The Structure of Hypergraph C*-Algebras

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Introduction

Graph C^*-algebras have attracted great interest in operator theory in the last 40 years. They are interesting due to their vast variety of examples and their entanglement of properties of C^*-algebras with geometric properties of the graph, which gives insights into the structure of C^*-algebras. Examples of graph C^*-algebras are the Toeplitz algebra, the matrix algebra, the compact operators and the continuous functions on the unit circle. A good overview of this is provided by the standard work [Rae05]. Another application of graph C^*-algebras is in classification theory. The link to K-theory is obtained through Morita equivalence as important equivalence relation for graph C^*-algebras.

Within the scope of this thesis the theory to hypergraph C^*-algebras is extended. Hypergraph C^*-algebras were first defined in the bachelor thesis of Dean Zenner in 2022 under the supervision of Prof. Moritz Weber and there is no literature already published on the subject. As an own research project the theory of hypergraph C^*-algebras is generalized within this thesis based on literature research on comparable objects, see [Tom03]. A hypergraph is a combinatorial object consisting of vertices which are connected by directed edges. As extension to graphs, edges in hypergraphs connect sets of vertices, not just individual vertices.

Figure 1: Different edges in graphs and hypergraphs.

Hence the path structure gets more complicated, as we have to deal with intersection of source and range sets. This is reflected in the relations of the corresponding C^*-algebra, which we define in the same manner as for graphs: the vertices are identified with projections and the edges with partial isometries. These are combined with further relations based on the underlying hypergraph. We study the C^*-algebras in detail for finite hypergraphs, but give first concrete starting points for generalization to infinite hypergraphs.

Cuntz-Krieger Relations	Hypergraph Relations
(CK1) $s_e^*s_f = \delta_{e,f}p_{r(e)}$	(HR1) $s_e^*s_f = \delta_{e,f}\sum_{v\in r(e)}p_v$
(CK2) $p_v = \sum_{e\in E^1,s(e)=v} s_es_e^*$	(HR2) $s_es_e^* \leqslant \sum_{v\in s(e)}p_v$
	(HR3) $p_v \leqslant \sum_{e\in E^1,v\in s(e)} s_es_e^*$

Table 1: Overview of the relations defining graph and hypergraph C^*-algebras.

Hypergraph C^*-algebras really extend the class of graph C^*-algebras. Indeed, while all graph C^*-algebras are nuclear, we proof that $C(\mathbb{T}) * \mathbb{C}^n$ is a non-nuclear hypergraph C^*-algebra. Using that nuclearity transfers to quotients we give a new method to construct further non-nuclear examples. By imposing concrete conditions on the hypergraph, we change this method such that the non-nuclearity can be read from the hypergraph alone. The hypergraph corresponding to a nuclear C^*-algebra is called amenable. Moreover, we define some kind of product of hypergraphs, to attach a non-amenable hypergraph to an arbitrary hypergraph to construct a new, non-amenable hypergraph. There are further restrictions we can place on the class of hypergraph C^*-algebras which contains non-nuclear C^*-algebras. We consider ultragraphs as special hypergraphs, in which an edge can have multiple vertices in its range, but just one in its source. The corresponding ultragraph C^*-algebras were defined by Tomforde in [Tom03]. Interestingly, his research showed, that all ultragraph C^*-algebras are Morita equivalent to graph C^*-algebras and hence nuclear. Thus, the class of hypergraph C^*-algebras also extends the class of ultragraph C^*-algebras and all non-amenable hypergraphs must have at least one edge with a multi valued source.

To better understand the connection between different hypergraph C^*-algebras, we generalize the proof of the previously mentioned Morita equivalence of ultragraphs. We show that each finite hypergraph can be transformed into a hypergraph with only single vertices in its ranges without changing the C^*-algebra (up to isomorphism). Applied to finite ultragraphs this shows, that each of these ultragraph C^*-algebras is isomorphic to a graph algebra. In particular, these results indicate that the new interesting phenomena of hypergraphs have their roots in sources with multiple elements. This also occurs when investigating the connection of hypergraph C^*-algebras to known C^*-algebras. The multi-valued source makes the crucial difference in the Gauge Uniqueness Theorem, which yields faithful representations of graph C^*-algebras. We found a hypergraph for which the theorem is not valid. For ultragraphs however it is still valid as shown in [Tom03]. Based on this observation we gave restrictions under which the theorem can be generalized.

Finally, we investigate how hypergraph C^*-algebras behave when the underlying hypergraph is modified. Based on manipulations of graphs in form of six concrete moves, graph C^*-algebras can be completely classified up to stable isomorphism, as a recent break through in the classification of graph C^*-algebras [ERRS21] showed. We extend four of these moves to hypergraphs and investigate, how the corresponding C^*-algebras behave and if the manipulations leave the the C^*-algebra invariant. In this context, further fields of research emerge.

Outline: As foundation for this thesis, we begin **chapter 1** by introducing universal C^*-algebras, which build the underlying concept of the definition of graph and hypergraph C^*-algebras. We have a closer look at the known results for graphs and introduce Morita equivalence. In **chapter 2** we formally define hypergraph C^*-algebras and introduce ultragraph C^*-algebras as specific examples. We examine the structural differences for hypergraphs more closely, including the path structure. Based on this we consider examples and give a way to construct representations. **Chapter 3** deals with the question of the nuclearity of hypergraph C^*-algebras. We define nuclearity for C^*-algebras and give an example of a non-nuclear hypergraph C^*-algebra, which shows, that the class of hypergraph C^*-algebras extends the class of graph C^*-algebras. Building on this example, we use that nuclearity is invariant under quotients to construct further non-nuclear examples. In **chapter 4** we show that we can transform each hypergraph into a hypergraph with single valued range map while leaving the corresponding C^*-algebra invariant. For specific cases we generalize the Gauge Uniqueness Theorem and give an example, that it does not hold for hypergraphs in general. In **chapter 5**, we discuss and generalize basic moves to manipulate hypergraphs and state their importance in the classification theory of graph C^*-algebras. We finish in **chapter 6** by stating further research topics, which came up in the scope of this thesis.

Background: As prerequisites, knowledge in functional analysis and algebra is required. The reader is furthermore expected to be familiar with operator theory and more specifically, with the theory of C^*-algebras. A good reference for this topic are the books [Bla06] and [Dav96].

1. Preliminaries

The basis of hypergraph C^*-algebras are universal C^*-algebras which we introduce right at the beginning. These give insights into the structure of C^*-algebras and build the underlying concept of graph and hypergraph C^*-algebras. Then we define graph C^*-algebras as particular universal C^*-algebras and give an overview about its structure. As examples we show that the matrix algebra $M_n(\mathbb{C})$, the compact operators $\mathcal{K}(\mathcal{H})$, the Toeplitz algebra $\mathcal{T}$ and the algebra $C(\mathbb{T})$ of continuous functions on the unit circle $\mathbb{T}$ are graph C^*-algebras. Finally we take a look at Morita equivalence.

1.1. Universal C*-Algebras

Universal C^*-algebras give an abstract way to construct C^*-algebras based on formal generating elements and concrete relations defined on the generators. This idea is similar to the construction of the free group with some generating set and relations given by the group axioms. As general reference for the upcoming construction see [Bla06, Section II.8.3] and [LVW21, Chapter 6].

The definition of universal C^*-algebras is based on free *-algebras. In the following we sketch the construction of them. Let a set of generators be given by an alphabet $E := \{x_i \mid i \in I\}$. We define non-commutative polynomial by the complex linear combination of words $y = x_{i_1} \cdots x_{i_m}$ in E. Together with the canonical addition, scalar multiplication and the multiplication of elements given by the concatenation of words, the set of non-commutative polynomials is an algebra called the *free (complex) algebra* on the generator set E. The algebra is called free, since there hold no equations between the elements except of the defining axioms of the algebraic structure. We add a copy of E denoted by $E^* := \{x_i^* \mid i \in I\}$ and consider the free algebra with generator set $E \cup E^*$. By enlarging the set of generators like this, we can artificially define an involution on the free algebra on the generator set $E \cup E^*$ and obtain the *free *-algebra* $P(E)$. To define more structure on the *-algebra let a set of relations be given, i.e. a set of polynomials $R \subseteq P(E)$. Let $J(R)$ be the two-sided *-ideal generated by R. Then the *universal *-algebra* is defined as the quotient

$$A(E \mid R) := P(E)/J(R).$$

To get a C^*-algebra we have to define a norm on the universal *-algebra. For that note, that a C^**-seminorm* p on a C^*-algebra A is a submultiplicative seminorm which fulfills the C^*-identity $p(a^*a) = p(a)^2$.

Definition 1.1. Let E be a set of generators and $R \subseteq P(E)$ be relations. Put

$$||x|| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E|R)\}.$$

If $||x|| \leqslant \infty$ for all $x \in A(E \mid R)$, $||\cdot||$ is a C^*-seminorm and we define the *universal* C^**-algebra* $C^*(E \mid R)$ as the completion with respect to $||\cdot||$:

$$C^*(E \mid R) := \overline{A(E \mid R)/\{x \in A(E \mid R) \mid ||x|| = 0\}}^{||\cdot||}.$$

The C^*-seminorm defined in the construction is only a C^*-norm if it is finite for all elements of the universal *-algebra. Thus the universal C^*-algebra only exists if this condition is given.

Lemma 1.2. *If there is a constant $C > 0$ such that $p(x_i) < C$ for all $i \in I$ and all C^*-seminorms p on the universal *-algebra, then the universal C^*-algebra exists.*

Besides the existence of universal C^*-algebras, we have to consider triviality, i.e. it might hold that $C^*(E \mid R) = 0$. To show non-triviality of universal C*-algebras we establish the universal property, which yields *-homomorphisms from universal C^*-algebras into known non-trivial C^*-algebras.

Proposition 1.3. *Let $E := \{x_i \mid i \in I\}$ be a set of generators and $R \subseteq P(E)$ be relations. Let B be a C^*-algebra containing a subset $E' := \{y_i \mid i \in I\}$. If the elements in E' satisfy the relations R, then there is a unique *-homomorphism $\pi : C^*(E \mid R) \to B$ sending x_i to y_i for all $i \in I$.*

In analogy to unitary operators and isometries in operator theory we generalize these terms to general C^*-algebras based on the relations that define unitary operators. In the context of universal C^*-algebras we call those elements universal unitaries and universal isometries respectively.

Definition 1.4. Let A be a C^*-algebra. An element $u \in A$ satisfying the relations $u^*u = uu^* = 1$ is called *unitary*. An element $v \in A$ with $v^*v = 1$ is called *isometry*.

Example 1.5. Multiple well known C^*-algebras can be expressed as universal C^*-algebras.

1. The universal C^*-algebra $C^*(e_{ij},\ i,j = 1,\dots,n \mid e_{ij}^* = e_{ji},\ e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i,j,k,l)$ is isomorphic to the matrix algebra $M_n(\mathbb{C})$. Indeed, the matrices E_{ij}, which are zero except for the ij-th entry given by 1, fulfill the relations. By the universal property there is thus a surjective *-homomorphism, which is injective by a dimension argument.
2. As infinite analog to the previous example, the C^*-algebra of compact operators $\mathcal{K}(\mathcal{H})$ on a Hilbertspace $\mathcal{H}$ is isomorphic to $C^*(e_{ij}, i,j \in \mathbb{N} \mid e_{ij}^* = e_{ji},\ e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i,j,k,l)$.
3. The universal C^*-algebra $C^*(u, 1 \mid u^*u = uu^* = 1)$ generated by a universal unitary element is isomorphic to the C^*-algebra of continuous functions on the unit circle $\mathbb{T}$, denoted by $C(\mathbb{T})$.
4. The Toeplitz algebra $\mathcal{T}$ is generated by the unilateral shift S on the space $l^2(\mathbb{N})$ of square-summable sequences, defined by $(a_n)_n \mapsto (0, a_1, a_2, \dots)$. The unilateral shift is an isometry. This hints that $\mathcal{T}$ is isomorphic to the universal C^*-algebra $C^*(v, 1 \mid v^*v = 1)$ generated by a universal isometry.

Remark 1.6. *We can also formulate C^*-algebraic constructions in the universal setting. The* free product *of C^*-algebras A and B is given as*

$$A * B \cong C^*(a \in A,\ b \in B \mid R_A,\ R_B),$$

where R_A and R_B are the C^*-relations of A and B respectively. The full crossed product C^*-algebra $A \rtimes_\alpha G$ *of a unital C^*-algebra A and a discrete group G acting on A by a group action $\alpha : G \to Aut(A)$ can be seen as universal C^*-algebra*

$$A \rtimes_\alpha G \cong C^*\left(a \in A,\ u_g \text{ unitaries for } g \in G \mid u_{gh} = u_g u_h,\ u_{g^{-1}} = u_g^*,\ u_g a u_g^* = \alpha_g(a)\right).$$

The tensor products of C^*-Algebras can also be expressed as universal C^*-Algebra. Here, however, caution is required, since there are different tensor products for C^*-algebras. We look at this more concretely in Section 3.1.

1.2. Graph C*-Algebras

The following introduction to graphs and their C^*-algebras is adapted from [Rae05, Chapter 1].

Definition 1.7. A *directed graph* $E = (E^0, E^1, r, s)$ is defined by two countable sets E^0 (vertices) and E^1 (edges), together with a range and source map $r, s : E^1 \to E^0$. We write graph instead of directed graph. We call a vertex v a *source* if and only if $r^{-1}(v) = \emptyset$ and we call it a *sink* if and only if $s^{-1}(v) = \emptyset$. A graph is called *finite* if the set of edges and vertices are finite. A graph is called *row-finite* if the each vertex receives at most finitely many edges.

The graph C^*-algebra is a universal C^*-algebra with generators associated to the vertices and edges of the graph and relations resembling the structure of the graph. Derived from the corresponding definitions for operator algebras we define projections and partial isometries for general C^*-algebras:

Definition 1.8. Let A be a C^*-algebra. We call an element $p \in A$ a *projection* if $p^2 = p = p^*$. Two projections p, q are *mutually orthogonal* if $pq = 0$. An element s which fulfills $ss^*s = s$ is called a *partial isometry*.

Remark 1.9. *With regard to the order relation on the set of projections on a closed Hilbertspace, we can define an order relation on the set of projections of a C^*-algebra by $p \leqslant q$ if and only if $pq = p = qp$. Based on this definition one can for example proof that the finite sum of projections is a projection if and only if the projections are mutually orthogonal. Another useful result is that $p_i \leqslant q$ and $q \leqslant \sum p_i$ for a projection q and finitely many mutually orthogonal projections p_i, implies $q = \sum p_i$.*

Definition 1.10. Let $E = (E^0, E^1, r, s)$ be a row-finite graph. The *graph C^*-algebra $C^*(E)$* is the universal C^*-algebra generated by mutually orthogonal projections p_v for all vertices $v \in E^0$ and partial isometries s_e for all edges $e \in E^1$ such that the following relations hold

(CK1) $s_e^* s_f = \delta_{e,f} p_{r(e)}$ for all $e, f \in E^1$;

(CK2) $p_v = \sum_{e \in E^1, s(e)=v} s_e s_e^*$ for all $v \in E^0$ in case v is not a sink.

The relations are called *Cuntz-Krieger relations*. Elements $\{S_e, P_v \mid e \in E^1, v \in E^0\}$ in a C^*-algebra A fulfilling the relations are called *Cuntz-Krieger E-family*. To simplify notation we abbreviate the notation of Cuntz-Krieger families throughout this thesis by $\{S_e, P_v\}$.

Remark 1.11. *Two different versions of the definition are used in the literature. Both differ only by swapping the range and source in the Cuntz-Krieger relations. This is for example done in Raeburns book [Rae05]. Mathematically, nothing serious changes: A graph with reversed edges yields the same C^*-algebra in Raeburns definition as the C^*-algebra of the initial graph based on our Cuntz-Krieger relations.*

Lemma 1.12. *Every universal C^*-algebra generated by projections and partial isometries exists.*

Proof. Let p be an arbitrary seminorm. Using the *-property of seminorms it holds for all projections x and partial isometries y

$$\begin{aligned} p(x) &= p(x^2) = p(x^*x) = p(x)^2, \\ p(y)^2 &= p(y^*y) = p(y^*yy^*y) = p(y^*y)^2 = p(y)^4. \end{aligned}$$

All seminorms are thus bounded by 1 on the generators and the C^*-algebra exists by Lemma 1.2. □

Remark 1.13. *The first Cuntz-Krieger relation ensures that the projections $s_e s_e^*$ are mutually orthogonal. Thus the sum in the second Cuntz-Krieger relation is a projection. Sometimes the first Cuntz-Krieger relation is just defined as $s_e^* s_e = p_{r(e)}$. In this case the argument turns around: since the sum in the second Cuntz-Krieger relation is a projection, the projections $s_e s_e^*$ must be orthogonal and thus $s_e^* s_f = s_e(s_e^* s_e)(s_f^* s_f)s_f^* = \delta_{e,f} p_{r(e)}$.*

Remark 1.14. *One can also generalize the definition of graph C^*-algebras to infinite graphs which are not row-finite, see [RS03]. As infinite sums of projections do not converge in norm, the second Cuntz-Krieger relation is only defined for vertices which are no sinks and emit at most finitely many edges. Furthermore, we have to add a new relation:*

(CK3) $s_e s_e^* \leqslant p_{s(e)}$ *for all* $e \in E^1$.

If $s(e)$ receives finitely many edges, this relation follows directly by the second Cuntz-Krieger relation using the definition of the order relation and the first Cuntz-Krieger relation.

Since the graph C^*-algebra is a universal C^*-algebra it has the following universal property:

Proposition 1.15. *Let E be a row-finite graph. For each C^*-algebra B which contains a Cuntz-Krieger E-family $\{S_e, P_v\}$ there is a *-homomorphism $\pi : C^*(E) \to B$ which maps the universal projections p_v to P_v and the universal partial isometries s_e to S_e.*

Applying the universal property, i.e. finding a Cuntz-Krieger family in a C^*-algebra, is the key step in the proofs of the following examples. Given the Cuntz-Krieger families, the proofs are straightforward. Exact calculations can be found in [Zen21, Chapter 2.1].

Example 1.16. The C^*-algebra of the graph E defined by a vertex v and an edge e with $s(e) = r(e) = v$

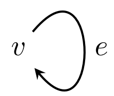


Figure 2: Graph generating $C(\mathbb{T})$.

is isomorphic to $C(\mathbb{T})$. To proof this one notes that the Cuntz-Krieger relations yield $s_e^* s_e = p_v = s_e s_e^*$. Thus, p_v is the identity element and s_e is a unitary element. Using the universal property twice one gets an isomorphism between the universal C^*-algebras $C^*(E)$ and $C^*(u, 1 \mid u^*u = uu^* = 1)$. As the latter is isomorphic to $C(\mathbb{T})$ by Example 1.5, we get the required isomorphism.

Example 1.17. This example illustrates, that the correspondence between graphs and their algebras are not one to one, as there can be multiple graphs which yield the same C^*-algebra. First we consider the following graph:

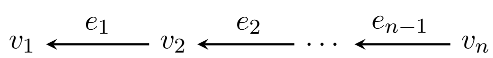


Figure 3: Graph generating $M_n(\mathbb{C})$.

The corresponding graph C^*-algebra is isomorphic to $M_n(\mathbb{C})$. To see this we use that by Lemma 1.5, $M_n(\mathbb{C})$ is isomorphic to $C^*(e_{ij}, i, j = 1, \dots, n \mid e_{ij}^* = e_{ji},\ e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i, j, k, l)$. We can define a Cuntz-Krieger family in $M_n(\mathbb{C})$ by $P_{v_i} := e_{ii}$ and $S_{e_i} := e_{i+1i}$. On the other hand we can define matrix units in the graph C^*-algebra by $E_{i,i} := p_{v_i}$ and $E_{i,j} := s_{e_{i_1}} \dots s_{e_j}$. Applying the universal property twice then yields the required inverse *-homomorphism. Instead of the above graph we could have also taken the following graphs:

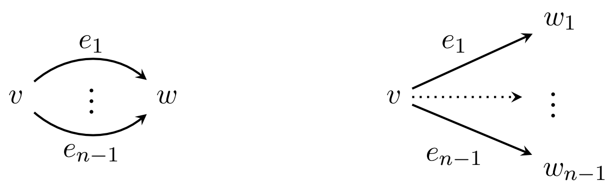


Figure 4: Further graphs generating $M_n(\mathbb{C})$.

Both graphs yield a graph C^*-algebra isomorphic to $M_n(\mathbb{C})$. Again we get the link to the compact operators. The same examples with infinite edges generate the C^*-algebra of compact operators.

Example 1.18. For $n \geqslant 2$ the C^*-algebra of the graph defined by a single vertex v and edges $e_1, \dots e_n$ with $s(e_i) = r(e_i) = v$

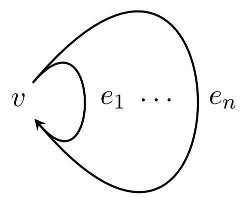


Figure 5: Graph generating the Cuntz algebra.

is isomorphic to the Cuntz algebra $O_n := C^*(t_1, \dots t_n, 1 \mid t_i$ isometry and $\sum_{i=1}^n t_i t_i^* = 1)$. Setting $p_v = 1$ and $s_{e_i} = t_i$ the Cuntz-Krieger relations of the graph directly represent the relations of O_n.

Example 1.19. The Toeplitz algebra, which is generated by a single isometry ν, is also a graph C^*-algebra. We consider the following graph

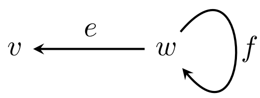


Figure 6: Graph generating the Toeplitz algebra.

and define a Cuntz-Krieger family in $\mathcal{T}$ by $P_v := 1 - \nu\nu^*$, $P_w := \nu\nu^*$, $S_e := \nu P_v$, $S_f := \nu P_w$. On the other hand, $s_e + s_f$ is an isometry and $p_v + p_w$ is the unit element. Applying the universal property twice yields the required isomorphism.

To understand more about the structure of graph C^*-algebras, we state important results regarding paths in graphs in connection with the corresponding elements in the graph C^*-algebra. In the following, we refer to the results from [Rae05, pp. 8-10].

Proposition 1.20. *Suppose that E is a graph and let $\{s_e, p_v\}$ be the universal Cuntz-Krieger E-family. Then*

1. *the projections $\{s_e s_e^* \mid e \in E^1\}$ are mutually orthogonal;*
2. *$s_e^* s_f \neq 0 \;\Rightarrow\; e = f$;*
3. *$s_e s_f \neq 0 \;\Rightarrow\; r(e) = s(f)$;*
4. *$s_e s_f^* \neq 0 \;\Rightarrow\; r(e) = r(f)$.*

This highlights the naturalness with which the partial isometries reflect the geometry of the graph.

Definition 1.21. A *path* of length n in a directed graph E is a sequence $\mu = \mu_1 \dots \mu_n$ of edges $\mu_j \in E$ such that $r(\mu_j) = s(\mu_{j+1})$. The length of a path μ is denoted by $|\mu|$. We define $s_\mu := s_{\mu_1} \dots s_{\mu_n}$ for a path μ in E and $s_v := p_v$ for $v \in E^0$.

Generalizing the proposition above one obtains the following useful characteristics of partial isometries corresponding to a path. With this, we also get an explicit, easy description of the graph C^*-algebra.

Corollary 1.22. *Suppose that E is a graph and let $\{s_e, p_v\}$ be a Cuntz-Krieger E-family. Let μ, ν be paths in E. Then*

1. *if $|\mu| = |\nu|$ and $\mu \neq \nu$, then $(s_\mu s_\mu^*)(s_\nu s_\nu^*) = 0$;*
2. $s_\mu^* s_\nu = \begin{cases} s_{\mu'} & \text{if } \mu = \nu\mu' \text{ for some path } \mu' \\ s_{\nu'} & \text{if } \nu = \mu\nu' \text{ for some path } \nu' \\ 0 & \text{otherwise}; \end{cases}$
3. *if $s_\mu s_\nu \neq 0$, then $\mu\nu$ is a path in E and $s_\mu s_\nu = s_{\mu\nu}$;*
4. *if $s_\mu s_\nu^* \neq 0$, then $r(\mu) = r(\nu)$.*

Corollary 1.23. *Suppose that E is a graph and let $\{s_e, p_v\}$ be a Cuntz-Krieger family. For paths μ, ν, α, β in E, we have*

$$(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \begin{cases} s_{\mu\alpha'} s_\beta^* & \text{if } \alpha = \nu\alpha' \\ s_\mu s_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise}. \end{cases}$$

In particular, it follows that every non-zero finite product of partial isometries s_e and s_f^* has the form $s_\mu s_\nu^*$ for some paths μ, ν in E. Hence

$$C^*(E) = \overline{span}\big(s_\mu s_\nu^* \mid \mu,\ \nu \text{ paths in } E,\ r(\mu) = r(\nu)\big).$$

1.3. Morita Equivalence and K-Theory of Graph C*-Algebras

An important equivalence relation in the scope of graph C^*algebras is Morita equivalence. Due to its connection to K-theory it is a frequently used tool in the classification of C^*-algebras. We give a short overview of the definition and the main property needed in this thesis. More details and proofs of the statements below can be found in [RW98, Chapter 3].

As first step we sketch the generalization of Hilbertspaces to C^*-Algebras. Let A be a C^*-algebra. A *right A-module* X_A is a vector space X together with a right multiplication by elements of A defined by a bilinear map $X \times A \to X$, $(x, a) \mapsto x \cdot a$. To achieve a Hilbert like structure on a these modules we generalize inner products to the C^*-algebra setting, i.e. we define a scalarproduct over the C^*-algebra A instead of $\mathbb{C}$. A map $\langle \cdot, \cdot \rangle_A : X \times X \to A$ is called an *A-valued inner product* if the following relations are fulfilled

1. $\langle x, \lambda y + \mu z \rangle_A = \lambda \langle x, y \rangle_A + \mu \langle x, z \rangle_A$;
2. $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$;
3. $\langle x, y \rangle^* = \langle y, x \rangle_A$;
4. $\langle x, x \rangle_A \geqslant 0$;
5. $\langle x, x \rangle_A = 0$ implies that $x = 0$.

Based on these A-valued inner products we define a norm on the underlying Hilbert A-module, using the norm on the C^*-algebra A:

$$\|x\|_A := \|\langle x, x \rangle_A\|^{\frac{1}{2}}.$$

A *full right Hilbert A-module* is a Hilbert A-module which is complete with regard to this norm. Using a left multiplication instead of a right multiplication and adjusting the inner product accordingly one similarly defines *full left Hilbert A-modules*. This construction is the key base point to define Morita equivalence.

Definition 1.24. Let A, B be C^*-algebras. An *$A - B$-imprimitivity bimodule* is an $A - B$-bimodule X such that

1. X is a full left Hilbert A-module and a full right Hilbert B-module;
2. for all $x, y \in X$, $a \in A$ and $b \in B$

$$\langle a \cdot x, y \rangle_B = \langle x, a^* \cdot y \rangle_B \qquad \text{and} \qquad {}_A\langle x \cdot b, y \rangle = {}_A\langle x, y \cdot b^* \rangle;$$

3. for all $x, y, z \in X$

$${}_A\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_B.$$

Two C^*-Algebras are *Morita equivalent* if there exists an $A - B$-imprimitivity bimodule.

Proposition 1.25. *Morita equivalence is an equivalence relation on C^*-algebras.*

Proof Sketch. Reflexivity follows since A is an $A - A$-imprimitivity bimodule, for transitivity one uses the internal tensor product of the corresponding imprimitivity bimodules and symmetry involves the so called dual module. The full proof can be found in [RW98, Prop. 3.18]. □

Within the scope of this thesis we mostly use the characterization of Morita equivalence by full corners. Note, that in case of unital C^*-algebras the multiplier algebra is the same as the initial C^*-algebra.

Definition 1.26. Let A be a C^*-algebra and p be a projection in the multiplier algebra $M(A)$. The corner pAp is *full* if it is not contained in a non-trivial closed two-sided ideal of A.

