their structure and classification.

Chapter I.

Foundations and Preliminary Concepts

1 Operator Algebras

We start with a little introduction to Operator algebras and Operator topologies and [6] can be referred to for a deeper understanding.

Definition I.1:

The subalgebra of bounded linear operators on a complex Hilbert Space H which are closed under the adjoint operation $A \to A^*$ is called an **Operator Algebra**.

There are two main classes of Operator Algebras that are worth considering: C^***-Algebras** and **Von Neumann Algebras** which are closed under locally convex operator toplogies which are defined next. Moreover, according to Fernando Lledo's paper *"Operator Algebras:An Informal Overview"* [6], these classes of Operator Algebras can be thought of as a rich algebraic structure on which we impose analytical conditions and this union of 'Algebra' and 'Analysis' is essential in proving some fundamental theorems. Some examples of statements that can be interpreted as a way of having algebraic characterization of analytical properties (or vice versa) are also worth reading from [6].

Definition I.2 Locally Convex Operator Topologies:

Let $(H, \langle ., . \rangle)$ be a Hilbert Space over $\mathbb{K}$.

1. The **weak operator topology (WOT)** on $B(H)$ is the locally convex topology that is defined by the family of seminorms
$$p_{x,y} : B(H) \to [0,\infty), T \mapsto |\langle Tx, y\rangle|$$
for all $x, y \in H$.
Weak operator topology (WOT) $\rightsquigarrow$ Von Neumann Algebras

2. The **strong operator topology (SOT)** on $B(H)$ is the locally convex topology induced by the family of seminorms
$$p_x : B(H) \to [0,\infty), T \mapsto \|Tx\|$$
for all $x \in H$.
Strong operator topology (SOT) $\rightsquigarrow$ Von Neumann Algebras

3. The **operator norm topology (ONT)** on $B(H)$ is the topology induced by the operator norm $\|.\|$.
Operator norm topology (ONT) $\rightsquigarrow$ C^*-Algebras

2 C^*- Algebras

In the realm of Operator Theory, C^*-algebras serve as a crucial framework, providing a rich structure for the study of operators and abstract algebraic systems. This section includes the definition, examples and some fundamental theorems for C^*-algebras and mostly these references can be referred [1],[3] and [8]. Before delving into the intricate properties and applications of C^*-algebras, let's establish some foundational concepts.

Definition I.3:

A **Banach Algebra** A is a $\mathbb{C}$-algebra that is complete and normed such that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

Definition I.4:

A C^***- Algebra** is a Banach Algebra with an involution i.e. an antilinear map $* : A \to A$, $x \mapsto x^*$ such that $(x+y)^* = x^* + y^*, x^{**} = x, (\lambda x)^* = \bar{\lambda} x^*$ and $(xy)^* = y^*x^*$ and $\|x^*x\| = \|x\|^2$. A is a **unital** C^***-Algebra** if it is unital as an Algebra (i.e. $1 \in A$).

Such an axiomatic definition of a C^*-Algebra allows for an abstract definition of a C^*-Algebra i.e. take some *-algebra and find a good norm. $\rightsquigarrow$ **Universal** C^***-Algebra**

Moreover, from such an axiomatic definition it follows that the norm is completely determined by the Algebraic Structure and is thus unique.
$\|\mathbf{x}^*\mathbf{x}\| = \|\mathbf{x}\|^{\mathbf{2}}$ is often referred as the C^***-identity** and one obvious consequence of this identity is that if A is a C^*-algebra and $x \in A$ such that $x^*x = 0$ then it implies that $x = 0$.

Let us now look at some examples to get a better understanding of what C^*-algebras might look.

Example I.5:

The scalar field $\mathbb{C}$ is a unital C^*-algebra with involution given by complex conjugation $\lambda \mapsto \overline{\lambda}$.

Example I.6:

The set of Bounded operators on any Hilbert space H i.e.

$$B(H) = \{T : H \to H : T \text{ is linear and bounded}\}$$

is a unital C^*-algebra with the usual operator norm and involution.

Example I.7:

If H is finite dimensional then the set of $n \times n$ matrices with complex entries i.e. $M_n(\mathbb{C})$ is a C^*-algebra with conjugate transpose as the involution and operator norm.

Example I.8:

The set of Compact operators on any Hilbert space H i.e.

$$\mathbb{K}(H) = \{T : H \to H : T \text{ is linear and compact}\}$$

is a C^*-algebra with the usual operator norm and involution. $\mathbb{K}(H)$ is non-unital if $\dim(H) = \infty$ and $\mathbb{K}(H) = B(H)$ if $\dim(H) < \infty$.

Example I.9:

Let X be a compact Hausdorff Space then the set of continuous functions i.e.

$$C(X) = \{f : X \to \mathbb{C} \text{ continuous }\}$$

is a unital commutative C^*-algebra.

Example I.10:

Let X be a locally compact space then

$$C_0(X) = \{f : X \to \mathbb{C} \text{ continuous}, \ f^*(z) = f(\bar{z}), f(\infty) = 0\}$$

is a non-unital C^*-algebra.

Example I.11:

If $(A_\lambda)_{\lambda \in \Lambda}$ is a family of C^*-algebras , then the direct sum $\oplus_\lambda A_\lambda$ is a C^*-algebra with the pointwise defined involution.

Example I.12:

Let S is a set, then $l^\infty(S)$ is a C^*-algebra with involution $f \mapsto \overline{f}$.

Example I.13:

Let (Ω, μ) be a measure space, then $L^\infty(\Omega, \mu)$ is a C^*-algebra with involution $f \mapsto \overline{f}$.

Example I.14:

Let Ω be a topological space, then $C_b(\Omega)$ is a C^*-algebra with involution $f \mapsto \overline{f}$.

Example I.15:

Let Ω be a measurable space, then $B_\infty(\Omega)$ is a C^*-algebra with involution $f \mapsto \overline{f}$.

A C^*-algebra may not always be unital as we have seen from the examples above($\mathbb{K}(H)$ and $C_0(X)$) and so the process of unitization comes into action by which we could reduce many aspects to the unital case to have a better understanding of the theory of C^*-algebras.

Definition I.16:

Let A be a non-unital C^*-algebra then we define

$$\tilde{A} := \{(a, \lambda) : a \in A, \lambda \in \mathbb{C}\}.$$

$\tilde{A}$ is a **unital $*$-algebra** with respect to involution, addition, scalar multiplication (entrywise) and multiplication $[(a, \lambda).(b, \mu) = (a + \lambda).(b + \mu) = ab + \lambda b + \mu a + \lambda\mu]$.

The following theorem is II.1.2.1 in [1] and Proposition 2.20 in [2].

Theorem I.17:

Let A be a C^*-algebra. Then there exists a norm on $\tilde{A}$ such that $\tilde{A}$ is a C^*-algebra and so we have that $A \lhd \tilde{A}, a \mapsto (a, 0) = a + 0$ and $\|a\|_A = \|a\|_{\tilde{A}}$ for all $a \in A$. This is known as the **minimial unitization** of A.

In the realm of Banach Algebras, homomorphisms play a pivotal role. An algebra homomorphism preserves algebraic structure, while a $*$-homomorphism additionally respects the involution.

Definition I.18:

Let A and B be Banach algebras then:

- $\varphi : A \to B$ is an **algebra homomorphism** $\Leftrightarrow \varphi$ is linear and multiplicative.
- $\varphi : A \to B$ is a **$*$-homomorphism** $\Leftrightarrow \varphi$ is linear, multiplicative and $\varphi(x)^* = \varphi(x^*)$.
- φ is **isometric** $\Leftrightarrow \|\varphi(x)\| = \|x\|$.
- φ is **unital** $\Leftrightarrow \varphi(1) = 1$.

Now let us first define the spectrum of a Bnanach algebra and then the spectrum of a C^*-algebra. The spectrum of an element in a C^*-algebra, which varies based on the algebra's structural properties, plays a crucial role in analyzing its behavior and characteristics.

Definition I.19:

Let A be a Banach algebra and $x \in A$. The **Spectrum** of x in A is

$$Sp_A(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ is not invertible in } A\}.$$

Definition I.20:

Let A be a C^*-algebra and $x \in A$. Then we define the **spectrum** of x as

$$Sp(x) = \begin{cases} Sp_A(x) & \text{if } A \text{ is unital} \\ Sp_{\tilde{A}}(x) & \text{if } A \text{ is not unital.} \end{cases}$$

Now there is a need to define what a character is in the context of a Banach algebra A in order to understand the relevance of $Spec(A)$ which is indeed the set of all characters of A.

Definition I.21:

Let A be a Banach Algebra, $\varphi : A \to \mathbb{C}$ is a **character** if and only if φ is an algebraic homomorphism and $\varphi \neq 0$.

