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Quantum versions of metric spaces and their symmetries

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Eidesstattliche Erklärung

Ich versichere hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Saarbrücken, den 28.08.2019

(Andreas Widenka)

When life gives you lemons
compute their quantum symmetry.

FELIX LEID

Contents

Introduction

The purpose of this thesis is the study of quantum versions of classical compact metric spaces and isometric quantum group actions on these spaces from different viewpoints. We will also give a formal definition of these quantum versions and present some new examples.

Quantum groups have been an object of study for many years now and especially the C^*-algebraic compact quantum groups (CQG) introduced by Woronowicz in [Wor87] possess a very powerful representation theory. Actions of quantum groups on C^*-algebras dualize the idea of group actions and describe the symmetries of an object in the non-commutative case; one gets some kind of "quantum symmetry". Wang showed in [Wan98] that even classical objects can have quantum symmetry unseen by restricting to classical groups. For example the set of n points gives rise to a commutative C^*-algebra but has genuine quantum symmetry. Its quantum symmetry group is the famous quantum group S_n^+ which is not a group for $n \geq 4$.

But allowing arbitrary quantum group actions on compact spaces often ignores too much information of the compact space; since we are working with C^*-algebras, only the topology of the space is taken into consideration. In that sense a square and a rectangle have the same quantum symmetry group. Alain Connes defined with the help of spectral triples quantum group actions on Riemannian Manifolds that preserve the differential structure of the manifold, see [Con94]. Generalizing this further, Goswami([Gos15]), Banica([Ban05]), Bichon and Collins have defined and studied isometric quantum group actions on classical finite and compact metric spaces discovering for example the non-commutative version of the hyperoctahedral group H_n^+ in search for the quantum isometry group of the n-dimensional hypercube in [BBC07]. Rieffel also introduced the notion of a non-commutative metric space in [Rie99] and Quaegebeur and Sabbe then defined isometric quantum group actions on such non-commutative metric spaces in [QS12]. It is not yet clear if the two notions of isometric actions introduced by Goswami and Quaegebeur-Sabbe respectively are equivalent on classical spaces.

In this thesis we give a new formal framework for the notion of a "non-commutative" or "quantum" version of a classical compact metric space $X \subset \mathbb{R}^n$ and their quantum symmetries. We also present some new examples of quantum versions of some compact metric spaces and calculate their quantum isometry.

In the first section we recall the definition of a CQG and a compact matrix quantum group (CMQG). We give some classic examples, show that every compact group is also

a CQG and how to dualize the concept of a group action on a compact space.
In the second section the theory of isometric quantum group actions on classical compact spaces is presented. We see that dualizing an isometric group action gives rise to an isometric quantum group action. First we deal with the case of a finite metric space where the notion of an isometric action was defined by Banica in [Ban05]. Then we pass to arbitrary compact metric spaces. If the compact space is actually a subset of $\mathbb{R}^n$ we present the surprising result of Goswami ([Gos15]) that, as in the classical case, an isometric quantum group action is automatically affine and orthogonal and hence the quantum group is a subgroup of the O_n^+. This characterization of isometric quantum group actions is later used to *define* an isometric quantum group action on the quantum version of a space. We list some examples of compact metric spaces and their quantum symmetries.
The third section consists mostly of our own research. We present the definition of a (maximal) quantum version of a classical compact metric spaces and their quantum symmetries. The definitions are new although the idea of quantum versions of the sphere, the hypercube or the torus is well-known to the experts. As a first example we give the definition of a maximal quantum version of the d-dimensional lemon and calculate its quantum symmetries. Then we present some other examples and compare them to the half-classical case.
The fourth and last section gives a brief overview of a different approach to noncommutative metric spaces, the compact quantum metric spaces (CMQS) as defined by Rieffel in [Rie99]. We show that every compact metric space is also a CQMS and motivate the definition of an isometric action on CQMS as given by Quaegebeur and Sabbe in [QS12].

Notation

The majority of the notation is introduced when needed. The symbol $A \otimes B$ denotes the minimal tensor product if A and B are both C^*-algebras and the algebraic tensor product if one of them (or both) is just an algebra. We write $C(A, B)$ for the continuous functions from A to B and $C(A) := C(A, \mathbb{C})$ for the continuous complex valued functions on A. By $C^*(x_1, \dots, x_n \,|\, \mathcal{R})$ with $\mathcal{R}$ being a set of polynomial relations in the elements x_i and $x_i{}^*$ we denote the universal C^*-algebra with generators x_i and relations $\mathcal{R}$ (see for example [Bla85]).

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1 Preliminaries

First we recall some definitions of the theory of compact quantum groups (CQG), starting with the definition of such a CQG. For more details see for example [Web17], [Tim08], [NT13] or [MVD98].

1.1 Compact quantum groups

Definition 1.1
A *compact quantum group* (A, Δ) is a unital C^*-algebra A together with a comultiplication, i.e. a unital $*$-homomorphism

$$\Delta : A \longrightarrow A \otimes A$$

such that it is *coassociative*, i.e.

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$$

and the sets $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are linearly dense in $A \otimes A$.

Remark 1.2
The continuous functions on any compact group form a CQG. Let G be a compact group and $C(G)$ the continuous functions on G. We identify $C(G) \otimes C(G)$ with $C(G \times G)$ by mapping $f \otimes g$ to $(s, t) \mapsto f(s)g(t)$. Then the multiplication on the group G defines a comultiplication on $C(G)$ by

$$\begin{aligned} \Delta : C(G) &\longrightarrow C(G \times G) \\ f &\longmapsto \Delta(f) \end{aligned}$$

where $\Delta(f)(s, t) := f(st)$. In this way the associativity of the group multiplication corresponds to the coassociativity of the comultiplication and the cancellation law of the group corresponds to the denseness of the span of the sets $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$. So every compact group corresponds to a CQG in a natural way. In this way the algebra A is of course commutative. On the other hand by the theorem of Gelfand-Naimark we get that every CQG given by a commutative algebra corresponds to a $C(G)$ for a fitting compact group G. For this reason we sometimes write $C(G)$ for the non-commutative algebra of a CQG and call G the CQG even though there is not really a compact group G underneath the algebra A.

We will mostly care about a particular kind of CQGs in this thesis, the *compact matrix quantum groups* (CMQG) introduced by Woronowicz in [Wor87]. We will see in the third section that the restriction to the CMQGs is not really a restriction at all concerning isometric actions of CQGs.

Definition 1.3
A *compact matrix quantum group* (A, a, Δ) is a unital C^*-algebra A together with a matrix $a = (\alpha_{ij})$ of elements of A and a $^*-$homomorphism $\Delta : A \to A \otimes A$ such that

1. $\Delta(\alpha_{ij}) = \sum_{k=1}^{n} \alpha_{ik} \otimes \alpha_{kj}$
2. the matrices a and a^t are invertible
3. the elements α_{ij} for $i, j = 1, \ldots, n$ generate A as a C^*-algebra.

Remark 1.4

- The comultiplication Δ is uniquely determined by the first and the third property of a CMQG. Hence, we can drop the Δ in the notation of the CMQG and just write (A, a) with the underlying Δ being implied.
- Every CMQG is actually a CQG as in Definition 1.1. A proof of that statement can be found in [NT13].
- Most of the CMQG that are known can be given as universal C^*-algebras, with the matrix entries α_{ij} being the generators subject to some algebraic relations.

Example 1.5
Let us list two famous examples of CMQGs.

- The CMQG O_n^+ is given by the universal C^*-algebra

$$C(O_n^+) := C^* \left(u_{ij},\ i, j = 1, \ldots, n \,\middle|\, u_{ij} = u_{ij}^*, \sum_{k=1}^{n} u_{ik} u_{jk} = \sum_{k=1}^{n} u_{ki} u_{kj} = \delta_{i,j} 1 \right).$$

  This algebra is non-commutative for all $n \geq 2$. Note that these relations, together with commutativity of all the u_{ij}, define the classical function algebra $C(O_n)$. So in some sense O_n^+ is the non-commutative version of O_n, see [Wan95].

- The CMQG S_n^+ is given by the universal C^*-algebra

$$C(S_n^+) := C^*\left(u_{ij},\, i,j = 1,\dots,n \,\middle|\, u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{kj} = \sum_{k=1}^n u_{ik} = 1 \right).$$

  Here the classical function algebra $C(S_n)$ is given by the same relations including the commutativity of the generators and for $n \geq 4$ we in fact have $C(S_n^+) \neq C(S_n)$ because $C(S_n^+)$ is non-commutative in these cases, see [Wan98].

Remark 1.6

In classical topological group theory it is a well known fact that every compact group admits a (up to a factor) unique left- (and right-) invariant measure on the group, called the *Haar measure*. This translates into our theory to the fact that for each CQG (A, Δ) there is a unique state h on A called the *Haar state* that is left- and right-invariant on A in the sense that

$$(id \otimes h)\Delta(a) = (h \otimes id)\Delta(a) = h(a) \cdot 1.$$

If now the GNS-construction of A is done with respect to the Haar state h the resulting CQG A_r is called the *reduced* CQG corresponding to A. If the Haar state is even tracial, i.e. $h(ab) = h(ba)$ for all $a, b \in A$, the CQG is said to be of *Kac type*.

We will also need occasionally the notion of a Hopf *-algebra, so we briefly give the definition and some results connecting it to CQGs here without proof.

Definition 1.7

A *Hopf *-algebra* $(A, \Delta, \epsilon, \kappa)$ is a *-algebra A together with a comultiplication Δ and maps $\epsilon : A \to \mathbb{C}$ and $\kappa : A \to A$ fulfilling the conditions

$$(\epsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id$$
$$m(\kappa \otimes id)\Delta = m(id \otimes \kappa)\Delta = \epsilon \cdot 1.$$

Here m denotes the map $m : A \otimes A \to A$ induced by the condition $m(x \otimes y) = xy$. The maps ϵ and κ are called the *counit* and the *antipode* respectively.

Remark 1.8
For every CQG (A, Δ) there exists a dense *-subalgebra $A_0 \subset A$ and maps ϵ and κ such that $(A_0, \Delta\big|_{A_0}, \epsilon, \kappa)$ is a Hopf *-algebra. In general the counit and antipode on the Hopf algebra A_0 cannot be extended to A. However, it is known that if A is of Kac type then the antipode κ can be extended in a bounded manner to the reduced CQG A_r with the property $\kappa^2 = \mathrm{id}$.

Example 1.9
In the case of a CMQG (A, a) it is easy to determine the underlying Hopf algebra. It is the *-algebra generated by the entries of the matrix a; the counit ϵ is given by $\epsilon(\alpha_{ij}) = \delta_{i,j}$ and the antipode is given by $\kappa(\alpha_{ij}) = \alpha^*_{ji}$. In the general case the Hopf algebra can be defined in a similar way but one has to use all the finite dimensional representations of a given CQG. For details see [MVD98].

1.2 Quantum group actions

Groups are used to encode and analyse the symmetry of spaces: the classical permutation group S_n is the symmetry group of n points in random positions, O_n is the symmetry group of the n-dimensional sphere in $\mathbb{R}^n$ and so on. This is made precise by the notion of *group actions*. An action of a group G on a space X is a group homomorphism $\tilde{\alpha}$ from G to the group of automorphisms of the space X. Depending on the structure of the space X (for example set, topological, metric) the automorphisms can be just bijections, bijective continuous functions or surjective isometries. If now $\tilde{\alpha}$ is a bijection we say that G is the symmetry group of X. A group action can also be defined in a different way, defining the action to be a map

$$\begin{aligned} \alpha : X \times G &\longrightarrow X \\ (x, g) &\longmapsto gx, \end{aligned}$$

where the gx are defined in such a way that α fulfills conditions making the corresponding $\tilde{\alpha}$ defined by

$$\begin{aligned} \tilde{\alpha} : G &\longrightarrow Aut(X) \\ g &\longmapsto \alpha(\cdot, g) \end{aligned}$$

a well-defined group homomorphism.
Since by Remark 1.2 CQGs are generalizations of classical groups, they should encode some kind of "quantum symmetry" of topological spaces that is unseen by the classical group actions. Therefore we dualize the notion of group action to quantum groups acting on C^*-algebras which are, by Gelfand-Naimark, the non-commutative analogues of classical compact spaces.

Definition 1.10 ([Pod95])
Let (A, Δ) be a CQG and B be a unital C^*-algebra. A *quantum group action* α of A on B is a *-homomorphism $\alpha : B \to B \otimes A$ fulfilling the conditions

1. $(\alpha \otimes id) \circ \alpha = (id \otimes \Delta) \circ \alpha$
2. $\alpha(B)(1 \otimes A)$ is linearly dense in $B \otimes A$.

An action is called *faithful* if there is no subalgebra A' of A such that (A', Δ) is a CQG and α is an action of A' on B.

The proof of the following lemma is clear so we omit it.