Proposition 1.27. *Let A be a C^*-algebra and p be a projection in the multiplier algebra $M(A)$ such that pAp is a full corner of A. Then A is Morita equivalent to pAp.*

Proof. We consider the space Ap and define the following A and pAp-valued inner products on it:

$$\langle x, y\rangle_A := x^*y, \qquad {}_{pAp}\langle x, y\rangle := xy^*$$

for $x, y \in Ap$. The multiplication in A defines a left multiplication on Ap by A and a right multiplication by pAp. Thus Ap is a left Hilbert A-module and a right Hilbert pAp-module. Using the completeness of A we get completeness of Ap with regard to the respective norms defined by the inner products. Thus Ap is a full left Hilbert A-module and a full right Hilbert pAp-module. Since for all $x, y \in Ap$, $a \in A$ and $b \in pAp$ we have

$$\langle a \cdot x, y\rangle_A = (a \cdot x)^*y = x^*(a^* \cdot y) = \langle x, a^* \cdot y\rangle_A \ ,$$
$${}_{pAp}\langle x \cdot b, y\rangle = (x \cdot b)y^* = x(y \cdot b^*)^* =_{pAp} \langle x, y \cdot b^*\rangle$$

and for all $x, y, z \in Ap$

$${}_{pAp}\langle x, y\rangle z = xy^*z = x\langle y, z\rangle_A$$

the scalar products fulfill the second and third relation of Definition 1.24. Thus Ap is an $A-pAp$-imprimitivity bimodule and A is Morita equivalent to pAp. □

Definition 1.28. Two C^*-algebras A and B are *stably isomorphic* if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ were $\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space.

Theorem 1.29 (Brown-Green-Rieffel). *[RW98, 5.55] Two C^*-algebras with countable approximate units are stably isomorphic if and only if they are Morita equivalent.*

By functoriality, isomorphic C^*-algebras have similar K-theories Furthermore, both K-groups are stable, that is $K_0(A) \cong K_0(A \otimes \mathcal{K})$ and $K_1(A) \cong K_1(A \otimes \mathcal{K})$ [RLL00, Prop. 6.4.1 and 8.2.8]. Thus, K-theory is a Morita equivalent invariant, as stated in the following.

Theorem 1.30. *Let A, B be Morita equivalent C^*-algebras with countable approximate units. Then their K-theory is similar.*

Graph C^*-algebras have a countable approximate unit given by $(\sum_{i=1}^{n} p_{v_i})_{n=|E^0|}$. Thus, two Morita equivalent graph C^*-algebras have a similar K-theory. For graph C^*-algebras there is a concrete way to calculate its K-theory.

Theorem 1.31. *[Rae05, Thm. 7.16] Let E be a row-finite graph with no sinks, and let A_E be the adjacency matrix of E. The $K_1(C^*(E))$ is isomorphic to the kernel of $1 - A_E^t$ and $K_0(C^*(E))$ is isomorphic to the cokernel.*

The result can be generalized to row-finite graphs with sinks as done in [RS03, Thm. 3.2].

2. Hypergraph C*-Algebras

In this chapter we introduce hypergraphs and their corresponding C^*-algebras based on the work of Dean Zenner in his bachelor thesis [Zen21], which is not published. We establish their relation to graph C^*-algebras and reference to the theory of ultragraphs as another important special case of hypergraphs, from which we can derive several results for hypergraphs. By introducing generalized vertices, we pave the way for the definition of C^*-algebras for infinite hypergraphs. In the following section we look more closely on the structure of hypergraph C^*-algebras and emphasize on the generalization of paths. We finish the chapter by stating examples and giving a general construction for representations.

2.1. Definition and Properties

Definition 2.1. A *(directed) hypergraph* $H\Gamma = (E^0, E^1, r, s)$ is defined by countable sets E^0 (vertices) and E^1 (edges), together with a range and source map $r, s : E^1 \to \mathcal{P}(E^0)$. We write hypergraph instead of directed hypergraph. We call a vertex v a *source* iff $v \notin r(e)$ for all $e \in E^1$ and we call it a *sink* iff $v \notin s(e)$ for all $e \in E^1$.

Example 2.2. As an example we consider the graph defined by vertices $E^0 = \{v_1, v_2, v_3\}$ and edges $E^1 = \{e, f\}$ with range and source map given by $s(e) = \{v_1\}$, $r(e) = \{v_1, v_3\}$ and $s(f) = \{v_2, v_3\}$, $r(f) = \{v_1, v_2\}$. The hypergraph can be visualized as follows:

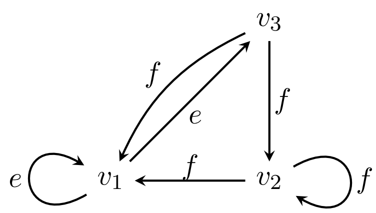


Figure 7: Example of a hypergraph.

We mainly restrict ourselves to finite hypergraphs, i.e. hypergraphs were the set of vertices and edges are both finite. This prevents us from having to worry about infinite sums while already illustrating the problems we have to deal with in the hypergraph setting. Our results can be used as a starting point for the generalization to infinite hypergraphs.

Definition 2.3. Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. The *hypergraph C^*-algebra $C^*(H\Gamma)$* is the universal C^*-algebra generated by mutually orthogonal projections p_v for all vertices $v \in E^0$ and partial isometries s_e for all $e \in E^1$ such that the following relations hold

(HR1) $s_e^* s_f = \delta_{e,f} \sum_{v \in r(e)} p_v$ for all $e, f \in E^1$;

(HR2) $s_e s_e^* \leqslant \sum_{v \in s(e)} p_v$ for all $e \in E^1$;

(HR3) $p_v \leqslant \sum_{e \in E^1, v \in s(e)} s_e s_e^*$ for all $v \in E^0$ with v not a sink.

We call the relations *hypergraph relations*. Elements $\{S_e, P_v\}$ in a C^*-algebra A fulfilling the hypergraph relations are called *Cuntz-Krieger $H\Gamma$-family*. Throughout this thesis we denote the canonical generating elements of $C^*(H\Gamma)$ by $\{s_e, p_v\}$. In general all universal elements are denoted by small letters.

Remark 2.4. *Every hypergraph C^*-algebra exists by Lemma 1.12. Furthermore, since the hypergraph C^*-algebra is a universal C^*-algebra we have a similar universal property as in Proposition 1.15.*

We note that each graph can be interpreted as a hypergraph by defining $r' : E^1 \to \mathcal{P}(E^0)$ via $r'(e) = \{r(e)\}$. Similarly for the source map. This identification is straight forward and we get an isomorphism of the corresponding hypergraph and graph C^*algebras.

Proposition 2.5. *[Zen21, Prop 3.8] Let $\Gamma = (E^0, E^1, r, s)$ be a graph. We interpret Γ as a hypergraph $H\Gamma = (E^0, E^1, r`, s')$. Then $C^*(\Gamma) \cong C^*(H\Gamma)$.*

Proof. The proof is a straightforward calculation that the generators of $C^*(\Gamma)$ fulfill the hypergraph relations and that the generators of $C^*(H\Gamma)$ fulfill the Cuntz-Krieger relations of the graph C^*-algebra. Then applying the universal property twice yields inverse *-homomorphisms. □

Thus, the class of hypergraph C^*-algebras contains the class of graph C^*-algebras and is non trivial, as we have a bunch of concrete examples of graph algebras. As an intermediate step between graphs and hypergraphs one can also consider ultragraphs. These were defined by Tomforde in his paper [Tom03] as graphs which allow an edge to have multiple vertices in its range but in contrast to hypergraphs, the source of edges still consists of a single vertex.

Definition 2.6. A *(directed) ultragraph* $\mathcal{G} = (G^0, G^1, r, s)$ is defined by two countable sets G^0 (vertices) and G^1 (edges), together with a source map $s : G^1 \to G^0$ and a range map $r : G^1 \to \mathcal{P}(G^0)$. We write ultragraph instead of directed ultragraph. We call the corresponding C^*-algebra *ultragraph C^*-algebra*.

To clarify the differences, we illustrate the possible edges in graphs, ultragraphs and hypergraphs in the following figure.

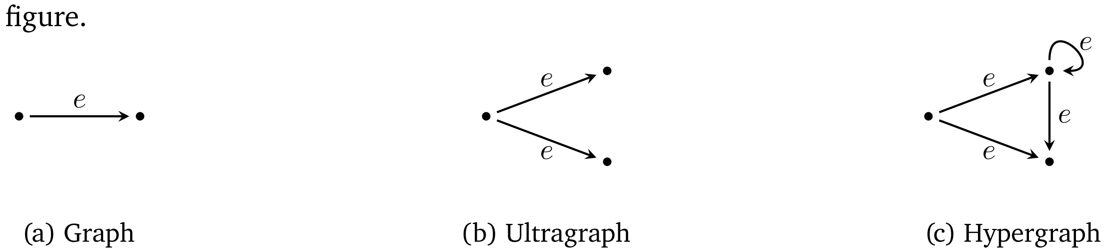


Figure 8: Different edges in graphs, ultragraphs and hypergraphs.

The research about ultragraph C^*-algebras already covers a lot. The main idea of the construction was to find a unified approach to graph C^*-algebras and Exel-Laca algebras, which has been found to be an approach to generalize Cuntz-Krieger algebras to the infinite setting. The interesting thing is, that each ultragraph C^*-algebra is Morita equivalent to a graph C^*-algebra [KMST10, Thm. 5.22]. The construction made in the paper is quite elaborated, but broken down to finite ultragraphs it just splits the range of each edge and creates a graph with one edge for each vertex in the range. And in the finite case the resulting *-homomorphism even strengthens to be an isomorphism. In section 4.1 we will see more on this construction and show how to extend it to hypergraphs.

We go back to the definition of hypergraph C^*-algebras and introduce a slightly different approach. This will on one hand simplify the notation and on the other hand give a first hint how the definition can be generalized to infinite hypergraphs. For $A \subseteq E^0$ we define $p_A := \sum_{v \in A} p_v$. Since all projections p_v are mutually orthogonal, p_A is again a projection. With this notation the first hypergraph relation simplifies to $s_e s_f^* = \delta_{e,f} p_{r(e)}$ and the second hypergraph relation can be expressed as $s_e s_e^* \leqslant p_{s(e)}$. We generalize this concept and define generalized vertices.

Definition 2.7. Let $H\Gamma = (E^0, E^1, r, s)$ be a hypergraph. Let

$$\mathcal{E}' := E^0 \cap \{s(e), r(e) \mid e \in E^1\}.$$

Let $\mathcal{E}^0$ be the smallest subcollection of $\mathcal{P}(E^0)$ containing $\mathcal{E}'$ which is closed under finite unions and finite intersections. We call the sets $A \in \mathcal{E}^0$ *generalized vertices*.

With this connotation we can give a slightly different definition of a Cuntz-Krieger family involving the generalized vertices. This definition is adapted from [Tom03, Def. 2.7].

Definition 2.8. Let $H\Gamma = (E^0, E^1, r, s)$ be a hypergraph. A *generalized Cuntz-Krieger $H\Gamma$-family* is a collection of partial isometries $\{s_e \mid e \in E^1\}$ and orthogonal projections $\{p_A \mid A \in \mathcal{E}^0\}$ such that

- (GR0) $p_\emptyset = 0$, $p_A p_B = p_{A \cap B}$ and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ for all $A, B \in \mathcal{E}^0$;
- (GR1) $s_e^* s_f = \delta_{e,f} p_{r(e)}$ for all $e, f \in E^1$;
- (GR2) $s_e s_e^* \leqslant p_{s(e)}$ for all $e \in E^1$;
- (GR3) $p_v \leqslant \sum_{e \in E^1, v \in s(e)} s_e s_e^*$ for all $v \in E^0$ which emit at least one and at most finitely many edges.

Remark 2.9. *The clue is, that this definition also makes sense for infinite hypergraphs. In the infinite case, we have to ensure that all sums of projections are well defined. Thus, all infinite sums must be avoided. In the third Cuntz-Krieger relation this can be done by just defining it for vertices which emit at most finitely many edges, as it is done for non row-finite graphs. But for edges with infinitely many vertices in their source or range, we have to adjust the other hypergraph relations as well, due to the possibly infinite sums*

$$\sum_{v \in r(e)} p_v \qquad \text{and} \qquad \sum_{v \in s(e)} p_v.$$

The approach involving generalized vertices forces the existence of the required projections in a natural way. And it turns out that in the finite case, both definitions still coincide.

Lemma 2.10. *Let $H\Gamma$ be a finite hypergraph. Then the Cuntz-Krieger families in Definition 2.3 and Definition 2.8 generate the same C^*-algebra.*

Proof. Since the hypergraph $H\Gamma = (E^0, E^1, r, s)$ is finite, the sum $p_A := \sum_{v \in A} p_v$ is a projection for all generalized vertices $A \in \mathcal{E}^0$. Leaving the partial isometries invariant we have a generalized Cuntz-Krieger $H\Gamma$-family. On the other hand, $E^0 \subseteq \mathcal{E}$. Hence by taking only projections corresponding to vertices, and leaving again the partial isometries invariant, we get a Cuntz-Krieger $H\Gamma$-family. Applying the universal property twice thus yields the required isomorphism between the generated universal C^*-algebras. □

Remark 2.11. *It is worth pointing out that there are other constructions in the literature, which are also called hypergraph C^*-algebras, but differ from our construction, see [AFLS15] and [Fri20]. The general idea of combining the geometric properties of hypergraphs graphs with C^*-algebras already finds application in quantum physics in the study of test spaces in quantum logic. In the physical interpretation of the hypergraph the vertices represent outcomes and the edges represent measurements. The definition of the corresponding hypergraph C^*-algebra used there is also based on universal C^*-algebras, but uses different relations corresponding to the physical interpretation. In this respect, this definition differs from the considerations in the following thesis.*

2.2. General Structure and Relations

Proposition 2.12. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. For each Cuntz-Krieger $H\Gamma$-family $\{p_v, s_e\}$ it holds*

$$p_{s(e)}s_e = s_e = s_e p_{r(e)}.$$

Proof. The second equation follows by the first hypergraph relation and the definition of the partial isometry:

$$s_e p_{r(e)} = s_e s_e^* s_e = s_e.$$

By the second hypergraph relation we know that $s_e s_e^* \leqslant p_{s(e)}$. Hence it follows by the definition of the order relation that $p_{s(e)} s_e s_e^* = s_e s_e^*$ and thus

$$p_{s(e)}s_e = p_{s(e)} s_e s_e^* s_e = s_e s_e^* s_e = s_e.$$

□

The next proposition is quite useful, as it gives us a unit at hand while working with finite hypergraphs.

Proposition 2.13. *[Zen21, Thm. 3.9] Let $H\Gamma = (E^0, E^1, r, s)$ be a hypergraph with finite number of vertices. Then the hypergraph C^*-Algebra is unital and $\sum_{v\in E^0} p_v = 1$.*

Proof. We show that $\sum_{v\in E^0} p_v$ behaves as unit on the set of generators, which yields the claim. For the projections we use the mutual orthogonality to get

$$\Big(\sum_{v\in E^0} p_v\Big)p_w = p_w = p_w\Big(\sum_{v\in E^0} p_v\Big).$$

For the partial isometries we use Proposition 2.12 and follow again by the mutal orthogonality of the projections, that

$$\Big(\sum_{v\in E^0} p_v\Big)s_e = \Big(\sum_{v\in E^0} p_v\Big)p_{s(e)}s_e = p_{s(e)}s_e = s_e.$$

Analogously we get the right multiplication using $p_{r(e)}$. □

Remark 2.14. *In case of an infinite number of vertices one has to be a bit careful, as an infinite sum of projections cannot converge in norm. Indeed, $\sum_{i=m}^{n} p_{v_i}$ is a projection, which has thus norm one. Hence the sequence of partial sums $\sum_{i=1}^{n} p_{v_i}$ cannot be Cauchy and it is thus impossible to converge in norm. Motivated* *from the corresponding result for ultragraphs in [Tom03, Lem. 3.2], we can show as generalization of the above theorem, that $C^*(H\Gamma)$ is unital if and only if*

$$E^0 \in \left\{ \bigcup_{i=1}^{n} \left(\bigcap_{e \in X_i} r(e) \right) \cup \bigcup_{i=1}^{m} \left(\bigcap_{e \in Y_i} s(e) \right) \cup F \;\middle|\; X_i, Y_i \;\subseteq E^1 \textit{ finite},\; F \subseteq E^0 \textit{ finite} \right\}.$$

The proof of this follows in exactly the same way as in [Tom03, Lem. 3.2], except of the definition of the constructed approximate unit. We provide it for completion in the appendix A.

Proposition 2.15. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph and $\{p_v, s_e\}$ be a Cuntz-Krieger $H\Gamma$-family. Then*

1. *$\{s_e s_e^* \mid e \in E^1\}$ consists of mutually orthogonal projections;*
2. *$s_e^* s_f \neq 0 \;\Rightarrow\; e = f$;*
3. *$s_e s_f \neq 0 \;\Rightarrow\; r(e) \cap s(f) \neq \emptyset$;*
4. *$s_e s_f^* \neq 0 \;\Rightarrow\; r(e) \cap r(f) \neq \emptyset$.*

Proof. 1. Using the first hypergraph relation we get by the definition of partial isometries

$$(s_e s_e^*)(s_f s_f^*) = s_e(s_e^* s_f)s_f = \delta_{ef} s_e s_e^* s_e s_e^* = \delta_{ef} s_e s_e^*.$$

2. This follows directly by the first hypergraph relation.
3. Applying Proposition 2.12 we get

$$s_e s_f = s_e p_{r(e)} p_{s(e)} s_f = \begin{cases} 0 & r(e) \cap s(f) = 0 \\ s_e p_{r(e) \cap s(f)} s_f & r(e) \cap s(f) \neq 0. \end{cases}$$

4. This follows by an analogous argument as before using that $s_f^* = (s_f p_{r(f)})^* = p_{r(f)} s_f^*$.

□

We want to elaborate a bit more on commutativity between elements in the hypergraph C^*-algebra.

Lemma 2.16. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph and $\{p_v, s_e\}$ be a Cuntz-Krieger $H\Gamma$-family. Then*

1. *$\{s_e^* s_e \mid e \in E^1\}$ consists of commutative projections;*
2. *$p_A(s_e^* s_e) = (s_e^* s_e)p_A$ for all $e \in E^1$ and $A \subseteq E^0$.*

Proof. 1. For $e \in E^1$ we have by the first hypergraph relation

$$(s_e^* s_e)(s_f^* s_f) = p_{r(e)} p_{r(f)} = p_{r(e) \cap r(f)} = p_{r(f)} p_{r(e)} = (s_f^* s_f)(s_e^* s_e).$$

2. This follows by a similar argument as above.

□

Remark 2.17. *In general we cannot recover $p_A(s_e s_e^*) = (s_e s_e^*)p_A$ for $e \in E^1$ and $A \subseteq E^0$ from the hypergraph relations. In case of ultragraphs the statement is true since $s_e s_e^* = p_{s(e)}$. More general we have commutativity if $s(e) \subseteq A$ or $s(e) \cap A = \emptyset$. The noncommutativity arises solely if $s_e s_e^* < p_{s(e)}$ is given. The question remains open, which impact this has and if representations loose information about this noncommutativity.*

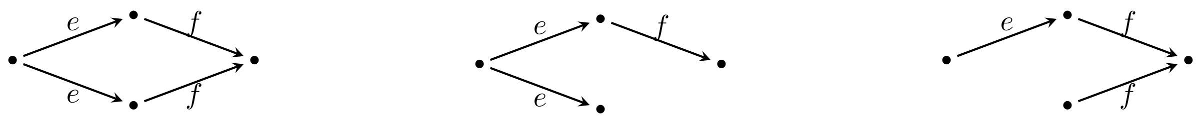
With regard to the second relation in Proposition 2.15 we define paths in hypergraphs.

Definition 2.18. Edges $\mu_1, ..., \mu_n \in E^1$ form a *path* $\mu = \mu_1...\mu_n$ in a finite hypergraph if $r(\mu_j) \cap s(\mu_{j+1}) \neq \emptyset$. Vertices are regarded as paths of length zero. The set of paths is denoted by E^*. Generalizing the range and source maps to E^* yields $s(\mu) := s(\mu_1)$ and $r(\mu) := r(\mu_n)$ for $|\mu| > 1$ and $s(v) = v = r(v)$ for $v \in E^0$. We then define $s_\mu := s_{\mu_1}...s_{\mu_n}$ and $s_v := p_v$.

It turns out that the paths are of different quality. This has implications for the structure of the hypergraph C^*-algebra, which we examine hereafter.

Definition 2.19. Let $\mu = \mu_1...\mu_n$ be a path in $H\Gamma$. Then we call μ

1. *perfect*, if $s(\mu_{j+1}) = r(\mu_j)$ for all $j \in \{1, ..., n\}$;
2. *quasi perfect*, if $s(\mu_{j+1}) \subseteq r(\mu_j)$ for all $j \in \{1, ..., n\}$;
3. *partial*, if $s(\mu_{j+1}) \cap r(\mu_j) \neq \emptyset$ for all $j \in \{1, ..., n\}$.



(a) Perfect path (b) Quasi perfect path (c) Partial path

Figure 9: Visualization of different paths in hypergraphs.

For hypergraphs with only quasi perfect paths, and thus also for hypergraphs with perfect paths, we get a similar result as in Corollary 1.22.

Proposition 2.20. *Let μ, ν be quasi perfect paths in a finite hypergraph $H\Gamma$. Then it holds*

1. $|\mu| = |\nu|,\ \mu \neq \nu \ \Rightarrow\ (s_\mu s_\mu^*)(s_\nu s_\nu^*) = 0$;
2. $$s_\mu^* s_\nu = \begin{cases} s_{\mu'}^* & \text{if } \mu = \nu\mu' \text{ for some } \mu' \in E^* \\ s_{\nu'} & \text{if } \nu = \mu\nu' \text{ for some } \nu' \in E^* \\ p_{r(\mu)} & \mu = \nu \\ 0 & \text{else}; \end{cases}$$
3. $s_\mu s_\nu \neq 0 \ \Rightarrow\ \mu\nu$ *is a path in* $H\Gamma$ *and* $s_\mu s_\mu = s_{\mu\nu}$;
4. $s_\mu s_\nu^* \neq 0 \ \Rightarrow\ r(\mu) \cap r(\nu) \neq \emptyset$.

Proof. Let $\mu = \mu_1...\mu_n$ and $\nu = \nu_1...\nu_m$ be quasi perfect paths in $H\Gamma$. The first hypergraph relation implies

$$\begin{aligned} s_\mu^* s_\nu &= s_{\mu_n}^* ... s_{\mu_1}^* s_{\nu_1} ... s_{\nu_m} \\ &= \delta_{\mu_1 \nu_1} s_{\mu_n}^* ... s_{\mu_2}^* p_{r(\mu_1)} s_{\nu_2} ... s_{\nu_m}. \end{aligned}$$

Applying Proposition 2.12 twice and using that we have quasi perfect paths, i.e. $s(\mu_2) \subseteq r(\mu_1)$, we get

$$\begin{aligned} &= \delta_{\mu_1 \nu_1} s_{\mu_n}^* ... s_{\mu_2}^* p_{s(\mu_2)} p_{r(\mu_1)} s_{\nu_2} ... s_{\nu_m} \\ &= \delta_{\mu_1 \nu_1} s_{\mu_n}^* ... s_{\mu_2}^* p_{s(\mu_2)} s_{\nu_2} ... s_{\nu_m} \\ &= \delta_{\mu_1 \nu_1} s_{\mu_n}^* ... s_{\mu_2}^* s_{\nu_2} ... s_{\nu_m}. \end{aligned}$$

We continue in this fashion and get the result of part (2). Part (1) follows by a similar argument using that $s_{\mu_j}^* s_{\nu_j} = 0$ for $\mu_j \neq \nu_j$. Part (3) follows by Proposition 2.12 and the definition of paths and part (4) is a result of (4) in Proposition 2.15. □

Remark 2.21. *In the case of graph C^*-algebras, the element s_μ for a path $\mu = \mu_1 \dots \mu_n$ is always a partial isometry. This is not the case if we deal with hypergraph C^*-algebras. The problem is, that in general $s_\mu^* s_\mu$ does not collapse to $p_{r(\mu)}$, as it is the case for graph C^*-algebras. Nevertheless, if the path is "nice enough" we can recover that s_μ is an isometry. Applying the last proposition we get that s_μ is a partial isometry, if the path μ is at least quasi perfect.*

For hypergraphs with partial paths we do not obtain the same results. Generalizing the results in Proposition 2.15 we get the following:

Proposition 2.22. *Let $\mu = \mu_1...\mu_n$, $\nu = \nu_1...\nu_m$ be paths in a finite hypergraph $H\Gamma$. Then*

1. $s_\mu^* s_\nu \neq 0 \;\Rightarrow\; \mu_1 = \nu_1$ *and* $s(\mu_2) \cap r(\mu) \cap s(\nu_2) \neq \emptyset$;
2. $s_\mu s_\nu \neq 0 \;\Rightarrow\; r(\mu) \cap s(\nu) \neq \emptyset$;
3. $s_\mu s_\nu^* \neq 0 \;\Rightarrow\; r(\mu) \cap r(\nu) \neq \emptyset$.

Proof. The proof is just an application of the definition of s_μ and Proposition 2.15. Since in general $s_e p_A \neq s_e$ for $A \subsetneq r(e)$ and $p_A s_e \neq s_e$ for $A \subsetneq s(e)$ the words do not collapse completely as in the case of quasi perfect paths. □

We go back in the setting of quasi perfect paths and investigate longer chains of partial isometries corresponding to paths. the next corollary is a direct consequence of Proposition 2.20.

Corollary 2.23. *Let μ, ν, α, β be quasi perfect paths in a finite hypergraph $H\Gamma$. Then we have*

$$(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \begin{cases} s_{\mu\alpha'} s_\beta^* & \alpha = \nu\alpha' \\ s_\mu s_{\nu'\beta}^* & \nu = \alpha\nu' \\ s_\mu p_{r(e)} s_\beta^* & \nu = \alpha \\ 0 & \textit{else}. \end{cases}$$

Using these results we can describe the hypergraph C^*-algebra more precisely - at least in the case of quasi perfect paths.

Corollary 2.24. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph with Cuntz-Krieger $H\Gamma$-family $\{p_v, s_e\}$. It holds*

$$C^*(H\Gamma) = \overline{span}\left\{s_{\mu_1}^{\epsilon_1}...s_{\mu_n}^{\epsilon_n} \mid \mu_1,...,\mu_n \in E^*,\ \epsilon_1,...\epsilon_n \in \{1,*\},\ \epsilon_j \neq \epsilon_{j+1},\ n \in \mathbb{N}\right\}.$$

If all paths in $H\Gamma$ are either perfect or quasi perfect it holds:

$$C^*(H\Gamma) = \overline{span}\left\{s_\mu s_\nu^* \mid \mu,\nu \in E^*,\ r(\mu)\cap r(\nu) \neq \emptyset\right\}.$$

Proof. The first part follows from the definition of paths (note that $p_v = s_v$), Proposition 2.22 and the definition of the hypergraph C^*-algebra as universal C^*-algebra. In the second case we can rely on Proposition 2.20 and copy the respective proof of [Rae05, Cor. 1.16]. We just have to notice, that since $s_\mu p_{r(e)} s_\beta^* = \sum_{v\in r(e)} s_\mu p_v s_\beta^* = \sum_{v\in r(e)} s_{\mu v} s_\beta^*$, we still get by Corollary 2.23 that

$$span\{s_\mu s_\nu^* \mid \mu,\nu \in E^*, r(\mu)\cap r(\nu) \neq \emptyset\}$$

is a subalgebra of $C^*(H\Gamma)$. This is a * subalgebra which implies that the closure is a C^*-subalgebra of $C^*(H\Gamma)$ which contains the generators of $C^*(H\Gamma)$. Hence the closure equals $C^*(H\Gamma)$. □

Remark 2.25. *We can give an equivalent description of a hypergraph C^*-algebra which explicitly includes the projections corresponding to the generalized vertices as defined in Definition 2.7. In case of hypergraphs with only quasi perfect paths we have*

$$C^*(H\Gamma) = \overline{span}\left\{s_\mu p_A s_\nu^* \mid \mu,\ \nu \in E^*, A \in \mathcal{E},\ A \cap r(\mu)\cap r(\nu) \neq \emptyset\right\}.$$

We finish this section with some lemmata which indicate the problems that arise in the hypergraph setting. As we will later see, the multiple elements in the ranges are nothing to worry about. The key complications are related to multiple vertices in the source and intersecting sources. This results in a higher relevance of the order relation of projections.