Definition I.22:

Let A be a Banach Algebra, then define:

$$\mathbf{Spec(A)} = \{\varphi : \varphi \text{ is a character}\}.$$

$Spec(A)$ is basically the set of non-zero algebraic homomorphism from the Banach algebra A to the complex space. It serves as a powerful tool particularly when A is commutative, revealing the algebra's intrinsic spectral properties. Moreover, during the application of the First Fundamental Theorem For C^*-Algebras, also known as the Gelfand-Naimark theorem (which links commutative Banach algebras to the algebra of continuous functions over a compact metric space), $Spec(A)$ serves as a fundamental tool.

Definition I.23:

Let A be a commutative unital Banach algebra. The **Gelfand Transform** is defined as

$$\kappa : A \to C(Spec(A)), x \mapsto \hat{x} \text{ i.e. } \hat{x}(\varphi) = \varphi(x).$$

When dealing with commutative unital Banach algebras, the Gelfand Transform emerges as a powerful tool. This transform, denoted as κ, maps elements of the algebra to continuous functions on the set of characters, offering a profound connection between algebraic structures and functional analysis. This connection between algebraic elements and continuous functions enriches our understanding of the underlying algebraic structure by providing a functional-analytic perspective.

Theorem I.24 First Fundamental Theorem For C^*-Algebras:

Let A be a commutative and unital C^*-algebra then $A \cong C(X)$ for some compact metric space X. In fact $X = Spec(A)$ i.e. the Gelfand transform is an isometric $*$-isomorphism for commutative unital C^*-algebras.

This theorem is II.2.2.4 in [1] and is also known as Gelfand-Naimark Theorem. It is indeed a Fundamental Theorem because it acts as a bridge between operator algebras and the study of functions on spaces, as it connects commutative unital C^*-algebras to function algebras on compact spaces. This link facilitates the study of properties and behaviors of operators on Hilbert spaces through the study of functions on compact metric spaces, making it easier to analyze and manipulate operators in certain cases.

Now there exists a very close connection between the C^*-algebras and Hilbert Space and it is essential to understand this connection as it motivates the study of Operator Theory and of course it's applications. This connection is made via two ideas: States and Representations.

Definition I.25:

Let A be a C^*-algebra and $x \in A$, then x is **positive** if and only if $x = x^*$ and $Sp(x) \subseteq [0, \infty)$.

Positivity of an element not only reflects its self-adjoint nature but also ensures that its spectrum lies in the non-negative real axis.

Definition I.26:

Let A be a C^*-algebra then $\varphi : A \to \mathbb{C}$ is a linear functional. φ is **positive** if and only if $\varphi(x) \geq 0$ for all $x \geq 0$.

Positivity in functionals captures a notion of positivity that aligns with the positivity of elements, establishing a link between algebraic structures and linear functionals.

Definition I.27:

Let A be a C^*-algebra and $\varphi : A \to \mathbb{C}$ is a **state** if and only if φ is positive linear and $\|\varphi\| = 1$.

Theorem I.28 Hahn-Banach For C^*-Algebras:

Let A be a C^*-algebra and $x \in A$ is normal. Then there is a state $\varphi : A \to \mathbb{C}$ with $|\varphi(x)| = \|x\|$.

The above result is Proposition II.6.3.3 in 1 and it is a generalization of the classical Hahn-Banach theorem and so it seems logical to call it Hahn-Bnanach for C^*-algebras. It establishes the existence of states by achieving the norm of a normal element.

Definition I.29:
Let A be a C^*-algebra :

- H is a Hilbert space then $\pi : A \to B(H)$ is a $*$-homomorphism and π is called a **representation**.
- Let $\pi_1 : A \to B(H_1)$ and $\pi_2 : A \to B(H_2)$ are representations, then π_1 and π_2 are **equivalent** if and only if there exists a unitary $U : H_1 \to H_2$ such that $\pi_2(x) = U\pi_1(x)U^*$ for all $x \in A$.
- $\pi : A \to B(H)$ is **non-degenerate** if and only if $\overline{\pi(A)H} = H$ where $\pi(A)H = \{\pi(a)x : a \in A, x \in H\}$.
- π is **faithful** if and only if π is injective.
- $\pi : A \to B(H)$ is **cyclic** if and only if there exists a cyclic vector $x \in H$ such that $\overline{\pi(A)x} = H$.
- A **subrepresentation** of a representation π on H is the restriction of H to a closed invariant subspace of H.
- A representation is **irreducible** if it has no nontrivial closed invariant sub-spaces. (Detail in section 6 of Chapter I)

These concepts deepen our understanding of how C^*-algebras can be faithfully represented on Hilbert spaces, providing a bridge between abstract algebraic structures and concrete operator spaces.

Now since we are aware of the States and Representations and the connection between C^*-algebras and Hilbert Space, here comes another main and fundamental idea of Operator Theory which provides a method of creating these representations of C^*-Algebras, **"The GNS Construction"**.

Theorem I.30 GNS(Gelfand–Naimark–Segal) Construction:
If A is a C^*-algebra and $\varphi : A \to \mathbb{C}$ is a state. Then there exist the following:

- a **Hilbert space** H_φ
- a **representation** $\pi_\varphi : A \to B(H_\varphi)$
- a **cyclic vector** $x_\varphi \in H_\varphi$

such that $\varphi(a) = \langle \pi_\varphi(a)x_\varphi, x_\varphi \rangle$.

The above result is II.6.4 in [1]. GNS construction is one of the main and fundamental ideas in Operator Theory as it provides a method of creating the representations of C^*-algebras.

Theorem I.31 Second Fundamental Theorem For C^*-Algebras:

Any C^*-algebra A possesses a faithful representation $\pi : A \to B(H)$ on some Hilbert Space H. Thus, A is isomorphic to $\pi(A) \subseteq B(H)$, a C^*-subalgebra of $B(H)$. If A is separable, H can also be chosen to be separable.

The above result is Corollary II.6.4.10 in [1]. The significance of the Second Fundamental Theorem (also known as Gelfand-Naimark theorem) lies in the fact that it ensures the existence of a concrete and faithful representation of any C^*-algebra on a Hilbert space, thereby connecting abstract algebraic structures to concrete operators on a well-understood functional space. This representation is helpful in the study and analysis of C^*-algebras, making it possible to apply techniques and tools from functional analysis and Hilbert space theory to gain insights into the original algebraic properties of A.

3 Group C^*- Algebras

In this section, we delve a little into Group C^*-algebras for which [7] can be referred. So, Group C^*- algebras helps to study the representation theory of groups with the help of C^*- algebra theory. In other words, the C^*- algebra of a group helps to encode all the information about unitary representations of the group.

Definition I.32:

Let G be a locally compact abelian group. Then the **dual** of G is defined as

$$\hat{G} := \{\phi : G \to \mathbb{T} \subseteq \mathbb{C} | \phi \text{ is a group homomorphism}\}$$

$\hat{G}$ is also locally compact abelian group.

Example I.33:

- $\hat{\mathbb{Z}} = \mathbb{T}$
- $\hat{\mathbb{R}} = \mathbb{R}$
- $\widehat{\mathbb{Z} \setminus n\mathbb{Z}} = \mathbb{Z} \setminus n\mathbb{Z}$

Moreover, Pontryagin duality states that $G \to \hat{\hat{G}}, x \mapsto ev_x$ is an isomorphism of topological groups where $ev_x : \hat{G} \to \mathbb{T}$ is the evaluation map such that $ev_x(\phi) = \phi(x)$.

Also, we have that if G is an abelian group then the group C^*-algebra denoted as $C^*(G) \cong C(\hat{G})$ as suppose we take the example where $G = \mathbb{Z}$ then we have

$$C^*(\mathbb{Z}) \cong C(\mathbb{T}) = C(\hat{\mathbb{Z}}).$$

In the case of abelian compact groups, the dual contains enough information to fully recover the original group. However, when dealing with non-abelian compact groups, the situation is different. The dual of a non-abelian compact group may not be sufficient to uniquely recover the original group. This is because it can happen that there are different non-isomorphic non-abelian compact groups with the same dual i.e. multiple non-abelian compact groups can have identical sets of group homomorphisms.

Then the question arises: **"How to define the dual of non-abelian compact group?"**
The answer is: **"By considering the group C^*-algebra i.e. $C^*(G)$"** because we already know that in the abelian case it is isomorphic to the continuous functions on the dual. Therefore, it is worth studying the group C^*-algebras because it is able to give some insights on how the dual could be defined for such non-abelian compact groups.

Now since we are clear why we need to study Group C^*-Algebras, let us move further towards its formal definition but for that we first need to define a few things which would lay the foundation of its definition.