Lemma 1.11
Let B have generators $x_1, \dots, x_n$ and suppose that the quantum group action of a CQG (A, Δ) on B can be written as

$$\alpha(x_i) = \sum_{j=1}^{n} x_j \otimes \alpha_{ij}$$

for some $\alpha_{ij} \in A$. Then the action is faithful if and only if A is generated by the α_{ij}.

Remark 1.12
The classical notion of group actions fits nicely in this framework. Let $\alpha : X \times G \to X$ be an action of a compact group G on a compact space X. We identify $C(X) \otimes C(G)$ with $C(X \times G)$ as before. Then the group action α induces a quantum group action α' by putting

$$\begin{aligned} \alpha' : C(X) &\longrightarrow C(X) \otimes C(G) \\ f &\longmapsto \alpha'(f) = f \circ \alpha. \end{aligned}$$

Then α' is a quantum group action; the two requirements in Definition 1.10 are fulfilled because of the axioms of the ordinary group action ($g(hx) = (gh)x$ and $ex = x$).

Remark 1.13
It is clear that instead of looking at a classical case we can also look at "half-classical" cases, where only one of the C^*-algebras is the algebra of continous functions on some group G or on some compact space X. Then we can either investigate the quantum symmetries of a classical space or the classical symmetries of a "quantum space". This will be important in the next section.

Example 1.14
Let $X = \{1, \ldots, n\}$ be a finite set with n points. The function algebra on X can be written as a universal C^*-algebra as

$$C(X) = C^* \left(x_1, \ldots, x_n \middle| x_i = x_i^* = x_i^2, \sum_{i=1}^{n} x_i = 1 \right).$$

Then $C(S_n^+)$ (see Example 1.5) acts on $C(X)$ by putting

$$\alpha(x_i) = \sum_{j=1}^{n} x_j \otimes u_{ji}.$$

Moreover, Wang showed in [Wan98] that S_n^+ is the biggest CQG acting on X in the sense that every other CQG action on $C(X)$ factorizes through $C(S_n^+)$.

Remark 1.15
It is shown in [Sol11] that for any action α of a CQG (A, Δ) on some C^*-algebra B one can find a *-dense unital subalgebra $B_0 \subset B$, called the *spectral subalgebra* such that α maps B_0 to $B_0 \otimes A_0$, where A_0 is the dense Hopf-algebra as in Remark 1.8.

2 Isometric quantum group actions on classical spaces

In this section we will introduce the notion of an *isometric* quantum group action on classical spaces. These actions have been studied to some extent by Connes [Con94], Banica [Ban05], Goswami [Gos15] and others.

If we allow any kind of quantum group action to be considered as a notion of symmetry, then the square given by $C^*(x_1, x_2 \,|\, x_1x_2 = x_2x_1, x_i = x_i^*, x_i^2 = 1)$ has the same quantum symmetry group as the rectangle given by $C^*(x_1, x_2 \,|\, x_1x_2 = x_2x_1, x_i = x_i^*, x_1^2 = 1, 2x_2^2 = 1)$ since the two C^*-algebras are clearly isomorphic (the isomorphism is given by $x_1 \mapsto x_1$ and $x_2 \mapsto \frac{1}{\sqrt{2}}x_2$) even though intuitively we want to distinguish these spaces regarding their quantum symmetries. The reason for the same quantum symmetry group is that by passing to the continuous functions on the respective space we lose the metrical information and only have the topological information available.

In this section we will introduce a way to let a coaction on the space $C(X)$ still respect the metrical information of the underlying space X. So in the following we will look at compact metric spaces (X, d) and the space of continuous functions from X to $\mathbb{C}$, $C(X)$. The metric on X can be for example (but not necessarily) the euclidean metric on $X \subset \mathbb{R}^n$. We will start with the classical case of a group acting on a finite metric space.

2.1 Finite metric spaces

Definition 2.1

Let $X = \{1, \dots, n\}$ be a finite metric space with metric d. We call the matrix defined by $D = (d(i, j))$ the *distance matrix* of (X, d). Since all the metrical information of the space is included in the matrix D, we will also write (X, D) instead of (X, d) for a finite metric space.

Let G be a finite group acting isometrically on a finite space (X, d), i.e. $d(x, gy) = d(g^{-1}x, y)$ for all $g \in G$ and $x, y \in X$. How does this translate into the C^*-algebraic setting?

The function algebras on X and G are generated by the point functions δ_i and δ_g given by

$$\delta_i(j) = \begin{cases} 1, & i = j \\ 0 & \text{else} \end{cases}, \quad \delta_g(h) = \begin{cases} 1, & g = h \\ 0 & \text{else} \end{cases}$$

for all $i = 1, \dots, n$ and $g \in G$ on X and G respectively. The following result is already stated in [Ban05], for convenience of the reader we give an explicit proof.

Lemma 2.2
Let (X, D) be a finite metric space and G a group acting on X.

i) The coaction $\alpha : C(X) \to C(X) \otimes C(G)$ of $C(G)$ on $C(X)$ is given by

$$\alpha(\delta_i) = \sum_{j \in X} \left(\delta_j \otimes \sum_{g^{-1}i=j} \delta_g \right).$$

ii) The action of G on X is isometric if and only if $aD = Da$ holds where D is the distance matrix of X and a is the matrix defined by $a = (\alpha_{ij})$ with

$$\alpha_{ij} := \sum_{g^{-1}i=j} \delta_g.$$

Proof. As we have seen in Remark 1.12 the coaction α of $C(G)$ on $C(X)$ is given by $f \mapsto ((x, g) \mapsto f(gx))$. So we have

$$\alpha(\delta_i)(j,h) = \begin{cases} 1, & h^{-1}i = j \\ 0, & \text{else} \end{cases}.$$

On the other hand we have

$$\left(\sum_{g \in G} \delta_{g^{-1}i} \otimes \delta_g \right)(j,h) = \sum_{g \in G} \delta_{g^{-1}i}(j)\delta_g(h) = \begin{cases} 1, & h^{-1}i = j \\ 0, & \text{else} \end{cases},$$

for all $h \in G$ and $j \in X$, so the coaction is given by

$$\alpha(\delta_i) = \sum_{g \in G} \delta_{g^{-1}i} \otimes \delta_g.$$

By collecting terms with the same δ_i we can rearrange the right hand side to

$$\alpha(\delta_i) = \sum_{j \in X} \left(\delta_j \otimes \sum_{g^{-1}i=j} \delta_g \right)$$

which shows the first part of the lemma.

For the second part write

$$\alpha_{ij} = \sum_{g^{-1}i=j} \delta_g, \quad a = (\alpha_{ij}).$$

Now we can multiply this matrix from the left and right by the distance matrix D and get

$$(aD)_{ij} = \sum_k \alpha_{ik} d(k,j) = \sum_k d(k,j) \sum_{g^{-1}i=k} \delta_g = \sum_{g \in G} d(g^{-1}i, j)\delta_g$$

and

$$(Da)_{ij} = \sum_k d(i,k)\alpha_{kj} = \sum_k d(i,k) \sum_{gj=k} \delta_g = \sum_{g \in G} d(i, gj)\delta_g.$$

By evaluating these functions on all $g \in G$ we see that $Da = aD$ is equivalent to $d(i, gj) = d(g^{-1}i, j)$ which is just the definition of the group action being isometric. □

Since every coaction of a CQG G on a finite space is given by

$$\alpha(\delta_i) = \sum_{j \in X} \left(\delta_j \otimes \alpha_{ij}\right)$$

for some coefficients $\alpha_{ij} \in G$, Banica gave in [Ban05] the following definition of an isometric quantum group action on a finite space.

Definition 2.3
Let (X, D) be a finite metric space with n points and G be a CQG acting on $C(X)$ via

$$\alpha(\delta_i) = \sum_{j \in X} \left(\delta_j \otimes \alpha_{ij}\right).$$

Denote by $a = (\alpha_{ij})$ the coefficient matrix of the action. Then we call the action α *isometric* if $aD = Da$ holds.

Proposition 2.4 ([BBC07])
Every action of a CQG on a simplex (i.e. a finite metric space where all points have the same distance) is isometric.

Proof. Let (X, D) be a simplex with n points and G be a CQG acting on X via

$$\alpha(\delta_i) = \sum_{j \in X} \left(\delta_j \otimes \alpha_{ij} \right).$$

As before, let $a = (\alpha_{ij})$ be the matrix of coefficients of the action. As seen in Example 1.14 the action α has to factorize through the quantum group S_n^+ which means that the α_{ij} satisfy the relations of the generators of the universal C^*-algebra $C(S_n^+)$, i.e.

$$\alpha_{ij} = \alpha_{ij}^* = \alpha_{ij}^2, \quad \sum_{k=1}^{n} \alpha_{kj} = \sum_{k=1}^{n} \alpha_{ik} = 1. \tag{1}$$

Now since (X, D) is a simplex its distance matrix is given by $D = \lambda(F - I_n)$ where I_n is the $n \times n$-identity matrix and F is the matrix with all 1's. Obviously a commutes with I_n so the action is isometric if $aF = Fa$ holds. But every entry in the matrices aF and Fa is just the sum over a row or column of a so by the relations (1) we see that $aF = Fa = F$ holds. So the action α is isometric. □

Definition 2.5
Let (X, D) be a finite metric space with n points. The *quantum isometry group* of (X, D) is the quotient of $C(S_n^+)$ by the ideal generated by the relations $uD = Du$.

Example 2.6
The quantum isometry group of the simplex with n points is the quantum group $C(S_n^+)$. This is just a reformulation of Example 1.14.

Example 2.7
The set of vertices of the n-dimensional hypercube has quantum isometry group O_n^{-1}. This quantum group is given by the quotient of $C(O_n^+)$ by the relations

$$u_{ik}u_{ij} = -u_{ij}u_{ik}, \quad u_{ki}u_{ji} = -u_{ji}u_{ki}, \quad k \neq j$$
$$u_{ij}u_{kl} = u_{kl}u_{ij}, \quad i \neq k, j \neq l.$$

Note that it seems like O_n^{-1} has the "wrong dimension" since the quantum isometry group of the n-dimensional hypercube is by definition a subgroup of $S_{2^n}^+$. This is a consequence of the fact that the hypercube can also be regarded as a Cayley graph of the group $\mathbb{Z}_2^n$ and the action can be defined on the n generators of the group, see [BBC07] for details. But we will see at the end of this section in Proposition 2.21 that we can recover the quantum group O_n^{-1} also by looking at the actual metric space.

2.2 Compact metric spaces

Now we turn to arbitrary compact metric spaces, not necessarily finite. Since the metrical information of the space is not given by a matrix now, we cannot use Definition 2.3 anymore and need a different definition of isometry.
In this section (X, d) will always be a compact metric space. In the following we will often identify $C(X) \otimes A$ for some C^*-algebra A (usually a CQG) with $C(X, A)$, the continuous functions from X to A, via

$$f \otimes a \mapsto (x \mapsto f(x)a).$$

We also restrict ourselves to faithful actions of CQGs. Intuitively that means that there is no part of the action that is trivial so we cannot factor out a subgroup and still capture the full symmetry. In search for the "isometry" of a given metric space it thus makes sense to just look at faithful actions. It is shown in [Hua16] that if a CQG A acts faithfully on $C(X)$ for a compact Hausdorff space (so in particular for a metric space) then A is of Kac type, i.e. the bounded antipode κ of the underlying Hopf algebra can be extended to the reduced CQG A_r in a norm-bounded way. This allowed Goswami in [Gos15] to define the following.

Definition 2.8
Let (X, d) be a compact metric space and A be a CQG acting on $C(X)$ via the action α. Let also A_r be the reduced CQG with respect to A with GNS representation π_r. Denote by $\alpha_r := (\mathrm{id} \otimes \pi_r) \circ \alpha$ the *reduced* action of A_r on $C(X)$. Then we call α isometric if and only if

$$\alpha_r(d_x)(y) = \kappa(\alpha_r(d_y)(x)) \qquad (2)$$

for all $x, y \in \mathrm{X}$. Here $d_x(z) := d(x, z)$ and κ is the extension of the antipode to A_r.

Let us first check that in the classical case this definition coincides with the classical definition of an isometric action of a compact group G. This is already noted by Goswami in [Gos15], we give an explicit proof here.

Proposition 2.9
Let G be a compact group acting on a compact metric space (X, d) and let α be the corresponding coaction in the sense of Remark 1.12. Then the action of G on X is isometric if and only if α is isometric in the sense of Definition 2.8.

Proof. Let α be the coaction of $C(G)$ on $C(X)$; it is given by

$$f \mapsto ((x, g) \mapsto f(gx)).$$

The antipode κ on $C(G)$ is given by $\kappa(f)(g) := f(g^{-1})$. Then for all $x, y \in X$ and $g \in G$

$$\alpha(d_x(y))(g) = d(gx, y), \quad \kappa(\alpha(d_y)(x))(g) = d(g^{-1}y, x) = d(x, g^{-1}y),$$

so we have $\alpha(d_x(y)) = \kappa(\alpha(d_y)(x))$ for all $x, y \in X$ if and only if $d(gx, y) = d(x, g^{-1}y)$ for all $x, y \in X$ and $g \in G$. □

Since all finite metric spaces are especially compact we can compare Definition 2.8 to Definition 2.3. Goswami gave a proof for the following proposition using unitary representations, we give a more direct proof.