Lemma 2.26. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. If for some vertex $w \in E^0$ the inequality in (HR3) is strict, i.e.*

$$p_w < \sum_{e\in E^1, w\in s(e)} s_e s_e^*$$

it follows, that there must be an edge $e \in E^1$ such that $w \in s(e)$ but $w \neq s(e)$.

Proof. We proof the contraposition. Assume that for all $e \in E^1$ with $w \in s(e)$ it follows that $w = s(e)$. Then we get by the second hypergraph relation

$$s_e s_e^* \leqslant p_w \qquad \forall e \in E^1 \text{ with } w = s(e).$$

Combining this with the third hypergraph relation we get that $p_w = \sum_{e\in E^1, w\in s(e)} s_e s_e^*$. □

If $w \in s(e)$ implies $w = s(e)$ for all $e \in E^1$, the contraposition of the previous Lemma implies that $p_w = \sum_{e \in E^1, w \in s(e)} s_e s_e^*$. This can be extended to a set of vertices $A \subseteq E^0$.

Lemma 2.27. *For each finite hypergraph $H\Gamma = (E^0, E^1, r, s)$ and $A \subseteq E^0$ such that $s(e) \cap A$ implies $s(e) \subseteq A$ for all $e \in E^1$. Then $p_A = \sum_{e \in E^1, s(e) \subseteq A} s_e s_e^*$. This implies in particular, that $\sum_{e \in E^1} s_e s_e^* = 1$*

Proof. By the second hypergraph relation we know

$$s_e s_e^* \leqslant p_{s(e)} \leqslant p_A \qquad \text{for all} e \in E^1 \text{ with } s(e) \subseteq A$$

using that $(s_e s_e^*) p_{A \setminus s(e)} = 0$. On the other hand we get by the third hypergraph relation

$$p_v \leqslant \sum_{e \in E^1, v \in s(e)} s_e s_e^* \leqslant \sum_{e \in E^1, s(e) \subseteq A} s_e s_e^* \qquad \text{for all } v \in A,$$

since $p_v \sum_{e \in E^1, v \notin s(e)} s_e s_e^* = 0$. Combining both inequalities and using that we deal with finite sums, the result follows. □

Lemma 2.28. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. If for some edge $e \in E^1$ the inequality in (HR2) is strict, i.e.*

$$s_e s_e^* < \sum_{v \in E^0, v \in s(e)} p_v,$$

it follows, that there must be an edge $f \in E^1$ such that $s(e) \cap s(f) \neq \emptyset$.

Proof. We assume by contraposition, that $s(e) \cap s(f) = \emptyset$ for all $f \in E^1$. Then we get by the third hypergraph relation for each $v \in s(e)$:

$$p_{s(e)} \leqslant s_e s_e^*.$$

Along with the second hypergraph relation we get equality, which shows the claim. □

2.3. Examples and Representations

In the following we have a look at some interesting examples of hypergraphs, which are no graphs. We first revisit the Cuntz algebra O_n. We already know by Example 1.18 that it is a graph algebra, but we can also describe it by a hypergraph.

Proposition 2.29. *[Zen21, Prop. 3.11] For $n \in N$ with $n \geqslant 2$ let $H\Gamma = (E^0, E^1, r, s)$ be the hypergraph defined by vertices $\{v_1, \dots, v_n\}$ and edges $\{e_1, \dots, e_n\}$ with $s(e_i) = \{v_i\}$ and $r(e_i) = \{v_1, \dots, v_n\}$ for $i = 1, \dots, n$. Then $C^*(H\Gamma) \cong O_n$.*

Proof. Let $\{s_{e_i}, p_v\}$ be the canonical generators of $C^*(H\Gamma)$, and $\{t_i\}$ be the canonical generators of O_n. By the first hypergraph relation we get $s_{e_i}^* s_{e_i} = p_{r(e_i)} = p_{E^0}$ for each $i = 1, \dots, n$. By Proposition 2.13, p_{E^0} is the unit. Thus, s_{e_i} is an isometry. By Lemma 2.27, we furthermore have $\sum_{e \in E^1} s_e s_e^* = 1$. Hence, the elements s_{e_i} fulfill the relations of the Cuntz algebra and by the universal property we get the *-homomorphism

$\pi : O_n \to C^*(H\Gamma)$ which maps $t_i \mapsto s_{e_i}$. On the other hand, by defining $S_{e_i} := t_i$ and $P_{v_i} := t_i t_i^*$ we get a Cuntz-Krieger $H\Gamma$-family in O_n. Indeed, since the elements t_i are isometries, we have by the relations of O_n

$$S_{e_i}^* S_{e_i} = t_i^* t_i = 1 = \sum_{i=1}^{n} t_i t_i^* = \sum_{i=1}^{n} P_{v_i} = P_{r(e_i)}.$$

Since the sum of $t_i t_i^*$ is a projection, the projections $P_{v_i} = t_i t_i^*$ must be mutually orthogonal. Hence $S_{e_i}^* S_{e_j} = S_{e_i} P_i P_j S_{e_j} = 0$ for $i \neq j$. The second and third hypergraph relation follow directly by the definition of P_{v_i}. Again by the universal property we get a *-homomorphism $\tilde{\pi} : C^*(H\Gamma) \to O_n$ sending $s_{e_i} \mapsto S_{e_i}$ and $p_{v_i} \mapsto P_{v_i}$. Both *-homomorphisms are inverse, which proofs the claim. □

We can also express the Toeplitz algebra as hypergraph C^*-algebra. We can even consider two different hypergraphs as we see in the following example.

Proposition 2.30. *The C^*-algebras generated by the following hypergraphs are both isomorphic to the Toeplitz algebra $\mathcal{T}$.*

Figure 10: Hypergraphs generating the Toeplitz algebra.

Proof. For the first hypergraph see [Zen21, Proposition 3.9]. For the second hypergraph, lets call it $H\Gamma$, we get by the hypergraph relations

$$s_e^* s_e = p_w, \qquad s_e s_e^* = p_v + p_w.$$

Since the sum of all projections is the unit, s_e^* is an isometry. By the universal property of the Toeplitz algebra we get a *-homomorphism $\phi : \mathcal{T} \to C^*(H\Gamma)$ defined by

$$u \mapsto s_e^*, \quad 1 \mapsto p_v + p_w = s_e s_e^*.$$

On the other hand we can define a Cuntz-Krieger $H\Gamma$-family in $\mathcal{T}$ as follows

$$S_e := u^*, \qquad P_w := uu^*, \qquad P_v := 1 - uu^*.$$

Simple calculations show that S_e is a partial isometry and $P_w,\ P_v$ are mutually orthogonal projections. One can also easily check, that they fulfill the hypergraph relations. The universal property then gives us a *-homomorphism $\psi : C^*(H\Gamma) \to \mathcal{T}$ defined by

$$s_e \mapsto S_e, \quad p_w \mapsto P_w, \quad p_v \mapsto P_v.$$

On the generators of both algebras we can check that $\phi \circ \psi = id_{C^*(H\Gamma)}$ and $\psi \circ \phi = id_{\mathcal{T}}$. Thus we get the required isomorphism. □

For universal C^*-algebras the question of non-triviality crucial. Since all graph C^*-algebras are non-trivial as already stated in [Rae05], we have already some non-trivial hypergraph C^*-algebras. To get non-triviality of universal C^*-algebras we have to construct concrete representations.

Example 2.31. We consider the hypergraph $H\Gamma$ defined by vertices $\{w, v_1, v_2\}$ and edges $\{e, f\}$ with $s(e) = \{w_1, w_2\},\ r(e) = \{w_1, w_2\}$ and $s(f) = \{w_1, w_2\},\ r(f) = \{v\}$.

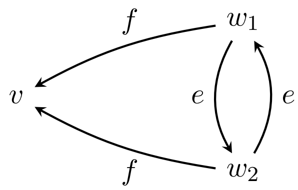


Figure 11: Visualization of the hypergraph in Example 2.31.

A representation of $C^*(H\Gamma)$ on the Hilbertspace $l^2(\mathbb{N}_0)$ is given as follows. Let $x = (x_n)_n$ be a sequence in $l^2(\mathbb{N}_0)$. Then the following projections and partial isometries define a Cuntz-Krieger $H\Gamma$-family:

$$\begin{aligned}
P_v x &:= (x_0, 0, \dots),\\
P_{w_1} x &:= (0, x_1, 0, x_3, 0, \dots),\\
P_{w_2} x &:= (0, 0, x_2, 0, x_4, \dots),\\
S_e x &:= (0, 0, x_1, x_2, \dots),\\
S_f x &:= (0, x_0, 0, 0, \dots).
\end{aligned}$$

Applying the universal property of hypergraph C^*-algebras yields the representation.

Multiple representations of hypergraph C^*-algebras on the Hilbertspace $l^2(\mathbb{Z}^2)$ can be seen in [Zen21, Section 3.3]. Since all these examples include only non-intersecting sources, we now state another example with a slight modification of the construction compared to [Zen21], which we highlight in the upcoming example.

Example 2.32. Consider the Hilbertspace $l^2(\mathbb{Z}^2)$ with basis $e_{(x,y)}$ for $x, y \in \mathbb{Z}$. Let $H\Gamma$ be the hypergraph defined by $s(e) = \{v_1, v_2\}$, $r(e) = v_3$ and $s(f) = \{v_2\}$, $r(f) = \{v_1\}$:

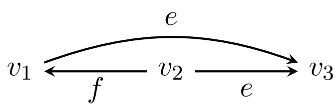


Figure 12: Visualization of the hypergraph in Example 2.32.

Consider subspaces of $l^2(\mathbb{Z}^2)$ by

$$\begin{aligned}
H_1 &:= \mathbb{N}_0 \times \mathbb{N}_0,\\
H_2 &:= \mathbb{Z} \setminus \mathbb{N}_0 \times \mathbb{N}_0,\\
H_3 &:= \mathbb{Z} \times \mathbb{Z} \setminus \mathbb{N}_0,
\end{aligned}$$

and let P_i be the projection onto H_i for $i = 1, 2, 3$. Since the subspaces are disjoint, the projections are mutually orthogonal. Using bijections

$$\begin{aligned} g &: H_3 \to (\mathbb{Z} \setminus \mathbb{N}_0 \times \{2n \mid n \in \mathbb{N}_0\}) \cup (\mathbb{N}_0 \times \mathbb{N}_0)\,, \\ h &: H_1 \to \mathbb{Z} \setminus \mathbb{N}_0 \times \{2n+1 \mid n \in \mathbb{N}_0\} \end{aligned}$$

we can define partial isometries

$$\begin{aligned} S_e e_{(x,y)} &:= \delta_{(x,y)\in H_3}\ e_{g((x,y))}, \\ S_f e_{(x,y)} &:= \delta_{(x,y)\in H_1}\ e_{h((x,y))}. \end{aligned}$$

These elements build a Cuntz-Krieger $H\Gamma$-family in $B(l^2(\mathbb{Z}^2))$, the space of bounded operators on $l^2(\mathbb{Z}^2)$. Note that we deviate at this point from the method in [Zen21]. There, the bijections were defined separately for the coordinates x and y, while we use only one bijection defined on a product space. To check the hypergraph relations we note that

$$\begin{aligned} S_e^* e_{(x,y)} &:= \delta_{(x,y)\in(\mathbb{Z}\setminus\mathbb{N}_0\times\{2n \mid n\in\mathbb{N}_0\})\cup(\mathbb{N}_0\times\mathbb{N}_0)}\ e_{g^{-1}((x,y))}, \\ S_f^* e_{(x,y)} &:= \delta_{(x,y)\in\mathbb{Z}\setminus\mathbb{N}_0\times\{2n+1 \mid n\in\mathbb{N}_0\}}\ e_{h^{-1}((x,y))}. \end{aligned}$$

With this we get for the first hypergraph relation

$$\begin{aligned} (S_e^* S_e) e_{(x,y)} &= \delta_{(x,y)\in H_3}\ S_e^* e_{g((x,y))} = \delta_{(x,y)\in H_3}\ e_{(x,y)} = P_3 e_{(x,y)}, \\ (S_f^* S_f) e_{(x,y)} &= \delta_{(x,y)\in H_1}\ S_f^* e_{h((x,y))} = \delta_{(x,y)\in H_1}\ e_{(x,y)} = P_1 e_{(x,y)}. \end{aligned}$$

Since the ranges of g and h are disjoint we get

$$\begin{aligned} (S_e^* S_f)\, e_{(x,y)} &= \delta_{(x,y)\in H_1} S_e^*\ e_{h((x,y))} \\ &= \delta_{(x,y)\in H_1} \delta_{h(x,y)\in(\mathbb{Z}\setminus\mathbb{N}_0\times\{2n \mid n\in\mathbb{N}_0\})\cup(\mathbb{N}_0\times\mathbb{N}_0)}\ e_{g^{-1}(h((x,y)))} \\ &= 0 \end{aligned}$$

and with the same argument we get $S_f^* S_e = 0$. For the second hypergraph relation we have

$$\begin{aligned} (S_e S_e^*) e_{(x,y)} &= \delta_{(x,y)\in(\mathbb{Z}\setminus\mathbb{N}_0\times\{2n \mid n\in\mathbb{N}_0\})\cup(\mathbb{N}_0\times\mathbb{N}_0)} e_{(x,y)} \\ &= \left(\delta_{(x,y)\in\mathbb{Z}\setminus\mathbb{N}_0\times\{2n \mid n\in\mathbb{N}_0\}} + \delta_{(x,y)\in(\mathbb{N}_0\times\mathbb{N}_0)}\right) e_{(x,y)}. \end{aligned}$$

Since $\mathbb{Z} \setminus \mathbb{N}_0 \times \{2n \mid n \in \mathbb{N}_0\} \cup \mathbb{N}_0 \times \mathbb{N}_0 \subseteq H_2 \cup H_1$ we get $S_e S_e^* \leqslant P_2 + P_1$. Furthermore we have for the edge f

$$(S_f S_f^*) e_{(x,y)} = \delta_{(x,y)\in\mathbb{Z}\setminus\mathbb{N}_0\times\{2n+1 \mid n\in\mathbb{N}_0\}} e_{(x,y)}$$

which proves $S_f S_f^* \leqslant P_2$. With respect to the expressions used above we get that $P_1 \leqslant S_e S_e^*$ and $P_2 \leqslant S_f S_f^*$ which proves the third hypergraph relation. By the universal property we thus get the representation on $l^2(\mathbb{Z}^2)$ given by the *-homomorphism $\pi : C^*(H\Gamma) \to B(l^2(\mathbb{Z}^2))$ defined by

$$p_i \mapsto P_i, \quad s_e \mapsto S_e, \quad s_f \mapsto S_f.$$

Hence we can represent $C^*(H\Gamma)$ on $l^2(\mathbb{Z}^2)$.

Remark 2.33. *We systematize the previous example and give a way to define representations on $l^2(\mathbb{Z}^2)$ of hypergraph C^*-Algebras. Base of the construction is the decomposition of $l^2(\mathbb{Z}^2)$. Let $H\Gamma$ be a finite hypergraph given by edges $\{e_1, ..., e_n\}$ and vertices $\{v_1, .., v_m\}$ for $n, m \in \mathbb{N}$. We decompose $l^2(\mathbb{Z}^2)$ into m closed subspaces $H_1, \dots, H_m$ and let P_i for $i = 1, ..., m$ be the corresponding projections onto H_i. Let A_j, B_j, C_j, D_j closed subspaces of $\mathbb{Z}$. Let*

$$f_j : A_j \times B_j \to C_j \times D_j$$

be a bijective function. Then we define a partial isometry S_j as follows

$$S_j e_{(x,y)} := \delta_{(x,y) \in A_j \times B_j} \; e_{f_j(x,y)}.$$

The adjoint operator is given by

$$S_j^* e_{(x,y)} = \delta_{(x,y) \in C_j \times D_j} \; e_{f_j^{-1}(x,y)}.$$

Combining both we get

$$S_j^* S_j = P_{A_j \times B_j},$$
$$S_j S_j^* = P_{C_j \times D_j}.$$

The concrete choice of the subspaces A_j, B_j, C_j, D_j depends on the given hypergraph relations:

1. *By the first hypergraph relation we know that $A_j \times B_j$ has to match the subset on which the projection $P_{r(e_j)}$ maps. Hence $A_j \times B_j = \bigcup_{i \in \{1,\dots,m\}, v_i \in r(e_j)} H_i$.*
2. *To completely match the first hypergrpah relation we have to ensure that $S_j^* S_k = 0$ for $j \neq k$. To achieve this we have to ensure that the ranges of the bijections f_j are disjoint.*
3. *The second hypergraph relation implies that $C_j \times D_j$ must be a subset of $\bigcup_{i \in \{1,\dots,m\}, v_i \in s(e_j)} H_i$.*
4. *The third hypergraph relation on the other hand implies that we have to ensure, that H_i is a subset of $\bigcup_{j \in \{1,\dots,n\}, v_i \in s(e_j)} C_j \times D_j$. Thus, it can be necessary to divide each segment of $l^2(\mathbb{Z}^2)$ in different subspaces, if different edges have similar vertices in their source.*

Note, that with the given method we will always get commutativity between P_i and $S_j S_j^*$.

3. Non-Nuclear Hypergraph C*-Algebra

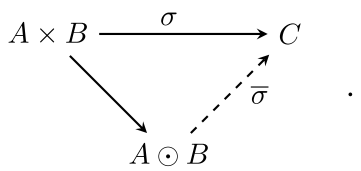
The definition of nuclear C^*-algebras by Takesaki in the 1960's deals with the uniqueness of the norm on tensor products of C^*-algebras. This property is interesting in the context of hypergraph C^*-algebras as it allows us to show that the class of hypergraph C^*-algebras is indeed larger than the class of graph C^*-algebras. We construct a hypergraph which generates a non-nuclear C^*-algebra. Building up on this example we develop techniques to identify and construct further non-nuclear hypergraph C^*-algebras.

3.1. Definition of Nuclearity

The definition of nuclearity is based on tensor products of C^*-algebras A and B. We give a short overview of the construction. For more detailed information see [BO08, Chapter 3]. Similar as for vector spaces we can construct the *algebraic tensor product* $A \odot B$ spanned by elementary tensors $a \otimes b$ for $a \in A$ and $b \in B$, which fulfill the tensor calculus

1. $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$ and $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$;
2. $\lambda(a \otimes b) = (\lambda a) \otimes b = a \otimes (\lambda b)$ for all $\lambda \in \mathbb{C}$.

The tensor product fulfills a universal property: for any C^*-algebra C and any bilinear map $\sigma : A \times B \to C$, there exists a unique linear map $\overline{\sigma} : A \odot B \to C$ such that the following diagram commutes:



By defining a multiplication and involution as

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2, \qquad (a \otimes b)^* := a^* \otimes b^*,$$

the algebraic tensor product becomes a *-algebra. To get a C^*-algebra we need a C^*-norm. The completion with respect to such a norm is then a C^*-algebra. One can proof that there is always a C^*-norm on the algebraic tensor product [Bla06, II.9.1.3]. But there can be multiple different norms on the algebraic tensor product. Since norms on C^*-algebras are unique, the completions with respect to these norms yield different C^*-algebras. The main idea to construct norms on the algebraic tensor product of C^*-algebras is to use representations on tensor products of Hilbertspaces. These have a unique norm turning them into Hilbertspaces [BO08, Prop. 3.2.1].

Definition 3.1. Let $\mathcal{H}, \mathcal{K}$ be Hilbertspaces and $B(\mathcal{H}), B(\mathcal{K})$ denote the corresponding sets of bounded operators. Let $\pi : A \to B(\mathcal{H})$ and $\sigma : B \to B(\mathcal{K})$ be faithful representations of C^*-algebras A and B on $\mathcal{H}$ and $\mathcal{K}$. Then the *spatial C^*-norm* an $A \odot B$ is defined as

$$\|\sum a_i \otimes b_i\|_{\min} = \|\sum \pi(a_i) \otimes \sigma(b_i)\|_{B(\mathcal{H} \otimes \mathcal{K})}$$

for $a_i \in A$ and $b_i \in B$. The completion of $A \odot B$ with respect to $\|\cdot\|_{\min}$ is denoted by $A \otimes B$. The *spatial tensor product* is also called *minimal tensor product*.

Definition 3.2. Given A and B, we define the *maximal C^*-norm* on $A \odot B$ to be

$$||x||_{\max} = sup\{||\pi(x)|| \mid \pi : A \odot B \to B(\mathcal{H}) \text{ a cyclic *-homomorphism}\}$$

for $x \in A \odot B$. We let $A \otimes_{\max} B$ denote the completion of $A \odot B$ with respect to $|| \cdot ||_{\max}$. It is called that *maximal tensor product*.

Definition 3.3. A C^*-algebra A is called *nuclear*, if for every C^*-algebra B, there is a unique C^*-norm on $A \odot B$.

One can indeed show that, as the names suggest, $|| \cdot ||_{\min} \leqslant || \cdot || \leqslant || \cdot ||_{\max}$. The proof goes back to a theorem of Takesaki which can be found in [BO08, Cor. 3.3.8, Thm. 3.4.8]. Thus to show nuclearity it is enough to prove that the spatial and the maximal norm coincide.

Example 3.4.

1. The matrix algebra $M_n(\mathbb{C})$ is nuclear for each $n \in \mathbb{N}$ as for each C^*-algebra A the algebraic tensor product $M_n(\mathbb{C}) \odot A$ is isomorphic to $M_n(A)$ which is a C^*-algebra and thus has a unique norm.
2. All finite dimensional C^*-algebras are nuclear as by Wedderburns Theorem, each finite dimensional C^*-algebra is isomorphic to a finite direct sum of matrix algebras over $\mathbb{C}$.
3. All graph C^*-algebras are nuclear. A sketch of the proof was outlined in [Rae05, Rem. 4.3]: The unit circle acts on each C^*-algebra by the gauge action γ. The crossed product $C^*(E) \rtimes_\gamma \mathbb{T}$ with regard to the gauge action is isomorphic to a C^*-algebra of the so called skew product graph. The corresponding graph C^*-algebra is AF. By the Takesaki-Takai Duality Theorem $C^*(E)$ is thus stably isomorphic to $(C^*(E) \rtimes_\gamma \mathbb{T}) \rtimes_{\hat{\gamma}} \mathbb{Z}$ which is nuclear. Nuclearity is invariant under stable isomorphism which then shows nuclearity of the graph C^*-algebra.

Nuclearity does not transfer from C^*-algebras to *-subalgebras. Neither are *-subalgebras of nuclear C^*-algebras always nuclear nor are *-subalgebras of non-nuclear C^*-algebras non-nuclear. If we restrict ourselves to hereditary subalgebras, then we see that nuclearity passes to hereditary subalgebras [Bla06, Cor. IV. 3.1.14]. Since hereditary subalgebras correspond to ideals this hints to the fact that ideals and quotients of nuclear C^*-algebras are again nuclear. Especially the quotient condition will be crucial in the following. The proof of this proposition is non-trivial and involves the second dual of a C^*-algebra. Therefore we refer for the proof to the given reference.

Proposition 3.5. *[Bla06, Cor. IV.3.1.13, Prop. II.9.6.3] Quotients and closed ideals of a nuclear C^*-algebra are nuclear.*

As already indicated in Section 1.2 it is also possible to view the tensor product as universal C^*-algebra.

Proposition 3.6. *[Bla85, Ex. 1.3g] The tensor product of a unital, nuclear C^*-Algebra A and a unital C^*-algebra B is isomorphic to the universal C^*-algebra*

$$A \otimes B \cong C^*(a \in A,\ b \in B \mid R_A,\ R_B,\ ab = ba\ \forall a \in A, b \in B),$$

where R_A and R_B are the normal C^*-relations on A and B.

Proof Sketch. Since A is nuclear, there is a unique norm on the tensor product and we have not to distinguish between different tensor products. We note that for non-nuclear C^*-algebras the given universal C^*-algebra is isomorphic to the maximal tensor product. We consider the elementary tensors $a \otimes 1$ and $1 \otimes b$. These commute clearly by the definition of the multiplication. Thus the universal property yields a *-homomorphism from the universal C^*-algebra onto the tensor product mapping $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$. The *-homomorphism is clearly surjective, as all elementary tensors lie in the image and with some more work it can be shown that it is also injective. □

We now turn to non-nuclear examples. This will later on be useful, as we will see that not all hypergraph C^*-algebras are nuclear. We only give an overview of the results in order to focus afterwards on the application to hypergraph C^*-algebras. For the proofs we therefore refer to the stated sources.

Key to the example below is the relation between amenability of discrete groups and nuclearity of the corresponding group C^*-algebra.

Definition 3.7. A group Γ is called *amenable*, if there exists a state μ on $l^\infty(\Gamma)$ which is invariant under left group action, i.e. for all $s \in \Gamma$ and $f \in l^\infty(\Gamma)$, $\mu(s \cdot f) = \mu(f)$.

Definition 3.8. Let G be a locally compact group. The *group C^*-algebra* is defined as

$$C^*(G) := \mathbb{C} \rtimes_\alpha G = C^*\left(u_g \text{ unitaries for } g \in G \mid u_{gh} = u_g u_h,\ u_{g^{-1}} = u_g^* \text{ for all } g, h \in G\right).$$

This definition of the group C^*-algebra is in line with the definition as norm closure of the Banach algebra $L^1(G)$ with respect to the full norm. If the group is discrete we get the following:

Theorem 3.9. *[BO08, Thm. 2.6.8] A discrete group G is amenable if and only if the group C^*-algebra $C^*(G)$ is nuclear.*

There are many other equivalent characterizations of it, but this one is the crucial one for us.

Proposition 3.10. *[BO08, Ex. 2.6.7] Let $\mathbb{F}_2$ be the free group of two generators. Then the group C^*-algebra $C^*(\mathbb{F}_2)$ is non-nuclear.*

Proof Sketch. The proof uses a paradoxical decomposition of the free group with generators a and b. Let A^+ be the set of all words starting with a, A^- be the set of words starting with a^{-1}. Similarly we define B^+ and B^-. Then we can express the free group by

$$\mathbb{F}_2 = A^+ \cup A^- \cup B^+ \cup B^- = A^+ \cup a \cdot A^- = B^+ \cup b \cdot B^-.$$

Assuming $\mathbb{F}_2$ would be amenable, there would be a left invariant mean μ. Using the left invariance combined with the above decomposition leads to a contradiction. □

In contrast to nuclearity, amenability can be transferred to subgroups.

Proposition 3.11. *[Run20, Thm. 1.2.7] All closed subgroups of amenable, locally compact groups are amenable.*

These two proposition combined give the base point of the creation of multiple non-nuclear C^*-algebras.

Corollary 3.12. *Let G be a discrete group. If the free group of two generators is isomorphic to a subgroup of G, then G is non-amenable and $C^*(G)$ is non-nuclear.*

3.2. Non-Nuclear Hypergraph C*-Algebras

We now use nuclearity to show that the class of hypergraph C^*-algebras is indeed larger as the class of graph C^*-algebras. All graph C^*-algebras are nuclear as stated in Example 3.4. And even all ultragraph C^*-algebras are nuclear as each ultragraph C^*-algebra is Morita equivalent to a graph C^*-algebra [KMST10, Thm. 5.22]. For hypergraph C^*-algebras this is not the case anymore as we see in the following example from [Zen21, Prop. 3.12].

Proposition 3.13. *Let $n \in \mathbb{N}$ and consider the hypergraph $H\Gamma$ with vertices $\{v_1, \dots, v_n\}$ and edge $\{e\}$ with $s(e) = \{v_1, \dots, v_n\} = r(e)$. Then $C^*(H\Gamma) \cong C(\mathbb{T}) * \mathbb{C}^n$. We denote this fully connected hypergraph with n vertices in the following by $\tilde{H\Gamma}_n$.*

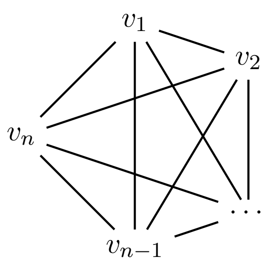


Figure 13: Hypergraph $\tilde{H\Gamma}_n$ generating $C(\mathbb{T}) * \mathbb{C}^n$. To simplify the visualization of the hypergraphs, we omit the arrowheads for edges that point in both directions. To be a bit more accurate, one should also include selfloops at each vertex.