Definition I.34:

Let G be a locally compact group. A **unitary representation** of G is a group homomorphism

$$\pi : G \to \mathcal{U}(H) := \{u \in B(H) \text{ is a unitary}\}$$

such that

$$G \to H, g \mapsto \pi(g)h$$

is continuous for all $h \in H$ i.e. π is continuous in the Strong Operator Topology.

Lemma I.35:

On $\mathcal{U}(H)$, the strong and the weak operator topology (I.2) coincide i.e. if a net $(u_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{U}(H)$ converges to a unitary $u \in \mathcal{U}(H)$ in the Weak Operator Topology then it also converges in the Strong Operator Topology .

Proof. Let $(u_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathcal{U}(H)$ such that $u_\lambda \to u$ in the Weak Operator Topology. Now consider

$$\begin{aligned}
\|u_\lambda h - uh\|^2 &= \|u_\lambda h\|^2 + \|uh\|^2 - \langle u_\lambda h, uh\rangle - \langle uh, u_\lambda h\rangle \\
&\to \|h\|^2 + \|uh\|^2 - \langle uh, uh\rangle - \langle uh, uh\rangle \\
&= \|h\|^2 + \|uh\|^2 - \|uh\|^2 - \|h\|^2 = 0
\end{aligned}$$

i.e. $\|u_\lambda h - uh\|^2 \to 0$. Therefore, $u_\lambda \to u$ in the Strong Operator Topology. □

Definition I.36:

Every locally compact group G has a distinguished representation called the **left regular representation** on $L^2(G)$. This is defined by:

$$\lambda(s)g(t) = g_s(t) = g(s^{-1}t).$$

This map is unitary.

Definition I.37:

The **reduced group C^*-algebra** of G is defined to be $C^*_r(G) = \overline{\lambda(L^1(G))}$.

Definition I.38:

The **group C^*-algebra** is the closure of the universal representation of $L^1(G)$.

In other words, take π_u to be a direct sum of all irreducible representations (upto unitary equivalence) of G. Then $C^*(G)$ is the norm closure of $\pi_u(L^1(G))$. Equivalently define a C^*- norm on $L^1(G)$ by

$$\|f\| = \sup\{\|\pi(f)\| : \pi \text{ is a } * \text{-representation of } L^1(G)\}.$$

This collection of representations is non-empty because any irreducible representation of $C_r^*(G)$ provides an irreducible representation of G and such representations exist by GNS construction. Also, since $\|f\| \leq \|f\|_1$, this supremum is well defined.

Now let us have a look at two examples of the group C^*-algebras.

Example I.39:

Let $\mathbb{Z}$ be the group of integers, then $C^*(\mathbb{Z}) \cong C(\hat{\mathbb{Z}}) = C(\mathbb{T})$ is the group C^*-algebra of $\mathbb{Z}$ and it is isomorphic to the continuous functions on the Torus.

Example I.40:

Let $\mathbb{Z}_2$ be the cyclic group of order 2, then $C^*(\mathbb{Z}_2) \cong C(\hat{\mathbb{Z}}_2) = C(\mathbb{Z}_2)$ is the group C^*-algebra of $\mathbb{Z}_2$ and it is isomorphic to the continuous functions on the cyclic group $\mathbb{Z}_2$.

4 Universal C^*- Algebras

The significance of universal constructions in the theory of C*-algebras has grown notably. Universal C*-algebras, defined by sets of generators and relations, offer a simple and natural expression for many important C*-algebras. This construction, while extremely general, provides a universal framework as every C*-algebra can be represented, albeit in an uninteresting manner, as the universal C*-algebra on a specific set of generators and relations.

Now let's slowly build towards the definition of a universal C^*-algebras with [2] being the reference to this section.

Definition I.41:

Let $E = \{x_i | i \in I\}$ be a set of elements, I is the index set:

- $x_{i_1}x_{i_2}...x_{i_m}$ is a **non-commutative monomial** where $i_1, ..., i_m \in I$.
- A **non-commutative polynomial** is $\mathbb{C}$-linear combination $\sum \alpha_{i_1...i_m} x_{i_1...i_m}$.
- **Multiplication by concatenation** is defined as: $(x_{i_1}x_{i_2}...x_{i_m})(x_{j_1}x_{j_2}...x_{j_m}) := x_{i_1}x_{i_2}...x_{i_m}x_{j_1}x_{j_2}...x_{j_m}$.
- **Involution** is defined as: $(\alpha x_{i_1}^{\epsilon_1}x_{i_2}^{\epsilon_2}.....x_{i_m}^{\epsilon_m})^* := \bar{\alpha} x_{i_1}^{\bar{\epsilon_1}}x_{i_2}^{\bar{\epsilon_2}}.....x_{i_m}^{\bar{\epsilon_m}}, \alpha \in \mathbb{C}$
  $\epsilon_1, ..., \epsilon_m \in \{1, *\}$
  $$\bar{\epsilon} = \begin{cases} * & \text{if } \epsilon = 1 \\ 1 & \text{if } \epsilon = *. \end{cases}$$
- The **Free *-algebra** $P(E)$ consisting of non-commutative polynomials over the set $\{x_i | i \in I\} \cup \{x_i^* | i \in I\}$ with multiplication by concatenation and involution as defined above.

Definition I.42:
Let $E = \{x_i | i \in I\}$, the set of generators and $R \subseteq P(E)$, the relations be given. Then $\mathbf{A}(\mathbf{E}|\mathbf{R}) = \mathbf{P}(\mathbf{E})/\mathbf{J}(\mathbf{R})$ is defined as the **universal *-algebra** with generators E and relations R and $J(R)$ is the two sided *-ideal generated by R.

Definition I.43:
Let A be a *-algebra, $p : A \to [0, \infty)$ is a C^***-seminorm** if for all $\lambda \in \mathbb{C}$ and for all $x \in A$:

1. $p(\lambda x) = |\lambda| p(x)$ and $p(x + y) \leq p(x) + p(y)$
2. $p(xy) \leq p(x)p(y)$
3. $p(x^*x) = p(x)^2$.

Now, of course this is not a norm yet, as "$p(x) = 0 \Rightarrow x = 0$" is still missing!

Definition I.44:
Let $E = \{x_i | i \in I\}$, the set of generators and $R \subseteq P(E)$, the relations be given. Put

$$\|x\| := \sup\{p(x) : p \text{ is a } C^*\text{-seminorm on } A(E|R)\}.$$

If $\|x\| < \infty$ for all $x \in A(E|R)$, it is a C^*- norm and $\{x \in A(E|R) : \|x\| = 0\}$ is a two sided *-ideal. Then we define,

$$C^*(E|R) := \overline{A(E|R) \setminus \{x \in A(E|R) : \|x\| = 0\}}^{\|.\|(\text{completion})}$$

as the **universal C^*-algebra** with generators E and relations R.

Now that we have defined the Universal C^*-Algebra, we now want to make sure whether any such algebra would exist which means we need to be sure that the norm is finite.

Lemma I.45:

Let E, R be given:

- If $\|x\| < \infty$ for all $x \in A(E|R)$, then $C^*(E|R)$ is a C^*-Algebra and we say that the universal C^*-Algebra of E and R exists.
- If there is a constant $c > 0$ such that $p(x_i) < c$ for all $i \in I$ and for all C^*-seminorms p on $A(E|R)$ then $\|x\| < \infty$ for all $x \in A(E|R)$.

Hence, it can be checked using this lemma if the given norm is finite for all the elements in the universal $*$-algebra which ultimately means that this specific universal $*$-algebra is a C^*-algebra. This finiteness of the norm further implies the existence of this Universal C^*-algebra.

Lemma I.46 Universal Property:

Let E, R be given and assume $C^*(E|R)$ exists. Let B be a C^*-algebra containing $E' = \{y_i : i \in I\} \subseteq B$ and satisfying the relations R. Then there is a unique $*$-homomorphism $\varphi : C^*(E|R) \to B$, $x_i \mapsto y_i$ for all $i \in I$.

This property is fundamental as it allows us to study and characterize C^*-algebras by their generators and relations, simplifying the analysis and providing a universal framework for comparing different C^*-algebras with similar algebraic structures.

Now, the following section provides two examples of Universal C^*-Algebras and it also deals with the existence and non-triviality of these Universal C^*-Algebras.

5 Universal C^*-Algebra Generated By One Projection

In this subsection, the existence and non-triviality of the Universal C^*-algebra generated by one projection is shown through own computations.

Definition I.47:

Universal Orthogonal Projection i.e. the C^*-algebra generated by one projection denoted by $C^*(p : p = p^* = p^2)$ is a universal C^*-algebra.