Proposition 2.10
Let (X, D) be a finite metric space and A a CQG acting faithfully on $C(X)$ via the action α with coefficient matrix a. Then $Da = aD$ if and only if $\alpha(d_x)(y) = \kappa(\alpha(d_y)(x))$ holds for all $x, y \in X$.

Proof. The action α is given by

$$\alpha(\delta_i) = \sum_{j=1}^{n} \left(\delta_j \otimes \alpha_{ij}\right)$$

for some coefficients α_{ij}. We write $d_{ij} := d(i, j) = D_{ij}$ and with this we can write the elements $d_i = d(\cdot, i) \in C(X)$ as

$$d_i = \sum_{k=1}^{n} d_{ik}\delta_k$$

so the action on the d_i is given by

$$\begin{aligned}\alpha(d_i) = \sum_{k=1}^{n} d_{ik}\alpha(\delta_k) &= \sum_{k=1}^{n} d_{ik} \sum_{j=1}^{n} \delta_j \otimes \alpha_{kj} \\ &= \sum_{j=1}^{n} \delta_j \otimes \left(\sum_{k=1}^{n} d_{ik}\alpha_{kj} \right).\end{aligned}$$

Note that $\alpha(d_i) = \alpha_r(d_i)$ for all i since $\alpha(d_i) \in C(X)\otimes A_0$ and the representation $\pi_r : A \to A_r$ is just the identity on the underlying Hopf algebra A_0. Thus on the left hand side of (2) in Definition 2.8 we have

$$\alpha(d_i)(j) = \sum_{k=1}^{n} d_{ik}\alpha_{kj}$$

for all $i, j \in X$. The antipode κ on A is given by $\kappa(\alpha_{ij}) = \alpha_{ji}{}^*$ and since the α_{ij} fulfil the relations of $C(S_n^+)$, the α_{ij} are selfadjoint and we even have $\kappa(\alpha_{ij}) = \alpha_{ji}$. So altogether we have

$$\kappa(\alpha(d_j)(i)) = \kappa\left(\sum_{k=1}^{n} d_{jk}\alpha_{ki} \right) = \sum_{k=1}^{n} d_{jk}\alpha_{ik} = \sum_{k=1}^{n} \alpha_{ik} d_{kj}$$

using the symmetry of D. Comparing the two terms we established the equivalence of the definitions. □

2.3 Compact metric spaces with the euclidean metric

The main goal of this section is to show that every isometric action of a CQG on a compact metric space $X \subset \mathbb{R}^n$ equipped with the euclidean metric is already affine in a sense to be defined. This motivates our definition of isometric actions on quantum versions of metric spaces in the next section. We follow the arguments of Goswami in [Gos15], reorganising the results to fit our needs and extend some of the proofs for increased readability. Let us fix some notation first. All our compact metric spaces X are now subsets of $\mathbb{R}^n$ equipped with the euclidean metric. We denote by $X_1, \dots, X_n$ the restrictions of the coordinate functions of $\mathbb{R}^n$ to X. For a point $x \in X$ we write $x_i := X_i(x)$ for the i-th component of x.

Let us first establish two preparatory lemmata.

Lemma 2.11
Let $X \subset \mathbb{R}^n$ be a compact metric space containing the 0 and suppose that the X_i are linearly independent. Let A be a C^*-algebra and $F_1, \dots F_n, G_1, \dots, G_n$ be functions from X to A such that

$$\sum_{i=1}^{n}(F_i(x) - q_i)y_i = \sum_{i=1}^{n}(G_i(y) - q_i')x_i$$

for all $x, y \in X$ where $q_i = F_i(0)$ and $q_i' = G_i(0)$. Then the F_i and G_i are affine, i.e. there are $\alpha_{ij} \in A$ such that $F_i(x) = q_i + \sum_{j=1}^{n} x_j \alpha_{ji}$, similarly for the G_i.

Proof. By replacing F_i with $F_i - q_i$ and G_i with $G_i - q_i'$ we can assume that $F_i(0) = G_i(0) = 0$. Take a linear combination of elements in X that is 0, i.e. $\sum_{k=1}^{l} c_k x^{(k)} = 0$ for some $c_k \in \mathbb{R}$ and $x^{(k)} \in X$. Then we have for any $y \in X$

$$\sum_{k=1}^{l} c_k \left(\sum_{i=1}^{n} F_i(x^{(k)}) y_i \right) = \sum_{k=1}^{l} c_k \left(\sum_{i=1}^{n} G_i(y) x_i^{(k)} \right) = \sum_{i=1}^{n} G_i(y) \sum_{k=1}^{l} c_k x_i^{(k)} = 0.$$

That means that for $Q_i := \sum_{k=1}^{l} c_k F_i(x^{(k)})$ we have $\sum_{i=1}^{n} Q_i y_i = 0$ for all $y \in X$. Expressed in terms of the tensor product $C(X) \otimes A$ we get $\sum_{i=1}^{n} X_i \otimes Q_i = 0$. But since the X_i are linearly independent, we have $Q_i = 0$ for all i. So for two equal linear combinations of elements $\sum_{k=1}^{l} c_k x^{(k)} = \sum_{j=1}^{t} d_j y^{(j)}$ we have $\sum_{k=1}^{l} c_k F_i(x^{(k)}) = \sum_{j=1}^{t} d_j F_i(y^{(j)})$. Therefore the linear extension $F_i\left(\sum_{k=1}^{l} c_k x^{(k)}\right) := \sum_{k=1}^{l} c_k F_i(x^{(k)})$ is well-defined on the span of the X_i. So there is a linear extension of F_i to the whole $\mathbb{R}^n$. Putting $\alpha_{ji} := F_i(e_j)$, the e_j being the canonical basis vectors of the $\mathbb{R}^n$, we get $F_i(x) = \sum_{j=1}^{n} x_j \alpha_{ji}$ as desired. By symmetry the same proof works for the G_i. □

Lemma 2.12
Let $X \subset \mathbb{R}^n$ be a compact set equipped with the euclidean metric and let X_i be the restriction of the coordinate functions to X. Then we can find new coordinate functions Y_i such that $\{Y_1, \dots Y_k\}$ is linearly independent and $d(x, y) = \sqrt{\sum_{i=1}^{k} (Y_i(x) - Y_i(y))^2}$ for all $x, y \in X$. Hence we can always assume that the coordinate functions on a metric space are linearly independent.

Proof. Let the first k coordinate functions be linearly independent and put

$$X_j = \sum_{i=1}^{k} d_{ji} X_i$$

for $j = k+1, \dots, n$.
If we put $D = (d_{jl})$ we can now write the metric in a matrix way:

$$\begin{aligned} d^2(x,y) = \sum_{i=1}^{n} (X_i(x) - X_i(y))^2 &= \sum_{i=1}^{k} (X_i(x) - X_i(y))^2 + \sum_{j=k+1}^{n} (X_j(x) - X_j(y)) \\ &= \sum_{i=1}^{k} (x_i - y_i)^2 + \sum_{j=k+1}^{n} \left(\sum_{i=1}^{k} d_{ji}(x_i - y_i) \right)^2 \\ &= \sum_{i=1}^{k} (x_i - y_i)^2 + \sum_{i,l=1}^{k} \left((x_i - y_i) \sum_{j=k+1}^{n} d_{ji} d_{jl} (x_l - y_l) \right) \\ &= Z^t (I_k + D^t D) Z, \end{aligned}$$

where Z is the k-dimensional vector $(x_i - y_i)$ and I_k is the identity matrix. Since $(I_k + D^t D)$ is symmetric and positive-definite ($d^2(x,y) \geq 0$ for all x, y in the span of X), there exists an invertible, symmetric, positive-definite root $C := \sqrt{I_k + D^t D} = (c_{ij})$. So choosing new coordinate functions $Y_i := \sum_{j=1}^{k} c_{ij} X_j$ for $i = 1, \dots, k$ we get

$$\begin{aligned} \sum_{i=1}^{k} (Y_i(x) - Y_i(y))^2 &= \sum_{i=1}^{k} \left(\sum_{j=1}^{k} c_{ij}(x_j - y_j) \right)^2 \\ &= \sum_{j,l=1}^{k} \left((x_j - y_j) \sum_{i=1}^{k} c_{ij} c_{il} (x_l - y_l) \right) \\ &= Z^t C^2 Z = Z^t (I_k + D^t D) Z = d^2(x,y) \end{aligned}$$

for all $x, y \in X$. □

From now on we always assume that the X_i are linearly independent. Also, by translating the set X, we can assume that $0 \in X$. Then the linear independence of $\{X_1, \dots, X_n\}$ implies the linear independence of $\{1, X_1, \dots, X_n\}$.
Now we can show that every isometric action is already affine. In [Gos15] this is done in the proof of Theorem 4.5.

Proposition 2.13
Let (X, d) be a compact metric space and $\alpha : C(X) \to C(X) \otimes A$ be a faithful isometric coaction of a CQG A on $C(X)$. Write $F_i^r := \alpha_r(X_i)$ for the reduced action α_r. Then the F_i^r are affine, i.e. there are $\alpha_{ij}, q_i \in A$ such that

$$F_i^r = \sum_{j=1}^{r} X_j \otimes \alpha_{ji} + 1 \otimes q_i.$$

Proof. Suppose that the action α is isometric, i.e. $\alpha_r(d_x)(y) = \kappa(\alpha_r(d_y)(x))$ for all $x, y \in X$. We can write

$$d_x^2 = \sum_{i=1}^{n} (x_i 1 - X_i)^2$$

so we have

$$\sum_{i=1}^{n} (x_i 1 - F_i^r(y))^2 = \alpha_r(d_x^2)(y) = \kappa(\alpha_r(d_y^2)(x)) = \sum_{i=1}^{n} (\kappa(F_i^r(x) - y_i 1))^2 \tag{3}$$

for all $x, y \in X$. Now we define $G_i(x) := \kappa(F_i^r(x))$, $q_i = F_i^r(0)$ and $q_i' = G_i(0) = \kappa(q_i)$. Putting $x = 0$ resp. $y = 0$ in (3) we get

$$\sum_{i=1}^{n} F_i^r(y)^2 = \sum_{i=1}^{n} (q_i' - y_i 1)^2, \quad \sum_{i=1}^{n} G_i^r(x)^2 = \sum_{i=1}^{n} (q_i - x_i 1)^2$$

and setting $x = 0 = y$ we get

$$\sum_{i=1}^{n} q_i^2 = \sum_{i=1}^{n} q_i'^2.$$

So, if we expand (3) we get

$$\begin{aligned}
&\sum_{i=1}^{n} (x_i 1 - F_i^r(y))^2 = \sum_{i=1}^{n} (\kappa(F_i^r(x) - y_i 1))^2 \\
\Leftrightarrow &\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} F_i^r(y)^2 - 2\sum_{i=1}^{n} x_i 1 F_i^r(y) = \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} G_i(x)^2 - 2\sum_{i=1}^{n} y_i 1 G_i(x) \\
\Leftrightarrow &\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} (q_i' - y_i 1)^2 - 2\sum_{i=1}^{n} x_i 1 F_i^r(y) = \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} (q_i - x_i 1)^2 - 2\sum_{i=1}^{n} y_i 1 G_i(x)
\end{aligned}$$

$$\Leftrightarrow -2\sum_{i=1}^{n}(q_i' y_i 1) - 2\sum_{i=1}^{n} x_i 1 F_i^r(y) = -2\sum_{i=1}^{n}(q_i x_i 1) - 2\sum_{i=1}^{n} y_i 1 G_i(x)$$

$$\Leftrightarrow \sum_{i=1}^{n} x_i 1(F_i^r(y) - q_i) = \sum_{i=1}^{n} y_i 1(G_i(x) - q_i').$$

So an application of Lemma 2.11 gives the result. □

Before we come to the main theorem of this section we need one more technical lemma which we do not prove here. A proof is given in [Gos15, Theorem 4.5]

Lemma 2.14

Let (X, d) be a compact metric space and $\alpha : C(X) \to C(X) \otimes A$ be a faithful coaction of a CQG A on $C(X)$. Write as before $F_i := \alpha(X_i)$ and $F_i^r := \alpha_r(X_i)$ for the corresponding reduced action. Denote by C_0 the spectral subalgebra of $C(X)$ and by A_0 the Hopf algebra associated to A (See Remarks 1.15 and 1.8). Then the following are equivalent:

i) The action α is isometric.

ii) We have $F_i \in C_0 \otimes A_0$ and

$$\sum_{i=1}^{n}(x_i 1 - F_i(y))^2 = \sum_{i=1}^{n}(\kappa(F_i(x) - y_i 1))^2$$

for all $x, y \in X$.

iii) We have $F_i \in C_0 \otimes A_0$ and

$$\sum_{i=1}^{n}(F_i(x)^2 + F_i(y)^2 - 2F_i(x)F_i(y)) = d^2(x, y)1$$

for all $x, y \in X$.

iv) We have $F_i \in C_0 \otimes A_0$ and

$$\sum_{i=1}^{n}(F_i(x) - F_i(y))^2 = d^2(x, y)1$$

for all $x, y \in X$.