Proof. We consider the universal C^*-algebras

$$C(\mathbb{T}) \cong C^*(u, 1 \mid u^*u = uu^* = 1) \quad \text{and} \quad \mathbb{C}^n \cong C^*(p_1, \dots, p_n \mid p_j^2 = p_j^* = p_j, \sum_{j=1}^n p_j = 1)$$

and show that $C^*(H\Gamma) \cong C(\mathbb{T}) * \mathbb{C}^n$. By the hypergraph relations we obtain $s_e^* s_e = p_{E^0}$ and $s_e s_e^* = p_{E^0}$. Hence s_e is a unitary by Proposition 2.13 and fulfills the relations of $C(\mathbb{T})$. Furthermore it is free of the projections, which fulfill the relations of $\mathbb{C}^n$. The universal property thus yields a *-homomorphism

$$\phi : C(\mathbb{T}) * \mathbb{C}^n \to C^*(H\Gamma)$$

sending u to s_e and p_i to p_{v_i}. Conversely we can define a Cuntz-Krieger $H\Gamma$-family in $C(\mathbb{T}) * \mathbb{C}^n$ via $S_e := u$ and $P_{v_i} = p_i$. Short calculations show that the hypergraph relations are fulfilled. The universal property yields then a *-homomorphism

$$\tilde{\phi} : C^*(H\Gamma) \to C(\mathbb{T}) * \mathbb{C}^n$$

which maps s_e to S_e and p_{v_i} to P_{v_i}. Both *-homomorphism are inverse to each other and give us thus the required isomorphism. □

Proposition 3.14. *The C^*-algebra $C(\mathbb{T}) * \mathbb{C}^n$ is non-nuclear.*

To prove this proposition we need a few lemmata.

Lemma 3.15. *The group C^*-algebra $C^*(\mathbb{Z})$ is isomorphic to $C(\mathbb{T})$.*

Proof. The C^*-algebra $C(\mathbb{T})$ is isomorphic to the universal C^*-algebra $C^*(u, 1 \mid u^*u = uu^* = 1)$. This C^*-algebra is isomorphic to $C^*(u_n, 1\ n \in \mathbb{N} \mid u_n^*u_n = u_nu_n^* = 1, u_{nm} = u_nu_m, u_{n^{-1}} = u_n^*$ for $n, m \in \mathbb{N})$. This follows by identifying u_j with the monom u^j and using the universal property twice. The latter is the universal C^*-algebra defining the group C^*-algebra $C^*(\mathbb{Z})$. □

Lemma 3.16. *The group C^*-algebra $C^*(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to $\mathbb{C}^n$.*

Proof. By Definition 3.8 the group C^*-algebra $C^*(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to the universal C^*-algebra $C^*(u, 1 \mid u^*u = uu^* = 1, u^n = 1)$. Since $u^n = 1$ the C^*-algebra consists of exactly n monoms and is thus at most n-dimensional. By Wedderbuns Theorem all finite dimensional C^*-algebras are isomorphic to a finite direct sum of Matrix algebras. Since the C^*-algebra is a commutative C^*-algebra, it is isomorphic to $\mathbb{C}^m = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ for some $m \leqslant n$. There is a unitary element $v \in \mathbb{C}^n$ with $v^n = 1$. i.e. consider the vector with entries $e^{2\pi i/n}$. Thus, by the universal property we get a *-homomorphism $\phi : \mathbb{C}^m \to \mathbb{C}^n$. The *-homomorphism is surjective since its image contains the unit. Thus by dimensional reasons $m = n$ and $C^*(\mathbb{Z}/n\mathbb{Z})$ is isomorphic to $\mathbb{C}^n$. □

Lemma 3.17. *Let G_1 and G_2 be non-trivial groups and G_2 be of order strictly greater than 2. The free group on two generators $\mathbb{F}_2$ is a subgroup of $G_1 * G_2$.*

Proof. Let x, y be the generators of $\mathbb{F}_2$. Let $1 \neq a \in G_1$ and $1 \neq b, c \in G_2$ such that $b^{-1}c \neq 1$ and define the *-homomorphism

$$\phi : \mathbb{F}_2 \to G_1 * G_2, \qquad x \mapsto (ab)^2,\ y \mapsto (ac)^2.$$

The elements $(ab)^2$ and $(ac)^2$ are free of each other, i.e. there are no relations between them, since a prevents interaction between the elements in G_2 and since we use the square, there is also no cancellation when multiplying with inverses. It remains to show that the *-homomorphism is injective which can be done by concrete calculations as executed in [Web11, Lemma 3.1.7]. □

Proof of 3.14. Combining the previous lemmata we get that $C(\mathbb{T}) * \mathbb{C}^n$ is isomorphic to $C^*(\mathbb{Z}) * C^*(\mathbb{Z}/n\mathbb{Z})$. As stated in Remark 1.6 and considering the expression of the group C^*-algebras as universal C^*-algebras, we have $C^*(\mathbb{Z}) * C^*(\mathbb{Z}/n\mathbb{Z}) \cong C^*(\mathbb{Z} * \mathbb{Z}/n\mathbb{Z})$. By Lemma 3.17 we know that $\mathbb{F}_2$ is a subgroup of $\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$. Thus by Proposition 3.12 it follows that $C^*(\mathbb{Z}*\mathbb{Z}/n\mathbb{Z})$ is non-nuclear and hence $C(\mathbb{T})*\mathbb{C}^n$ is non-nuclear. □

In analogy to Theorem 3.9 we want to define amenable hypergraphs to be those hypergraphs whose C^*-algebra is nuclear.

Definition 3.18. Let $H\Gamma$ be a hypergraph. We call $H\Gamma$ *amenable*, if the corresponding hypergraph C^*-algebra is nuclear.

3.3. Construction of Non-Nuclear Hypergraph C*-Algebras

We can now use the above non-amenable hypergraph $\tilde{H\Gamma}_n$ to construct further non-amenable hypergraphs. The main idea is to use, that nuclearity transfers to quotients. This can be achieved by extending the hypergraph $\tilde{H\Gamma}_n$ appropriately. The canonical generators of the corresponding hypergraph C^*-algebra $C^*(\tilde{H\Gamma}_n)$ are in the following given by $\{t_f\} \cup \{q_{v_1}, \dots, q_{v_n}\}$.

Proposition 3.19. *Let $n \geqslant 3$. Let $H\Gamma$ be the hypergraph defined by $E^0 = \{w, v_1, \dots, v_n\}$ and $E^1 = \{e, f\}$ with*

$$\begin{aligned} s(e) &= \{w\}, & r(e) &= \{v_n\}, \\ s(f) &= \{v_1, \dots, v_n\}, & r(f) &= \{v_1, \dots, v_n\}. \end{aligned}$$

Then $C^*(H\Gamma)$ is non-nuclear.

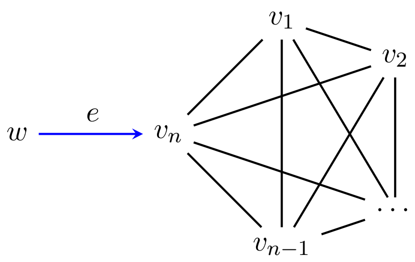


Figure 14: Non-amenable hypergraph of Proposition 3.19.

Proof. The idea of the proof relies on the fact that quotients of nuclear C^*-algebra are again nuclear, see Proposition 3.5. Thus, by proving that the hypergraph C^*-algebra has a non-nuclear quotient, the hypergraph C^*-algebra itself must be non-nuclear. We define a Cuntz-Krieger $H\Gamma$-family in $C^*(\tilde{H\Gamma}_{n-1})$ by

$$\begin{aligned} P_w &:= 0, \\ P_{v_n} &:= 0, \\ P_{v_i} &:= q_{v_i} \qquad \text{for } i \leqslant n-1, \\ S_e &:= 0, \\ S_f &:= t_f. \end{aligned}$$

Indeed, for the first hypergraph relation we have

$$\begin{aligned} &S_e^* S_e = 0 = P_{v_n}, \\ &S_f^* S_f = t_f^* t_f = \sum_{i=1}^{n-1} q_{v_i} = \sum_{i=1}^{n-1}, P_{v_i} = \sum_{i=1}^{n-1} P_{v_i} + P_{v_n} = P_{r(f)}. \end{aligned}$$

For the second hypergraph relation it follows that

$$\begin{aligned} &S_e S_e^* = 0 = P_w, \\ &S_f S_f^* = t_f t_f^* \geqslant \sum_{i=1}^{n-1} q_{v_i} = \sum_{i=1}^{n-1} P_{v_i} = \sum_{i=1}^{n-1} P_{v_i} + P_{v_n} = P_{s(f)}. \end{aligned}$$

And finally the third hypergraph relation follows since

$$P_w = 0 = S_e S_e^*,$$
$$P_{v_n} = 0 \leqslant S_f S_f^*$$
$$P_{v_i} = q_{v_i} \leqslant t_f t_f^* = S_f S_f^*.$$

This leads to a *-homomorphism $\pi : C^*(H\Gamma) \to C^*(\tilde{H\Gamma}_{n-1})$ that maps the canonical generators $p_{v_i} \mapsto P_{v_i}$ for $i \leqslant n-1$, $s_f \mapsto S_f$ and sends the other generators to 0. The *-homomorphism is surjective, since all generators of $C^*(\tilde{H\Gamma}_{n-1})$ lie in the range. Thus we get that $C^*(H\Gamma)/Ker(\pi) \cong C^*(\tilde{H\Gamma}_{n-1})$. By Proposition 3.13 we know that the C^*-algebra $C^*(\tilde{H\Gamma}_{n-1})$ is non nuclear. Thus, $C^*(H\Gamma)$ has a non-nuclear quotient and is thus also non-nuclear. □

Remark 3.20. *If we consider the above example for the case $n = 2$, the previous procedure does not lead to a non-nuclear quotient. Indeed, the quotient is given as $C(\mathbb{T})$ which is known to be nuclear. Thus we cannot say something about the nuclearity of the hypergraph $H\Gamma$. It could be interesting to investigate this further.*

Example 3.21. We can extend the above example by adding further edges to the hypergraph in Proposition 3.13 in the same manner as in the previous example. In the following, we have $n \geqslant 4$.

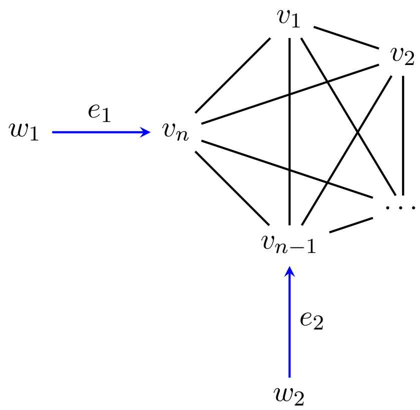


Figure 15: Non-amenable hypergraph of Example 3.21.

The only restriction is, that the remaining quotient must still be a non-nuclear C^*-algebra. In our setting this is the case, if at most $n-2$ vertices are connected to a source. Thus for $m \leqslant n-2$ the following hypergraph is non-amenable and the corresponding hypergraph algebra is thus non-nuclear:

$$E^0 := \{w_1, \dots, w_m, v_1, \dots v_n\},$$
$$E^1 := \{e_1, \dots, e_m, f\},$$
$$s(e_i) := \{w_i\}, \qquad r(e_i) = \{v_{n-i+1}\},$$
$$s(f) := \{v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\}.$$

Sending all partial isometries s_{e_i} and projections corresponding to their source or range to zero, and sending $p_{v_i} \mapsto q_{v_i}$ for $i \leqslant n-m$ and $s_f \mapsto t_f$ we get a surjective *-homomorphism $\pi : C^*(H\Gamma) \to C^*(\tilde{H\Gamma}_{n-m})$ and hence that $C^*(H\Gamma)/Ker(\pi) \cong C^*(\tilde{H\Gamma}_{n-m})$. Thus $C^*(H\Gamma)$ is non-nuclear.

There are multiple ways to create further non-nuclear hypergraph C^*-algebras with the above technique. We could add multiple vertices to the source/range of e, attache more edges to the hypergraph, and so on. The main idea would be always to set the partial isometry corresponding to the new edge equal to zero and use the hypergraph relations to determine which projections must be zero. Then we consider the hypergraph $\tilde{H\Gamma}_m$ with edge f, were we delete all vertices, whose projection is zero, from the range and source. The resulting Cuntz-Krieger family leads to a surjective *-homomorphism which then leads to an isomorphism between the quotient and the non-nuclear C^*-algebra $C^*(\tilde{H\Gamma}_m)$. We added a bunch of examples in Appendix C.

Nevertheless, the above illustrations are somewhat misleading. The examples originate as manipulations of the non-amenable hypergraph. Thus, at first glance, the non-nuclear subhypergraph $\tilde{H\Gamma}_n$ seems to be decisive. But in fact the quotient given by $C^*(\tilde{H\Gamma}_m)$ for some $m < n$ is crucial.

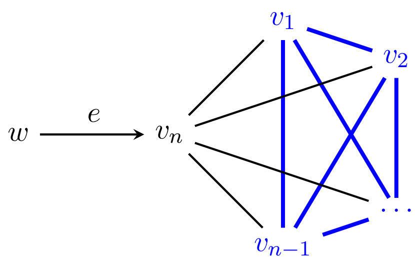


Figure 16: Visualization of the part of the hypergraph in Proposition 3.19 corresponding to the non-nuclear quotient. It is highlighted in blue with bold edges.

To get a better graphical understanding we express the above technique by concrete requirements on the hypergraph. The idea is to extract a non-amenable part of the hypergraph with slight modifications on the edges by deleting vertices from its source and range. This gives an easy way to check non-amenability for a given hypergraph without using the corresponding C^*-algebra.

Proposition 3.22. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. If there exist $N^0 \subsetneq E^0$ and $N^1 \subsetneq E^1$ such that $N^0 \cap r(e) \neq \emptyset$ and $N^0 \cap s(e) \neq \emptyset$ holds if and only if $e \in N^1$ and the hypergraph defined by $\tilde{H\Gamma} = (N^0, N^1, r_N, s_N)$ with*

$$r_N(e) := r(e) \cap N^0,$$
$$s_N(e) := s(e) \cap N^0$$

is non-amenable, then $H\Gamma$ is non-amenable.

Proof. We show that $\tilde{H\Gamma}$ is a quotient of $C^*(H\Gamma)$. We can define a Cuntz-Krieger $H\Gamma$-family in $C^*(\tilde{H\Gamma})$ by

$$P_v := \begin{cases} q_v & \text{for } v \in N^0 \\ 0 & \text{for } v \in E^0 \setminus N^0, \end{cases}$$
$$S_e := \begin{cases} t_e & \text{for } e \in N^1 \\ 0 & \text{for} e \in E^1 \setminus N^1. \end{cases}$$

Since all edges whose sources and sinks intersect with N^0 lie in N^1, the hypergraph relations of the Cuntz-Krieger family directly follow from the hypergraph relations in $C^*(\tilde{H\Gamma})$. Indeed, as the projections and partial isometries corresponding to vertices and edges not in $\tilde{H\Gamma}$ are 0, they can be added without changing any relations. By the universal property we get *-homomorphism π from $C^*(H\Gamma)$ onto $C^*(\tilde{H\Gamma})$ which is surjective as all generators of $C^*(\tilde{H\Gamma})$ lie in its range. Hence $C^*(H\Gamma)/Ker(\pi) \cong C^*(\tilde{H\Gamma})$. Thus $C^*(H\Gamma)$ contains a non-nuclear quotient and is hence non-nuclear. In other words, $H\Gamma$ is non-amenable. □

Example 3.23. We apply the above proposition to the hypergraph defined by vertices $\{v_1, \ldots, v_n\}$ and edges e, f, g with $s(e) = \{v_5\}$, $r(e) = \{v_5\}$, $s(f) = \{v_4\}$, $r(f) = \{v_5\}$, $s(g) = \{v_1, v_2, v_3\}$, $r(g) = \{v_1, v_2, v_3, v_4\}$.

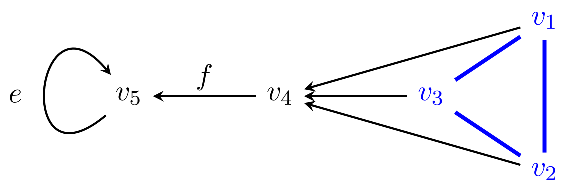


Figure 17: Visualization of the hypergraph in Example 3.23. The non-amenable part of the hypergraph is highlighted in blue with bold edges.

By defining $N^0 := \{v_1, v_2, v_3\}$ and $N^1 = \{f\}$ it is easy to check, that both sets fulfill the requirements of the proposition. Since $s_N(g) = r_N(g) = \{v_1, v_2, v_3\}$, the resulting hypergraph is given by $\tilde{H\Gamma}_3$, which is non-amenable. Thus the initial hypergraph is also non-amenable.

The previous constructions have the objective to check if a given hypergraphs is non-amenable by deleting and manipulating edges and vertices. The emerging question now is, how to attache a non-amenable hypergraph to an arbitrary hypergraph to receive a non-amenable hypergraph. The technique below defines some kind of product between two hypergraph C^*-algebras.

Proposition 3.24. *Let $H\Gamma = (E^0, E^1, r_\Gamma, s_\Gamma)$ and $H\Delta = (F^0, F^1, r_\Delta, s_\Delta)$ be finite hypergraphs. For fixed $f \in E^1$ and $w \in F^0$ we define a linked hypergraph $H\Theta$ by*

$$
\begin{aligned}
G^0 &:= E^0 \cup F^0, \\
G^1 &:= E^1 \cup F^1, \\
r(e) &:= \begin{cases} r_\Gamma(e) & \text{for } e \in E^1 \\ r_\Delta(e) & \text{for } e \in F^1, \end{cases} \\
s(e) &:= \begin{cases} s_\Gamma(e) & \text{for } e \in E^1 \setminus \{f\} \\ s_\Gamma(f) \cup \{w\} & \text{for } e = f \\ r_\Delta(e) & \text{for } e \in F^1. \end{cases}
\end{aligned}
$$

If $H\Gamma$ is non-amenable, then $H\Theta$ is non-amenable.

Proof. We show that $C^*(H\Gamma)$ is a quotient of $C^*(H\Theta)$. To do this we define a Cuntz-Krieger $H\Theta$-family in $C^*(H\Gamma)$ by letting all projections and partial isometries corresponding to vertices and edges in $H\Delta$ be zero and identify the elements corresponding to vertices and edges in $H\Gamma$ with the generators of $C^*(H\Gamma)$. The clue is, that by letting all elements corresponding to $C^*(H\Delta)$ be zero, we "delete" the new vertex in the source and obtain $H\Gamma$. We denote the elements in the constructed Cuntz-Krieger $H\Theta$-family by T_e and Q_v. This is indeed a Cuntz-Krieger $H\Theta$-family. The crucial part is the linking edge f and the vertex w. If we consider the hypergraph relations for these we get:

$$\begin{aligned} T_f^* T_f &= s_f^* s_f = p_{r_\Gamma(f)} = p_{r(f)} = Q_{r(f)}, \\ T_f T_f^* &= s_f s_f^* \leqslant p_{s_\Gamma(f)} = p_{s_\Gamma(f)} + 0 = Q_{s_\Gamma(f)} + Q_w = Q_{s(f)}, \\ Q_w &= 0 \leqslant \sum_{e \in G^1, w \in s(e)} T_e T_e^*. \end{aligned}$$

The *-homomorphism π given by the universal property is clearly surjective, as all generators of $C^*(H\Gamma)$ are in the range. Thus we get that $C^*(H\Gamma)$ is isomorphic to $C^*(H\Theta)/Ker(\pi)$. Thus $C^*(H\Theta)$ has a non-nuclear quotient, since $H\Gamma$ is non-amenable. Hence $C^*(H\Theta)$ is non-nuclear and $H\Theta$ is non-amenable. □

In the previous proposition, we added the vertex w to the source of the edge f. Similarly we could have also added the vertex w to the range of f. In either cases the quotient deletes the further vertex in the source/range. Furthermore, we must not restrict ourselves to a single connection. Using the same idea of the proof we could extend to multiple linking edges and multiple new vertices in their sources/ranges.

Example 3.25. We consider the hypergraph $\tilde{H\Gamma}_n$ and the hypergraph generating $C(\mathbb{T})$.

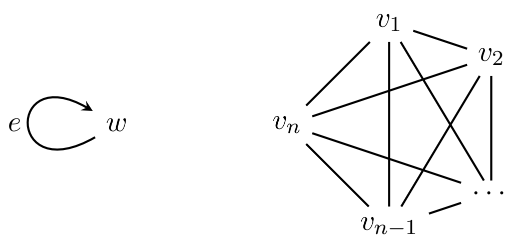


Figure 18: Hypergraph generating $C(\mathbb{T})$ and hypergraph $\tilde{H\Gamma}_n$.

We extend the edge f in $\tilde{H\Gamma}_n$ by adding the vertex w in its source.

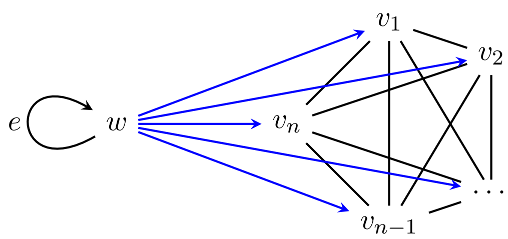


Figure 19: Non-amenable hypergraph created by linking the hypergraphs of Example 3.25.

Then by the last proposition the emerging hypergraph is non-amenable.

Besides for non-nuclearity the above technique is also a nice feature to calculate quotients in general. We wont deepen this topic as it is would go beyond the scope of this thesis and leave it for future research.

4. Connections Between Hypergraph C*-Algebras

Multiple hypergraphs can have a similar corresponding C^*-algebra. By decomposing the range of an edge, we give a concrete way to construct new hypergraphs while leaving the corresponding C^*-algebra invariant. This furthermore gives us information about the relation between graph and ultragraph C^*-algebras and shows the crucial differences of hypergraphs. Besides the connection between different hypergraph C^*-algebras we investigate the connection of hypergraph C^*-algebras to known C^*-algebras. We consider the Gauge Uniqueness Theorem in the second section as tool to identify C^*-algebras isomorphic to hypergraph C^*-algebras. We construct an example, which proves that the Gauge Uniqueness Theorem does not hold in the general case. Given this constraint, we develop restrictions on the hypergraph and its corresponding C^*-algebra under which we can generalize the Gauge Uniqueness Theorem.

4.1. Decomposition of Ranges

In [KMST10] it is shown that each ultragraph C^*-algebra is Morita equivalent to a graph algebra. In the finite case this even strengthens to isomorphisms, as we will see in the following. The key is to take the range of an edge apart and form new edges, according to each vertex in the range. The source remains unchanged. We can adapt this idea and generalize it for hypergraphs.

Theorem 4.1. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. Define the hypergraph $\tilde{H\Gamma} = (\tilde{E}^0, \tilde{E}^1, \tilde{r}, \tilde{s})$ as*

$$\begin{aligned} \tilde{E}^0 &:= E^0, \\ \tilde{E}^1 &:= \{(e, v) \mid e \in E^1,\ v \in r(e)\}, \\ \tilde{r}((e, v)) &:= v, \\ \tilde{s}((e, v)) &:= s(e). \end{aligned}$$

The corresponding hypergraph C^*-algebras are isomorphic, i.e. $C^*(H\Gamma) \cong C^*(\tilde{H\Gamma})$. In particular it holds that $\tilde{r} : \tilde{E}^1 \to \tilde{E}^0$.

Figure 20: Visualization of the decomposition of ranges. The different colors and thickness of the edges indicate different edges.

Proof. Let $\{q_v \mid v \in \tilde{E}^0\}$, $\{t_\alpha \mid \alpha \in \tilde{E}^1\}$ be the universal Cuntz-Krieger $\tilde{H\Gamma}$-family. We define

$$\begin{aligned} P_v &:= q_v & \forall v \in E^0, \\ S_e &:= \sum_{v \in r(e)} t_{(e,v)} & \forall e \in E^1. \end{aligned}$$

The elements $\{P_v \mid v \in E^0\}$ are mutually orthogonal projections and a quick calculation shows that $\{S_e \mid e \in E^1\}$ are partial isometries with mutually orthogonal ranges. Together they form a Cuntz-Krieger $H\Gamma$-family in $C^*(\tilde{H}\Gamma)$, as we see in the following.

(HR1): Using the first hypergraph relation for the universal Cuntz-Krieger $\tilde{H}\Gamma$-family we get

$$\begin{aligned} S_e^* S_f &= \sum_{v \in r(e)} t_{(e,v)}^* \sum_{w \in r(f)} t_{(f,w)} \\ &= \sum_{v \in r(e)} \sum_{w \in r(f)} t_{(e,v)}^* t_{(f,w)} \\ &= \delta_{e,f} \sum_{v \in r(e)} q_v \\ &= \delta_{e,f} \sum_{v \in r(e)} P_v. \end{aligned}$$

(HR2): Using that the ranges of (e,w) and (e,z) for distinct vertices w, z are disjoint we get using Proposition 2.12 that $t_{(e,w)}t_{(e,z)}^* = 0$ for $w \neq z$. Since the second hypergraph relation implies that $t_{(e,w)}t_{(e,w)}^* \leqslant \sum_{v \in s(e)} q_v$ for all $w \in r(e)$ we get

$$\begin{aligned} S_e S_e^* &= \sum_{w \in r(e)} t_{(e,w)} \sum_{z \in r(e)} t_{(e,z)}^* \\ &= \sum_{w \in r(e)} t_{(e,w)} t_{(e,w)}^* \\ &\leqslant \sum_{v \in s(e)} q_v \\ &= \sum_{v \in s(e)} P_v. \end{aligned}$$

(HR3): Using the third hypergraph relation for $C^*(\tilde{H}\Gamma)$ and the orthogonality of the ranges of (e,w) and (e,z) for distinct vertices w and z, we get

$$\begin{aligned} P_v &= q_v \\ &\leqslant \sum_{\alpha \in \tilde{E}^1, v \in s(\alpha)} t_\alpha t_\alpha^* \\ &= \sum_{e \in E^1, v \in s(e)} \sum_{w \in r(e)} t_{(e,w)} t_{(e,w)}^* \\ &= \sum_{e \in E^1, v \in s(e)} \sum_{w \in r(e)} t_{(e,w)} \sum_{z \in r(e)} t_{(e,z)}^* \\ &= \sum_{e \in E^1, v \in s(e)} S_e S_e^*. \end{aligned}$$

Hence all hypergraph relations are fulfilled and we thus get a *-homomorphism $\phi : C^*(H\Gamma) \to C^*(\tilde{H}\Gamma)$ which maps the canonical generators $s_e \mapsto S_e$ and $p_v \mapsto P_v$.

To construct the inverse map we define the elements

$$\begin{aligned} Q_v &:= p_v && \forall v \in \tilde{E}^0, \\ T_{(e,v)} &:= s_e p_v && \forall (e,v) \in \tilde{E}^1. \end{aligned}$$

Clearly, the elements Q_v are mutually orthogonal projections and a short calculation confirms that $T_{(e,v)}$ is a partial isometry for each $(e,v) \in \tilde{E}^1$. By construction these elements have orthogonal ranges. We check that these elements form a Cuntz-Krieger $\widetilde{H\Gamma}$-family in $C^*(H\Gamma)$.