Lemma I.48:

$C^*(p : p = p^* = p^2)$ exists and is non-trivial.

Proof. **Existence** can be shown using the Lemma I.45. Let r be some C^*-seminorm. Then,

$$r(p)^2 = r(p^*p) = r(p.p) = r(p^2) = r(p) \Rightarrow r(p) \in \{0, 1\}$$

which means that $r(x) \leq 1 := C$ for all $x \in C^*(p : p = p^* = p^2)$ and for all C^*-seminorms r. Hence, $\|x\| < \infty$ and therefore, $C^*(p : p = p^* = p^2)$ exists. Since, $C^*(p : p = p^* = p^2)$ contains just one monomial "**p**" [as $(p^*)^n = p^n = p$ and $(p^*p)^n = (pp)^n = (p^2)^n = p^n = p$] therefore the dimension of $C^*(p : p = p^* = p^2)$ is atmost 1 and so it is **isomorphic to** $\mathbb{C}$ and hence, **non-trivial**. □

Definition I.49:

Unital C^*-algebra generated by one projection is a Universal C^*-Algebra and is denoted by

$$C^*(p, 1 : p \text{ is a projection i.e. } p^2 = p = p^* \text{ and } 1 \text{ is a unit}).$$

Lemma I.50:

$C^*(p, 1 : p$ is a projection i.e. $p^2 = p = p^*$ and 1 is a unit$)$ exists and is non-trivial.

Proof. **Existence** can be shown in a similar fashion as above. **Non-triviality** can be shown if we can find some B with $s \in B$ such that s is a projection and $s \neq 0$. Let $B = C^*(\mathbb{Z}_2) = C^*(u, 1 : u$ is a unitary i.e. $u^*u = uu^* = 1$ and $u^2 = 1)$ and so let

$s = \dfrac{1-u}{2}$. Therefore, we have

$$s^2 = \frac{(1-u)^2}{4} = \frac{1-2u+u^2}{4} = \frac{2-2u}{4} = \frac{1-u}{2} = s.$$

Therefore, s is a projection in B. Hence, by the Universal Property, there is a unique $*$-homomorphism $\varphi : C^*(p,1) \to C^*(\mathbb{Z}_2)$ and so it is non-trivial. $\square$

Lemma I.51:

$C^*(p,1)$ is isomorphic to $C^*(\mathbb{Z}_2)$.

Proof. Consider the maps:

- $\varphi : C^*(p,1) \to C^*(\mathbb{Z}_2), p \mapsto \dfrac{1-u}{2}, 1 \mapsto 1$
- $\psi : C^*(\mathbb{Z}_2) \to C^*(p,1), u \mapsto 1-2p, 1 \mapsto 1$

Now by the Universal Property of C^*-algebras these maps exists because we have already shown in the above lemma that $\frac{1-u}{2}$ is a projection in $C^*(\mathbb{Z}_2)$ and indeed $1-2p$ is a unitary in $C^*(p,1)$ as

$$(1-2p)(1-2p)^* = (1-2p)(1-2p^*) = 1-2p^*-2p+4pp^* = 1-4p+4p = 1$$

and

$$(1-2p)^*(1-2p) = (1-2p^*)(1-2p) = 1-2p-2p^*+4p^*p = 1-4p+4p = 1$$

because p is a projection. Also, since p is a projection we have,

$$(1-2p)^2 = 1-2p-2p+4p^2 = 1-4p+4p = 1.$$

So, it verifies that $1-2p$ is a unitary in $C^*(p,1)$.
For showing isomorphism it can be easily checked that:

$$\varphi \circ \psi(u) = \varphi(\psi(u)) = \varphi(1-2p) = \varphi(1) - 2\varphi(p) = 1 - 2\frac{(1-u)}{2} = u$$

$$\psi \circ \varphi(p) = \psi(\varphi(p)) = \psi(\frac{1-u}{2}) = \frac{1}{2}\psi(1) - \frac{1}{2}\psi(u) = \frac{1}{2} - \frac{1-2p}{2} = p.$$

Since,

$$\varphi \circ \psi = I_{C^*(\mathbb{Z}_2)} \text{ and } \psi \circ \varphi = I_{C^*(p,1)}$$

therefore we have, $\psi = \varphi^{-1}$.

Hence, $\varphi : C^*(p,1) \to C^*(\mathbb{Z}_2)$ is an isomorphism i.e. $\mathbf{C^*(p,1) \cong C^*(\mathbb{Z}_2)}$. $\square$

Furthermore, $C^*(p,1)$ contains just 2 monomials "1" and "p" [as $1^* = 1 = 1^2$ and $1p = p = p1$], so the dimension is atmost 2 so it is also isomorphic to $\mathbb{C}^2$ i.e. $\mathbf{C^*(p,1) \cong \mathbb{C}^2}$.

6 Irreducible Representations

In this section, we try to understand irreducible representations firstly via an example of reducible representation and then via the formal definition and fundamental properties, where [7] and [8] being the references.

Let A be a C^*-algebra and $\pi : A \to B(H)$ be a representation of A. Let $H = H_1 \oplus H_2$ with $\pi(A)H_1 \subset H_1$ and $\pi(A)H_2 \subset H_2$ such that $\pi = \pi_1 + \pi_2$. If π is cyclic then it can be reduced further.

Example I.52:
Let

$$A = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\} \subset M_2(\mathbb{C}).$$

Then $(1,1)^T$ is a cyclic vector.
$\pi(A)$ is invariant under $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
$\Rightarrow \pi = \pi_1 + \pi_2$
where π_1 and π_2 are one dimensional representations that act on $\mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ such that $A = \mathbb{C} \oplus \mathbb{C}$.

Definition I.53:

A representation (H, π) of A is called **irreducible** if $\{0\}$ and H are the only subspaces of H which are invariant under $\pi(A)$ i.e. if $K \subset H$ is a proper subspace of H and $\pi(A)K \subset K$ then it implies that either $K = \{0\}$ or $K = H$.

If $\dim \mathbf{H} = \mathbf{1}$, then every representation $\pi : A \to B(H) = \mathbb{C}$ is irreducible and $\pi \equiv 0$ because H is generated by a single vector so there are no non-trivial proper subspaces.

The following result is Theorem 4.1.12 in 8.

Theorem I.54:

Let (H, π) is a representation of A with $\pi \not\equiv 0$, then the following are equivalent;

1. π is irreducible.

2. $\pi(A)' := \{x \in B(H) : \pi(a)x = x\pi(a) \text{ for all } a \in A\} = \mathbb{C} \cdot 1$.

3. $\pi(A)$ is strongly dense in $B(H)$.

Proof. $1 \Leftrightarrow 2$:
If $\pi(A)' \supset \mathbb{C} \cdot 1$ then it should contain a non-scalar positive operator and thus it contains a proper projection P. And thus, PH will be an invariant subspace for $\pi(A)$ which is a contradiction as π is irreducible.

Conversely, suppose that M is a proper invariant subspace for $\pi(A)$ and let P be the orthogonal projection onto M. Now consider

$$P\pi(a) = (\pi(a^*)P)^* = (P\pi(a^*)P)^* = P\pi(a)P = \pi(a)P$$

for every $a \in A$. Thus, P is a non-scalar operator in $\pi(A)'$.

$2 \Leftrightarrow 3$:
Let $\pi(A)' = \mathbb{C} \cdot 1$ then

$$(\pi(A) + \mathbb{C} \cdot 1)' = \mathbb{C} \cdot 1$$

and therefore

$$(\pi(A) + \mathbb{C} \cdot 1)" = B(H).$$

However, since π is irreducible and non-degenerate and if $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $\pi(A)$, it is strongly convergent to 1. Therefore, $1 \in \overline{\pi(A)}$ and

$$\overline{\pi(A)} = \overline{\pi(A) + \mathbb{C}\cdot 1} = B(H).$$

Hence, $\pi(A)$ is strongly dense in $B(H)$.
Conversely, if $\pi(A)$ is strongly dense in $B(H)$, then

$$\pi(A)' = B(H)' = \mathbb{C}.$$

□

This theorem would also be helpful to show that if we have an irreducible representation of a commutative C^*-algebra then this representation is indeed a character because the Hilbert Space on which this C^*-algebra will be represented will turn out to be $\mathbb{C}$.

Lemma I.55:

Let (H, π) is an irreducible representation of A with $\pi \not\equiv 0$, then every vector $x \in H$ with $x \neq 0$ is cyclic for π.