In particular, $X_i \in C_0$, $\alpha_{ij} \in A_0$ and $F_i = F_i^r$ holds for all i and j.

Proof. See Theorem 4.5 in [Gos15]. □

We can now prove the main result of this section, namely that an isometric quantum group action on a compact space in euclidean space is automatically affine and orthogonal.

Theorem 2.15 ([Gos15, Corollary 4.6])
Let as before (X, d) be a compact metric space (containing the 0) with the euclidean metric and A a CQG acting faithfully on $C(X)$ via the action α. Then α is isometric if and only if the following conditions hold:

i) For all i we have
$$F_i := \alpha(X_i) = \sum_{j=1}^{n} X_j \otimes \alpha_{ji} + 1 \otimes \zeta_i$$
for some self-adjoint $\alpha_{ij}, \zeta_i \in A_0$.

ii) The matrix $a = (\alpha_{ij})$ is orthogonal.

iii) There are real numbers c_i such that
$$\zeta_i = c_i 1 - \sum_j c_j \alpha_{ji},$$
i.e.
$$\alpha(X_i - c_i) = \sum_j (X_j - c_j) \otimes \alpha_{ji}.$$

iv) (A, a) is a quantum subgroup of O_n^+, i.e. there is a surjective morphism from $C(O_n^+)$ to A sending the generators u_{ij} of O_n^+ to α_{ij}.

Proof. Let us first check the isometry of the action if the conditions i) - iv) are fulfilled. This follows directly from the calculation

$$\begin{aligned}\sum_{i=1}^{n}(F_i(x) - F_i(y))^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \alpha_{ji}(x_j - y_j)\right)^2 &= \sum_{j,l=1}^{n}(x_j - y_j)(x_l - y_l)\sum_{i=1}^{n} \alpha_{ji}\alpha_{li} \\ &= \sum_{j=1}^{n}(x_j - y_j)^2 = d^2(x, y),\end{aligned}$$

which is exactly condition iv) in Lemma 2.14.
Now let the action α be isometric. Statement i) we already established in Proposition 2.13 for the reduced action F_i^r and Lemma 2.14 gives the result for the F_i. Let us prove statement iii) next. We have to find numbers c_i such that $\zeta_i = c_i 1 - \sum_j c_j \alpha_{ji}$ holds for all i. Take any faithful state ϕ on $C(X)$, let h be the Haar state of A and consider the state $\tilde{\phi} := (\phi \otimes h) \circ \alpha$ on $C(X)$. Remember that h is α-invariant, i.e. $(\mathrm{id} \otimes h) \circ \Delta(a) = (h \otimes \mathrm{id}) \circ \Delta(a) = h(a)1$. Then also $\tilde{\phi}$ is α-invariant in the sense that $(\tilde{\phi} \otimes \mathrm{id}) \circ \alpha = \tilde{\phi}1$. Indeed,

$$\begin{aligned}(\tilde{\phi} \otimes \mathrm{id}) \circ \alpha &= ((\phi \otimes h) \circ \alpha \otimes \mathrm{id}) \circ \alpha \\ &= (\phi \otimes h \otimes \mathrm{id}) \circ (\alpha \otimes \mathrm{id}) \circ \alpha \\ &= (\phi \otimes h \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \alpha \\ &= (\phi \otimes (h \otimes \mathrm{id} \circ \Delta)) \circ \alpha \\ &= (\phi \otimes h1) \circ \alpha = \tilde{\phi}1,\end{aligned}$$

where we used the α-invariance of h and the associativity of α. So defining $c_i := \tilde{\phi}(X_i)$ we get

$$c_i 1 = (\tilde{\phi} \otimes \mathrm{id}) \circ \alpha(X_i) = (\tilde{\phi} \otimes \mathrm{id})(\sum_{j=1}^n X_j \otimes \alpha_{ji} + 1 \otimes \zeta_i) = \sum_{j=1}^n c_j \alpha_{ji} + \zeta_i$$

or equivalently

$$\zeta_i = c_i 1 - \sum_{j=1}^n c_j \alpha_{ji}$$

as desired.
Now let us check statement ii). We take again iv) of Lemma 2.14 with $y = 0$ to get

$$\sum_{i=1}^n (F_i(x) - F_i(y))^2 = \sum_{i=1}^n \left(\sum_{j=1}^n \alpha_{ji} x_j \right)^2 = \sum_{i=1}^n x_i^2 1. \tag{4}$$

If we expand condition iii) of Lemma 2.14 we get

$$\sum_{i=1}^n (F_i(x)^2 + F_i(y)^2 - 2F_i(x)F_i(y)) = \left(\sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 - 2 \sum_{i=1}^n x_i y_i \right) 1.$$

Now expanding the left hand side using the fact that the F_i are affine and using (4) we get after some rearranging

$$\sum_{j=1}^{n}(x_j - y_j)\left(\sum_{i=1}^{n}(\zeta_i\alpha_{ji} - \alpha_{ji}\zeta_i)\right) - 2\sum_{j,k=1}^{n} x_j y_k \left(\sum_{i=1}^{n}\alpha_{ji}\alpha_{ki}\right) = -2\left(\sum_{j=1}^{n} x_j y_j\right)1 \tag{5}$$

for all $x, y \in X$. Putting again $y = 0$ we have

$$\sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n}(\zeta_i\alpha_{ji} - \alpha_{ji}\zeta_i)\right) = 0$$

and since the coordinate functions X_i are linearly independent we can conclude as in the proof of Lemma 2.11 that $\sum_{i=1}^{n}(\zeta_i\alpha_{ji} - \alpha_{ji}\zeta_i) = 0$ for all j. This means that (5) is now

$$\sum_{j,k=1}^{n} x_j y_k \left(\sum_{i=1}^{n}\alpha_{ji}\alpha_{ki}\right) = \left(\sum_{j=1}^{n} x_j y_j\right)1.$$

But using the linear independence of the X_i again for fixed $y \in X$ this means that

$$\sum_{k=1}^{n} y_k \left(\sum_{i=1}^{n}\alpha_{ji}\alpha_{ki}\right) = y_j 1$$

for all $y \in X$. Now, using the linear independence of the X_i a third time we get

$$\left(\sum_{i=1}^{n}\alpha_{ji}\alpha_{ki}\right) = \delta_{jk}1, \tag{6}$$

which is the first part of the orthogonality of the matrix (α_{ij}). But statement iii) now shows that α acts linearly on the shifted space generated by $Y_1 := X_1 - c_1, \ldots, Y_n = X_n - c_n$ and thus induces a non-degenerate finite dimensional representation with invertible matrix (α_{ji}). (See [Gos15]). But (6) shows that $(\alpha_{ji})^t$ is a one-sided inverse, so it has to be the two-sided inverse, too. That means that also

$$\left(\sum_{i=1}^{n}\alpha_{ij}\alpha_{ik}\right) = \delta_{jk}1$$

holds.

Finally, statement iv) follows directly from the universality of $C(O_n^+)$. □

Remark 2.16
Goswami showed that the numbers c_i do not depend on the action α or even the CQG A but only on the metric space (X, d).

Intuitively we would expect the CQG A to be a compact matrix quantum group as it acts affine on $C(X)$ and is a subgroup of the O_n^+. The next lemma, which we could not find in the literature, shows that this is actually the case.

Lemma 2.17
Let (A, Δ) be a CQG acting faithfully on a C^*-algebra B via the action α given by

$$\alpha(X_i) = \sum_{j=1}^{n} X_j \otimes \alpha_{ji}$$

for some finitely many linear independent generators X_i of B. Furthermore, let the matrix (α_{ij}) be orthogonal. Then (A, Δ) is a CQMG and especially a quantum subgroup of O_n^+.

Proof. Since the α_{ij} satisfy the orthogonality relations and generate A as a C^*-algebra we only have to check that

$$\Delta(\alpha_{ij}) = \sum_{k=1}^{n} \alpha_{ik} \otimes \alpha_{kj}.$$

But using the coassociativity we get

$$\begin{aligned}
0 &= ((\alpha \otimes \mathrm{id}) \circ \alpha)(x_i) - ((\mathrm{id} \otimes \Delta) \circ \alpha)(X_i) \\
&= (\alpha \otimes \mathrm{id})(\sum_{j=1}^{n} X_j \otimes \alpha_{ji}) - (\mathrm{id} \otimes \Delta)(\sum_{k=1}^{n} X_k \otimes \alpha_{ki}) \\
&= \sum_{j,k=1}^{n} X_k \otimes \alpha_{kj} \otimes \alpha_{ji} - \sum_{k=1}^{n} X_k \otimes \Delta(\alpha_{ki}) \\
&= \sum_{k=1}^{n} X_k \otimes \left(\sum_{j=1}^{n} \alpha_{kj} \otimes \alpha_{ji} - \Delta(\alpha_{ki}) \right).
\end{aligned}$$

Since $\{X_1, \ldots, X_n\}$ is linearly independent, we get $\sum_{j=1}^{n} \alpha_{kj} \otimes \alpha_{ji} = \Delta(\alpha_{ki})$ for all $i = 1, \ldots n$, so (A, Δ) is CMQG. The existence of the surjective morphism from $C(O_n^+)$ to A follows from the universality of $C(O_n^+)$. □

Remark 2.18

Note that the characterization of isometry in Theorem 2.15 also holds for metric spaces $X \subset \mathbb{R}^n$ that do not contain the 0. We can simply shift one point of the metric space into the origin, use Theorem 2.15 there and shift it back. This will still result in an affine action.

Definition 2.19

Let (X, d) be a compact metric space. The *quantum isometry group* of (X, d) is a CQG (A, Δ) that acts isometrically on $C(X)$ with the action α and fulfils the following universal property: If there is any CQG (B, Δ') acting isometrically on $C(X)$ via the action β there is a morphism of quantum groups $\phi : A \to B$ such that $\beta = (\mathrm{id} \otimes \phi) \circ \alpha$, i.e. every isometric quantum group action factorizes through A.

As an example let us look at the finite space given by the vertices of the d-dimensional hypercube and see if we get the same quantum isometry group O_d^{-1} as Banica, Bichon and Collins in [BBC07] (Proposition 2.9 tells us that this should be the case.)

Definition 2.20

The d-dimensional hypercube $X \subset \mathbb{R}^n$ is given by all the points $x \in \mathbb{R}^n$ with entries $x_i = 0$ or $x_i = 1$ for all i. With that choice, $C(X) = C^*(X_1, \ldots, X_d | X_i X_j = X_j X_i, X_i^* = X_i, X_i^2 = X_i)$. Note that in this way the centre of symmetry is not the origin.

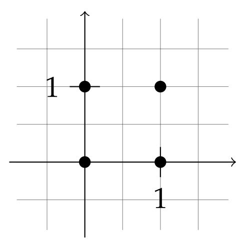

Figure 1: The 2-dimensional hypercube: the square

Proposition 2.21
The CMQG O_d^{-1} is the quantum isometry group of the d-dimensional hypercube X via the action

$$\alpha(X_i) = \sum_{j=1}^{d} X_j \otimes \alpha_{ji} + 1 \otimes \zeta_i$$

with $\zeta_i := \frac{1}{2} - \sum_{j=1}^{d} \frac{1}{2}\alpha_{ji}$.