(HR1): The first hypergraph relation of $C^*(H\Gamma)$ yields

$$\begin{aligned} T^*_{(e,v)} T_{(f,w)} &= p_v s_e^* s_f p_w \\ &= \delta_{e,f} p_v s_e^* s_e p_w \\ &= \delta_{e,f} p_v \Big(\sum_{z \in r(e)} p_z \Big) p_w \\ &= \delta_{e,f} \delta_{v,w} p_v \\ &= \delta_{(e,v),(f,w)} Q_v. \end{aligned}$$

(HR2): Using the definition of partial isometries and the order relation of projections we get by applying the second hypergraph relation of $C^*(H\Gamma)$

$$\begin{aligned} T_{(e,v)} T^*_{(e,v)} &= s_e p_v p_v s_e^* \\ &\leqslant s_e s_e^* \\ &\leqslant \sum_{w \in s(e)} p_v \\ &= \sum_{w \in s((e,v)))} Q_v. \end{aligned}$$

(HR3): We recall, that $p_{r(e)} = \sum_{v \in r(e)} p_v$. With this we get by the third hypergraph relation of $C^*(H\Gamma)$

$$\begin{aligned} Q_v &= p_v \\ &\leqslant \sum_{e \in E^1, v \in s(e)} s_e s_e^* \\ &\leqslant \sum_{e \in E^1, v \in s(e)} s_e p_{r(e)} s_e^* \\ &\leqslant \sum_{e \in E^1, v \in s(e)} \sum_{v \in r(e)} s_e p_v s_e^* \\ &= \sum_{e \in E^1, v \in s(e)} \sum_{v \in r(e)} T_{(e,v)} T^*_{(e,v)} \\ &= \sum_{\alpha \in \tilde{E}^1, v \in s(\alpha)} T_\alpha T_\alpha^*. \end{aligned}$$

The universal property gives us thus the *-homomorphism $\psi : C^*(\tilde{H\Gamma}) \to C^*(H\Gamma)$ which maps the canonical generators $t_e \mapsto T_e$ and $q_v \mapsto Q_v$. One can easily check, that the *-homomorphisms ϕ and ψ are inverse to each other which yields the claim. □

Instead of a complete decomposition of the range into its single vertices we could have also disassembled it into a disjoint union of nonempty sets, i.e. $r(e) = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n$ and associate to each set $\mathcal{E}_j$ the edge $(e, \mathcal{E}_j)$.

Corollary 4.2. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. For each $e \in E^1$ let $r(e) = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{n_e}$ for nonempty disjoint sets $\mathcal{E}_j$ and $n_e \in \mathbb{N}$. Define the hypergraph $\tilde{H\Gamma} = (\tilde{E}^0, \tilde{E}^1, \tilde{r}, \tilde{s})$ as*

$$\begin{aligned}\tilde{E}^0 &:= E^0,\\ \tilde{E}^1 &:= \{(e, \mathcal{E}_j) \mid e \in E^1,\ j = 1, \ldots, n_e\},\\ \tilde{r}((e, \mathcal{E}_j)) &:= \mathcal{E}_j,\\ \tilde{s}((e, \mathcal{E}_j))) &:= s(e).\end{aligned}$$

The corresponding hypergraph algebras are isomorphic, i.e. $C^*(H\Gamma) \cong C^*(\tilde{H\Gamma})$.

Remark 4.3. *Sadly we only get the decomposition for ranges. The same approach for sources is not possible. For example the element $p_v s_e$ for a vertex $v \in s(e)$ is in general no partial isometry. Furthermore, this is also clear since otherwise all C^*-algebras of finite hypergraphs would be isomorphic to graph C^*-algebras, contradicting the existence of non-amenable hypergraphs and the fact that nuclearity is an invariant under isomorphism.*

Using the previous Theorem we can now show, that in the finite case, ultragraph algebras are isomorphic to graph C^*-algebras and do thus not extend the class of graph algebras. For the infinite case we refer to [KMST10, Thm. 5.22] where it is shown, that, up to Morita equivalence, ultragraph C^*-algebras and graph C^*-algebras are the same.

Corollary 4.4. *The C^*-algebra of a finite ultragraph is isomorphic to a graph C^*-algebra.*

Proof. We apply Theorem 4.1 to the ultragraph to receive a graph whose C^*-algebra is isomorphic to the C^*-algebra corresponding to the ultragraph. □

Example 4.5. Using the decomposition of the range, one can give an alternative proof, that the C^*-algebra of the hypergraph defined by $s(e) = w$, $r(e) = \{v, w\}$ is the Toeplitz algebra. A straight forward calculation using the Cuntz-Krieger families is given in [Zen21, Prop. 3.9]. Applying Theorem 4.1 instead, leads to the graph on the right.

Figure 21: Decomposition of ranges applied to the hypergraph generating the Toeplitz algebra.

The corresponding graph algebra of the right graph is the Toeplitz algebra and as we get an isomorphism between both hypergraph C^*-algebras by Theorem 4.1, the same holds true for the hypergraph algebra of the initial hypergraph.

Instead of taking the range apart we can reverse the above construction to merge edges with similar sources and disjoint ranges. We state the corollary in case of two edges, but it can be directly generalized to any finite number of edges by iteration.

Corollary 4.6. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. Consider $e, f \in E^1$ with $s(e) = s(f)$ and $r(e) \cap r(f) = \emptyset$. The hypergraph $\tilde{H\Gamma}$ given by*

$$\begin{aligned}
\tilde{E}^0 &:= E^0, \\
\tilde{E}^1 &:= (E^1 \setminus \{e, f\}) \cup g, \\
\tilde{s}(h) &:= s(h) \quad \forall h \in E^1 \setminus \{e, f\}, \qquad \tilde{s}(g) := s(e), \\
\tilde{r}(h) &:= r(h) \quad \forall h \in E^1 \setminus \{e, f\}, \qquad \tilde{r}(g) := r(e) \cup r(f)
\end{aligned}$$

generates an isomorphic hypergraph C^*-algebra.

Proof. We apply Proposition 4.1 to $H\Gamma$ and $\tilde{H\Gamma}$. Both yield the same hypergraph, which gives the required isomorphism. □

This Corollary is interesting, as it gives us a concrete way to construct a hypergraph out of a graph without changing the corresponding C^*-algebra.

Example 4.7. We consider the matrix algebra $M_n(\mathbb{C})$. As stated in Example 1.17 the C^*-algebra of the graph with $s(e_j) = v$ and $r(e_j) = w_j$ for $j = 1, \dots, n-1$ is isomorphic to $M_n(\mathbb{C})$. Applying the above corollary we get, that the hypergraph defined by one edge e with $s(e) = \{v\}$ and $r(e) = \{w_1, \dots, w_{n-1}\}$ also generates the matrix algebra $M_n(\mathbb{C})$.

Figure 22: Merging of ranges applied to the hypergraph generating $M_n(\mathbb{C})$.

Besides of new examples, the decomposition and merge of ranges yields interesting insights about the parts in which hypergraph C^*-algebras show a different behaviour compared to graph and ultragraph C^*-algebras. Since ranges can be decomposed, the crucial differences seem to occur solely by multiple vertices in the source. The connection to the order relation seen in Lemma 2.26 and Lemma 2.28 could be interesting to investigate further.

4.2. Gauge Uniqueness Theorem

Hypergraph C^*-algebras are universal C^*-algebras. Thus, one possibility to identify a C^*-algebra isomorphic to the hypergraph C^*-algebra, is to check that it fulfills the universal property. Nevertheless, this can be a bit nasty. To expand our toolbox, we take a closer look at the Gauge Uniqueness Theorem, which yields faithful representations of graph C^*-algebras. It is probably one of the most used theorems in the context of graph C^*-algebras because of its simple application. Under specific assumptions on the hypergraph, we can extend it. But we will see, that it does not completely generalizes to hypergraphs.

The basis of the Gauge Uniqueness Theorem for graph algebras is the existence of a continuous action of the unit circle $\mathbb{T}$ on the graph C^*-algebra, which leaves the projections invariant and rotates the partial isometries. This action is called *gauge action* and the existence of such an action for graph C^*-algebras follows by the universal property. Adapting the proof we get the existence of the gauge action for hypergraphs.

Proposition 4.8. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph with universal Cuntz-Krieger $H\Gamma$-family $\{s_e, p_v\}$. Then there exists a continuous action of $\mathbb{T}$ on $C^*(H\Gamma)$ such that*

$$\gamma_z(s_e) = zs_e \quad \forall e \in E^1, \qquad \gamma_z(p_v) = p_v \quad \forall v \in E^0.$$

The action is called gauge action*.*

Proof. The proof is similar as for graphs [Rae05, Prop. 2.1]. For fixed $z \in \mathbb{T}$ the collection $\{zs_e, p_v\}$ is again a Cuntz-Krieger $H\Gamma$-family. Thus, the universal property of hypergraph C^*-algebras leads to the existence of an isomorphism $\gamma_z : (C^*(E), \{s_e, p_v\}) \to (C^*(E), \{zs_e, p_v\})$ which maps s_e to zs_e and p_v to p_v. On generators, $\gamma_w \circ \gamma_z$ agrees with γ_{wz} for $z, w \in \mathbb{T}$. Thus they agree on all of $C^*(H\Gamma)$ and γ is a *-homomorphism. We still need to show continuity of $z \mapsto \gamma_z(a)$ for all, but fixed $a \in C^*(H\Gamma)$. The argument again follows as in [Rae05, Prop. 2.1] using an $\epsilon/3$-argument, with the exception, that we now choose elements $c = \sum \lambda_{\mu_1 \ldots \mu_n, \epsilon_1, \ldots, \epsilon_n} s_{\mu_1}^{\epsilon_1} \ldots s_{\mu_n}^{\epsilon_n}$ for each fixed element $a \in C^*(H\Gamma)$ and use, that scalar multiplication is continuous and the composition of continuous functions remain continuous to conclude that $w \mapsto \gamma_w(c)$ is continuous. □

In the context of graph C^*-algebras, the existence of such an action yields a faithful representation of the graph C^*-algebra. We state this in the following Theorem.

Theorem 4.9 (Gauge Uniqueness Theorem). *[BHRS02, Thm. 2.1] Let E be an arbitrary graph, let $\{S_e, P_v\}$ be a Cuntz-Krieger E-family in a C^*-Algebra A, and let π be the representation of $C^*(E)$ such that $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$. Suppose that each P_v is non-zero, and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(S_e, P_v)$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for $z \in \mathbb{T}$. Then π is faithful.*

Proof Sketch. The idea of the proof of the Gauge Uniqueness Theorem is to project the graph C^*-algebra $C^*(E)$ onto the fixed point algebra

$$C^*(E)^\gamma := \{a \in C^*(E) \mid \gamma_z(a) = a \text{ for all } z \in \mathbb{T}\}$$

by averaging over the gauge action. Then since

$$||\int_{\mathbb{T}} \gamma_z(a)dz|| \leqslant ||\pi(a)|| \qquad \text{for all } a \in C^*(E),$$

faithfulness of the *-homomorphism on $C^*(E)^\gamma$ leads to faithfulness on $C^*(E)$. The key point is the faithfulness on the fixed point algebra. For graphs, $C^*(E)^\gamma$ is an AF-Algebra: by Lemma 1.22 the elements $s_\mu^* s_\nu$ for $|\mu| = |\nu| = k$ and $r(\mu) = r(\nu) = v$ are matrix units and with this, one gets that the graph C^*-algebra is the inductive limit of C^*-algebras of compact operators. Using this, and the fact that every matrix unit has non-zero image under the representation π we get the required faithfulness. □

Remark 4.10. *For hypergraphs, we cannot construct these matrix units, since the elements to not cancel as nicely as for graphs, see Corollary 2.23. Hence we cannot imitate this proof directly.*

In general, the Gauge Uniqueness Theorem does not apply to hypergraphs, as we see in the next example.

Example 4.11. Recalling the expression of the free product and the tensor product of two C^*-algebras A and B as universal C^*-Algebra by

$$A * B := C^*(a \in A, b \in B \mid R_A,\ R_B),$$
$$A \otimes B := C^*(a \in A,\ b \in B \mid R_A,\ R_B,\ ab = ba\ \forall a \in A,\ b \in B),$$

we directly get by the universal property, that there is always a surjective *-homomorphism of the free product C^*-algebra onto the tensor product C^*-algebra. We consider now the hypergraph $\tilde{H\Gamma}_n$ defined by $s(e) = r(e) = \{v_1, \dots, v_n\}$. The corresponding hypergraph C^*-Algebra is given by $C^*(\tilde{H\Gamma}_n) \cong C(\mathbb{T}) * \mathbb{C}^n$, see Proposition 3.13. Let $\{S_e, P_v\}$ be the Cuntz-Krieger $\tilde{H\Gamma}_n$-family in $C(\mathbb{T}) * \mathbb{C}^n$ given by the isomorphism. Using the observation above we get a surjective *-homomorphism

$$\pi : C^*(\tilde{H\Gamma}_n) \cong C(\mathbb{T}) * \mathbb{C}^n \to C(\mathbb{T}) \otimes \mathbb{C}^n$$

defined by $s_e \mapsto S_e$, $p_{v_i} \mapsto P_{v_i}$. The tensor product $C(\mathbb{T}) \otimes \mathbb{C}^n$ has a gauge action $\tilde{\gamma}$ with the property, that $\pi \circ \gamma = \tilde{\gamma} \circ \pi$, while γ is the gauge action on $C^*(\tilde{H\Gamma}_n)$. Thus, the preliminaries for the gauge uniqueness theorem would be given. Nevertheless, the *-homomorphism π cannot be injective: $C(\mathbb{T}) * \mathbb{C}^n$ is not nuclear as seen in Proposition 3.14. On the other hand, since $C(\mathbb{T})$ and $\mathbb{C}^n$ are both nuclear, we get that the tensor product $C(\mathbb{T}) \otimes \mathbb{C}^n$ is nuclear [Bla06, IV. 3.1.1]. Nuclearity is an invariant under isomorphism, and since we saw above, that π is surjective it cannot be injective.

Remark 4.12. *We have a closer look at the structure of the hypergraph in the counterexample. The hypergraph $\tilde{H\Gamma}_n$ has only perfect paths. The corresponding hypergraph C^*-algebra is not generated by the canonical partial isometry s_e, as $\overline{span}(s_e) = \{1, s_e^n, (s_e^*)^n \mid n \in \mathbb{N}\}$ does not contain the canonical projection p_v. On the other hand, we can manipulate this hypergraph by decomposing the range of the edge e to construct a hypergraph whose C^*-algebra is generated by the canonical partial isometries but which now has partial paths instead of (quasi) perfect paths.*

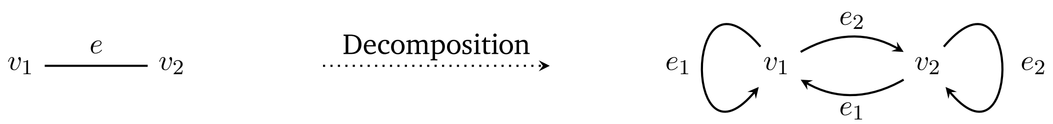


Figure 23: Decomposition of ranges applied to $\tilde{H\Gamma}_2$.

This already hints, what we will see later: It is possible to prove gauge uniqueness if both is give, i.e. the hypergraph has only (quasi) perfect paths and the corresponding C^*-algebra is generated by the partial isometries.

Keeping this remark in mind we try to define specific conditions under which we can extend the Gauge Uniqueness Theorem to hypergraphs. The main idea of the proof is based on an isomorphism between the C^*-algebras of the graph and the dual graph as shown in [Rae05, Ex. 2.7] to make use of the Gauge Uniqueness Theorem for graphs.

Definition 4.13. Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. The *dual graph* $\tilde{\Gamma}$ is defined as

$$\begin{aligned}
\tilde{E^0} &:= \{e \mid e \in E^1\}, \\
\tilde{E^1} &:= \{(e, f) \mid e, f \in E^1,\ s(f) \cap r(e) \neq \emptyset\}, \\
\tilde{s}((e, f)) &:= e, \\
\tilde{r}((e, f)) &:= f.
\end{aligned}$$

Example 4.14. To given an example of the dual graph we consider the following hypergraph and its dual.

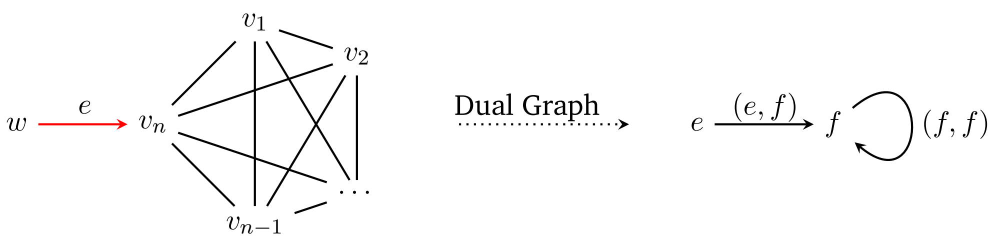


Figure 24: Dual Graph of the hypergraph of Example 3.19.

Remark 4.15. *The dual graph is really a graph, not a hypergraph anymore. Hence it will not be possible to recover the result of graph algebras, where the graph and the dual graph generated isomorphic C^*-Algebras if the graph has no sinks. This would not be in line with the non-nuclear examples of hypergraph-algebras. Nevertheless, it could be interesting to investigate if there are other connections between the hypergraph and its dual. Do hypergraphs with similar dual graphs share any properties? Are all non-nuclear hypergraph C^*-algebras isomorphic to the C^*-algebra of its dual graph?*

Lemma 4.16. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph with only quasi perfect paths and no sinks. Then it holds for all $e \in E^1$*

$$p_{r(e)} = \sum_{f \in E^1,\ s(f) \subseteq r(e)} s_f s_f^*.$$

Proof. This is a direct consequence of Lemma 2.27, since by definition of quasi perfect paths we have $s(f) \cap r(e)$ implies $s(f) \subseteq r(e)$ for all $e, f \in E^1$. □

Proposition 4.17. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph with only quasi perfect paths and no sinks. $\tilde{\Gamma}$ be its dual graph. Then $C^*(\tilde{\Gamma})$ is canonically isomorphic to the C^*-subalgebra of $C^*(H\Gamma)$ generated by $\{s_e \mid e \in E^1\}$.*

Proof. We define a Cuntz-Krieger $\tilde{\Gamma}$-family in $C^*(H\Gamma)$ by

$$\begin{aligned}
Q_e &:= s_e s_e^* & \forall e \in \tilde{\Gamma}^0, \\
T_{(e,f)} &:= s_e Q_f & \forall (e, f) \in \tilde{\Gamma}^1.
\end{aligned}$$

To check that this is a Cuntz-Krieger $\tilde{\Gamma}$-family, a short calculation using the hypergraph relations of $C^*(H\Gamma)$ shows, that the elements Q_e are mutually orthogonal projections and the elements $T_{(e,f)}$ are partial isometries.

It remains to check the Cuntz-Krieger relations for graphs, since the dual graph is a graph. Since the hypergraph has only quasi perfect paths, all paths ef fulfill $s(f) \subseteq r(e)$ which implies by Lemma 2.12 and the definition of Q_f, that $Q_f p_{r(e)} = Q_f$. With this we get the first Cuntz-Krieger relation:

$$\begin{aligned}
T^*_{(e,f)} T_{(g,h)} &= (s_e Q_f)^* (s_g Q_h) \\
&= Q_f s_e^* s_g Q_h \\
&= \delta_{e,g} Q_f p_{r(e)} Q_h \\
&= \delta_{e,g} Q_f Q_h \\
&= \delta_{e,g} \delta_{f,h} Q_f \\
&= \delta_{(e,f),(g,h)} Q_{r((e,f))}.
\end{aligned}$$

For the second Cuntz-Krieger relation we need, that for quasi perfect paths with no sinks Lemma 4.16 applies and we get:

$$\begin{aligned}
Q_e &= s_e s_e^* \\
&= s_e p_{r(e)} s_e^* \\
&= s_e \left(\sum_{f \in E^1, s(f) \subseteq r(e)} s_f s_f^* \right) s_e^* \\
&= s_e \left(\sum_{f \in E^1, s(f) \subseteq r(e)} Q_f \right) s_e^* \\
&= \sum_{f \in E^1, s(f) \subseteq r(e)} s_e Q_f s_e^* \\
&= \sum_{f \in E^1, s(f) \subseteq r(e)} T_{(e,f)} T^*_{(e,f)} \\
&= \sum_{x \in \tilde{\Gamma}^1, s(x) = e} T_x T_x^*.
\end{aligned}$$

Thus, by the universal property, we get the canonical *-homomorphism $\pi : C^*(\tilde{\Gamma}) \to C^*(H\Gamma)$ defined by $q_e \mapsto Q_e$ and $t_{(e,f)} \mapsto T_{(e,f)}$. Since the dual graph is really a graph, we are back in the familiar area and can use the Gauge Uniqueness Theorem 4.9. Let γ and $\tilde{\gamma}$ be the gauge action on $C^*(H\Gamma)$ and $C^*(\tilde{\Gamma})$ respectively. Then a short calculation shows, that $\pi \circ \tilde{\gamma}_z = \gamma_z \circ \pi$ for all $z \in \mathbb{T}$. Thus the requirements for the Gauge Uniqueness Theorem are given (note that all projections are non-zero) and the *-homomorphism π is injective. By definition of the Cuntz-Krieger family $\{T_{(e,f)}, Q_f\}$ we know that $Im(\pi) \subseteq C^*(s_e \mid e \in E^1)$. Using again Lemma 4.16 we get

$$s_e = s_e p_{r(e)} = s_e \sum_{f \in E^1, s(f) \subseteq r(e)} s_f s_f^* = \sum_{f \in E^1, s(f) \subseteq r(e)} s_e Q_f = \sum_{f \in E^1, s(f) \subseteq r(e)} T_{(e,f)}.$$

Hence the Cuntz-Krieger family $\{T_{(e,f)}, Q_f\}$ generates $\{s_e \mid e \in E^1\}$. Thus, $Im(\pi) = C^*(s_e \mid e \in E^1)$ and π is an isomorphism between $C^*(\tilde{\Gamma})$ and $C^*(s_e \mid e \in E^1)$. □

For specific hypergraphs we thus get an isomorphism between the hypergraph C^*-algebra and the graph C^*-algebra of its dual graph.

Corollary 4.18. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph with only quasi perfect paths, no sinks and $C^*(H\Gamma)$ be generated by $\{s_e \mid e \in E^1\}$. $\tilde{\Gamma}$ be its dual graph. Then $C^*(\tilde{\Gamma}) \cong C^*(H\Gamma)$.*

Remark 4.19. *As stated in Remark 4.15, the isomorphism of $C^*(\tilde{\Gamma})$ and $C^*(H\Gamma)$ cannot hold for non-amenable hypergraphs. Thus, there cannot be a non-amenable hypergraph with only quasi perfect paths and no sinks such that $C^*(H\Gamma)$ be generated by $\{s_e \mid e \in E^1\}$.*

Theorem 4.20. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph with only quasi perfect paths, no sinks and $C^*(H\Gamma)$ be generated by $\{s_e \mid e \in E^1\}$. Let $\{P_v, S_e\}$ be a Cuntz-Krieger $H\Gamma$-family in a C^*-algebra B with each $P_v \neq 0$. If there is a continuous action $\beta : \mathbb{T} \to Aut(B)$ such that the gauge action γ commutes with the canonical *-homomorphism $\pi : C^*(H\Gamma) \to B$, i.e. $\pi \circ \gamma_z = \beta_z \circ \pi$ for all $z \in \mathbb{T}$. Then π is faithful.*

Proof. Since $C^*(H\Gamma)$ is generated by $\{s_e \mid e \in E^1\}$, we get using Proposition 4.17, that $C^*(\tilde{\Gamma})$ is isomorphic to $C^*(H\Gamma)$ by an isomorphism ϕ. Furthermore, the *-homomorphism $\pi \circ \phi$ generates a Cuntz-Krieger $\tilde{\Gamma}$-family in B with nonzero projections. Since the canonical isomorphism ϕ is equivariant for the gauge actions and $\pi \circ \gamma_z = \beta_z \circ \pi$ by assumption, we get:

$$(\pi \circ \phi) \circ \gamma_z = \beta_z \circ (\pi \circ \phi) \qquad \forall z \in \mathbb{T}.$$

Applying the Gauge Uniqueness Theorem for graphs we get, that $\pi\circ\phi$ is faithful and since ϕ is an isomorphism, we get that π is faithful. □

We gave some negative results for which Theorem 4.20 cannot be applied and in which gauge uniqueness is not valid. Lets turn to the cases in which it is valid.

Corollary 4.21. *Let $\mathcal{G}$ be an ultragraph without sinks. Then the Gauge Uniqueness Theorem is valid.*

Proof. We check that ultragraphs fulfill the requirements of Theorem 4.20. Since the source of each edge in an ultragraphs is given by one vertex, combining the second and third hypergraph relations yield $p_v = \sum_{e \in E^1, v = s(e)} s_e s_e^*$. Since all vertices emit at least one edge, this equality is valid for all vertices and hence, the C^*-algebra is generated by the partial isometries. Furthermore, each path is automatically quasi perfect, as $r(e) \cap s(f) \neq 0$ implies that $r(e) \cap s(f) = s(f)$, as $s(f)$ consists of just one vertex. □

Without changing the C^*-algebra we can transform each hypergraph into a hypergraph whose partial isometries generate the corresponding hypergraph C^*-algebra. However, the method below leads to hypergraphs with partial paths in general.

Lemma 4.22. *For each C^*-algebra $C^*(H\Gamma)$ corresponding to a finite hypergraph $H\Gamma = (E^0, E^1, r, s)$ without sinks, exists a hypergraph $\tilde{H\Gamma} = (\tilde{E}^0, \tilde{E}^1, \tilde{r}, \tilde{s})$ such that $C^*(H\Gamma) \cong C^*(\tilde{H\Gamma})$ and $C^*(H\Gamma)$ is generated by the canonical partial isometries $\{t_{\tilde{e}} \mid x \in \tilde{E}^1\}$ of $C^*(\tilde{H\Gamma})$.*

Proof. Let $\tilde{H\Gamma}$ be the hypergraph obtained by decomposing the range of $H\Gamma$, see Theorem 4.1. Since $H\Gamma$ has no sinks, $\tilde{H\Gamma}$ has no sinks. Hence for each vertex $v \in E^0$ there is an edge $\tilde{e} \in \tilde{E}^1$ such that $v = \tilde{r}(\tilde{e})$. The first hypergraph relation thus yields $p_v = t_{\tilde{e}}^* t_{\tilde{e}}$. □

Remark 4.23. *It is not yet clear if there exist hypergraphs which are not ultragraphs with the required properties. Or, if we can describe ultragraphs without sinks to be exactly the hypergraphs with quasi perfect paths whose C^*-algebra is generated by its canonical partial isometries. This could be interesting for further research.*

Till know, we only considered hypergraphs without sinks. For ultragraphs, the Gauge Uniqueness Theorem can be extended to ultragraphs with sinks.

Theorem 4.24. *[Tom03, Thm. 6.8] Let $\mathcal{G}$ be an ultragraph, $\{s_e, p_A\}$ the canonical generators in $C^*(\mathcal{G})$ and γ the gauge action on $C^*(\mathcal{G})$. Also let B be a C^*-algebra and $\phi : C^*(\mathcal{G}) \to B$ be a *-homomorphism for which $\phi(p_A) \neq 0$ for all nonempty $A \in \mathcal{E}^0$. If there exists a continuous action β of $\mathbb{T}$ on B such that $\beta_z \circ \phi = \phi \circ \gamma_z$ for all $z \in \mathbb{T}$, then ϕ is faithful.*

Proof Sketch. We give a rough sketch of the idea of the proof: First, one generalizes the setting to infinite ultragraphs [Tom03, Prop. 5.5]. The idea there is, to approximate the ultragraph C^*-algebra by C^*-algebras of finite graphs, for which the Gauge Uniqueness Theorem applies. The construction of these finite graphs is based on the dual graph of subgraphs, but with some technicalities to account for "boundary vertices". Then by adding a tail to each sink v_0, i.e. attaching a chain

$$v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3 \cdots\cdots\cdots\rightarrow \cdots$$

Figure 25: Adding a tail to a sink.

one can embed the ultragraph C^*-algebra into the C^*-algebra of the infinite ultragraph without sinks and use that for these the Gauge Uniqueness Theorem holds. For more details see [Tom03, Thm. 6.8] □

Remark 4.25. *Hypergraphs build a nice and straightforward generalisation of graphs and ultragraphs. Nevertheless they lack of some nice properties which are useful to work with, such as gauge uniqueness. Another generalization of ultragraphs was done in [BP07] by Bates and Pask. The idea traces back to shift spaces. The edges of a graph are labelled with elements of an alphabet. In this setting paths correspond to words in the alphabet which defines a language of the shiftspace. The representation reminds of the hypergraph construction, as the source and range maps defined on the language of the shiftspace allow multiple valued sources and ranges and two elements α, β in the language connect to a path, if the range of α and the source of β have nonempty intersection. The same holds true in the hypergraph case, as we stated in Definition 2.18. We can reformulate hypergraphs as labelled graphs: Let $H\Gamma = (E^0, E^1, r, s)$ be a hypergraph. We construct a corresponding labelled graph (L, π) as follows. The graph L is defined as*

$$\begin{aligned} L^0 &:= E^0, \\ L^1 &:= \{e_{(v,w)} \mid v \in s(e), w \in r(e)\}, \\ s_L(e_{(v,w)}) &:= v, \\ r_L(e_{(v,w)}) &:= w \end{aligned}$$

and the labelling is given by

$$\pi : L^1 \to E^1, \qquad e_{(v,w)} \mapsto e.$$

So far so good. But this labelled space is not left resolving [BP07, Definition 3.2], i.e. the restriction of the labelling π to $r_L^{-1}(w)$ is not injective for some $w \in L^0$, if we have multiple valued sources. The upcoming theory in the paper of Bates and Pask restricts to the case of left resolving labelled graphs, which thus excludes our hypergraph construction.