Proof. Let π be irreducible and $x \in H$ is a non-zero vector. The space $\pi(A)x$ is invariant for $\pi(A)$ and so it should be equal to either 0 or H. Since, $\pi \not\equiv 0$ there exists $y \in H$ and $a \in A$ such that $\pi(a)y \neq 0$. Hence, $\pi(A)y = H$, so π is non-degenerate. Therefore, it follows that $\pi(A)x$ is not the zero space but it is the whole of H i.e x is a cyclic vector for π. □

Remarks:

1. Let (H, π) be a non-degenerate representation such that $\dim H < \infty$ then it implies that $\pi = \oplus\pi_i$ where π_i are irreducible representations.

2. When the $\dim H = \infty$, then we do not have such an easy description like in the former case where $\dim H < \infty$.

3. Let (H, π) is an irreducible representation of a commutative C^*-algebra A, then we have,

$$\pi(A) \underset{\text{A Comm.}}{\subset} \pi(A)' = \mathbb{C}\cdot 1 \text{ (by Theorem } I.54)$$
$$\Rightarrow \pi(A) \subset \mathbb{C}\cdot 1 \Rightarrow \pi(A)' = B(H)$$
$$\text{i.e. } B(H) = \mathbb{C} \Rightarrow H = \mathbb{C}$$

so $\pi : A \to \mathbb{C}$ is a $*$-homomorphism i.e. a Character. Therefore, if (H, π) is irreducible and A is commutative then it implies that $\dim H = 1$ and π is a character.

Theorem I.56:

Let A be a C^*-algebra such that $x \in A$ and $x \neq 0$ then there exists an irreducible representation $\pi : A \to B(H)$ such that $\pi(x) \neq 0$.

The above result is II.6.4.9 in [1] and it will play a crucial role later when proving the injectivity in the Main Theorem III.1 of the thesis.

Chapter II.

Definition Of Universal C^*-Algebra Generated By Two Projections

In this chapter we are going to deal with another example of Universal C^*-Algebras which is the most relevant for our main result represented in Chapter III.

Definition II.1:

The **Universal C^*-Algebra generated by two projections** is denoted as

$C^*(p, q, 1 : p, q$ are projections and 1 is a unit such that $p^* = p^2 = p$ and $q^* = q^2 = q)$.

Notation:

From now on we denote the above defined Universal C^*-Algebra generated by two projections as $C^*(p, q, 1)$.

1 Existence and Non-triviality

Lemma II.2:

$C^*(p, q, 1)$ exists and is non-trivial.

Proof. First thing is to find if such a Universal C^*-Algebra **exists** using the Lemma I.45: Let r be some C^*-seminorm. Then,

$$r(p)^2 = r(p^*p) = r(pp) = r(p^2) = r(p) \Rightarrow r(p) \in \{0, 1\}.$$

Similarly $r(q) \in \{0,1\}$ which means that $r(x) \leq 1 := C$ for all $x \in C^*(p,q,1)$ and for all C^*-seminorms r. Hence, $\|x\| < \infty$ and therefore, $\mathbf{C^*(p,q,1)}$ **exists**.

Second thing is to find if it is **non-trivial**, and for that we can find some B with $s,t \in B$ such that s and t are projections and $s,t \neq 0$.
Let $B = M_2(\mathbb{C})$ then

$$s = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } t = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are the two projections such that $s,t \in B$. By the universal property, there exists a *-homomorphism $\phi : C^*(p,q) \to B$. Hence, $\mathbf{C^*(p,q,1)}$ **is non-trivial**. □

Now, to study the structure of $C^*(p,q,1)$ we should know whether the two projections p and q commute with each other or not.

2 Non-Commutativity

Lemma II.3:

Let p,q are two projections in $C^*(p,q,1)$ which generate this universal C^*-algebra then p and q do not commute with each other.

Proof. Let $\psi : C^*(p,q,1) \to M_2(\mathbb{C})$ be a $*$-homomorphism such that

$$p \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p' \text{ and } q \mapsto \begin{pmatrix} c^2 & sc \\ sc & s^2 \end{pmatrix} = q'$$

where $c = \cos t$ and $s = \sin t$ where $t \neq \pi$ and $t \neq 0$.
Then, we have,

- $\psi(pq) = \psi(p)\psi(q) = p'q' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c^2 & sc \\ sc & s^2 \end{pmatrix} = \begin{pmatrix} c^2 & sc \\ 0 & 0 \end{pmatrix}$
- $\psi(qp) = \psi(q)\psi(p) = q'p' = \begin{pmatrix} c^2 & sc \\ sc & s^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c^2 & 0 \\ sc & 0 \end{pmatrix}.$

Hence, $\psi(pq - qp) = \psi(pq) - \psi(qp) \neq 0 \Rightarrow pq - qp \neq 0.$ □

So, now the question is whether $C^*(p, q, 1 : pq = qp)$ exists or not and whether it is non-trivial or not!

Lemma II.4:

$C^*(p, q, 1 : pq = qp)$ exists and is isomorphic to $\mathbb{C}^4$.

Proof. To find if such a Universal C^*-Algebra **exists** we can use the Lemma I.45: Let r be some C^*-seminorm. Then,

$$r(p)^2 = r(p^*p) = r(pp) = r(p^2) = r(p) \Rightarrow r(p) \in \{0, 1\}.$$

Similarly $r(q) \in \{0, 1\}$. Also,

$$\begin{aligned} r(pq)^2 &= r((pq)^*(pq)) = r(q^*p^*pq) \\ &= r(qppq) = r(qp^2q) = r(qpq) \\ &= r(qqp) = r(q^2p) = r(qp) = r(pq) \end{aligned}$$

which implies $r(pq) \in \{0, 1\}$. Hence, $r(x) \leq 1 := C$ for all $x \in C^*(p, q, 1)$ and for all C^*-seminorms r. Hence, $\|x\| < \infty$ and therefore, $\mathbf{C^*(p, q, 1 : pq = qp)}$ **exists**.
Moreover, $C^*(p, q, 1 : pq = qp)$ contains four monomials $1, p, q$ and pq, so the dimension is atmost 4.
Now let

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ und } q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are projections in $M_4(\mathbb{C})$. It can be verified that $p^2 = p = p^*$ and $q^2 = q = q^*$. Moreover, these projections commute.

$$pq = p \cdot q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$qp = q \cdot p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $pq = qp$, we conclude that p and q commute.
Now the elements $\{1, p, q, pq\}$ are clearly linearly independent, span a 4-dimensional subspace and includes the identity element. Hence, they generate a 4-dimensional C^*-algebra. Since the algebra generated by $\{1, p, q, pq\}$ is closed under addition, multiplication, and adjoints, and is spanned by these four matrices, it forms a 4-dimensional C^*-algebra. Each of these matrices corresponds to an element of $\mathbb{C}^4$ and therefore, it is isomorphic to $\mathbb{C}^4$. □

Now, obviously since in $C^*(p', q', 1 : p'q' = q'p')$ we have an extra relation $p'q' = q'p'$, therefore there can not exist one-to-one mapping between $C^*(p, q, 1)$ and $C^*(p', q', 1 : p'q' = q'p')$ but it can be shown that there exists a surjection from $C^*(p, q, 1)$ to $C^*(p', q', 1 : p'q' = q'p')$.

Lemma II.5:

The map $\Phi : C^*(p, q, 1) \to C^*(p', q', 1 : p'q' = q'p')$ is surjective.

Proof. We know that by the Universal Property, there is a unique $*$-homomorphism $\Phi : C^*(p, q, 1) \to C^*(p', q', 1 : p'q' = q'p')$ which maps the generators of $C^*(p, q, 1)$ to the generators of $C^*(p', q', 1 : p'q' = q'p')$ i.e.

$$\Phi(p) = p', \Phi(q) = q' \text{ and } \Phi(1) = 1.$$

This means that every generator in $C^*(p', q', 1 : p'q' = q'p')$ has a pre-image in $C^*(p, q, 1)$ which proves that Φ is a surjective map. □

3 Isomorphism to $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$

Lemma II.6:

$C^*(p, q, 1)$ is isomorphic to $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$.

Proof. We know that,

$C^*(\mathbb{Z}_2 * \mathbb{Z}_2) = C^*(u, v, 1 : u, v$ are unitaries i.e. $u^*u = uu^* = 1, v^*v = vv^* = 1$ and $u^2 = v^2 = 1)$.

Now, let $Z = C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ and let $s, t \in Z$ such that

$$s = \frac{1-u}{2} \text{ and } t = \frac{1-v}{2}.$$

If s and t are projections then by the universal property, there exists a $*$-homomorphism $\phi : C^*(p, q, 1) \to Z$.
To show s and t are projections, consider

$$s^2 = \frac{(1-u)^2}{4} = \frac{1-2u+u^2}{4} = \frac{2-2u}{4} = \frac{1-u}{2} = s$$

and

$$t^2 = \frac{(1-v)^2}{4} = \frac{1-2v+v^2}{4} = \frac{2-2v}{4} = \frac{1-v}{2} = t.$$

Therefore, there exists a unique $*$-homomorphism $\phi : C^*(p, q, 1) \to C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$.