Proof. Let α be the action of a CQG A acting isometrically on X. By Theorem 2.15 there are selfadjoint coefficients $\alpha_{ij} \in A$ and $\zeta_i \in A$ such that

$$\alpha(X_i) = \sum_{j=1}^{d} X_j \otimes \alpha_{ji} + 1 \otimes \zeta_i$$

for all i. Note that the coordinate functions X_i of X fulfil the relations $X_i^2 = X_i$ and $X_iX_j = X_jX_i$. So $\alpha(X_i) = \alpha(X_i)^2$ has to hold, i.e.

$$\sum_{j=1}^{d} X_j \otimes \alpha_{ji} + 1 \otimes \zeta_i = \sum_{j,k=1}^{d} X_jX_k \otimes \alpha_{ji}\alpha_{ki} + \sum_{j=1}^{d} X_j \otimes (\alpha_{ji}\zeta_i + \zeta_i\alpha_{ji}) + 1 \otimes \zeta_i^2. \tag{7}$$

Using $X_iX_j = X_jX_i$ and $X_i^2 = X_i$ we can transform the right hand side to

$$\sum_{j<k}^{d} X_jX_k \otimes (\alpha_{ji}\alpha_{ki} + \alpha_{ki}\alpha_{ji}) + \sum_{j=1}^{d} X_j \otimes (\alpha_{ji}\zeta_i + \zeta_i\alpha_{ji} + \alpha_{ji}^2) + 1 \otimes \zeta_i^2. \tag{8}$$

The set $\{X_jX_k, X_k, 1 \mid j < k = 1, \ldots, d\}$ is linearly independent: Let $\sum_{j<k} \lambda_{jk}X_jX_k + \sum_{k=1}^{d} \lambda_k X_k + \lambda = 0$ for some $\lambda_i \in \mathbb{R}$. Plugging in the point $0 \in \mathbb{R}^d$ gives $\lambda = 0$ and the canonical basis vectors $e_k \in X$ provide $\lambda_k = 0$. Finally plugging in the point x with $x_k = x_j = 1$ and $x_i = 0$ otherwise, we get $\lambda_{jk} = 0$. So comparing (8) with the left hand side of (7) we get the relations

$$\alpha_{ji}\alpha_{ki} + \alpha_{ki}\alpha_{ji} = 0, \quad \forall i, j \neq k \tag{9}$$

$$\alpha_{ji}\zeta_i + \zeta_i\alpha_{ji} + \alpha_{ji}^2 = \alpha_{ji}, \quad \forall i, j \tag{10}$$

$$\zeta_i^2 = \zeta_i, \quad \forall i. \tag{11}$$

Now looking at the commutativity relations we have

$$\begin{aligned}\alpha(X_iX_j) &= \alpha(X_i)\alpha(X_j)\\ &= \sum_{k,l}^{d} X_kX_l\otimes\alpha_{ki}\alpha_{lj} + \sum_{k=1}^{d} X_k\otimes\alpha_{ki}\zeta_j + \sum_{l=1}^{d} X_l\otimes\zeta_i\alpha_{lj} + 1\otimes\zeta_i\zeta_j\\ &= \sum_{k<l} X_kX_l\otimes(\alpha_{ki}\alpha_{lj}+\alpha_{li}\alpha_{kj}) + \sum_{l} X_l\otimes(\alpha_{li}\zeta_j+\zeta_i\alpha_{lj}+\alpha_{li}\alpha_{lj}) + 1\otimes\zeta_i\zeta_j.\end{aligned}$$

Now using $X_iX_j = X_jX_i$ and again the linear independence of $\{X_jX_k, X_k, 1 \mid j < k = 1,\dots,d\}$ we get for $i \neq j$ the relation

$$\alpha_{ki}\alpha_{lj} + \alpha_{li}\alpha_{kj} = \alpha_{kj}\alpha_{li} + \alpha_{lj}\alpha_{ki}, \quad \forall k < l \tag{12}$$

but since the expression is symmetric in k, l it is true for all $k \neq l$ as well. The antipode on the dense Hopf-Algebra of the CMQG acting on the lemon is given by $\kappa(\alpha_{ij}) = \alpha_{ji}$ and is antimultiplicative. Using the antipode on (12) gives

$$\alpha_{jl}\alpha_{ik} + \alpha_{jk}\alpha_{il} = \alpha_{il}\alpha_{jk} + \alpha_{ik}\alpha_{jl}, \quad \forall k \neq l, i \neq j \tag{13}$$

Interchanging k and i, resp. l and j in (13) we get

$$\alpha_{lj}\alpha_{ki} + \alpha_{li}\alpha_{kj} = \alpha_{kj}\alpha_{li} + \alpha_{ki}\alpha_{lj}$$

and subtracting this from (12) gives

$$\alpha_{ki}\alpha_{lj} - \alpha_{lj}\alpha_{ki} = \alpha_{lj}\alpha_{ki} - \alpha_{ki}\alpha_{lj} = -(\alpha_{ki}\alpha_{lj} - \alpha_{lj}\alpha_{ki}) = 0, \quad \forall i \neq j, k \neq l, \tag{14}$$

so together with relation (9) the α_{ij} fulfil the relations of $C(O_d^{-1})$ and thus the CQG acting on X is a quantum subgroup of O_d^{-1}.
An isometric action of O_d^{-1} on X is given by

$$\alpha(X_i) = \sum_{j=1}^{d} X_j\otimes\alpha_{ji} + 1\otimes\zeta_i,$$

so we have to find matching elements $\zeta_i \in C(O_d^{-1})$ or, equivalently, by the third isometry condition in Theorem 2.15, find numbers c_i such that with $\zeta_i := c_i 1 - \sum_j c_j\alpha_{ji}$ an action can be defined.

Let us look at relation (10) and let us compute the term $\alpha_{ji}\zeta_i + \zeta_i\alpha_{ji}$. We get

$$\begin{aligned}\alpha_{ji}\zeta_i + \zeta_i\alpha_{ji} &= c_i\alpha_{ji} - \sum_{k=1}^{d} c_k\alpha_{ji}\alpha_{ki} + c_i\alpha_{ji} - \sum_{k=1}^{d} c_k\alpha_{ki}\alpha_{ji} \\ &= 2c_i\alpha_{ji} - 2c_j\alpha_{ji}^2 - \sum_{k\neq j}^{d} c_k\alpha_{ji}\alpha_{ki} - c_k\alpha_{ki}\alpha_{ji} \\ &= 2c_i\alpha_{ji} - 2c_j\alpha_{ji}^2,\end{aligned}$$

where we used $\alpha_{ji}\alpha_{ki} + \alpha_{ki}\alpha_{ji} = 0$ for $k \neq j$ in the last step. So now we have the relation $2c_i\alpha_{ji} - 2c_j\alpha_{ji}^2 + \alpha_{ji}^2 = \alpha_{ji}$ which can be fulfilled by putting $c_i = \frac{1}{2}$ for all i. With this choice we also have $\zeta_i = \zeta_i^2$, so we have an action of O_d^{-1} on the d-dimensional hypercube which finishes the proof. □

Another example of a quantum isometry group on a classical metric space was calculated by Goswami. We only present the result here, for the proof we refer to [Gos15].

Example 2.22
Let $T_d \subset \mathbb{R}^d$ be the metric space given by gluing d-copies of the intervall $[-1, 1]$ together at the origin. The quantum isometry group of T_d is the hyperoctahedral quantum group H_d^+ which is given by the universal C^*-algebra

$$C(H_d^+) := C^*\left(\alpha_{ij}, 1 \leq i, j \leq d \,\middle|\, \alpha_{ij} = \alpha_{ij}^*, (\alpha_{ij}) \text{ is orthogonal, } \alpha_{ij}\alpha_{ik} = \alpha_{ji}\alpha_{ki} = 0, k \neq j\right)$$

as defined in [BBC07].

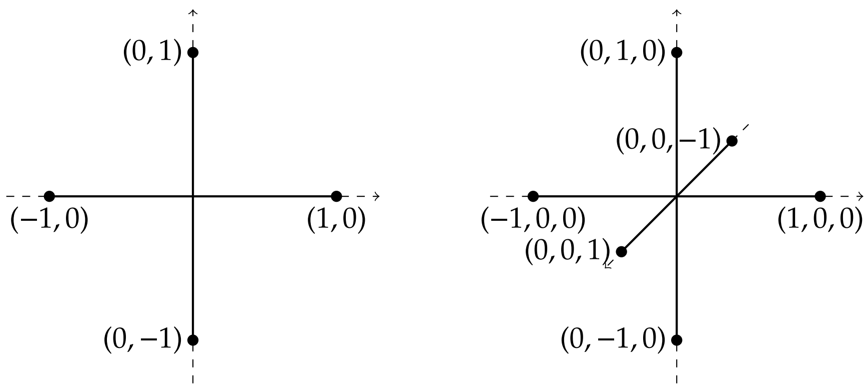

Figure 2: The metric spaces T_2 and T_3

Remark 2.23

Note that T_d and the d-dimensional hypercube have the same isometry group H_d but different quantum isometry groups.

3 Quantum versions of classical spaces and their symmetries

We now switch our focus to quantum actions on non-commutative C^*-algebras. Instead of defining the notion of an arbitrary quantum metric space (that is the focus of the next section) we start with a classical metric space $X \subset \mathbb{R}^n$ and define the *quantum version* of this space. In that way we still have the geometric intuition of the classical space to work with. The definition of quantum versions of metric spaces we give is new although the idea of dropping commutativity from the relations on the functions of a classical space is well-known. We give a formal framework to work with these kind of spaces and to calculate their symmetries. With Goswami's characterization of Theorem 2.15 of an isometric quantum action we define what an isometric quantum action on such a quantum version of a metric space should be. Then we calculate the quantum isometry group of some (old and new) quantum versions of classical spaces.

3.1 Quantum versions of metric spaces

Definition 3.1
Let $X \subset \mathbb{R}^n$ be a compact metric space in an n-dimensional space (equipped with the euclidean metric) defined by some algebraic relations $\mathcal{R}$ on the coordinate functions not explicitly imposing commutativity. Then we call the universal C^*-algebra

$$C(X^+) := C^*(1, x_1, x_2, \ldots, x_n | x_i = x_i^*, \mathcal{R})$$

a *quantum version* of the metric space X (if it exists). It is called *genuine quantum* if $C(X^+)$ is not a commutative algebra.
We call a quantum version $C(X^+)$ *maximal* if the set of relations $\mathcal{R}$ is minimal in the sense that any subset of relations $\mathcal{R}' \nsubseteq \mathcal{R}$ defines a different metric space than $C(X)$.

Remark 3.2

- There may exist different quantum versions of the same metric space since the set of relations $\mathcal{R}$ on the coordinate functions is not uniquely defined.
- It might happen that all quantum versions of a classical space are already commutative, i.e. $C(X^+) = C(X)$, independent of the choice of the relations $\mathcal{R}$ (for example

in the 1-dimensional case).

- Since the quantum versions of metric spaces are given by universal C^*-algebras the existence of the quantum version might by unclear. But for a given space there always exists at least one universal C^*-algebra with the given relations: Since X is a compact metric space, there exists a constant $C \in \mathbb{R}$ such that $x_i < C$ holds for all coordinate functions x_i. If we add the relations $x_i < C$ to the set of relations R then the norms of the generators x_i are bounded by C and the universal C^*-algebra with generators x_i exists (of course it may still be commutative).

- Intuitively we want to consider maximal quantum versions so we do not assume "unnecessary" relations on the generators and get the largest quantum isometry group of a non-commutative version of a space.

Remark 3.3
In Lemma 2.12 we showed that for a compact metric space the coordinate functions can always be chosen linearly independent. Thus also in the quantum version of the metric space we can assume that the set $\{1, x_1, \ldots, x_n\}$ is linearly independent.

Example 3.4
The functions on the d-dimensional real sphere S^{d-1} are given by the universal C^*-algebra

$$C(S^{d-1}) = C^*\left(x_1, \ldots, x_d \,\middle|\, x_i = x_i^*,\, x_i x_j = x_j x_i,\, \sum_{i=1}^{d} x_i^2 = 1\right).$$

The *free* d-dimensional sphere S_+^{d-1} is given by the universal C^*-algebra

$$C(S_+^{d-1}) = C^*\left(x_1, \ldots, x_d \,\middle|\, x_i = x_i^*,\, \sum_{i=1}^{d} x_i^2 = 1\right)$$

and the *half-liberated* d-dimensional sphere S_*^{d-1} is given by the universal C^*-algebra

$$C(S_*^{d-1}) = C^*\left(x_1, \ldots, x_d \,\middle|\, x_i = x_i^*,\, x_i x_j x_k = x_k x_j x_i,\, \sum_{i=1}^{d} x_i^2 = 1\right).$$

Both were defined by Banica and Goswami in [BG10] and are quantum versions of the classical d-dimensional sphere in the sense of Definition 3.1, S_+^{d-1} being a maximal quantum version.

3.2 Isometric actions on quantum versions

Now we want to define an isometric action on such a quantum version of a classical compact space. Since we are not defining some kind of metric on the space (see the next section for this approach) we use Goswami's characterization of Theorem 2.15 of isometric quantum group actions from the previous section to *define* isometric actions on quantum versions.

Definition 3.5
Let $C(X^+)$ be a quantum version of a compact space X. We call an action α of a CQG A on $C(X^+)$ *affine isometric*, if the following conditions hold:

1. $\alpha(x_i) = \sum_j x_j \otimes \alpha_{ji} + 1 \otimes \zeta_i$ for some (necessarily selfadjoint) $\alpha_{ij}, \zeta_i \in A$.

2. The matrix $a = (\alpha_{ij})$ is orthogonal.

3. There are real numbers c_i such that

$$\zeta_i = c_i 1 - \sum_j c_j \alpha_{ji}$$

   i.e.

$$\alpha(x_i - c_i 1) = \sum_j (x_j - c_j 1) \otimes \alpha_{ji}.$$

Remark 3.6
The first property basically fixes a coordinate system for the non-commutative space while the second property ensures that the action is isometric. Because of the ζ_i the action can be affine so the centre of the coordinate system can be chosen freely.

Remark 3.7
By Lemma 2.17 we know that if α is a faithful action, then A is actually a CMQG.