5. Moves on Hypergraphs

In this section we discuss basic moves to manipulate hypergraphs. These moves play an important role in the classification of graph C^*-algebras. We introduce four of these moves, adapt them to the hypergraph setting and investigate the corresponding C^*-algebras. For readers familiar with the theory of symbolic dynamics it is to be noted that the moves we consider are closely related to flow equivalence of shifts spaces [LM21]. The four basic moves are

(Move S) Removing a source,
(Move R) Reduction at a non-sink,
(Move O) Outsplitting,
(Move I) Insplitting.

The equivalence relation generated by the moves S, R, O and I is called *move equivalence* and is denoted by $\sim_{ME}$. We will see, that in the special case of graphs, move S, move R and move I preserve Morita equivalence. For move O we even get isomorphic C^*-algebras. Recalling that Morita equivalence implies stable isomorphism for C^*-algebras with approximate unit by Theorem 1.29, this essentially proves an important theorem, which we state here to emphasize the power of the moves.

Theorem 5.1. *[Sør13, Thm. 4.8] Let E and F be graphs such that $E \sim_{ME} F$. Then the graph C^*-algebras are stably isomorphic, i.e. $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$.*

Besides stable isomorphism we even get, that K-theory is an invariant under move equivalence by Theorem 1.30. Nevertheless, the shown moves alone do not completely classify the unital graph C^*-algebras. That means, there are graphs which are stably isomorphic, but they are not move equivalent. A recent breakthrough in the classification of C^*-algebras shows that one can indeed classify graph C^*-algebras by moves. In the paper [ERRS21] of Søren Eilers et al. they introduce a new move P which, together with the Cuntz splice (move C), completes the set of moves necessary to classify graph C^*-algebras up to stable isomorphism [ERRS21, Thm. 3.1]. The proof of this result is highly non-trivial and we make no attempt to completely replicate it for hypergraphs. Our purpose in the following chapter is to take first steps to understand the behavior of moves on hypergraphs and to work out the challenges that come with it. The construction of the moves is motivated by [ERRS18, Def. 2.14-2.17]. Since the upcoming proofs are often quite technical, we remind the reader that $\{s_e, p_v\}$ are the canonical generating elements of $C^*(H\Gamma)$.

5.1. Move S - Removing a Source

Definition 5.2 (Move S). Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. Let $w \in E^0$ be a source. The hypergraph $H\Gamma_S$ obtained by application of *move S* is defined as

$$E_S^0 := E^0 \setminus \{w\}, \quad E_S^1 := E^1 \setminus \{e \mid w \in s(e)\}, \quad s_S := s_{|E_S^1}, \quad r_S := r_{|E_S^1}.$$

We call $H\Gamma_S$ the hypergraph obtained by removing the source w from the hypergraph $H\Gamma$.

For graphs this move leads to Morita equivalent C^*-algebras [Sør13, Prop. 3.1]. The question now is if we still obtain Morita equivalence between $C^*(H\Gamma)$ and $C^*(H\Gamma_S)$ and thus similar K-Theories. The main idea is, that the restriction the of the Cuntz-Krieger $H\Gamma$-family is a Cuntz-Krieger $H\Gamma_S$-family.

Figure 26: Illustration of the application of move S for hypergraphs.

Remark 5.3. *As seen in the figure above, removing the source is not a "local" property anymore. By deleting the edges emerging from the source, we possibly also delete connections between other vertices. This impacts the third hypergraph relation. If there is an edge $f \in E^1$ with $w \in s(f)$, such that there exists another vertex $v \neq w$ with $v \in s(f)$ we get*

$$p_v \leqslant \sum_{e \in E^1, v \in s(e)} s_e s_e^* > \sum_{e \in E_S^1, v \in s(e)} s_e s_e^*$$

as $s_f s_f^*$ is not contained in the second sum. Thus we cannot say anything about the validity of the third hypergraph relation. To omit this, we restrict us to the special case, where for all edges $e \in E^1$ starting at the source w it holds that $w \in s(e)$ implies $w = s(e)$, i.e. the source is a one-point set. Other edges can still have multiple vertices as source. This restrictions ensures that no "edges" between vertices in E_S^0 are deleted. Hence, the hypergraph must look locally like an ultragraph.

Proposition 5.4. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph with source w such that for all $e \in E^1$ with $w \in s(e)$ it holds $w = s(e)$. $H\Gamma_S$ be the hypergraph obtained by removing the source w from $H\Gamma$. Then $\{p_v,\ s_e \mid v \in E_S^0,\ e \in E_S^1\}$ is a Cuntz-Krieger $H\Gamma_S$-family in $C^*(H\Gamma)$ and there exists a canonical *-homomorphism $\pi : C^*(H\Gamma_S) \to C^*(H\Gamma)$ sending the canonical generators $q_v \mapsto p_v$ for all $v \in E_S^0$ and $t_e \mapsto s_e$ for all $e \in E_S^1$.*

Proof. We check that $\{p_v, s_e \mid v \in E_S^0, e \in E_S^1\}$ fulfills the hypergraph relations for $H\Gamma_S$. The first two relations hold in general, even without the restriction on the source w since the move does not change anything at the corresponding edges and vertices. For the third hypergraph relation we note that by the given restriction for each $v \neq w$ it follows that $e \in E_S^1$ for each edge $e \in E^1$ with $v \in s(e)$ since $w \notin s(e)$. Thus we get by the third hypergraph relation of $C^*(H\Gamma)$

$$p_v \leqslant \sum_{e \in E^1, v \in s(e)} s_e s_e^* = \sum_{e \in E_S^1, v \in s(e)} s_e s_e^* \qquad \forall v \in E_S^0.$$

Hence the universal property yields the required *-homomorphism. □

Proposition 5.5. *The *-subalgebra $Im(\pi)$ is a full corner in $C^*(H\Gamma)$.*

Proof. We define the projection $p = \sum_{v \in E_S^0} p_v$ and claim first that $Im(\pi) = pC^*(H\Gamma)p$. Since

$$\begin{aligned} p_v &= p p_v p && \forall v \in E_S^0, \\ s_e &= \Big(\sum_{v \in s(e)} p_v\Big) s_e \Big(\sum_{v \in r(e)} p_v\Big) = p\Big(\sum_{v \in s(e)} p_v\Big) s_e \Big(\sum_{v \in r(e)} p_v\Big) p && \forall e \in E_S^1, \end{aligned}$$

the image of the canonical generators is contained in $pC^*(H\Gamma)p$. Hence, $Im(\pi) \subseteq pC^*(H\Gamma)p$.

On the other hand it holds for all paths μ in $H\Gamma$ by Proposition 2.12

- $ps_\mu = \begin{cases} 0 & \text{if } s(\mu) = w \\ s_\mu \in C^*(p_v, s_e) & \text{else,} \end{cases}$
- $ps^*_\mu = s^*_\mu$,
- $s_\mu p = s_\mu$,
- $s^*_\mu p = \begin{cases} 0 & \text{if } s(\mu) = w \\ s^*_\mu \in C^*(p_v, s_e) & \text{else.} \end{cases}$

We consider a general Element $s^{\epsilon_1}_{\mu_1} ... s^{\epsilon_n}_{\mu_n} \neq 0$ with paths μ_j in $H\Gamma$, $\epsilon_j \in \{1, *\}$ and $\epsilon_j \neq \epsilon_{j+1}$. Since w is a source, using the relations given in Proposition 2.22, we get that only the first and last isometries in $s^{\epsilon_1}_{\mu_1} ... s^{\epsilon_n}_{\mu_n}$ can correspond to edges with source w. Thus we can use the relations above and get that $ps^{\epsilon_1}_{\mu_1} ... s^{\epsilon_n}_{\mu_n} p \in span\{p_v,\ s_e | v \in E^0_S,\ e \in E^1_S\} \subseteq Im(\pi)$. Hence $pC^*(H\Gamma)p \subseteq Im(\pi)$. Combining both parts we get the claimed equality.

To show that the corner $pC^*(H\Gamma)p$ is full, let I be a closed two-sided ideal containing the corner. Thus I contains $\{p_v, s_e \mid v \in E^0_S, e \in E^1_S\}$ by definition of p and Proposition 2.12. Then we note, that for all $e \in E^1$ with $s(e) = w$ we have $p_{r(e)} \in I$ and hence $s_e = s_e p_{r(e)} \in I$. Given our special case, we get by Lemma 2.27 that $p_w = \sum_{e \in E^1, s(e)=w} s_e s^*_e$. Thus $p_w \in I$ as linear combination of elements in the ideal. Hence, I contains all generators of $C^*(H\Gamma)$ and must thus be equal to it. ☐

Corollary 5.6. *For finite graphs and ultragraphs, move S yields Morita equivalent C^*-algebras.*

Proof. In the special case of graphs and ultragraph we can apply the Gauge Uniqueness Theorem. The canonical *-homomorphism π is thus injective and the C^*-algebra $C^*(H\Gamma_S)$ is isomorphic to a full corner of $C^*(H\Gamma)$. Proposition 1.27 then implies Morita equivalence of the C^*-algebras. ☐

For general hypergraphs we could not proof injectivity of the *-homomorphism π. It is easy to deduce injectivity on $span\{s_e, p_v \mid e \in E^1_S,\ v \in E^0_S\}$, but this does not directly imply injectivity on the closure. Thus, we cannot say something about the Morita equivalence.

Remark 5.7. *We revisit the example from Remark 3.20 once again. Let $H\Gamma$ be the hypergraph with $s(e) = \{w\}$, $r(e) = \{v_2\}$ and $s(f) = r(f) = \{v_1, v_2\}$.*

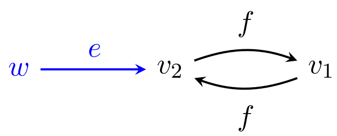


Figure 27: Hypergraph of Example 3.19 for $n = 2$. The application of move S is marked blue.

*We can link the question of nuclearity of $C^*(H\Gamma)$ to the question of injectivity of the *-homomorphism of move S. Applying move S to the hypergraph $H\Gamma$ yields the non-amenable hypergraph $H\tilde{\Gamma}_2$. Thus, if the *-homomorphism π obtained from Proposition 5.4 is injective, we get non-amenability of $H\Gamma$, since nuclearity is an invariant of*

Morita equivalence. On the other hand, if we can proof that the hypergraph $H\Gamma$ is non-amenable, we have a counterexample and know that move S does not produce Morita equivalent C^*-algebras in general.

Example 5.8. There are examples were move S yields Morita equivalent C^*-algebras even if the restriction $w \in s(e)$ implies $w = s(e)$ is not given. We consider the following application of move S. The C^*-Algebra of

Figure 28: Move S applied to the hypergraph generating the Toeplitz algebra.

the hypergraph on the left is the Toeplitz Algebra, and the C^*-Algebra of the right graph is $\mathbb{C}$. For both the K-theory is given by $K_0 = \mathbb{Z}$ and $K_1 = 0$. Hence by Theorem 1.30 both hypergraph C^*-algebras are Morita equivalent.

5.2. Move R - Reduction at a Non-Sink

Definition 5.9 (Move R). Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph. Let $w \in E^0$ be a vertex that emits exactly one edge f and only one vertex $x \neq w$ emits to w. The graph E_R obtained by application of *move R* is defined as

$$
\begin{aligned}
&E^0_R := E^0 \setminus \{w\},\\
&E^1_R := E^1 \setminus \left(r^{-1}(\{w\}) \cup \{f\}\right) \cup \left\{e_f \mid e \in E^1, r(e) = \{w\}\right\},\\
&s_R(e) = s(e), \qquad s_R(e_f) = s(e),\\
&r_R(e) = r(e), \qquad r_R(e_f) = r(e) \setminus \{w\} \cup r(f).
\end{aligned}
$$

Figure 29: Illustration of the application of move R for hypergraphs. Each color/thickness represents one edge.

Remark 5.10. *For graphs the above definition of the edge e_f just yields the path ef. The generalization to hypergraphs using paths would mean "losing" the other elements in the range of e. Thus, we modify the idea of paths to account for further vertices in the range. Nevertheless, to simplify the upcoming proofs, we restrict ourselves to the case when $w \in r(e)$ implies $w = r(e)$. In this case we have $e_f = ef$. This is not really a restriction, since we can transform any finite hypergraph into a hypergraph with this condition using the decomposition of ranges in Theorem 4.1. Then we can apply the propositions before re-transforming the hypergraphs. Note that we could have also used this method involving the decomposition of ranges to define the moves for ultragraphs based on the results from graphs.*

Similar as for move S we restrict ourselves to the special case were the edge f has no further vertices in its source other than w. In this case we can define a Cuntz-Krieger $H\Gamma_R$-family in $C^*(H\Gamma)$.

Proposition 5.11. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph with vertex $w \in E^0$ that emits exactly one edge f and only one vertex x emits to w. For all edges $e \in E^1$ with $w \in r(e)$ it holds $w = r(e)$. The elements $\{Q_v \mid v \in E^0_R\}$ and $\{T_y \mid y \in E^1_R\}$ defined as*

$$Q_v := p_v,$$
$$T_y := \begin{cases} s_e & \text{if } y = e \in E^1 \setminus \big(r^{-1}(\{w\}) \cup \{f\}\big) \\ s_{ef} & \text{if } y = e_f \in \big\{e_f \mid e \in E^1, r(e) = \{w\}\big\} \end{cases}$$

*form a Cuntz-Krieger $H\Gamma_R$-family in $C^*(H\Gamma)$. Then there exists a *-homomorphism $\pi : C^*(H\Gamma_R) \to C^*(H\Gamma)$ which maps the generators $q_v \mapsto Q_v$ for all $v \in E^0_R$ and $t_y \mapsto T_y$ for all $y \in E^1_R$.*

Proof. The elements Q_v are clearly mutually orthogonal projections and the elements T_y are partial isometries since ef are a quasi perfect paths. The first hypergraph relation for $e \in E^1_R$ follows directly from the first hypergraph relation for $H\Gamma$, since $w \notin r(e)$ implies that the range is completely contained in E^0_R. For $e_f \in E^1_R$ we have

$$T^*_{e_f} T_{e_f} = s^*_{ef} s_{ef} = s^*_f s^*_e s_e s_f = s^*_f p_{r(e)} s_f = s^*_f p_w p_{r(e)} s_f = s^*_f p_w s_f = s^*_f s_f = p_{r(f)} = Q_{r_R(e_f)}.$$

The delta-condition $T^*_y T_z = 0$ for $y \neq z$ follows directly from $s^*_e s_g = 0$ for $e \neq g$. The second hypergraph relation is again clear for $e \in E^1_R$. For $e_f \in E^1_R$ we get

$$T_{e_f} T^*_{e_f} = s_{ef} s^*_{ef} = s_e s_f s^*_f s^*_e \leqslant s_e p_{s(f)} s^*_e = s_e p_w p_{s(f)} s^*_e = s_e s^*_e \leqslant p_{s(e)} = Q_{s_R(e_f)}$$

while we used in the last step that $w \notin s(e)$ since $e \neq f$. It remains to check the third hypergraph relation. We have for all vertices $v \in E^0_R$

$$\begin{aligned} Q_v &= p_v \\ &\leqslant \sum_{e \in E^1, v \in s(e)} s_e s^*_e \\ &= \sum_{e \in E^1, v \in s(e), w \notin r(e)} s_e s^*_e + \sum_{e \in E^1, v \in s(e), w \in r(e)} s_e s^*_e. \end{aligned}$$

At this stage we need the restriction that $s(f) = w$ to get $s_f s^*_f = p_w$ and thus by Lemma 2.12 that $s_e s^*_e = s_e s_f s^*_f s_f$. With this we get

$$\begin{aligned} &= \sum_{e \in E^1, v \in s(e), w \notin r(e)} T_e T^*_e + \sum_{e \in E^1, v \in s(e), w \in r(e)} T_{e_f} T^*_{e_f} \\ &= \sum_{y \in E^1_R, v \in s_R(y)} T_y T^*_y. \end{aligned}$$

Thus, we get the required canonical *-homomorphism by the universal property. □

Again we would want to achieve Morita equivalence as in the case of graph C^*-algebras. Similar as for move S, one step in this direction is to show that the image of π is a full corner of $C^*(H\Gamma)$.

Proposition 5.12. *Let π be the canonical *-homomorphism obtained from move R. Then $Im(\pi)$ is a full corner in $C^*(H\Gamma)$.*

Proof. We define the projection $p = \sum_{v \in E_R^0} p_v$ and show that $Im(\pi) = pC^*(H\Gamma)p$. We first show that the Cuntz-Krieger $H\Gamma_R$-family in $H\Gamma$ is contained in the corner, which proves that $Im(\pi) \subseteq pC^*(H\Gamma)p$. Indeed, we have

$$\begin{aligned} Q_v &= p_v = pp_vp, \\ T_e &= s_e = p_{s(e)}s_ep_{r(e)} = pp_{s(e)}s_ep_{r(e)}p, \\ T_{ef} &= s_es_f = p_{s(e)}s_es_fp_{r(f)} = pp_{s(e)}s_es_fp_{r(f)}p, \end{aligned}$$

where we used that $w \notin s(e)$ for $e \neq f$, $w \notin r(e)$ for $e \in E^1 \setminus \left(r^{-1}(\{w\}) \cup \{f\}\right)$ and that $w \notin r(f)$ as $s(f) = w \neq x$.

To show that the corner is contained in $Im(\pi)$ we first consider some properties of the crucial edges in $r^{-1}(\{w\}) \cup \{f\}$ and the interaction of the corresponding partial isometries. Let $\mu = \mu_1 \dots \mu_n$ be a path in $H\Gamma$.

- If $\mu_1 = f$ it holds by Lemma 2.12 that $ps_\mu = pp_ws_\mu = 0$ and similarly $s_\mu^*p = 0$. The first hypergraph relation furthermore gives that $s_f^*s_e = s_e^*s_f = \delta_{e,f}p_{r(f)} = Q_{r(f)}$ for all $e \in E^1$.
- If $\mu_j = f$ for $j = 2, \dots n$, by the definition of paths we must have $\mu_{j-1} \in r^{-1}(\{w\})$ and hence $s_{\mu_{j-1}}s_{\mu_j} = T_{\mu_{j-1}f}$.
- If $\mu_j \in r^{-1}(\{w\})$ for $j = 1, \dots, n-1$, we get again by the definition of paths that $\mu_{j+1} = f$ and hence $s_{\mu_j}s_{\mu_{j+1}} = T_{\mu_j f}$.
- If $\mu_n \in r^{-1}(\{w\})$ we get by Lemma 2.12 that $s_\mu p = s_\mu p_w p = 0$ and similarly $ps_\mu^* = 0$. Furthermore, by definition of paths and the fact that only one vertex emits to w, $s_{\mu_n}s_e^* \neq 0$ if and only if $e \in r^{-1}(\{w\})$. Since in our special case $s_fs_f^* = p_w$ and $r(\mu_n) = \{w\}$ we get for $e \in r^{-1}(\{w\})$ again by Lemma 2.12 that $s_{\mu_n}s_e^* = s_{\mu_n}p_ws_e^* = s_{\mu_n}s_fs_f^*s_e^* = T_{\mu_n f}T_{e_f}^*$. Similarly we can show that $s_es_f^* = T_{e_f}T_{\mu_n f}^*$.

Thus combining these properties we get that for a general element $S := s_{\mu_1}^{\epsilon_1} \dots s_{\mu_n}^{\epsilon_n} \in C^*(H\Gamma)$ where $\mu_1, \dots, \mu_n$ are paths in $H\Gamma$ and $\epsilon_j \in \{1, *\}$, $\epsilon_j \neq \epsilon_{j+1}$ that $pSp \in Im(\pi)$. Proposition 2.3 then yields the claim.

It remains to show that the corner is full. Let I be a closed two-sided ideal containing the corner $pC^*(H\Gamma)p$. Then I contains all projections corresponding to the vertices in E_R^0. Consider $e \in r^{-1}(\{w\})$. Then $e \neq f$ and $w \notin s(e)$ and hence $p_{s(e)} \in I$. Thus $s_e = p_{s(e)}s_e \in I$ by properties of the ideal. The first Cuntz-Krieger relation then gives $p_w = s_e^*s_e \in I$. Hence by Proposition 2.13, the ideal contains the unit and hence it must be all of $C^*(H\Gamma)$. Thus the corner is not contained in a proper closed two sided ideal and is thus full. □

Similar as for move S we get the following corollary by the Gauge Uniqueness Theorem.

Corollary 5.13. *For finite graphs and ultragraphs, move R yields Morita equivalent C^*-algebras.*

5.3. Move O - Outsplitting

Definition 5.14 (Move O). Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph and w be a vertex that is not a sink. We partition the set of outgoing edges in finitely many nonempty sets:

$$\{e \in E^1 \mid w \in s(e)\} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n.$$

The hypergraph $H\Gamma_O$ obtained by performing *move O* on $H\Gamma$ is defined by

$$E^0_O := E^0 \setminus \{w\} \cup \{w^1, \ldots, w^n\},$$
$$E^1_O := \{e^1 \mid e \in E^1, w \notin r(e)\} \cup \{e^1, \ldots, e^n \mid e \in E^1,\ w \in r(e)\},$$
$$r_O(e^i) := \begin{cases} r(e) & \text{if } i = 1 \text{ and } w \notin r(e) \\ (r(e) \setminus \{w\}) \cup \{w^1\} & \text{if } i = 1 \text{ and } w \in r(e) \\ w^i & \text{if } i > 1 \text{ and } w \in r(e), \end{cases}$$
$$s_O(e^i) := \begin{cases} s(e) & \text{if } w \notin s(e) \\ (s(e) \setminus \{w\}) \cup \{w^j\} & \text{if } w \in s(e) \text{ and } e \in \mathcal{E}_j. \end{cases}$$

We call $H\Gamma_O$ the hypergraph obtained by *outsplitting* $H\Gamma$ at w.

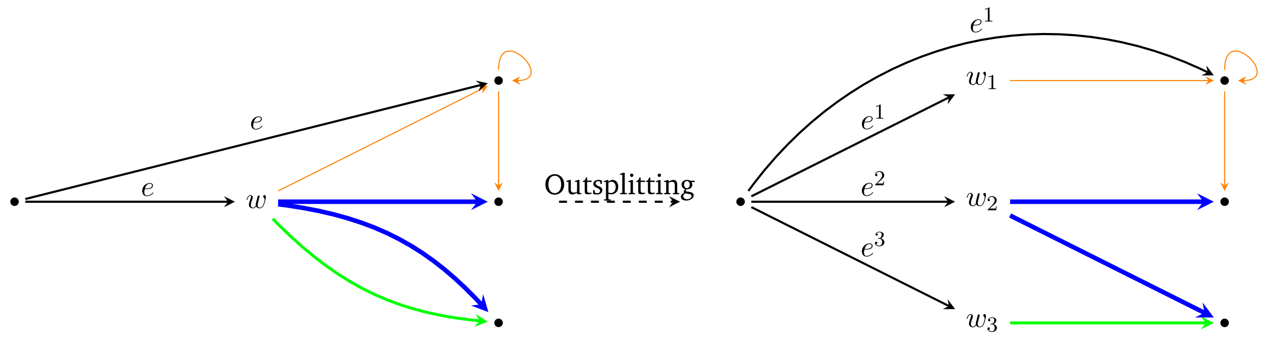


Figure 30: Illustration of the application of Move O for hypergraphs. Each color/thickness marks one edge and we partition the set of outgoing edges of w into one-point sets.

Remark 5.15. *The definition of the range was made like this to simplify notations and avoid distinction of cases. It does not matter if $r(e) \setminus \{w\}$ is merged with w^1 or some other w^j. We could have also split it up. It just has to be ensured that the ranges of e^i are disjoint.*

We show in the following, that the outsplitting produces isomorphic C^*-algebras even in the hypergraph setting. The idea is to construct a Cuntz Krieger $H\Gamma$-family in $H\Gamma_O$ and a Cuntz-Krieger $H\Gamma_O$-family in $H\Gamma$ and show that the *-homomorphisms given by the universal property are inverse to each other. The proofs are quite technical as they involve various case distinctions. But they consist mainly of combination of simple results seen in Section 2.2.

Proposition 5.16. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph, w be a vertex that is not a sink and $H\Gamma_O$ be the hypergraph obtained by outsplitting $H\Gamma$ at w. Let $\{q_v \mid v \in E^0_O\}$, $\{t_e \mid e \in E^1_O\}$ be the universal Cuntz-Krieger $H\Gamma_O$-family. Then $\{P_v \mid v \in E^0\}$, $\{S_e \mid e \in E^1\}$ defined as*

$$P_v := \begin{cases} q_v & \text{if } v \neq w \\ \sum_{i=1}^n q_{w^i} & \text{if } v = w, \end{cases}$$

$$S_e := \begin{cases} t_e & \text{if } w \notin r(e) \\ \sum_{i=1}^n t_{e^i} & \text{if } w \in r(e) \end{cases}$$

forms a Cuntz-Krieger $H\Gamma$-family in $H\Gamma_O$.

Proof. The elements P_v are nonzero mutually orthogonal projections since the projections q_v are nonzero mutually orthogonal projections. The elements S_e are clearly partial isometries if $w \notin r(e)$. For the other case we note that the ranges of $e^1, \dots, e^n$ are disjoint. Thus we get

$$S_e S_e^* = \left(\sum_{i=1}^n t_{e^i}\right)\left(\sum_{i=1}^n t_{e^i}\right)^* = \sum_{i=1}^n t_{e^i} t_{e^i}^*.$$

Using the first hypergraph relation for $H\Gamma_O$ we get

$$S_e^* S_e = \left(\sum_{i=1}^n t_{e^i}\right)^* \left(\sum_{i=1}^n t_{e^i}\right) = \sum_{i=1}^n t_{e^i}^* t_{e^i}.$$

Combining both and using that t_{e^i} are partial isometries we get that S_e are partial isometries.

(HR1): For the first hypergraph relation we first consider the case were $w \notin r(e)$:

$$S_e^* S_e = t_{e^1}^* t_{e^1} = q_{r_O(e^1)} = P_{r(e)}.$$

For $w \in r(e)$ we get using the second equation above:

$$S_e^* S_e = \sum_{i=1}^n t_{e^i}^* t_{e^i} = \sum_{i=1}^n q_{r_O(e^i)} = q_{r(e)\setminus w} + \sum_{i=1}^n q_{w^i} = P_{r(e)\setminus w} + P_w = P_{r(e)}.$$

By the hypergraph relations of $H\Gamma_O$ we know that $t_{e^i}^* t_{f^j} = 0$ for $e \neq f$ or $i \neq j$. With this at hand we directly get that $S_e^* S_f = 0$ for $e \neq f$.

(HR2): For the second hypergraph relation we again start with the case $w \notin r(e)$:

$$S_e S_e^* = t_{e^1} t_{e^1}^* \leqslant q_{s_O(e^1)} = \begin{cases} P_{s(e)} & e \notin \mathcal{E}_j \\ q_{s(e)\setminus w} + q_{w^j} \leqslant q_{s(e)\setminus w} + \sum_{j=1}^n q_{w^j} = P_{s(e)\setminus w} + P_w = P_{s(e)} & e \in \mathcal{E}_j. \end{cases}$$

For $w \in r(e)$ it follows using the first equation:

$$S_e S_e^* = \sum_{i=1}^{n} t_{e^i} t_{e^i}^* \leqslant \sum_{i=1}^{n} q_{s_O(e^i)}.$$

Similar as in the equation above we get $q_{s_O(e^i)} \leqslant P_{s(e)}$ for all $i = 1, \dots n$. Using the definition of the order relation we get especially, that

$$\sum_{i=1}^{n} q_{s_O(e^i)} \leqslant P_{s(e)}$$

which yields the required result.