Moreover, we consider the maps:

- $\phi : C^*(p, q, 1) \to C^*(\mathbb{Z}_2 * \mathbb{Z}_2), p \mapsto \dfrac{1-u}{2}, q \mapsto \dfrac{1-v}{2}, 1 \mapsto 1.$
- $\psi : C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \to C^*(p, q, 1), u \mapsto 1-2p, v \mapsto 1-2q, 1 \mapsto 1.$

We check if $1-2p$ and $1-2q$ are unitaries and $(1-2p)^2 = 1 = (1-2q)^2$ for the map ψ to exist.
So consider,

$$(1-2p)^2 = 1 + 4p^2 - 4p = 1 + 4p - 4p = 1$$

and

$$(1-2q)^2 = 1 + 4q^2 - 4q = 1 + 4q - 4q = 1.$$

Moreover we compute,

- $\phi \circ \psi(u) = \phi(\psi(u)) = \phi(1-2p) = 1 - 2(\dfrac{1-u}{2}) = 1 - 1 + u = u$
- $\psi \circ \phi(p) = \psi(\phi(p)) = \psi(\dfrac{1-u}{2}) = \dfrac{1-(1-2p)}{2} = p$

Since, we get $\phi \circ \psi = I_{C^*(\mathbb{Z}_2 * \mathbb{Z}_2)}$ and $\psi \circ \phi = I_{C^*(p,q,1)}$, we have that $C^*(p, q, 1) \cong C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$. □

Chapter III.

Isomorphism Between $C^*(p, q, 1)$ And Matrix Valued Continuous Functions

Now, using the paper, [4] **The C^*-Algebra Generated By Two Projections** by **Iain Raeburn** and **Allan M. Sinclair**, a C^*-algebraic approach can be adopted to further study $C^*(p, q, 1)$ as a subalgebra of matrix valued continuous functions on $[0, 1]$. In this chapter we prove the theorem which is the main focus of this thesis and the aim here is to make the proof of this theorem more detailed and clear.

We start by proving a series of lemmas which act as the ingredients of the main proof of this theorem which consists of three parts i.e. we first show a $*$-homomorphism from

$$\varphi : C^*(p, q, 1) \to A' = \{f \in C([0, 1], M_2(\mathbb{C})) : f(0) \text{ and } f(1) \text{ are diagonal matrices}\},$$

then we show that φ is surjective and lastly we show that φ is injective too.

The approach to show the injectivity involves representation theory as we use an irreducible representation of $C^*(p, q, 1)$ on $B(H)$ and show that it is a composition of a $*$-representation from $A' \to B(H)$ and the map φ. Following is the main theorem of our thesis and it is Theorem 1.3 in [4].

Theorem III.1:
$C^*(p, q, 1)$ is isomorphic to

$$A' = \{f \in C([0, 1], M_2(\mathbb{C})) : f(0) \text{ and } f(1) \text{ are diagonal matrices.}\}$$

This isomorphism carries the generating projections into the functions

$$p : x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q : x \mapsto \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}.$$

Lemma III.2:

The functions p and q which are in A' satisfy the following conditions:

a) p and q are projections contained in A'.

b) For $x \in (0,1), p(x)$ and $q(x)$ generate $M_2(\mathbb{C})$ as a C^*-Algebra.

c) For $x = 0$, $p(0)$ and $q(0)$ generate the diagonal algebra.

d) For $x = 1$, $p(1), q(1)$ and $\mathbb{1}$ generate the diagonal algebra.

Proof. We prove the above conditions:

a) First we have,

$$p(x)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p(x)$$

for all $x \in [0,1]$.
Now, taking the conjugate transpose of $p(x)$ i.e.

$$p(x)^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p(x)$$

for all $x \in [0,1]$.
Similarly we consider,

$$\begin{aligned} q(x)^2 &= \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix} \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix} \\ &= \begin{pmatrix} x^2 + x - x^2 & x\sqrt{x(1-x)} + (1-x)\sqrt{x(1-x)} \\ x\sqrt{x(1-x)} + (1-x)\sqrt{x(1-x)} & x - x^2 + 1 + x^2 - 2x \end{pmatrix} \\ &= \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix} \\ &= q(x) \end{aligned}$$

for all $x \in [0, 1]$.
Now, taking the conjugate transpose of $q(x)$ i.e

$$q(x)^* = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}^* = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix} = q(x)$$

for all $x \in [0, 1]$.
Hence, p and q are projections in A'.

b) Consider for $x \in (0, 1)$,

$p(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$\mathbb{1} - p(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = t(x)$,

$\dfrac{q(x)p(x) - xp(x)}{\sqrt{x(1-x)}} = \dfrac{1}{\sqrt{x(1-x)}} \left[\begin{pmatrix} x & \sqrt{x(1-x)} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = r(x)$,

$\left(\dfrac{q(x)p(x) - xp(x)}{\sqrt{x(1-x)}} \right)^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = s(x)$.

Now let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$ then we have,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a.p(x) + b.r(x) + c.s(x) + d.t(x).$$

Hence, for $x \in (0, 1)$, $p(x)$ and $q(x)$ generate $M_2(\mathbb{C})$.

c) For $x = 0 : p(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Now, let $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in$ the diagonal algebra where $a, b \in \mathbb{C}$ then we have,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a.p(0) + b.q(0).$$

Hence, for $x = 0$, $p(0)$ and $q(0)$ generate the diagonal algebra .

d) For $x = 1 : p(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $q(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbb{1}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now, let $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in$ the diagonal algebra where $a, b \in \mathbb{C}$ then we have,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -b & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a.p(1) + (-b).q(1) + b.\mathbb{1}_{2\times 2}.$$

Hence, $p(1), q(1)$ and $\mathbb{1}$ generate the diagonal algebra.

□

Lemma III.3:

Let $Q \in M_2(\mathbb{C})$ be a non-trivial projection then it implies that there exists a $t \in [0, 1]$ such that

$$Q = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Proof. First if we consider these two possibilities of Q, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ which are indeed non-trivial projections in $M_2(\mathbb{C})$. Then these can be represented as $Q = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$ when $t = 0$ and $t = 1$ respectively.

Now if $t \in (0, 1)$ then we let $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a projection in $M_2(\mathbb{C})$.

So,

$$Q = Q^* \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

and

$$Q^2 = Q \Rightarrow \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From the above two relations it follows that:

$a^2 + bc = a = a^*$

$b(a + d) = b = c^*$

$c(a+d) = c = b^*$
$bc + d^2 = d = d^*$
First let $a = t$ then it follows that

$$t^2 + bc = t \Rightarrow bc = t(t-1)$$

and

$$b(t+d) = b \Rightarrow t+d = 1 \Rightarrow d = 1-t.$$

Now, since $bc = t(t-1)$, $b = c^*$ and $c = b^*$, we get

$$c^*c = t(t-1) \text{ and } bb^* = t(t-1)$$

which implies that,

$$c = \sqrt{t(t-1)} \text{ and } b = \sqrt{t(t-1)}.$$

Therefore, we have $Q = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$ when $t \in (0,1)$.
Hence, if $Q \in M_2(\mathbb{C})$ is a projection then there would always exist a $t \in [0,1]$ such that

$$Q = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

□

Lemma III.4:

All the irreducible representations of $C^*(p,q,1)$ are finite dimensional.

Proof. Let $\pi : C^*(p,q,1) \to B(H)$ be an irreducible representation. Let $\dim H = \infty$. In an infinite-dimensional space, these projections can create invariant subspaces. For instance, if $\pi(p)$ and $\pi(q)$ are projections, their ranges and null spaces are invariant subspaces of H. Consider the projection $\pi(p)$. Let $L_p = \pi(p)H$ and $L_p^\perp = (\pi(p)H)^\perp$. These are both closed subspaces of H. Since $\pi(p)$ and $\pi(q)$ do not commute, the subspaces L_p and L_q interact in a non-trivial way, potentially creating further proper invariant subspaces under π. So, this would contradict the irreducibility π. Hence, if $\pi : C^*(p,q,1) \to B(H)$ is any irreducible representation then $\dim H$ has to be finite. □

Lemma III.5:

Let $\pi : C^*(p,q,1) \to B(H)$ be an irreducible representation then it implies that $\dim(H) \leq 2$ and every irreducible unital representation of $C^*(p,q,1)$ is unitarily equivalent to one of the following:

1. π_1 such that $p \mapsto 0, q \mapsto 0$ in $\mathbb{C}$.
2. π_2 such that $p \mapsto 0, q \mapsto 1$ in $\mathbb{C}$.
3. π_3 such that $p \mapsto 1, q \mapsto 0$ in $\mathbb{C}$.
4. π_4 such that $p \mapsto 1, q \mapsto 1$ in $\mathbb{C}$.
5. π_5 such that $p \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
6. π_6 such that $p \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q \mapsto \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$ for some $t \in (0,1)$.