3.3 Examples of quantum versions of metric spaces

Now we want to calculate the quantum isometry group of quantum versions of some classical compact spaces. Preferably these are objects which are already given by some algebraic relations on the coordinates of the space. The definition of a quantum isometry group is the same as in Definition 2.19. Our first example is new and is the d-dimensional lemon in $\mathbb{R}^d$. It is given by the equation

$$\sum_{i=1}^{d-1} x_i^2 = x_d^3(1 - x_d)^3.$$

With this relation the restrictions of the coordinate functions on the lemon are linearly independent. In this coordinate system the lemon is not centred at the origin but one of the cusps of the lemon is located at the origin. Note that the set of relations defining the space is of course minimal.

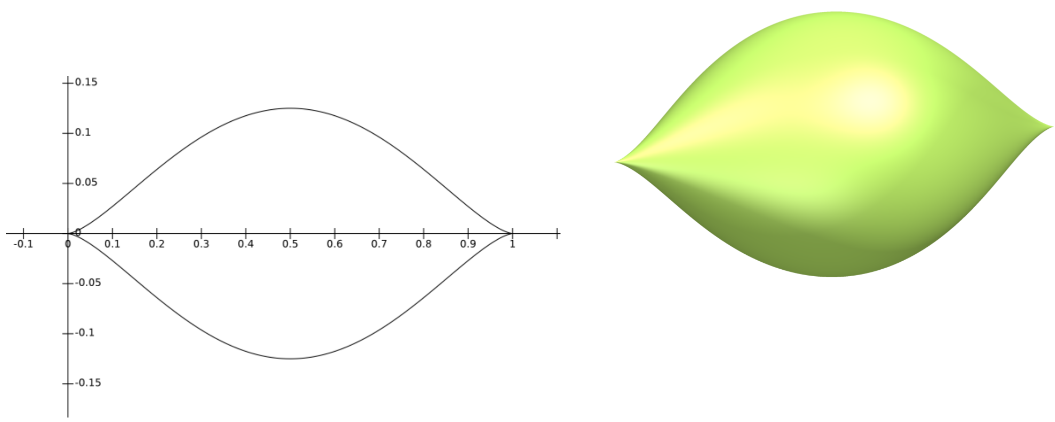

Figure 3: The lemons of dimension 2 and 3

Remark 3.8
The isometry group of the classical d-dimensional lemon is given by $O_{d-1} \times \mathbb{Z}_2$ (rotation around the x_d axis and flipping in the middle).

Lemma 3.9
The universal C^*-algebra

$$C(X^+) := C^*\left(x_1, x_2, \dots, x_d | x_i = x_i^*, \sum_{i=1}^{d-1} x_i^2 = x_d^3(1-x_d)^3\right)$$

exists and is a maximal quantum version of the classical d-dimensional lemon.

Proof. We have to show that semi-norms on the free algebra on d generators subject to the relation

$$\sum_{i=1}^{d-1} x_i^2 = x_d^3(1-x_d)^3$$

are bounded on the generators. On the left-hand side we are adding $d-1$ positive operators so the right-hand side is positive, too. Therefore

$$x_d(1-x_d) \geq 0 \Rightarrow x_d \geq x_d^2 \geq 0$$

and with x_d and $x_d(1-x_d)$ being positive we see that $1-x_d$ is positive as well. So $\|x_d\|$ is bounded by 1 and with $1 \geq 1-x_d \geq 0$ the norm of $1-x_d$ is bounded by 1 as well. So

$$\|x_j\|^2 = \|x_j^2\| \leq \left\|\sum_{i=1}^{d-1} x_j^2\right\| = \|(x_d(1-x_d))^3\| \leq \|x_d(1-x_d)\|^3 \leq 1$$

and the semi-norms are bounded (and by looking at the classical metric space we see that $\|x_i\| = 1$ for all i.) Since the set of relations is minimal (it only consists of one relation) $C(X^+)$ is a maximal quantum version of the d-dimensional lemon. □

Proposition 3.10
The quantum isometry group (A, Δ) of $C(X^+)$ is a quantum subgroup of the CMQG $O_{d-1}^+ * \mathbb{Z}_2$, i.e. there is a surjective morphism of quantum groups from $C(O_{d-1}^+) * C(\mathbb{Z}_2)$ to A.

Before proving the theorem we will recall the definition (or rather an equivalent description) of the free product of CQG given by universal C^*-algebras. For details see [Wan95].

Definition 3.11
Let $A = C^*\left(\alpha_{ij},\ i,j = 1,\dots,n \,\middle|\, \mathcal{R}\right)$ and $A' = C^*\left(\beta_{ij},\ i,j = 1,\dots,m \,\middle|\, \mathcal{R}'\right)$ be two unital CQGs given by universal C^*-algebras. Then the free product $A * A'$ is given by the universal C^*-algebra

$$A * A' = C^*\left(\alpha_{ij},\ \beta_{kl} \quad i,j = 1,\dots,n,\ k,l = 1,\dots,m \,\middle|\, \mathcal{R}, \mathcal{R}', 1_A = 1_{A'}\right)$$

and also has the structure of a CQG. If $(A, (\alpha_{ij}) =: u)$ and $(A', (\beta_{ij}) =: u')$ are CMQGs, then $A * A'$ is also a CMQG with matrix

$$\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}.$$

Now let us prove Proposition 3.10. The proof uses the diamond lemma of ring theory by Bergman [Ber78]. The diamond lemma provides a way to construct a basis for the universal algebra generated by finitely many generators subject to some algebraic relations. This is done by defining a partial order on the monomials and using the given relations to reduce all monomials to a minimal form. The in this sense irreducible monomials then form a basis of the algebra. We will not include the explicit calculations. For details on the diamond lemma and further applicatons see [Ber78].

Proof. Let (A, Δ) be a quantum group acting faithfully affine isometric on $C(X^+)$. Then by Lemma 2.17 we see that (A, Δ) is a CMQG with matrix (α_{ij}) and the action is given by

$$\alpha(x_i) = \sum_{j=1}^{d} x_j \otimes \alpha_{ji} + 1 \otimes \zeta_i$$

for some ζ_i in A.
The defining relation of the lemon gives that under the action

$$\alpha(\sum_{i=1}^{d-1} x_i^2) = \sum_{i=1}^{d-1} \alpha(x_i)^2 = \sum_{j,k} x_j x_k \otimes \sum_{i=1}^{d-1} \alpha_{ij}\alpha_{ik} + \sum_{l} x_l \otimes \sum_{i=1}^{d-1} (\alpha_{il}\zeta_i + \zeta_i\alpha_{il}) + 1 \otimes \sum_{i=1}^{d-1} \zeta_i^2$$

has to be equal to

$$\alpha(x_d)^3\alpha(1-x_d)^3 = -\sum_{i=1}^{d} x_i^6 \otimes \alpha_{di}^6 + \sum_{j\in J} z_j \otimes \beta_j$$

where the $z_j \neq x_i^6$ for all i, j are monomials in the x_i of degree ≤ 6 and the β_j are some elements of A. Using the relation $-x_d^6 = \sum_{i=1}^{d-1} x_i^2 - 3x_d^5 + 3x_d^4 - x_d^3$ and the diamond lemma of Bergman [Ber78] we can even get

$$\alpha(x_d)^3\alpha(1-x_d)^3 = -\sum_{i=1}^{d-1} x_i^6 \otimes \alpha_{di}^6 + \sum_{j\in J} z_j \otimes \beta_j$$

for some new z_j and β_j and an index set J chosen such that the set $\{x_i^6, z_j \,|1 \leq i \leq d-1, j \in J\}$ is linearly independent. So if we look at the x_i^6 summand for $i \neq d$, we can see that α_{di}^6 has to be 0. Since the α_{ij} are all self-adjoint we get $\alpha_{di} = 0$ for $i \neq d$. Because of the orthogonality of the matrix a we get $\alpha_{dd}^2 = 1$ and therefore $\alpha_{id} = 0$ for all $i \neq d$. So the coefficients in the upperleft matrix $a' = (\alpha_{ij})$ for $1 \leq i, j \leq d-1$ fulfil the relations of $C(O_{d-1}^+)$ and α_{dd} is a reflection. By the definition of the free product of quantum groups we get a surjective morphism

$$C(O_{d-1}^+) * C(\mathbb{Z}_2) \to A$$

So every CQG that acts faithfully affine isometric on $C(X^+)$ is a quantum subgroup of $O_{d-1}^+ * \mathbb{Z}_2$. □

Before we show that $O_{d-1}^+ * \mathbb{Z}_2$ acts on $C(X^+)$, we will establish two preparatory (and quite technical) lemmas.

Lemma 3.12
Let $a = (\alpha_{ij})$ be the canonical generating matrix of the CMQG O_{d-1}^+. Then the set $\{1, \alpha_{1l}, \dots, \alpha_{(d-1)l}\}$ is linearly independent for all $l = 1, \dots, (d-1)$.

Proof. Take a linear combination of the elements that is equal to 0, i.e. $\sum_{i=1}^{d-1} \lambda_i \alpha_{il} + \lambda_d = 0$. By the surjective morphism $C(O_{d-1}^+) \to C(O_{d-1})$, this relation also holds for the coefficients of all matrices in the classical matrix group O_{d-1}. But since there is an orthogonal matrix with the e_i unit vector as the l-th column for all $i = 1, \dots, d-1$, we get that $\lambda_i = 0$ for all $i = 1, \ldots, d-1$ and so $\lambda_d = 0$ as well. □

Lemma 3.13
Let $A = (C(O^+_{d-1}) * C(\mathbb{Z}_2), (\alpha_{ij}))$ be acting isometrically on the quantum version of the d-dimensional lemon $C(X^+)$ via

$$\alpha(x_i) = \sum_j x_j \otimes \alpha_{ji} + 1 \otimes \zeta_i$$

and $a = (\alpha_{ij})$ being the matrix for the free product as defined in Definition 3.11. Then we have for the action α the equations

$$\alpha\left(\sum_{i=1}^{d-1} x_i^2\right) = (-x_d^6 + 3x_d^5 - 3x_d^4 + x_d^3) \otimes 1 + \sum_l x_l \otimes \sum_{i=1}^{d-1} (\alpha_{il}\zeta_i + \zeta_i\alpha_{il}) + 1 \otimes \sum_{i=1}^{d-1} \zeta_i^2 \tag{15}$$

and

$$\begin{aligned}\alpha(x_d)^3\alpha(1-x_d)^3 = &- x_d^6 \otimes 1 + x_d^5 \otimes 3\hat{r}_d - x_d^4 \otimes 3\hat{r}_d^2 + x_d^4 \otimes 3\tilde{r} \\ &+ x_d^3 \otimes \hat{r}_d^3 - x_d^2 \otimes 3\tilde{r}^2 + x_d^2 \otimes 3\hat{r}_d^2\tilde{r} + x_d \otimes 3\hat{r}_d\tilde{r}^2 + 1 \otimes \tilde{r}^3\end{aligned} \tag{16}$$

with $\tilde{r} := \zeta_d(1-\zeta_d)$ and $\hat{r}_d := \alpha_{dd}(1-\zeta_d) - \zeta_d\alpha_{dd}$. Furthermore (15) and (16) are equal.

Proof. The equality of the terms of course follows from the defining relation of $C(X^+)$. Let us look at (15) first. By expanding we get

$$\alpha\left(\sum_{i=1}^{d-1} x_i^2\right) = \sum_{j,k} x_j x_k \otimes \sum_{i=1}^{d-1} \alpha_{ij}\alpha_{ik} + \sum_l x_l \otimes \sum_{i=1}^{d-1} (\alpha_{il}\zeta_i + \zeta_i\alpha_{il}) + 1 \otimes \sum_{i=1}^{d-1} \zeta_i^2.$$

Now the upper left $(d-1)\times(d-1)$ block in a is an orthogonal matrix, so $\sum_{i=1}^{d-1} \alpha_{ij}\alpha_{ik} = \delta_{jk}1$. The first term therefore collapses to $\sum_{j=1}^{d-1} x_j^2 \otimes 1$ and using $\sum_{j=1}^{d-1} x_j^2 = -x_d^6 + 3x_d^5 - 3x_d^4 + x_d^3$ we get (15).
For equation (16) first notice that $\alpha_{di} = \alpha_{id} = 0$ for all $i \neq d$. So the action on x_d is given by

$$\alpha(x_d) = x_d \otimes \alpha_{dd} + 1 \otimes \zeta_d, \qquad \alpha(1-x_d) = -x_d \otimes \alpha_{dd} + 1 \otimes (1-\zeta_d).$$

Now multiplying $\alpha(x_d)^3\alpha(1-x_d)^3$ out (we spare the details), using $\alpha_{dd}^2 = 1$ and setting $\tilde{r} := \zeta_d(1-\zeta_d)$ and $\hat{r}_d := \alpha_{dd}(1-\zeta_d) - \zeta_d\alpha_{dd}$ we get (16). □

Proposition 3.14
The CMQG $O^+_{d-1} * \mathbb{Z}_2$ acts affine isometrically on $C(X^+)$.