(HR3): We finally tackle the last hypergraph relation. For $v \neq w$ we have

$$\begin{aligned}
P_v &= q_v \\
&\leqslant \sum_{x \in E_O^1, v \in s_O(x)} t_x t_x^* \\
&= \sum_{e \in E^1, v \in s(e), w \notin r(e)} t_{e^1} t_{e^1}^* + \sum_{e \in E^1, v \in s(e), w \in r(e)} \sum_{i=1}^{n} t_{e^i} t_{e^i}^* \\
&= \sum_{e \in E^1, v \in s(e), w \notin r(e)} t_{e^1} t_{e^1}^* + \sum_{e \in E^1, v \in s(e), w \in r(e)} \left(\sum_{i=1}^{n} t_{e^i} \right) \left(\sum_{i=1}^{n} t_{e^i} \right)^* \\
&= \sum_{e \in E^1, v \in s(e), w \notin r(e)} S_e S_e^* + \sum_{e \in E^1, v \in s(e), w \in r(e)} S_e S_e^* \\
&= \sum_{e \in E^1, v \in s(e)} S_e S_e^*.
\end{aligned}$$

For $v = w$ we can duplicate the above calculation to get for each $i = 1, \dots, n$ that

$$q_{w^i} \leqslant \sum_{e \in \mathcal{E}_i} S_e S_e^*.$$

Using that $\{e \in E^1 \mid w \in s(e)\} = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_n$ and $S_e S_e^*$ are mutually orthogonal projections we get

$$P_w = \sum_{i=1}^{n} q_{w^i} \leqslant \sum_{e \in E^1, w \in s(e)} S_e S_e^*.$$

□

By the above proposition the universal property of $C^*(H\Gamma)$ yields a *-homomorphism

$$\pi : C^*(H\Gamma) \to C^*(H\Gamma_O), \quad p_v \mapsto P_v, \quad s_e \mapsto S_e.$$

We already know that the *-homomorphism is an isomorphism for graphs [BP04, Thm. 3.2]. The next proposition extends this to the case were $H\Gamma$ is a hypergraph which looks like an ultragraph at the vertex w.

Proposition 5.17. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph and w be a vertex that is not a sink such that for all $e \in E^1$ with $w \in s(e)$ it follows $w = s(e)$. Let $H\Gamma_O$ be the hypergraph obtained by outsplitting $H\Gamma$ at w. The elements $\{Q_v \mid v \in E^0_O\}$, $\{T_{e^i} \mid e^i \in E^1_O\}$ defined as*

$$Q_v := \begin{cases} p_v & \text{if } v \neq w^j \\ \sum_{e \in \mathcal{E}_j} s_e s_e^* & \text{if } v = w^j, \end{cases}$$

$$T_{e^i} := \begin{cases} s_e & \text{if } w^j \notin r_O(e^i) \\ s_e Q_{r_O(e^i)} & \text{if } w^j \in r_O(e^i) \end{cases}$$

form a Cuntz-Krieger $H\Gamma_O$-family in $H\Gamma$.

Proof. Since we assumed that for all $e \in E^1$ with $w \in s(e)$ it follows $w = s(e)$, it follows for all vertices $v \neq w$ that $(s_e s_e^*) p_v = 0 = p_v (s_e s_e^*)$. Hence, since the sets $\mathcal{E}_j$ are disjoint and the projections p_v are mutually orthogonal, we get using the first hypergraph relation of $C^*(H\Gamma)$ that the projections Q_v are mutually orthogonal. Furthermore we get that $Q_{w^i} p_w = Q_{w^i}$, which will be useful later on. It remains to check that T_{e^i} are partial isometries and that the elements fulfill the hypergraph relations. The first will be a consequence of the calculations in the first hypergraph relation.

(HR1): The first relation for $e^1 \in E^1_O$ with $w \notin r(e)$ is obvious. In case that $w \in r(e)$ we get for e^1

$$\begin{aligned} T_{e^1}^* T_{e^1} &= Q_{r_O(e^1)} s_e^* s_e Q_{r_O(e^1)} \\ &= Q_{r_O(e^1)} p_{r(e)} Q_{r_O(e^1)} \\ &= \left(Q_{w^1} + p_{r(e)\setminus\{w\}}\right) p_{r(e)} \left(Q_{w^1} + p_{r(e)\setminus\{w\}}\right) \\ &= Q_{w^1} + p_{r(e)\setminus\{w\}} \\ &= Q_{r_O(e^1)}, \end{aligned}$$

using that $Q_{w^j} p_v = \delta_{v,w} Q_{w^j}$. This explains also the case for $i = 2, \ldots, n$ since

$$\begin{aligned} T_{e^i}^* T_{e^i} &= Q_{r_O(e^i)} s_e^* s_e Q_{r_O(e^i)} \\ &= Q_{w^i} p_{r(e)} Q_{w^i} \\ &= Q_{w^i} \\ &= Q_{r_O(e^i)}. \end{aligned}$$

By the first hypergraph relation for $C^*(H\Gamma)$ and the orthogonality of the projections we get $T_{e^i}^* T_{f^j} = 0$ for $e^i \neq f^j$. Furthermore, using these results it is straightforward to see that the elements T_{e^i} are partial isometries.

(HR2): For the second hypergraph relation we again consider the case $i = 1$ and $w \in r(e)$ first. We have

$$T_{e^1} T_{e^1}^* = s_e s_e^* \leqslant \begin{cases} p_{s(e)} = Q_{s_O(e^1)} & \text{if } s(e) \neq w \\ \sum_{f \in \mathcal{E}_j} s_f s_f^* = Q_{w^j} = Q_{s_O(e^1)} & \text{if } s(e) = w, \end{cases}$$

where we used the assumption that either $w \notin s(e)$ or $w = s(e)$ for all $e \in E^1$ and that the elements $s_f s_f^*$

are mutually orthogonal projections. For $w = s(e)$ we have using $Q_{r_O(e^i)} \leqslant 1$

$$T_{e^i} T_{e^1}^* = s_e Q_{r_O(e^i)} s_e^* \leqslant s_e s_e^*$$

which can be estimated similar to the previous case.

(HR3): Finally we check the third hypergraph relation. We note, that the assumption that $w \in s(e)$ implies $w = s(e)$ leads to $p_w = \sum_{j=1}^n Q_{w^j}$ by Lemma 2.27. For $v \in E^0 \setminus \{w\}$ we then have using the third hypergraph relation for $C^*(H\Gamma)$

$$\begin{aligned}
Q_v &= p_v \\
&\leqslant \sum_{e \in E^1, v \in s(e)} s_e s_e^* \\
&= \sum_{e \in E^1, v \in s(e), w \notin r(e)} T_{e^1} T_{e^1}^* + \sum_{e \in E^1, v \in s(e), w \in r(e)} s_e p_{r(e)} s_e^* \\
&= \sum_{e \in E^1, v \in s(e), w \notin r(e)} T_{e^1} T_{e^1}^* + \sum_{e \in E^1, v \in s(e), w \in r(e)} s_e \left(\sum_{j=1}^n Q_{w^j} + p_{r(e) \setminus \{w\}} \right) s_e^* \\
&= \sum_{e \in E^1, v \in s(e), w \notin r(e)} T_{e^1} T_{e^1}^* + \sum_{e \in E^1, v \in s(e), w \in r(e)} \left(s_e (Q_{w^1} + p_{r(e) \setminus \{w\}}) s_e^* + \sum_{j=2}^n s_e Q_{w^j} s_e^* \right) \\
&= \sum_{e \in E^1, v \in s(e), w \notin r(e)} T_{e^1} T_{e^1}^* + \sum_{e \in E^1, v \in s(e), w \in r(e)} \sum_{j=1}^n T_{e^j} T_{e^j}^* \\
&= \sum_{e^i \in E_O^1, v \in s_O(e^i)} T_{e^i} T_{e^i}^*.
\end{aligned}$$

For $v = w^j$ we have by definition that $Q_{w^j} = \sum_{e \in E^1, w^j \in s(e)} s_e s_e^*$. Hence the same calculation as above yields the required result. □

Theorem 5.18. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph and w be a vertex that is not a sink such that for all $e \in E^1$ with $w \in s(e)$ implies $w = s(e)$. Let $H\Gamma_O$ be the hypergraph obtained by outsplitting $H\Gamma$ at w. Then $C^*(H\Gamma) \cong C^*(H\Gamma_O)$.*

Proof. By Proposition 5.16 we get the canonical *-homomorphism

$$\pi : C^*(H\Gamma) \to C^*(H\Gamma_O), \quad p_v \mapsto P_v, \quad s_e \mapsto S_e.$$

Proposition 5.17 yields the canonical *-homomorphism

$$\tilde{\pi} : C^*(H\Gamma_O) \to C^*(H\Gamma), \quad q_v \mapsto Q_v, \quad t_e \mapsto T_e.$$

Straightforward calculations show that both *-homomorphisms are inverse to each other on the generators. Thus they are inverse on the whole C^*-algebras and we get the required isomorphism. □

For graphs and ultragraphs the condition $w \in s(e)$ implies $w = s(e)$ is trivially fulfilled. Hence the theorem is particularly valid for these specific hypergraphs.

Corollary 5.19. *For finite graphs and ultragraphs, move O yields isomorphic C^*-algebras.*

5.4. Move I - Insplitting

Before we define move I, we have a look at the indelay which introduces vertices to delay the arrival of an edge on its range. One can define this even in a more general setting with a so called Drinen range vector as done in [BP04, Ch. 4]. We only consider the special case needed for the connection to move I, which we consider afterwards. The constructions and proofs in the following section are adapted from [BP04, Ch. 4 and 5] and extended to the hypergraph setting. The upcoming proofs are again quite technical and deal with similar case distinctions as seen for move O. We will thus only highlight the critical steps and provide the details in Appendix A.

Definition 5.20 (Indelay). Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph and w be a vertex that is not a source. We partition the set of incoming edges in finitely many nonempty sets:

$$\{e \in E^1 \mid w \in r(e)\} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n.$$

The hypergraph $H\Gamma_D$ obtained by an *indelay* of $H\Gamma$ at w is defined by

$$\begin{aligned}
E_D^0 &:= E^0 \setminus \{w\} \cup \{w^1, \dots, w^n\}, \\
E_D^1 &:= E^1 \cup \{f^1, \dots f^n\}, \\
r_D(e) &:= \begin{cases} r(e) & \text{if } w \notin r(e) \\ (r(e) \setminus \{w\}) \cup \{w^j\} & \text{if } w \in r(e), \end{cases} \\
r_D(f^j) &:= w^j, \\
s_D(e) &:= \begin{cases} s(e) & \text{if } w \notin s(e) \\ (s(e) \setminus \{w\}) \cup \{w^1\} & \text{if } w \in s(e), \end{cases} \\
s_D(f^j) &:= w^{j+1}.
\end{aligned}$$

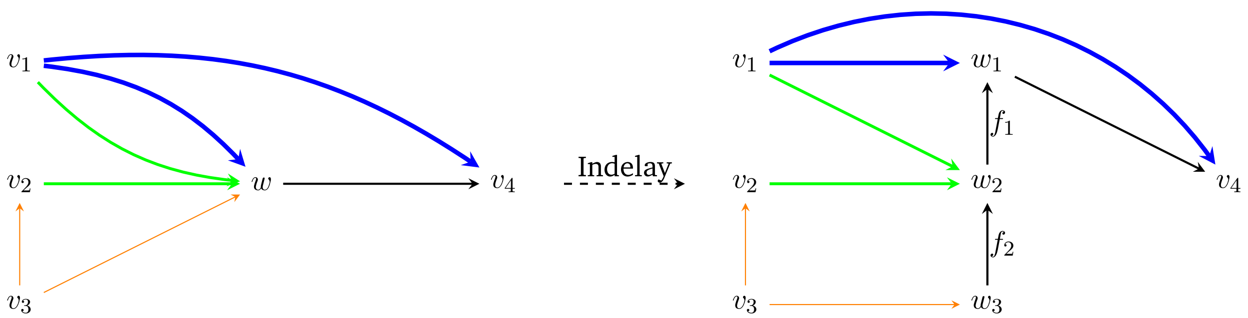


Figure 31: Illustration of the application of the indelay. The colored edges of different thickness symbolize one edge each. The incoming edges are partitioned into one-point sets.

Quite intuitively, one sees that the composition of he colored paths with the paths f_j in the right hypergraph yields the left hypergraph, when identifying w with w^1. We formalize this in form of a Cuntz-Krieger family in the following. Similar as in Proposition 5.11 we consider the case when $w \in r(e)$ implies $w = r(e)$. By applying the decomposition of ranges to the hypergraph we can recover this setting for each hypergraph without changing the hypergraph C^*-algebra (up to isomorphism).

Proposition 5.21. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph, w be a vertex that is not a source such that $w \in r(e)$ implies $w = r(e)$. $H\Gamma_D$ be the hypergraph obtained by an indelay of $H\Gamma$ at w. Let $\{q_v, t_e\}$ be the universal Cuntz-Krieger $H\Gamma_D$-family. Then $\{P_v \mid v \in E^0\}$, $\{S_e \mid e \in E^1\}$ defined as*

$$P_v := \begin{cases} q_v & \text{if } v \neq w \\ q_{w^1} & \text{if } v = w \end{cases}$$

$$S_e := \begin{cases} t_e & \text{if } w \notin r(e) \\ t_e t_{f_{j-1}} \dots t_{f_1} & \text{if } w \in \mathcal{E}^j \end{cases}$$

forms a Cuntz-Krieger $H\Gamma$-family in $H\Gamma_D$.

Proof. The hypergraph relations follow by mainly straightforward calculations, see AppendixA. We mention shortly the critical tricks. For the first hypergraph relations note that $f_j \dots f_1$ are perfect paths for all $j \in \{1, \dots n-1\}$. Thus, for all $e \in \mathcal{E}_j$ it follows by Proposition 2.20 that

$$S_e^* S_e = q_{r_D(f_1)} = q_{w_1} = P_w.$$

This result also explains, why we have to add the assumption that $w \in r(e)$ implies $w = r(e)$. The assumption implies furthermore, that $ef_j \dots f_1$ is a perfect path and since $q_{w_j} = t_{f_{j-1}} t_{f_{j-1}}^*$ by the construction of the edges $f_j \in E_D^1$ this leads to $S_e S_e^* = t_e t_e^*$ for all $e \in E^1$, which is crucial for the second hypergraph relation. For the third hypergraph relation we use that $v \in s_D(e)$ implies $v \in s(e)$ and $w_1 \in s_D(e)$ implies $w \in s(e)$. Combining this with the hypergraph relations of $H\Gamma_D$ shows that the given elements form a Cuntz-Krieger $H\Gamma$-family. □

Proposition 5.22. *There is a surjective *-homomorphism from $C^*(H\Gamma_D)$ onto a full corner of $C^*(H\Gamma)$.*

Proof. By the previous proposition we can apply the universal property and get the canonical *-homomorphism

$$\pi : C^*(H\Gamma) \to C^*(H\Gamma_D), \qquad p_v \mapsto P_v, \quad s_e \mapsto S_e$$

Let $F := E^0 \setminus \{w\} \cup \{w^1\} \subseteq E_D^0$. We show that the image of π is given by the corner $pC^*(H\Gamma_D)p$ with $p := q_F = \sum_{v \in E^0, v \neq w} q_v + q_{w^1}$. By definition of P_v we get $P_v = pP_v p$. For all $e \in E^1$ we have $s_D(e) \subseteq F$. For $e \in E^1$ with $w \notin r(e)$ we have $r_D(e) \subseteq E^0 \setminus \{w\} \subseteq F$ and for $e \in \mathcal{E}_j$ we have $r_D(ef_{j-1} \dots f_1) = w_1 \subseteq F$. Hence by applying Proposition 4.16 we get $S_e = pS_e p$. Thus, the image of the generators of $C^*(H\Gamma)$ is contained in $pC^*(H\Gamma_D)p$ and $Im(\pi) \subseteq pC^*(H\Gamma_D)p$.

To see the converse we consider a general element $S := t_{\mu_1}^{\epsilon_1} \dots t_{\mu_m}^{\epsilon_m}$ for paths $\mu_1, \dots, \mu_m$ in $H\Gamma_D$ and $\epsilon_1, \dots, \epsilon_m \in \{1, *\}$. We show that $pSp \in Im(\pi)$. We first have a deeper look at a path $\mu = e_1 \dots e_k$ in $H\Gamma_D$.

- If $e_j \in E^1$ for $j = 1, \dots, k$ we have $s_D(\mu) \subseteq F$. Hence $pt_\mu = t_\mu = S_\mu \in Im(\pi)$.

- If $e_j \in E^1$ for $j = 1, \dots, k$ and $w_i \notin r_D(\mu)$, we have $r_D(\mu) \subseteq E^0 \setminus \{w\}$. Hence $t_\mu p = t_\mu = S_\mu \in Im(\pi)$. On the other hand, if $w_i \in r_D(\mu)$ we have $w_i = r_D(\mu)$ by assumption and hence $t_\mu p = 0 \in Im(\pi)$.
- If $e_j = f_l$ for some $j \in \{1, \dots, k\}$ and $l \in \{1, \dots, n-1\}$, either the whole path $f_l \dots f_{n-1}$ is contained in μ or $e_k = f_{l+k-j}$. If $e_1 \neq f^l$ and $e_k \neq f_{l+k-j}$, the path $\nu := e_{j-1} f_l \dots f_{n-1}$ is contained in μ and hence $t_\mu = s_{e_1} \dots S_\nu \dots s_{e_k} \in Im(\pi)$. Hence it remains to consider the cases, when the path starts or ends with an element in $\{f_1, \dots, f_{n-1}\}$.
  - If $e_1 = f^l$ we have $s_D(\mu) = w_l \notin F$ and hence $pt_\mu = 0 \in Im(\pi)$. On the other hand, if we have a second path α such that $t_\alpha^* t_\mu \neq 0$ or $t\mu^* t_\nu \neq 0$, we must have $\alpha_j = f_{l+j-1}$ since we have perfect paths. Hence using properties of perfect paths the elements vanish and we are left with paths μ', α' which dies not contain the elements $\{f_1, \dots, f_n\}$. Hence $t_{\mu'} = s_{\mu'} \in Im(\pi)$.
  - If $e_k = f_{l+k-j}$ we have $t_\mu p = 0 \in Im(\pi)$. A similar argument as in the last step shows, that the interaction with another path α cancels the elements t_{f_j} and we are left with $t_{\mu'} = s_{\mu'} \in Im(\pi)$.

Combining these arguments it follows that $pSp \in Im(\pi)$.

It remains to show that the corner is full. Let $I \subseteq C^*(H\Gamma_D)$ be a closed two-sided ideal containing $pC^*(H\Gamma_D)p$. Then I contains the projections $q_v = pq_vp$ for all $v \in E^0 \setminus \{w\} \cup \{w_1\}$. Since $s_D(e) \subseteq E^0 \setminus \{w\} \cup \{w^1\}$ for all $e \in E^1$, we get $t_e = pt_e \in I$ for all $e \in E^1$. Since $\mathcal{E}_j \neq \emptyset$, for each vertex w_j there exists an edge $e \in E^1$ such that $w_j = r_D(e)$. Hence since $p_{w_j} = t_e^* t_e \in I$. Thus all canonical projections are contained in the ideal, and thus by Proposition 2.13 the unit is contained in the ideal. This shows that the ideal must be all of $C^*(H\Gamma_D)$ and the corner is full. □

Similar as seen for the previous moves, we could not recover a proof of injectivity. In case of graphs and ultragraphs, the Gauge Uniqueness Theorem can be applied to get injectivity of π. Hence in these cases the previous proposition provides an isomorphism of $C^*(H\Gamma)$ onto a full corner of $C^*(H\Gamma_D)$.

Corollary 5.23. *For finite graphs and ultragraphs, the indelay yields Morita equivalent C^*-algebras.*

Definition 5.24 (Move I). Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph and w be a vertex that is not a source. We partition the set of incoming edges in finitely many nonempty sets:

$$\{e \in E^1 \mid w \in r(e)\} = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_n.$$

The hypergraph $H\Gamma_I$ obtained by performing *move I* on $H\Gamma$ is defined by

$$\begin{aligned}
E_I^0 &:= E^0 \setminus \{w\} \cup \{w^1, \dots, w^n\}, \\
E_I^1 &:= \{e^1 \mid e \in E^1, w \notin s(e)\} \cup \{e^1, \dots, e^n \mid e \in E^1,\ w \in s(e)\}, \\
r_I(e^i) &:= \begin{cases} r(e) & \text{if } i = 1 \text{ and } w \notin r(e) \\ (r(e) \setminus \{w\}) \cup \{w^j\} & \text{if } e^i \in \mathcal{E}_j, \end{cases} \\
s_I(e^i) &:= \begin{cases} s(e) & \text{if } w \notin s(e) \\ (s(e) \setminus \{w\}) \cup \{w^1\} & \text{if } i = 1 \text{ and } w \in s(e) \\ \{w^i\} & i = 2, \dots, n. \end{cases}
\end{aligned}$$

We call $H\Gamma_I$ the hypergraph obtained by *insplitting* $H\Gamma$ at w.

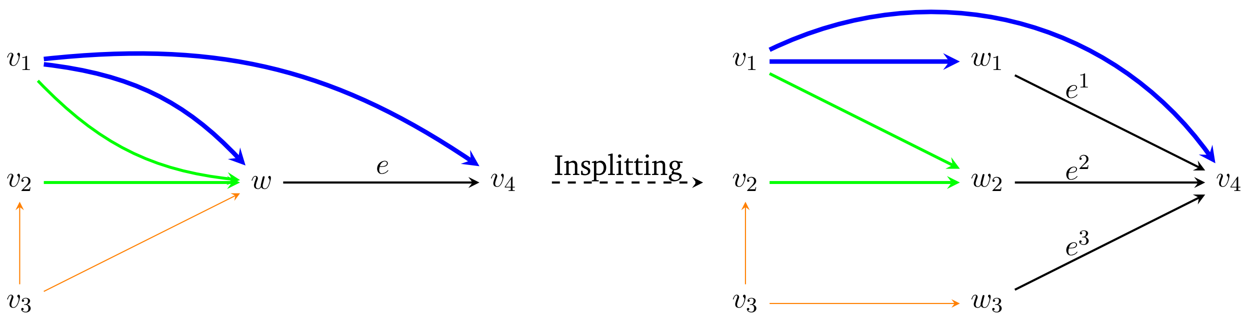


Figure 32: Illustration of the application of Move I for hypergraphs. Each color/thickness symbolizes one edge and $\mathcal{E}_j$ are one-point sets.

Now we can connect the indelay with Move I and receive isomorphic C^*-algebras.

Proposition 5.25. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph and w be a vertex that is not a source such that $w \in s(e)$ implies $w = s(e)$. The incoming edges of w be partitioned into disjoint sets $\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n$. Let $H\Gamma_D$ and $H\Gamma_I$ be the corresponding hypergraphs formed by an indelay and an insplitting respectively. Then $C^*(H\Gamma_D) \cong C^*(H\Gamma_I)$.*

Proof. Let $\{p_v, s_e\}$ and $\{q_v, t_e\}$ be the canonical generators of $C^*(H\Gamma_I)$ and $C^*(H\Gamma_D)$ respectively. We define a Cuntz-Krieger $H\Gamma_I$-family in $C^*(H\Gamma_D)$ by

$$P_v := q_v,$$
$$S_{e^i} := \begin{cases} t_e & \text{if } i = 1 \\ t_{f_{i-1}} \dots t_{f_1} t_e & \text{if } i = 2, \dots, n. \end{cases}$$

On the other hand we can define a Cuntz-Krieger $H\Gamma_D$-family in $C^*(H\Gamma_I)$ by

$$Q_v := p_v$$
$$T_e := s_{e^1}$$
$$T_{f_j} := \sum_{e \in E^1, w \in s(e)} s_{e^{j+1}} s_{e^j}^*.$$

The proof that these are really Cuntz-Krieger families is quite technical, but not very difficult. We provide it in Appendix A. Applying the universal property twice we get the *-homomorphisms

$$\begin{aligned} \pi : C^*(H\Gamma_I) &\to C^*(H\Gamma_D) \\ p_v &\mapsto P_v \\ s_{e^i} &\mapsto S_{e^i} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \tilde{\pi} : C^*(H\Gamma_D) &\to C^*(H\Gamma_I) \\ q_v &\mapsto Q_v \\ t_e &\mapsto T_e \end{aligned}$$

which are inverse to each other. To see this, we show that both are inverse to each other on the generators. This is clear for all projections and edges except of $e^i \in C^*(H\Gamma_I)$ with $i = 2, \dots, n$ and $f_j \in C^*(H\Gamma_D)$. For these we get

$$
\begin{aligned}
\tilde{\pi} \circ \pi(s_{e^i}) &= \tilde{\pi}(t_{f_{i-1}} \dots t_{f_1} t_e) \\
&= \left(\sum_{g \in E^1} s_{g^i} s_{g^{i-1}}^* \right) \dots \left(\sum_{g \in E^1} s_{g^2} s_{g^1}^* \right) s_{e^1} \\
&= s_{e^i} s_{e^{i-1}}^* \dots s_{e^2} s_{e^1}^* s_{e^1} \\
&= s_{e^i}.
\end{aligned}
$$

Since $w \in s(e)$ implies $w = s(e)$ we get by Lemma 2.27 hat $q_{w^1} = \sum_{e \in E^1, w \in s(e)} t_e t_e^*$. Using this it follows

$$
\begin{aligned}
\pi \circ \tilde{\pi}(t_{f_i}) &= \pi(\sum_{e \in E^1} s_{e^{i+1}} s_{e^i}^*) \\
&= \sum_{e \in E^1, w \in s(e)} t_{f_i} \dots t_{f_1} t_e t_e^* t_{f_1}^* \dots t_{f_{i-1}}^* \\
&= t_{f_i} \dots t_{f_1} \left(\sum_{e \in E^1, w \in s(e)} t_e t_e^* \right) t_{f_1}^* \dots t_{f_{i-1}}^* \\
&= t_{f_i} \dots t_{f_1} q_{w^1} t_{f_1}^* \dots t_{f_{i-1}}^* \\
&= t_{f^i},
\end{aligned}
$$

where we used the properties of perfect paths. □

Combining Proposition 5.22 and Proposition 5.25 we get that under the assumption, that the hypergraph looks locally like an ultragraph, the C^*-Algebra of the insplitted hypergraph is almost Morita equivalent to the C^*-Algebra of the initial hypergraph.

Corollary 5.26. *Let $H\Gamma = (E^0, E^1, r, s)$ be a finite hypergraph and w be a vertex that is not a source such that for all $e \in E^1$ with $w \in s(e)$ it follows $w = s(e)$. $H\Gamma_I$ be the hypergraph obtained by insplitting $H\Gamma$ at w. Then there is a surjective *-homomorphism from $C^*(H\Gamma_I)$ onto a full corner of $C^*(H\Gamma)$.*

Similar as for the other moves we again get Morita equivalence in the case of graphs and ultragraphs.

Corollary 5.27. *For finite graphs and ultragraphs, move I yields Morita equivalent C^*-algebras.*

Remark 5.28. *In order to approach the question whether a similar classification result as in [ERRS21] holds for hypergraph C^*-algebras, further research has to be done in this topic. One major problem concerns the injectivity of the canonical *-homomorphisms respectively the lack of the Gauge Uniqueness Theorem. Except of move O we could not prove it yet. But we could not construct a counterexample either, so maybe there is a workaround. On the other hand we had to restrict the moves to hypergraphs which look locally like an ultragraph. This reinforces the thesis that the large differences arise from the multi-valued sources and new tool-sets have to be developed to deal with these. We will list these issues together with further research topics in the upcoming final chapter.*

6. Further Research Topics

In the course of this thesis, questions and issues have arisen at several points that are of interest for future work on this and related topics. We summarize these as an overview below.