Proof. Let $\dim H < \infty$ and let P and Q be the projection operators corresponding to the finite-dimensional subspace onto which p and q projects i.e. $\pi(p) = P$ and $\pi(q) = Q$.

Now, let u is a unitary such that

$$uPu^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } uQu^* = \begin{pmatrix} c & a \\ b & d \end{pmatrix}.$$

So, uPu^* and uQu^* are operators in $B(H) = B(PH \oplus (PH)^\perp$.

Also, since P is a projection we have,

$$(uPu^*)^* = uP^*u^* = uPu^*$$

$$\text{and } (uPu^*)(uPu^*) = uP^2u^* = uPu^*$$

which ensures that uPu^* is a projection.

Similarly since Q is a projection we have,

$$(uQu^*)^* = uQ^*u^* = uQu^*$$

$$\text{and } (uQu^*)(uQu^*) = uQ^2u^* = uQu^*$$

which ensures that uQu^* is also a projection.
And we also have that
$a : (PH)^\perp \to PH$,
$b : PH \to (PH)^\perp$,
$c : PH \to PH$ and
$d : (PH)^\perp \to (PH)^\perp$ are operators.
Therefore, by finite dimensionality of H we have that $\dim(PH) < \infty$ which implies that c is matrix. So, let v be an eigenvector of c i.e. $cv = \lambda v$.

Consider the subspace say K spanned by $\begin{pmatrix} v \\ 0 \end{pmatrix}$ and $\pi(q)\begin{pmatrix} v \\ 0 \end{pmatrix}$,

$$K = \text{Span}\left\{\begin{pmatrix} v \\ 0 \end{pmatrix}, \pi(q)\begin{pmatrix} v \\ 0 \end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix} v \\ 0 \end{pmatrix}, Q\begin{pmatrix} v \\ 0 \end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda v \\ bv \end{pmatrix}\right\}$$

as $Q\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} c & a \\ b & d \end{pmatrix}\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} cv \\ bv \end{pmatrix} = \begin{pmatrix} \lambda v \\ bv \end{pmatrix}$.

By this we get that $\dim K \leq 2$ as K is spanned by atmost 2 vectors.

Also, we have that,

- $P\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} \in K$
- $PQ\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda v \\ bv \end{pmatrix} = \begin{pmatrix} \lambda v \\ 0 \end{pmatrix} = \lambda\begin{pmatrix} v \\ 0 \end{pmatrix} \in K$
- $Q\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda v \\ bv \end{pmatrix} \in K$
- $QQ\begin{pmatrix} v \\ 0 \end{pmatrix} = Q\begin{pmatrix} v \\ 0 \end{pmatrix} \in K$ (as Q is a projection)

Hence, we have that K reduces π as it is invariant under P and Q and since π is an irreducible representation therefore it implies that $K = H$ or $K = 0$.
Therefore, $\dim H \leq 2$.

Now, we have that if $\pi : C^*(p, q, 1) \to B(H)$ is an irreducible representation on then $\dim H = \{0, 1, 2\}$ and we need to show that π is unitarily equivalent to each of the given representations π_i where $i = 1, 2, ..., 6$.

Let $L = \pi(p)H$ and so $H = L \oplus L^{\perp}$ and

$$\pi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \pi(q) = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in B(L \oplus L^{\perp}).$$

First it is required to show that $\dim L \leq 1$ and $\dim L^{\perp} \leq 1$:

$$L = \pi(p)H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x : x \in H \right\}$$

and we know that $\dim H \leq 2$.

- When $\dim H = 0$, then clearly $\dim L = \dim L^{\perp} = 0$.
- When $\dim H = 1$, then also we clearly have either $\dim L = 1$ and $\dim L^{\perp} = 0$ or $\dim L = 0$ and $\dim L^{\perp} = 1$.
- When $\dim H = 2$ and suppose $\dim L = 2$ then clearly $\dim L^{\perp} = 0$ which would imply that $\pi(q) = 0$ and so we could find a one-dimensional reducing subspace of H which will in turn imply that π is a reducible representation which is a contradiction. Hence, when $\dim H = 2$ then $\dim L < 2$. Similarly we can show that when $\dim H = 2$ then $\dim L^{\perp} < 2$. So, the only possiblity left is that $\dim L = 1 = \dim L^{\perp}$ when $\dim H = 2$.

Hence, we have shown that $\dim L \leq 1$ and $\dim L^{\perp} \leq 1$. Now, we show how can π be unitarily equivalent to π_i where $i = 1, 2, ..., 6$.

1. When $\dim H = 0$ and $H = L \oplus L^{\perp}$, so we have $\dim L = 0 = \dim L^{\perp}$. This is only possible when $\pi(p) = 0 = \pi(q)$. Hence, it shows that π is equivalent to π_1.
2. When $\dim H = 1$ and $H = L \oplus L^{\perp}$, so we have further 2 cases:
    - $\dim L = 0$ and $\dim L^{\perp} = 1$. This is only possible when $\pi(p) = 0$and $\pi(q) = 1$. Hence, it shows that π is equivalent to π_2.

- $\dim L = 1$ and $\dim L^{\perp} = 0$. This is only possible when $\pi(p) = 1$and $\pi(q) = 0$. Hence, it shows that π is equivalent to π_3.

3. When $\dim H = 2$ and $H = L \oplus L^{\perp}$, so we have $\dim L = 1 = \dim L^{\perp}$ and we consider further 3 cases:

   - If $b = 0$ and $a \neq 0$ we will have $\pi(p) = 1$ and $\pi(q) = 1$. Hence, it shows that π is equivalent to π_4.
   - If $b = 0$ and $a = 0$ we will have

$$\pi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \pi(q) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Hence, it shows that π is equivalent to π_5.
   - If $b \neq 0$ we will have

$$\pi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \pi(q) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$$

   where $t \in [0,1]$. Hence, it shows that π is equivalent to π_6.

□

Lemma III.6:

For all irreducible representations $\pi \neq 0$ there exists a $*$-representation $\sigma : A' \to B(H)$ such that the following diagram of maps commute:

$$\begin{tikzcd} C^*(p,q,1) \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & A' \arrow[d, "\sigma"] \\ & B(H) \end{tikzcd}$$

Proof. Let π be an irreducible representation of $C^*(p,q,1)$ and we know that from Lemma III.5 that π is unitarily equivalent to one of the π_i's where $i = 1,2,...6$.
Also, we have that from Lemma III.2((b),(c) and (d)) that $\varphi : C^*(p,q,1) \to A'$ is surjective.
Also, let $ev_t : \{f : f \in C([0,1], M_2(\mathbb{C}))\} \to M_2(\mathbb{C})$ be the evaluation map where $t \in [0,1]$.
Moreover, let us define:

- $\sigma_0 := ev_0|_{A'}$ or $ev_1|_{A'}$.
- $\sigma_1 : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$ such that $(a,b) \mapsto a$.
- $\sigma_2 : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}$ such that $(a,b) \mapsto b$.

Hence, the below diagram of maps clearly shows that how are we supposed to define σ in each case.

$$\begin{tikzcd}
A' \arrow[d, "\sigma_0"] & \subseteq & \{f : f \in C([0,1], M_2(\mathbb{C}))\} \arrow[d, "ev_t"] \\
\mathbb{C} \oplus \mathbb{C} \arrow[d, "\sigma_1 \text{ or } \sigma_2"] & \subseteq & M_2(\mathbb{C}) \\
\mathbb{C} & &
\end{tikzcd}$$

Let us begin with the 6 cases and show that the diagram of maps commute:

1. Let $\pi_1 : C^*(p,q,1) \to M_2(\mathbb{C})$ be an irreducible representation of $C^*(p,q,1)$ such that $\pi_1(p) = 0$ and $\pi_1(q) = 0$.

$$\begin{tikzcd}
C^*(p,q,1) \arrow[r, "\varphi"] \arrow[dr, "\pi_1"'] & A' \arrow[d, "\sigma := \sigma_2 \circ ev_1|A'"] \\
& B(H)
\end{tikzcd}$$

   Therefore, we take here $\sigma := \sigma_2 \circ \sigma_0 = \sigma_2 \circ ev_1|_{A'}$ and check if the above diagram of maps commute.
   Consider first,

$$\sigma \circ \varphi(p) = \sigma_2(ev_1(\varphi(p))) = \sigma_2\left(ev_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) = \sigma_2(1,0) = 0 = \pi_1(p)$$

   and

$$\begin{aligned}\sigma \circ \varphi(q) = \sigma_2(ev_1(\varphi(q))) &= \sigma_2\left(ev_1\begin{pmatrix}x & \sqrt{x(1-x)}\\ \sqrt{x(1-x)} & 1-x\end{pmatrix}\right)\\ &= \sigma_2(1,0) = 0 = \pi_1(q).\end{aligned}$$

   Hence, we have shown that $\sigma \circ \varphi = \pi_1$.