Proof. To show that $O^+_{d-1} * \mathbb{Z}_2$ acts affine isometrically on $C(X^+)$ we have to find ζ_i such that

$$\begin{aligned} C(X^+) &\to C(X^+) \otimes \left(C(O^+_{d-1}) * C(\mathbb{Z}_2)\right) \\ x_i &\mapsto \sum_j x_j \otimes \alpha_{ji} + 1 \otimes \zeta_i \end{aligned}$$

is an action, the α_{ij} fulfilling again the relations that in the matrix $a = (\alpha_{ij})$ the upper left $(d-1)\times(d-1)$ block (denoted by a') is precisely the matrix of O^+_{d-1} and the α_{dd} entry is a reflection, i.e. $\alpha_{dd}^2 = 1$. By the third isometry condition we can set

$$\zeta_i = c_i 1 - \sum_{j=1}^{d} c_j \alpha_{ji},$$

so we have to find matching numbers c_i.
Step 1: Calculating c_d.
We calculated the action in Lemma 3.13. By subtracting the equations (15) and (16) we get

$$\begin{aligned} 0 =& 3x_d^5 \otimes (1-\hat{r}_d) - 3x_d^4 \otimes (1-(\hat{r}_d^2-\tilde{r})) + x_d^3 \otimes (1-\hat{r}_d^3) + 3x_d^2 \otimes (\tilde{r}^2 - \hat{r}_d^2\tilde{r}) \\ &+ \sum_{l=1}^{d-1} x_l \otimes \sum_{i=1}^{d-1} (\alpha_{il}\zeta_i + \zeta_i\alpha_{il}) - x_d \otimes 3\hat{r}_d\tilde{r}^2 + 1 \otimes \left(\sum_{i=1}^{d-1} \zeta_i^2 - \tilde{r}^3\right). \end{aligned} \tag{17}$$

Since all the x_i^k that are left are linearly independent (again by the diamond lemma) all the second components of the tensors have to be equal to zero. The x_d^5-term then gives

$$1 = \hat{r}_d = \alpha_{dd}(1-\zeta_d) - \zeta_d\alpha_{dd}$$

and by using $\zeta_d = c_d 1 - c_d\alpha_{dd}$ we get

$$\begin{aligned} 1 = \alpha_{dd} - 2\alpha_{dd}c_d + 2\alpha_{dd}^2 c_d &= \alpha_{dd}(1-2c_d) + 2c_d \\ \Rightarrow 1 - 2c_d &= \alpha_{dd}(1-2c_d) \end{aligned}$$

and since $a_{dd} \neq 1$ we need to have $c_d = \frac{1}{2}$ and thus

$$\zeta_d = \frac{1}{2} - \frac{1}{2}\alpha_{dd}.$$

Step 2: Calculating the other c_i.
One can check that now $\hat{r}_d = 1$ and $\tilde{r} = 0$ holds so every summand in equation (17) with a x_d-component is equal to 0. Now looking at the x_l-terms we can compute

$$\begin{aligned}\sum_{i=1}^{d-1}(\alpha_{il}\zeta_i + \zeta_i\alpha_{il}) &= \sum_{i=1}^{d-1}\alpha_{il}\zeta_i + \sum_{i=1}^{d-1}\zeta_i\alpha_{il}\\ &= \sum_{i=1}^{d-1}(\alpha_{il}c_i - \sum_{j=1}^{d-1}c_j\alpha_{il}\alpha_{ij}) + \sum_{i=1}^{d-1}(\alpha_{il}c_i - \sum_{j=1}^{d-1}c_j\alpha_{ij}\alpha_{il})\\ &= \sum_{i=1}^{d-1}\alpha_{il}c_i - \sum_{i,j=1}^{d-1}c_j\alpha_{il}\alpha_{ij} + \sum_{i=1}^{d-1}\alpha_{il}c_i - \sum_{i,j=1}^{d-1}c_j\alpha_{ij}\alpha_{il}.\end{aligned}$$

Using the orthogonality of the matrix a' we get

$$\begin{aligned}&\sum_{i=1}^{d-1}\alpha_{il}c_i - \sum_{i,j=1}^{d-1}c_j\alpha_{il}\alpha_{ij} + \sum_{i=1}^{d-1}\alpha_{il}c_i - \sum_{i,j=1}^{d-1}c_j\alpha_{ij}\alpha_{il}\\ =&\sum_{i=1}^{d-1}\alpha_{il}c_i - \sum_{j=1}^{d-1}c_j\delta_{jl} + \sum_{i=1}^{d-1}\alpha_{il}c_i - \sum_{j=1}^{d-1}c_j\delta_{jl}\\ =&\sum_{i=1}^{d-1}\alpha_{il}c_i - c_l + \sum_{i=1}^{d-1}\alpha_{il}c_i - c_l = 2\left(\sum_{i=1}^{d-1}\alpha_{il}c_i - c_l\right)\end{aligned}$$

Since the set $\{1, \alpha_{1l}, \ldots, \alpha_{(d-1)l}\}$ is linearly independent for all $1 \leq l \leq d-1$ by Lemma 3.12, we see that $c_i = 0$ for all $1 \leq i \leq d-1$. To summarize we have

$$\zeta_i = \begin{cases}\frac{1}{2} - \frac{1}{2}\alpha_{dd}, & i = d\\ 0, & else\end{cases},$$

which corresponds to a translation of $\frac{1}{2}$ along the x_d-axis. With this choice of the c_i we get an action of $O_{d-1}^{+} * \mathbb{Z}_2$ on $C(X^{+})$. □

But with a view to section 2 we can ask if even the classical d-dimensional lemon has some genuine quantum symmetry. We will see that the commutativity of $C(X)$ already implies that a quantum group acting faithfully and isometric on the commutative lemon is already commutative, i.e. a subgroup of $O_{d-1} \times \mathbb{Z}_2$.

Theorem 3.15

a) The quantum isometry group of the maximal quantum version $C(X^+)$ of the d-dimensional lemon is $O^+_{d-1} * \mathbb{Z}_2$.

b) The quantum isometry group of the classical d-dimensional lemon is $O_{d-1} \times \mathbb{Z}$.

In particular, the quantum version $C(X^+)$ of the d-dimensional lemon is genuine quantum.

Proof. The proof of a) is just Proposition 3.10 and Proposition 3.14 combined. So it only remains to prove part b).
The same arguments as before give that the quantum isometry group is a subgroup of the $O^+_{d-1} * \mathbb{Z}_2$. It remains to prove the commutativity of the algebra. We have

$$\begin{aligned}
\alpha(x_i x_j) &= \alpha(x_i)\alpha(x_j) \\
&= \sum_{k,l}^{d} x_k x_l \otimes \alpha_{ki}\alpha_{lj} + \sum_{k=1}^{d} x_k \otimes \alpha_{ki}\zeta_j + \sum_{l=1}^{d} x_l \otimes \zeta_i \alpha_{lj} + 1 \otimes \zeta_i\zeta_j \\
&= \sum_{k} x_k^2 \otimes \alpha_{ki}\alpha_{kj} + \sum_{k<l} x_k x_l \otimes (\alpha_{ki}\alpha_{lj} + \alpha_{li}\alpha_{kj}) + \sum_{l} x_l \otimes (\alpha_{li}\zeta_j + \zeta_i\alpha_{lj}) + 1 \otimes \zeta_i\zeta_j
\end{aligned}$$

Hence applying the commutativity relations $x_i x_j = x_j x_i$ gives on the α_{ij} the relations

$$\alpha_{ki}\alpha_{kj} = \alpha_{kj}\alpha_{ki}, \quad \forall i, j, k \tag{18}$$

$$\alpha_{ki}\alpha_{lj} + \alpha_{li}\alpha_{kj} = \alpha_{kj}\alpha_{li} + \alpha_{lj}\alpha_{ki}, \quad \forall i, j, k < l \tag{19}$$

but since the expression is symmetric in k, l it is true for all k, l as well. Here we used the linear independence of the set $\{1, x_j, x_i x_j \,|\, i \leq j\}$, which is another application of the diamond lemma. Now the commutativity of the algebra follows in the exact same way as in Proposition 2.21. The c_i can be chosen in the same way as in Proposition 3.14 to get an affine action. □

Remark 3.16
Goswami showed in [GJ18] that for a compact metric space, that is either a connected

smooth manifold or has nonempty interior in $\mathbb{R}^n$, any CQG that acts faithfully and isometric on $C(X)$ is already commutative. The result was not applicable here because the d-dimensional lemon as singularities at the cusps. But notice that in the proof of part b) of Theorem 3.15 the exact relations on the x_i were not important: the linear independence of $\{1, x_j, x_i x_j \,|\, i \leq j\}$ was the only condition needed to show the commutativity of the algebra. So if we have a classical compact space where the set $\{1, x_j, x_i x_j \,|\, i \leq j\}$ is linearly independent, we can expect the quantum isometry group to be classical as well.

As our next example let us look at a quantum version of the d-dimensional hypercube K_d given in Definition 2.20.

Definition 3.17
A quantum version of the d-dimensional hypercube K_d is given by the universal C^*-algebra

$$C(K_d^+) := C^* \left(x_1, \ldots, x_d \,|\, x_i^* = x_i,\, x_i^2 = x_i \right).$$

The existence of the C^*-algebra $C(K_d^+)$ is clear since the relations $x_i^* = x_i$ and $x_i^2 = x_i$ imply $\|x_i\| = 1$.

Proposition 3.18
The quantum isometry group of K_d^+ is the non-commutative hyperoctahedral group H_n^+.

Proof. Let (A, a) be the quantum isometry group of K_d^+ with action α defined by

$$\alpha(x_i) = \sum_{j=1}^{d} x_j \otimes \alpha_{ji} + 1 \otimes \zeta_i.$$

With the relation $x_i^2 = x_i$ we get $\alpha(x_i)^2 = \alpha(x_i)$ and thus

$$\sum_{j=1}^{d} x_j \otimes \alpha_{ji} + 1 \otimes \zeta_i = \sum_{j,k=1}^{d} x_j x_k \otimes \alpha_{ji}\alpha_{ki} + \sum_{j=1}^{d} x_j \otimes (\alpha_{ji}\zeta_i + \zeta_i\alpha_{ji}) + 1 \otimes \zeta_i^2.$$

Using the relation $x_i^2 = x_i$ in the first sum on the left hand side and using bilinearity of the tensor product we get the equation

$$\sum_{j=1}^{d} x_j \otimes \alpha_{ji} + 1 \otimes \zeta_i = \sum_{j \neq k}^{d} x_j x_k \otimes \alpha_{ji}\alpha_{ki} + \sum_{j=1}^{d} x_j \otimes (\alpha_{ji}\zeta_i + \zeta_i\alpha_{ji} + \alpha_{ji}^2) + 1 \otimes \zeta_i^2.$$

Using the diamond lemma we can conclude that the set $\{1, x_i, x_i x_j \,|\, 1 \leq i \neq j \leq d\}$ is linearly independent, so we get the relation

$$\alpha_{ji}\alpha_{ki} = 0, \quad \forall i, k \neq j.$$

By universality of $C(H_d^+)$ we get a surjective morphism $C(H_d^+) \to A$ so (A, a) is a quantum subgroup of H_d^+. On the other hand, choosing $\zeta_i := \frac{1}{2} - \sum_{j=1}^{d} \frac{1}{2}\alpha_{ji}$ we get as in Proposition 2.21 an affine action of H_d^+ on $C(K_d^+)$. So H_d^+ is the quantum isometry group of K_d^+. □

Example 3.19
The quantum isometry groups of the free sphere S_+^{d-1} is the CQG O_d^+ and the quantum isometry group of the half-liberated sphere S_*^{d-1} is the half-liberated orthogonal group O_d^* given by the universal C^*-algebra

$$C(O_d^*) := C^*\left(u_{ij},\, i, j = 1, \ldots, d \,\middle|\, u_{ij} = u_{ij}^*, \sum_{k=1}^{n} u_{ik}u_{jk} = \sum_{k=1}^{n} u_{ki}u_{kj} = \delta_{i,j}1, u_{ij}u_{kl}u_{st} = u_{st}u_{kl}u_{ij} \right)$$

constructed in [BS09].

We will not give the explicit calculation here, it uses similar arguments as the calculations before. But note that the same result was proven by Banica and Goswami in [BG10] where the isometry group was defined in terms of the spectral triples in the sense of the non-commutative geometry established by Connes.

This finishes our discussion of quantum versions of metric spaces in the sense of Definition 3.1. In the following last section of this thesis we briefly want to look at another approach to non-commutative metric spaces.

4 Compact quantum metric spaces

In this last section we want to take a brief look at the concept of arbitrary compact quantum metric spaces (CQMS) as defined by Rieffel in [Rie99]. CQMS do not originate in a classical metric space as the quantum versions we looked at in the last section and are therefore further generalizations of metric spaces. But it is not yet clear how to define a quantum version of a compact metric space that is also a CQMS. The results of this section can also be found in [Rie99] and [QS12]; we added left out proofs or extended existing ones for better readability.