- Based on our results for finite hypergraphs a next step could be to investigate infinite hypergraphs. We already shed some light on critical steps in the definition, see Definition 2.8. Another indication can be the results on infinite ultragraphs in [Tom03].
- Building on the specific characteristics of hypergraph C^*-algebras, one can investigate further implications of the path structure, commutativity of projections and the relevance of the order relation of projections. As starting points are for example the Lemmata 2.26 and 2.28 to be mentioned, which emphasize the relevance of the multi-valued source and intersecting sources, which we observed to be the crucial difference of hypergraphs at multiple stages throughout this thesis. Based on this it can be interesting to construct further counter examples and representations.
- The topic of non-nuclearity offers a broad field of research questions. One can investigate concrete conditions for nuclearity and try to describe them via properties of the hypergraph. Recalling our construction of further non-nuclear hypergraph C^*-algebras based on non-nuclear quotients, see Proposition 3.19, the visualization of quotients in hypergraphs is interesting to look at. Within this context one can examine the ideal structure of hypergraph C^*-algebras and their relation to saturated and hereditary subgraphs. This is especially interesting as hereditary subalgebras of nuclear C^*-algebras are nuclear. Thus, this could be used to further enlarge the number of examples of non-nuclear hypergraph C^*-algebras.
- Our counterexample in Example 4.11 has shown that a direct generalization of the Gauge Uniqueness Theorem is not possible. This raises several new research questions: Can the theorem be generalized with another action? Do the restrictions under which the Gauge Uniqueness Theorem holds already describe the ultragraph C^*-algebras (see Remark 4.23)? Are there other ways to proof injectivity of representations?
- In the realm of the Gauge Uniqueness Theorem we touched the dual graph of a hypergraph. We saw that it looses information, especially, its C^*-algebra is not isomorphic, not even Morita equivalent to the initial hypergraph C^*-algebra, see Remark 4.15. Multiple questions are interesting in this regard: Which information is lost? Have the hypergraphs with similar dual properties in common? Are there other constructions/generalizations of the dual graph which give more insights?
- The manipulation of hypergraphs by moves and the corresponding changes in the associated graphs remain an exciting area of research. Based on our results for the moves S, R, I, O, further investigations can be made. In particular, by constructing counterexamples. Of particular interest is also the observation that the hypergraphs must locally look like ultragraphs in order to apply the moves, see Remark 5.3. In the context of classification of hypergraph C^*-algebras, generalization and development of further moves may yield new results.
- In connection with Morita equivalence, which plays a decisive role in this thesis, stands the K-theory of hypergraph C^*-algebras. For graph C^*-algebras there is already an explicit way to compute the K-groups [RS03, Thm. 3.2]. This result can be applied to ultragraphs by Morita equivalence. With the structural insights into the theory of hypergraph C^*-algebras laid by this work, we have formed a foundation which allows a deeper entry into the subject of K-theory of hypergraph C^*-algebras.

A. Further Proofs

Proof Remark 2.14 - Unital C*-algebra for infinite hypergraphs

The proof follows exactly the steps from the corresponding proofs for ultragraphs[Tom03, Lem. 3.2] with exception of the definition of the sets A_n. We assume first that

$$E^0 \in \left\{ \bigcup_{i=1}^{n} \left(\bigcap_{e \in X_i} r(e) \right) \cup \bigcup_{i=1}^{m} \left(\bigcap_{e \in Y_i} s(e) \right) \cup F \;\middle|\; X_i, Y_i \subseteq E^1 \text{ finite},\; F \subseteq E^0 \text{ finite} \right\} =: M.$$

Then we have by Proposition 2.12 that p_{E^0} acts as unit on the elements $s_{\mu_1}^{\epsilon_1}...s_{\mu_n}^{\epsilon_n}$ for paths $\mu_1, \dots, \mu_n$ in $H\Gamma$, $\epsilon_1, ...\epsilon_n \in \{1, *\}$, $\epsilon_j \neq \epsilon_{j+1}$ and $n \in \mathbb{N}$. These span of these elements is dense in $C^*(H\Gamma)$ by Proposition 2.24. This implies that p_{E^0} the unit of $C^*(H\Gamma)$.

For the other direction we list the vertices $\{v_1, v_2, \dots\}$ and edges $\{e_1, e_2, \dots\}$ and define the following sets

$$A_n := \bigcup_{i=1}^{n} r(e_i) \cup \bigcup_{i=1}^{n} s(e_i) \cup \bigcup_{i=1}^{n} \{v_i\}.$$

For each $n \in \mathbb{N}$, $A_n \in M$ and we have $A_n \subseteq A_{n+1}$. By definition of A_n, there is an $n \in \mathbb{N}$ large enough for each element $s_{\mu_1}^{\epsilon_1}...s_{\mu_m}^{\epsilon_m}$, such that p_{A_n} acts as unit on $s_{\mu_1}^{\epsilon_1}...s_{\mu_m}^{\epsilon_m}$. Hence the elements p_{A_n} form an approximate unit on $C^*(H\Gamma)$. Since $C^*(H\Gamma)$ is unital, the approximate unit must converge in norm to the unit, which can only happen if the approximate unit is eventually constant. Indeed, assuming that it is not constant, i.e. for each $m \in \mathbb{N}$ there is some $n > m$ such that $A_m \subsetneq A_n$, there exists a vertex $v \in A_n \setminus A_m$ and hence $(p_{A_n} - p_{A_m})p_v = p_v = p_v(p_{A_n} - p_{A_m})$. This implies that $p_{A_n} - p_{A_m}$ is a non-zero projection and has thus norm one, contradicting the fact that it must be a Cauchy sequence. Thus, there is some $k \in \mathbb{N}$ such that $p_{A_k} = 1$. Since $p_{A_k} \geqslant p_{E^0} \geqslant 1$, we must have $p_{A_k} = p_{E^0}$ and hence $E^0 = A_k \in M$.

Proof Proposition 5.21 - Indelay

(HR1): For $e \in E_1$ with $w \notin r(e)$ we have

$$S_e^* S_e = t_e t_e^* = q_{r(e)} = P_{r(e)}.$$

For $e \in \mathcal{E}_j$, i.e. for $e \in E^1$ with $w \in r(e)$, it holds applying the relations for perfect paths

$$\begin{aligned} S_e^* S_e &= t_{f_1}^* \dots t_{f_{j-1}}^* t_e^* t_e t_{f_{j-1}} \dots t_{f_1} \\ &= t_{f_1}^* t_{f_1} \\ &= q_{r_D(f_1)} \\ &= q_{w_1} \\ &= P_w. \end{aligned}$$

Since by the first hypergraph relation $t_e^* t_g = 0$ for $e, g \in E_D^1$ with $e \neq g$, it follows $S_e^* S_g = 0$ for all $e, g \in E^1$ with $e \neq g$.

(HR2): For $e \in E_1$ with $w \notin r(e)$ we have since $P_w := q_{w_1}$

$$S_e S_e^* = t_e t_e^* \leqslant q_{s_D(e)} = \begin{cases} P_{s(e)} & w \notin s(e) \\ q_{w_1} + \sum_{v \in s_D(e), v \neq w_j} q_v = P_{s(e)} & w \in s(e). \end{cases}$$

The second case for $e \in \mathcal{E}_j$ follows similar, since $q_{w_j} = t_{f_{j-1}} t_{f_{j-1}}^*$ and the perfect path relations yield

$$S_e S_e^* = t_{f_{j-1}} \dots t_{f_1} t_{f_{j-1}}^* t_e^* t_e = t_e t_e^*.$$

(HR3): For $v \neq w$ we have

$$P_v = q_v \leqslant \sum_{e \in E_D^0, v \in s_D(e)} t_e t_e^* = \sum_{e \in E_D^0, v \in s(e)} S_e S_e^*$$

where we used $t_e t_e^* = S_e S_e^*$ for all $e \in E^1$ and $v \in s_D(e)$ if and only if $v \in s(e)$. For $v = w$ we use that $w_1 \in s_D(e)$ if and only if $w \in s(e)$. With this we get

$$P_w = q_{w_1} \leqslant \sum_{e \in E_D^0, w_1 \in s_D(e)} t_e t_e^* = \sum_{e \in E_D^0, w \in s(e)} S_e S_e^*.$$

Thus, all hypergraph relations are fulfilled.

Proof Proposition 5.25 - Indelay and Move I

We first prove that

$$\begin{aligned} P_v &:= q_v, \\ S_{e^i} &:= \begin{cases} t_e & \text{if } i = 1 \\ t_{f_{i-1}} \dots t_{f_1} t_e & \text{if } i = 2, \dots, n \end{cases} \end{aligned}$$

is in fact a Cuntz-Krieger $H\Gamma_I$-family in $C^*(H\Gamma_D)$.

(HR1): We first note that by definition, $r_D(e) = r_I(e^i)$ for all $e \in E^1$. Thus we get for $i = 1$

$$S_{e^1}^* S_{e^1} = t_e^* t_e = q_{r_D(e)} = P_{r_I(e^1)}.$$

For $i = 2, \dots, n$ we get using the fact that the f_j build perfect paths

$$S_{e^i}^* S_{e^i} = t_e^* t_{f_1}^* \dots t_{f_{i-1}}^* t_{f_{i-1}} \dots t_{f_1} t_e = t_e^* t_e = q_{r_D(e)} = P_{r_I(e^i)}.$$

It follows directly that for $e^i \neq g^j$ we have $S_{e^i}^* S_{g^j} = 0$.

(HR2): For $i = 1$ we get using that $s_D(e) = s_I(e^1)$ for all $e \in E^1$

$$S_{e^1} S_{e^1}^* = t_e t_e^* \leqslant q_{s_D(e)} = P_{s_I(e^1)}.$$

For $i = 2, \ldots, n$ we get, using again that we deal with perfect paths

$$\begin{aligned} S_{e^i} S_{e^i}^* &= t_{f_{i-1}} \ldots t_{f_1} t_e t_e^* t_{f_1}^* \ldots t_{f_{i-1}}^* \\ &\leqslant t_{f_{i-1}}^* t_{f_{i-1}} \\ &= q_{w^i} \\ &= P_{w^i} \\ &= P_{s_I(e^i)}. \end{aligned}$$

(HR3): Consider $v \neq w^i$ for $i > 1$. Then we have that $v \in s_D(e)$ if and only if $v \in s_I(e^1)$. Recalling that $S_{e^1} = t_e$ we get

$$P_v = q_v \leqslant \sum_{e \in E_D^1, v \in s_D(e)} t_e t_e^* \leqslant \sum_{e \in E_I^1, v \in s_I(e)} S_e S_e^*.$$

In case of $v = w^i$ for $i > 1$ we have using the perfect paths

$$\begin{aligned} P_{w^i} &= q_{w^i} \\ &= t_{f_{i-1}} t_{f_{i-1}}^* \\ &= t_{f_{i-1}} \ldots t_{f_1} t_{f_1}^* \ldots t_{f_{i-1}}^* \\ &= t_{f_{i-1}} \ldots t_{f_1} q_{w_1} t_{f_1}^* \ldots t_{f_{i-1}}^* \\ &\leqslant t_{f_{i-1}} \ldots t_{f_1} \left(\sum_{e \in E_D^1, v \in s_D(e)} t_e t_e^* \right) t_{f_1}^* \ldots t_{f_{i-1}}^* \\ &= \sum_{e \in E_D^1, v \in s_D(e)} t_{f_{i-1}} \ldots t_{f_1} t_e t_e^* t_{f_1}^* \ldots t_{f_{i-1}}^* \\ &= \sum_{e^i \in E_D^1, w^i \in s_I(e^i)} S_{e^i} S_{e^i}^*. \end{aligned}$$

Next we prove that

$$\begin{aligned} Q_v &:= p_v, \\ T_e &:= s_{e^1} \qquad \text{for } e \in E^1, \\ T_{f_j} &:= \sum_{w \in s(e)} s_{e^{j+1}} s_{e^j}^* \end{aligned}$$

is in fact a Cuntz-Krieger $H\Gamma_D$-family in $C^*(H\Gamma_I)$.

(HR1): For $e \in E^1$ we have using that $r_I(e^1) = r_D(e)$

$$T_e^* T_e = s_{e^1}^* s_{e^1} = p_{r_I(e^1)} = Q_{r_D(e)}.$$

For the remaining edges $\{f_1, \dots, f_{n-1}\}$ it holds using that $r_I(e^{j+1}) = r_I(e^j)$ and using that using that $s_I(e^i) = w^i$ for $i = 2, \dots, n$

$$\begin{aligned} T_{f_j}^* T_{f_j} &= \left(\sum_{w \in s(e)} s_{e^j} s_{e^{j+1}}^* \right) \left(\sum_{w \in s(e)} s_{e^{j+1}} s_{e^j}^* \right) \\ &= \sum_{w \in s(e)} s_{e^j} s_{e^{j+1}}^* s_{e^{j+1}} s_{e^j}^* \\ &= \sum_{w \in s(e)} s_{e^j} s_{e^j}^* \\ &= p_{w^j} \\ &= Q_{w^j} \\ &= Q_{r_D(f_j)}. \end{aligned}$$

(HR2): For $e \in E^1$ we have since $s_I(e^1) = s_D(e)$

$$T_e T_e^* = s_{e^1} s_{e^1} \leqslant p_{s_I(e^1)} = Q_{s_D(e)}.$$

Similar as for (HR1) we get for the remaining edges $\{f_1, \dots, f_{n-1}\}$ using that $s_I(e^i) = w^i$ for $i = 2, \dots, n$

$$\begin{aligned} T_{f_j} T_{f_j}^* &= \left(\sum_{w \in s(e)} s_{e^{j+1}} s_{e^j}^* \right) \left(\sum_{w \in s(e)} s_{e^j} s_{e^{j+1}}^* \right) \\ &= \sum_{w \in s(e)} s_{e^{j+1}} s_{e^j}^* s_{e^j} s_{e^{j+1}}^* \\ &= \sum_{w \in s(e)} s_{e^{j+1}} s_{e^{j+1}}^* \\ &= p_{w^{j+1}} \\ &= Q_{w^{j+1}} \\ &= Q_{s_D(f_j)}. \end{aligned}$$

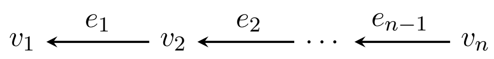
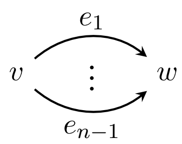
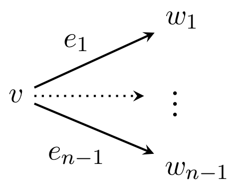
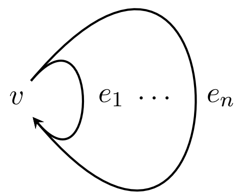
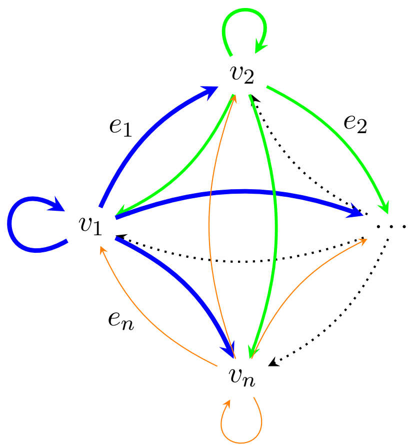
(HR3): For $v \neq w^j$ we get since $s_I(e^1) = s_D(e)$

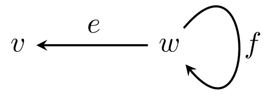
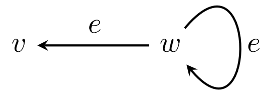
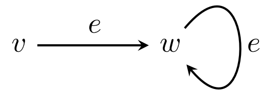
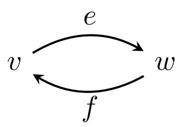
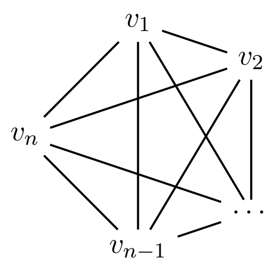
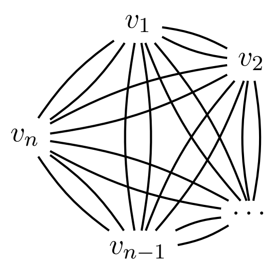
$$Q_v = p_v \leqslant \sum_{e^i \in E_I^1, v \in s_I(e^i)} s_{e^i} s_{e^i}^* = \sum_{e \in E_D^1, v \in s_D(e)} T_e T_e^*.$$

The case $v = w^j$ for $j = 2, \dots, n$ follows directly from the calculation in (HR2).

B. List of Graph and Hypergraph C*-Algebras

We list some examples of graph- and hypergraph C^*-algebras as overviews. We visualize them and use colored edges of different thickness to mark single edges if it simplifies the picture.

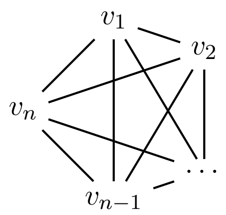
C^*-Algebra	Definition	Hypergraph
$\mathbb{C}$	$E^0 = \{v\}$, $E^1 = \emptyset$	•
$M_n(\mathbb{C})$	$E^0 = \{v_1, \dots, v_n\}$ $E^1 = \{e_1, \dots, e_{n-1}\}$ $s(e_j) = v_{j+1}$, $r(e) = v_j$	
	$E^0 = \{v, w\}$ $E^1 = \{e_1, \dots, e_{n-1}\}$ $s(e_j) = v$, $r(e_j) = w$	
	$E^0 = \{v, w_1, \dots, w_{n-1}\}$ $E^1 = \{e_1, \dots, e_{n-1}\}$ $s(e_j) = v$, $r(e_j) = w_j$	
O_n	$E^0 = \{v\}$ $E^1 = \{e_1, \dots, e_n\}$ $s(e_j) = v, r(e_j) = v$	
	$E^0 = \{v_1, \dots, v_n\}$ $E^1 = \{e_1, \dots, e_n\}$ $s(e_j) = v_j, r(e_j) = \{v_1, \dots, v_n\}$	

$\mathcal{T}$	$E^0 = \{v, w\}$ $E^1 = \{e, f\}$ $s(e) = w, r(e) = v$ $s(f) = w, r(f) = w$	
	$E^0 = \{v, w\}$ $E^1 = \{e, f\}$ $s(e) = \{w\}, r(e) = \{v, w\}$	
	$E^0 = \{v, w\}$ $E^1 = \{e, f\}$ $s(e) = \{v, w\}, r(e) = \{w\}$	
$M_2(C(\mathbb{T}))$	$E^0 = \{v, w\}$ $E^1 = \{e, f\}$ $s(e) = v, r(e) = w$ $s(f) = w, r(f) = v$	
$C(\mathbb{T}) * \mathbb{C}^n$	$E^0 = \{v_1, \ldots, v_n\}$ $E^1 = \{e\}$ $s(e) = \{v_1, \ldots, v_n\}$ $r(e) = \{v_1, \ldots, v_n\}$	
$O_2 * \mathbb{C}^n$	$E^0 = \{v_1, \ldots, v_n\}$ $E^1 = \{e_1, \ldots, e_m\}$ $s(e_j) = \{v_1, \ldots, v_n\}$ $r(e_j) = \{v_1, \ldots, v_n\}$	

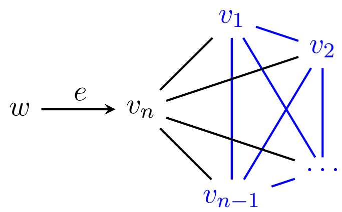
C. List of Non-Amenable Hypergraphs

In the following we list a bunch of non-amenable hypergraphs. Since we have to be ensure that the remaining quotient is non-nuclear, n must be chosen sufficiently large. The crucial non-nuclear part of the hypergraph is colored blue.

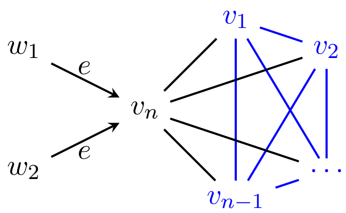
$$
\begin{aligned}
E^0 &:= \{v_1, \dots v_n\} \\
E^1 &:= \{f\} \\
s(f) &:= \{v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\}
\end{aligned}
$$



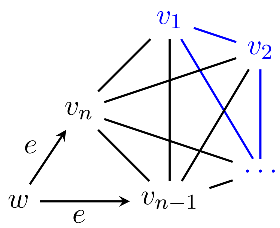
$$
\begin{aligned}
E^0 &:= \{w, v_1, \dots v_n\} \\
E^1 &:= \{e, f\} \\
s(e) &:= \{w\}, \qquad r(e) := \{v_n\} \\
s(f) &:= \{v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\}
\end{aligned}
$$



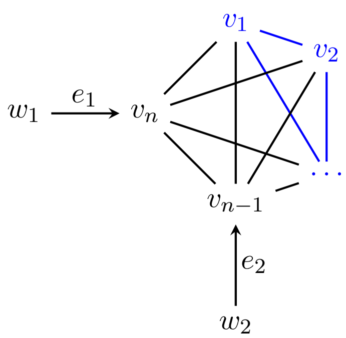
$$
\begin{aligned}
E^0 &:= \{w_1, w_2, v_1, \dots v_n\} \\
E^1 &:= \{e, f\} \\
s(e) &:= \{w_1, w_2\}, \qquad r(e) := \{v_n\} \\
s(f) &:= \{v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\}
\end{aligned}
$$



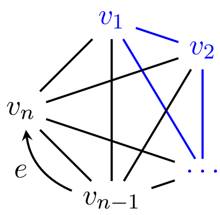
$$
\begin{aligned}
E^0 &:= \{w, v_1, \dots v_n\} \\
E^1 &:= \{e, f\} \\
s(e) &:= \{w\}, \qquad r(e) := \{v_{n-1}, v_n\} \\
s(f) &:= \{v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\}
\end{aligned}
$$



$$
\begin{aligned}
E^0 &:= \{w_1, w_2, v_1, \dots v_n\} \\
E^1 &:= \{e_1, e_2, f\} \\
s(e_1) &:= \{w, v_{n-1}\}, \qquad r(e_1) := \{v_{n-1}, v_n\} \\
s(e_2) &:= \{w, v_{n-1}\}, \qquad r(e_2) := \{v_{n-1}, v_n\} \\
s(f) &:= \{v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\}
\end{aligned}
$$



$$\begin{aligned} E^0 &:= \{v_1, \dots v_n\} \\ E^1 &:= \{e, f\} \\ s(e) &:= \{v_{n-1}\}, \qquad r(e) := \{v_n\} \\ s(f) &:= \{v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\} \end{aligned}$$



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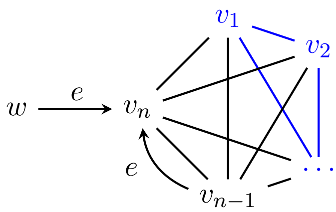
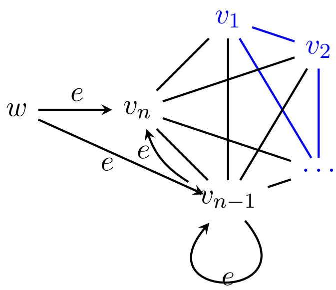
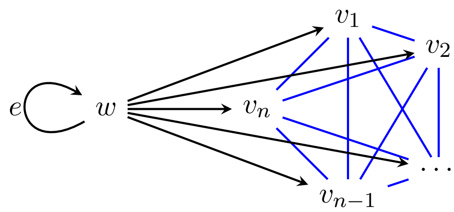


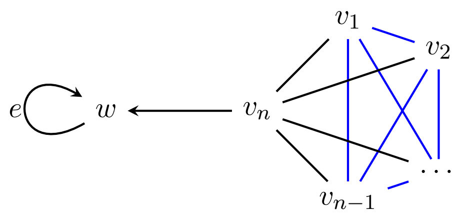
$$\begin{aligned} E^0 &:= \{w, v_1, \dots v_n\} \\ E^1 &:= \{e, f\} \\ s(e) &:= \{w, v_{n-1}\}, \qquad r(e) := \{v_{n-1}, v_n\} \\ s(f) &:= \{v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\} \end{aligned}$$



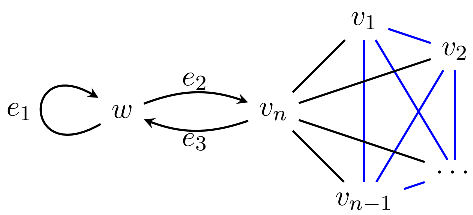
$$\begin{aligned} E^0 &:= \{w, v_1, \dots v_n\} \\ E^1 &:= \{e, f\} \\ s(e) &:= \{w\}, \qquad r(e) := \{w\} \\ s(f) &:= \{w, v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\} \end{aligned}$$



$$\begin{aligned} E^0 &:= \{w, v_1, \dots v_n\} \\ E^1 &:= \{e, f\} \\ s(e) &:= \{w, v_n\}, \qquad r(e_1) := \{w\} \\ s(f) &:= \{w, v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\} \end{aligned}$$



$$\begin{aligned} E^0 &:= \{w, v_1, \dots v_n\} \\ E^1 &:= \{e_1, e_2, e_3, f\} \\ s(e_1) &:= \{w\}, \qquad r(e_1) := \{w\} \\ s(e_2) &:= \{w\}, \qquad r(e_2) := \{v_n\} \\ s(e_3) &:= \{v_n\}, \qquad r(e_3) := \{w\} \\ s(f) &:= \{w, v_1, \dots, v_n\}, \qquad r(f) := \{v_1, \dots, v_n\} \end{aligned}$$



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List of Symbols

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{C}$	Complex numbers
$\mathbb{C}^n$	Complex coordinate space
$\mathbb{F}_2$	Free group on two generators
$\mathbb{Z}/n\mathbb{Z}$	Integers modulo n
$\mathcal{H}$	Hilbertspace
$\mathcal{B}(\mathcal{H})$	Bounded operators on a Hilbertspace $\mathcal{H}$
$\mathcal{K}(\mathcal{H})$	Compact operators on a Hilbertspace $\mathcal{H}$
$C(\mathbb{T})$	Continuous functions on the unit circle
$M_n(\mathbb{C})$	Matrix algebra
$\mathcal{T}$	Toeplitz algebra
O_n	Cuntz-algebra
$l^2(\mathbb{Z}^2)$	Square-summable sequences on $\mathbb{Z}^2$
$P(E)$	Free *-algebra with generator set $E \cup E^*$
$A(E \mid R)$	Universal *-algebra with generators E and relations R
$C^*(E \mid R)$	Universal C^*-algebra with generators E and relations R
$C^*(E)$	Graph C^*-algebra
$C^*(H\Gamma)$	Hypergraph C^*-algebra
$C^*(G)$	Group C^*-algebra
A, B	C^*-algebras
$A * B$	Free product of C*-algebras
$A \rtimes_\alpha B$	Full crossed product with action α
$A \times B$	Product C^*-algebra
$A \odot B$	Algebraic tensor product
$A \otimes B$	Spatial (minimal) tensor product
$A \otimes_{\max} B$	Maximal tensor product
$\|\| \cdot \|\|_{\min}$	Minimal tensor product norm
$\|\| \cdot \|\|_{\max}$	Maximal tensor product norm
$\langle \cdot \| \cdot \rangle_A,\ {}_A\langle \cdot \| \cdot \rangle$	A-valued inner products
$\|\| \cdot \|\|_A,\ {}_A\|\| \cdot \|\|$	Norms based on A-valued inner products
$X_A,\ {}_AX$	Full right/left Hilbert A-module
$K_0(A),\ K_1(A)$	K-Theory of C^*-algebra A
E	Graph
$H\Gamma$	Hypergraph
$H\Gamma_S$	Hypergraph obtained by move S

$H\Gamma_R$	Hypergraph obtained by move R
$H\Gamma_O$	Hypergraph obtained by move O
$H\Gamma_I$	Hypergraph obtained by move I
$H\Gamma_D$	Hypergraph obtained by indelay
E^0	Set of vertices
$\mathcal{E}^0$	Generalized vertices
E^1	Set of Edges
v, w	Vertices
$e,\ f,\ g$	Edges
$r(e)$	Range of e
$s(e)$	Source of e
$\mu,\ \nu,\ \alpha,\ \beta$	Paths
A_E	Adjacency matrix
$p,\ q$	Projections
s	Partial isometries
$\{s_e, p_v\}$	Canonical generators of $H\Gamma$
$p_{r(e)}$	$\sum_{v\in r(e)} p_v$
$p_{s(e)}$	$\sum_{v\in s(e)} p_v$
s_μ	Element corresponding to path μ
u	Universal unitary
v	Universal isometry
$\leqslant$	Order relation of projections
γ	Gauge action
π	*-Homomorphism
$x \mapsto x^*$	Involution

List of Figures

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