2. Let $\pi_2 : C^*(p,q,1) \to M_2(\mathbb{C})$ be an irreducible representation of $C^*(p,q,1)$ such that $\pi_2(p) = 0$ and $\pi_2(q) = 1$.

$$\begin{tikzcd} C^*(p,q,1) \arrow[r, "\varphi"] \arrow[dr, "\pi_2"'] & A' \arrow[d, "\sigma:=\sigma_2\circ ev_0|A'"] \\ & B(H) \end{tikzcd}$$

Therefore, we take here $\sigma := \sigma_2 \circ \sigma_0 = \sigma_2 \circ ev_0|_{A'}$ and check if the above diagram of maps commute.
Consider first,

$$\sigma \circ \varphi(p) = \sigma_2(ev_0(\varphi(p))) = \sigma_2 \left(ev_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \sigma_2(1,0) = 0 = \pi_2(p)$$

and

$$\begin{aligned}\sigma \circ \varphi(q) = \sigma_2(ev_0(\varphi(q))) = \sigma_2 &\left(ev_0 \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix} \right) \\ = \sigma_2(0,1) = 1 = \pi_2(q).&\end{aligned}$$

Hence, we have shown that $\sigma \circ \varphi = \pi_2$.

3. Let $\pi_3 : C^*(p,q,1) \to M_2(\mathbb{C})$ be an irreducible representation of $C^*(p,q,1)$ such that $\pi_3(p) = 1$ and $\pi_3(q) = 0$.

$$\begin{tikzcd} C^*(p,q,1) \arrow[r, "\varphi"] \arrow[dr, "\pi_3"'] & A' \arrow[d, "\sigma=\sigma_1\circ ev_0|A'"] \\ & B(H) \end{tikzcd}$$

Therefore, we take here $\sigma = \sigma_1 \circ \sigma_0 = \sigma_1 \circ ev_0|_{A'}$ and check if the above diagram of maps commute.
Consider first,

$$\sigma \circ \varphi(p) = \sigma_1(ev_0(\varphi(p))) = \sigma_1 \left(ev_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \sigma_1(1,0) = 1 = \pi_3(p)$$

and

$$\begin{aligned}\sigma \circ \varphi(q) = \sigma_1(ev_0(\varphi(q))) = \sigma_1 &\left(ev_0 \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix} \right) \\ = \sigma_1(0,1) = 0 = \pi_3(q).&\end{aligned}$$

Hence, we have shown that $\sigma \circ \varphi = \pi_3$.

4. Let $\pi_4 : C^*(p,q,1) \to M_2(\mathbb{C})$ be an irreducible representation of $C^*(p,q,1)$ such that $\pi_4(p) = 1$ and $\pi_4(q) = 1$.

$$\begin{tikzcd} C^*(p,q,1) \arrow[r, "\varphi"] \arrow[dr, "\pi_4"'] & A' \arrow[d, "\sigma=\sigma_1\circ ev_1|A'"] \\ & B(H) \end{tikzcd}$$

Therefore, we take here $\sigma := \sigma_1 \circ \sigma_0 = \sigma_1 \circ ev_1|_{A'}$ and check if the above diagram of maps commute.
Consider first,

$$\sigma \circ \varphi(p) = \sigma_1(ev_1(\varphi(p))) = \sigma_1\left(ev_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) = \sigma_1(1,0) = 1 = \pi_4(p)$$

and

$$\begin{aligned}\sigma \circ \varphi(q) = \sigma_1(ev_1(\varphi(q))) &= \sigma_1\left(ev_1\begin{pmatrix}x & \sqrt{x(1-x)}\\ \sqrt{x(1-x)} & 1-x\end{pmatrix}\right)\\ &= \sigma_1(1,0) = 1 = \pi_4(q).\end{aligned}$$

Hence, we have shown that $\sigma \circ \varphi = \pi_4$.

5. Let $\pi_5 : C^*(p,q,1) \to M_2(\mathbb{C})$ be an irreducible representation of $C^*(p,q,1)$ such that

$$\pi_5(p) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \text{ and } \pi_5(q) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Let $\sigma = ev_0|_{A'}$ be the evaluation map such that $ev_0(f) = f(0)$.

$$\begin{tikzcd} C^*(p,q,1) \arrow[r, "\varphi"] \arrow[dr, "\pi_5"'] & A' \arrow[d, "\sigma=ev_0|_{A'}"] \\ & B(H) \end{tikzcd}$$

Consider first,

$$\sigma \circ \varphi(p) = ev_0(\varphi(p)) = \varphi(p)(t) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_0 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \pi_5(p)$$

and

$$\sigma \circ \varphi(q) = ev_0(\varphi(q)) = \varphi(q)(t) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \pi_5(q).$$

Hence, we have shown that $\sigma \circ \varphi = \pi_5$.

6. Let $\pi_6 : C^*(p,q,1) \to M_2(\mathbb{C})$ be an irreducible representation of $C^*(p,q,1)$ such that

$$\pi_6(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \pi_6(q) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$$

for some $t \in (0,1)$.

Let $\sigma = ev_t|_{A'}$ be the evaluation map such that $ev_t(f) = f(t)$ where $t \in (0,1)$.

$$\begin{tikzcd} C^*(p,q,1) \arrow[r, "\varphi"] \arrow[dr, "\pi_6"'] & A' \arrow[d, "\sigma = ev_t|_{A'}"] \\ & B(H) \end{tikzcd}$$

Consider first,

$$\sigma \circ \varphi(p) = ev_t(\varphi(p)) = \varphi(p)(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \pi_6(p)$$

and

$$\sigma \circ \varphi(q) = ev_t(\varphi(q)) = \varphi(q)(t) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}_t$$
$$= \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix} = \pi_6(q).$$

Hence, we have shown that $\sigma \circ \varphi = \pi_6$.

Hence, we have that for every irreucible representation of $C^*(p,q,1)$ there exists a $*$-representation

$$\sigma : A' \to B(H) \text{ such that } \sigma \circ \varphi(x) = \pi(x)$$

for all $x \in C^*(p,q,1)$. $\square$

Proof of Theorem:

1. **$*$-Homomorphism:** By Lemma III.2(a), we have by the Universal Property of C^*-algebras that there exists a unique $*$-homomorphism

$$\varphi : C^*(p, q, 1) \to A', p \mapsto p(x), q \mapsto q(x).$$

2. **Surjectivity:** By Lemma III.2(b),(c),(d) we have that Range(φ) is a rich subalgebra of A', hence all of A' i.e.

$$\varphi : C^*(p, q, 1) \to A' \text{ is surjective.}$$

3. **Injectivity:** Let $x \in C^*(p, q, 1)$ such that $x \neq 0$ then by Theorem I.56, there exists an irreducible representation

$$\pi : C^*(p, q, 1) \to B(H)$$

such that $\pi(x) \neq 0$ and therefore by Lemma III.6, there exists a $*$-representation

$$\sigma = A' \to B(H)$$

such that $\pi = \sigma \circ \varphi$.
Since,

$$\pi(x) \neq 0 \Rightarrow \sigma \circ \varphi(x) \neq 0$$

which indeed says that $\varphi(x) \neq 0$ and hence

$$\varphi : C^*(p, q, 1) \to A' \text{ is injective.}$$

Now, since $\varphi : C^*(p, q, 1) \to A'$ is bijective and $*$-homomorphism therefore we have that

$$\mathbf{C^*(p, q, 1) \cong A'}.$$

$\square$

Hence, we have successfully shown that the Universal C^*-algebra generated by two projections is a subalgebra of matrix-valued continuous functions on $[0, 1]$. This theorem helped in providing a tangible description of the Universal C^*-algebra generated by two projections where the generating projections are represented as matrices in $M_2(\mathbb{C})$. This proof not only elucidates the structure of $C^*(p, q, 1)$ but also underscores the nuclearity of C^*-algebras, which is characterized by their representation's factorization properties and approximation by finite-dimensional representations. This nuclear property enhances the understanding of their K-theory, aiding in the classification and further study of such structures.

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