4.1 Compact quantum metric spaces

The concept of a CQMS was defined by Rieffel in the more general setting of order-unit spaces. We restrict ourselves to the C^*-algebra case.

Definition 4.1
Let A be a unital C^*-Algebra. A *Lipnorm* on A is a seminorm $L : A \to [0, \infty]$ such that

- $L(a) = L(a^*)$ for all $a \in A$
- $L(a) = 0 \Leftrightarrow a \in \mathbb{C}1$
- L is lower semicontinuous with respect to the C^*-norm on A
- the topology on the state space $S(A)$ given by the metric

  $$\rho_l(\mu, \nu) := \sup\left\{ |\mu(a) - \nu(a)| \,\middle|\, a \in A, L(a) \leq 1 \right\}$$

  for all $\mu, \nu \in S(A)$ coincides with the weak *-topology on $S(A)$.

The pair (A, L) is then called a *compact quantum metric space* or short CQMS.

Again, to motivate this definition we will look at the classical case first. Let (X, d) be a compact metric space and put $A = C(X)$. On A we can define a seminorm by putting

$$L_d(f) := \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)} \,\middle|\, x, y \in X, x \neq y \right\}$$

for a function $f \in C(X)$; this seminorm is called the *Lipschitz seminorm*. The next result is stated in [QS12] but not proved there.

Proposition 4.2
Let (X, d) be a compact metric space and put $A = C(X)$. Then A together with the Lipschitz seminorm L_d on A is a CQMS.

Proof. The first two properties of a Lipnorm are easily verified.
Step 1: Lower semicontinuity. We have to show that

$$\liminf_{g \to f} L_d(g) = \lim_{n \to \infty} \inf \left\{ L_d(g) \middle| \, \|f - g\|_\infty < \frac{1}{n} \right\} \geq L_d(f)$$

for all $f \in C(X)$. Note that, for any function $g \in C(X)$, $L_d(g)$ is the smallest Lipschitz constant or ∞ if g is not Lipschitz-continuous. Let $f \in C(X)$ be any continuous function on X.
First suppose that $L_d(f) = \infty$ and take any sequence (f_n) converging to f in the supremum norm. So for all $M > 0$ there exists a pair of points x, y such that $|f(x) - f(y)| > Md(x, y)$. For N big enough we see that also $|f_n(x) - f_n(y)| > Md(x, y)$ for all $n > N$, so $\lim_{n \to \infty} L_d(f_n) = \infty$.
Now suppose $L_d(f)$ is finite and thus f is Lipschitz with smallest Lipschitz constant $L_d(f)$. Set

$$K := \liminf_{g \to f} L_d(g)$$

and take any $\epsilon > 0$. For each $n \in \mathbb{N}$ there is a function g_n with $\|g_n - f\| < \frac{1}{n}$ and

$$L_d(g_n) < K + \epsilon.$$

So we have $g_n \xrightarrow{\|\cdot\|_\infty} f$ and

$$|g_n(x) - g_n(y)| < (K + \epsilon)d(x, y)$$

for all $x, y \in X$. Now we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - g_n(x)| + |g_n(x) - g_n(y)| + |g_n(y) - f(y)| \\ &\leq 2\|f - g_n\|_\infty + |g_n(x) - g_n(y)| \\ &< \frac{2}{n} + (K + \epsilon)d(x, y) \end{aligned}$$

for all $x, y \in X$.

Letting $\epsilon \to 0$ and $n \to \infty$ we get

$$|f(x) - f(y)| \leq Kd(x,y) = \liminf_{g \to f} L_d(g) d(x,y)$$

and therefore $\liminf_{g \to f} L_d(g)$ is a Lipschitz constant for f. So we have $\liminf_{g \to f} L_d(g) \geq L_d(f)$ and hence L_d is lower semicontinuous.

*Step 2: Weak *-topology.* We now have to show that the weak *-topology defined on the state space $S(A)$ is the same as the topology defined by the metric

$$\rho(\mu, \nu) := \sup\left\{ |\mu(f) - \nu(f)| \,\Big|\, f \in A, L_d(f) \leq 1 \right\}.$$

We will call this topology the *ρ-topology.* Remember that the weak *-topology on the state space is given by a neighbourhood basis defined by the sets

$$\{\nu \in S(A) \,|\, |\mu(f_i) - \nu(f_i)| < \epsilon, f_i \in A, i = 1, \ldots, n\}$$

for all $\mu \in S(A)$.

It is easy to see that the ρ-topology is finer than the weak *-topology: Let μ_n be a sequence in $S(A)$ converging in ρ-topology to $\mu \in S(A)$. We see that the supremum of all $|\mu_n(f) - \mu(f)|$ for $L_d(f) \leq 1$ tends to zero, so $\mu_n(f)$ converges to $\mu(f)$ for all f with $L_d(f) \leq 1$. But this means, by homogeneity, that $\mu_n(f)$ converges to $\mu(f)$ or all $f \in A$, so μ_n converges to μ in the weak *-topology.

For the other direction we have to show that every open ball around μ in the ρ-topology contains a weak *-neighbourhood of μ. So let $B(\mu, \epsilon)$ be such a ball and suppose first that the image of $L_1 := \{f \in A \,|\, L_d(f) \leq 1\}$ in the quotient $A/(\mathbb{C}1)$ is relatively compact. Then there are finitely many elements g_j such that $L_d(g_j) \leq 1$ and the balls of radius $\frac{\epsilon}{3}$ around the g_j in $A/(\mathbb{C}1)$ cover the whole space. We claim that

$$U := \left\{ \nu \in S(A) \,|\, |\mu(g_i) - \nu(g_i)| < \frac{\epsilon}{3}, \, i = 1, \ldots, n \right\}$$

is the desired neighbourhood of μ. Let $f \in L_1$ be arbitrary, then there exist a constant function c and a j such that

$$\|f - g_j - c\| < \frac{\epsilon}{3}.$$

So for any $\nu \in U$ we have

$$\begin{aligned} |\mu(f) - \nu(f)| &\le |\mu(f) - \mu(g_j + c)| + |\mu(g_j + c) - \nu(g_j + c)| + |\nu(g_j + c) - \nu(f)| \\ &= |\mu(f - g_j - c)| + |\mu(g_j) + \mu(c) - \nu(g_j) - \nu(c)| + |\nu(f - g_j - c)| \\ &< \frac{\epsilon}{3} + |\mu(g_j) - \nu(g_j)| + \frac{\epsilon}{3} < \epsilon, \end{aligned}$$

so $\rho(\mu, \nu) < \epsilon$ and $U \subset B(\mu, \epsilon)$ as claimed.
We still have to prove that the image of L_1 in $A/(\mathbb{C}1)$ is relatively compact. Let $f \in L_1$ be arbitrary. Since f is bounded we can find a constant function c such that $\tilde{f} := f + c$ has a zero x_0. Let $y \in X$ be a point with $\tilde{f}(y) = \|\tilde{f}\|$ (this is possible since X is compact). Then

$$\|\tilde{f}\| = |\tilde{f}(y) - \tilde{f}(x_0)| = |f(y) - f(x_0)| \le d(y, x_0) \le C$$

for some constant $C \in \mathbb{R}$ since the metric d is bounded. So the images of L_1 and the set $\{f \in L_1 \,|\, \|f\| \le C\}$ in the quotient space coincide. But by the theorem of Arzelà-Ascoli the second set is relatively compact and, since projections are continuous, its image is relatively compact, too. □

Remark 4.3
One can recover the original metric on X by identifying every point in X with the state that is given by the evaluation at that point, i.e. $\mu_x(f) = f(x)$, and then restricting the metric ρ to the set of this evaluations. The metric given by

$$\rho'(x, y) := \rho(\mu_x, \mu_y)$$

is then equal to the original metric d.
Even more, if we start with a commutative CQMS (A, L), we can find a compact set X such that $A = C(X)$. Then we can define as above a metric d on X and Rieffel has shown in [Rie99] that in that case $L = L_d$.

Remark 4.4
There may exist various Lipnorms on a given C^*-algebra that turn it into a CQMS. Rieffel provides a lot of examples in [Rie99] for constructions for special C^*-algebras, for example using spectral triples, actions of compact groups or graphs interpreted as circuits. If we have a metric ρ on the state space $S(A)$ for some C^*-algebra A that gives

the weak *-topology on this space we can define a Lipnorm on A by putting

$$L(a) := \sup \left\{ \frac{|\mu(a) - \nu(a)|}{\rho(\mu, \nu)} \middle| \mu \neq \nu \right\}.$$

None of these approaches seem to work for our definition of quantum versions of metric spaces which is why we did not turn them into CQMS.

4.2 Isometric actions on CQMS

The question now is: If we have a CQG acting on a C^*-algebra that also has the structure of a CQMS, when should we call the action isometric? We will motivate the definition again by looking at the classical case first.

Proposition 4.5
Let (X, d) be a compact metric space, L_d the corresponding Lipschitz seminorm and G a compact group acting on X. Let $\alpha : C(X) \to C(X \times G)$ be the coaction as in Remark 1.12. Then the action is isometric if and only if

$$L_d(\alpha(f)(\cdot, g)) = L_d(f)$$

for all $f \in C(X)$ and $g \in G$.

Proof. Let the action of G on X be isometric. Write f_g for the function $\alpha(f)(\cdot, g)$ which is given by $f_g(x) = f(gx)$. Then

$$\begin{aligned} L_d(f_g) &= \sup \left\{ \frac{|f(gx) - f(gy)|}{d(x, y)} \middle| x, y \in X, x \neq y \right\} \\ &= \sup \left\{ \frac{|f(gx) - f(gy)|}{d(gx, gy)} \middle| x, y \in X, x \neq y \right\} \\ &= \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \middle| x, y \in X, x \neq y \right\} = L_d(f), \end{aligned}$$

where the second equality uses the isometry of the action and the third equality the fact that multiplication with any g is a bijection on X.
Now let $L_d(f) = L_d(f_g)$ for all $f \in C(X)$ and $g \in G$. We will use the metric on the state

space of Remark 4.3. Then we have for all $x, y \in X$

$$\begin{aligned} d(gx, gy) = \rho(\mu_{gx}, \mu_{gy}) &= \sup\{f(gx) - f(gy) \,|\, L_d(f) \leq 1\} \\ &= \sup\{f_g(x) - f_g(y) \,|\, L_d(f) \leq 1\} = \rho(\mu_x, \mu_y) = d(x, y), \end{aligned}$$

since $f \mapsto f_g$ is a bijection that is isometric with respect to L_d. □

Now we will pass to the half-classical case and use the previous proposition to motivate the following definition.

Definition 4.6 ([QS12])
Let B be a CQMS and let G be a compact group acting on B in the sense that there is a coaction

$$\alpha : B \longrightarrow B \otimes C(G) \cong C(G, B),$$

where the isomorphism $B \otimes C(G) \cong C(G, B)$ is given by $b \otimes f \mapsto (g \mapsto f(g)b)$. Then we call α *isometric* if

$$L(\alpha(b)(g)) = L(b). \tag{20}$$

for all $b \in B$ and $g \in G$.

Remark 4.7
We have seen in Proposition 4.5 that this notion of isometry coincides with the classical one in the case that $B = C(X)$ for some compact metric space X.

Now we want to generalize this further to a CQG acting on the CQMS B. We will follow the arguments of [QS12]. Note that we can write condition (20) also as

$$L((\mathrm{id} \otimes \mu_g)\alpha(b)) = L(b)$$

for all $b \in B$ and $g \in G$, where $\mu_g \in S(C(G))$ is mapping a function to its evaluation at g. These states, called *pure states*, have by the theorem of Krein-Milman the property that their convex hull is weak *-dense in $S(C(G))$. Suppose we have a state μ that is a convex combination of pure states μ_i, i.e. $\mu = \sum_{i=1}^n \lambda_i \mu_i$ with $\sum_{i=1}^n \lambda_i = 1$. Then we have

$$L((\mathrm{id} \otimes \mu)\alpha(b)) \leq \sum_{i=1}^n \lambda_i L((\mathrm{id} \otimes \mu_i)\alpha(b)) = L(b).$$

Now, using the lower-semicontinuouity of the Lip-norm L, one can show that

$$L((\mathrm{id} \otimes \mu)\alpha(b)) \leq L(b)$$

holds for all states $\mu \in S(C(G))$. This led Quaegebeur and Sabbe in [QS12] to the following definition.

Definition 4.8
Let A be a CQG, (B, L) a CQMS and α a coaction of A on B. Then α is called *1-isometric* if

$$L((\mathrm{id} \otimes \mu)\alpha(b)) \leq L(b)$$

for all states $\mu \in S(A)$ and $b \in B$.

Remark 4.9
Chirvasitu showed in [Chi15] that for the half classical case of a CQG acting on a $C(X)$ for some compact space X, Goswami's definition of isometric quantum group action implies that of Definition 4.8. It is conjectured that they are even equivalent but there is no proof yet.

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