Hypergraph C^*-algebras

Bachelor's thesis

submitted by

Dean Zenner

in fulfillment of the thesis requirement for the degree of
Bachelor of Science

presented to the
Saarland University
Faculty of Mathematics and Computer Science

in the degree course
Mathematics

Supervisor: Prof. Dr. Moritz Weber
Second Referee: Prof. Dr. Michael Hartz

Saarbrücken, 2021

Statement in Lieu of an Oath

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

Saarbrücken, 4th January 2022

Dean Zenner

Acknowledgements

First of all, I would like to thank Prof. Dr. Weber for suggesting such an interesting subject. I am grateful for all his help and guidance during the process of writing this thesis.
Last but not least, I thank my family and friends for their constant support.

Contents

Introduction

In this thesis we deal with a new generalization of graph C^*-algebras: we use hypergraphs for this approach.
A graph is a structure that is made up of a set of vertices and edges. By adding a source and a range map to our structure we obtain directed graphs. Directed graphs are used to construct graph C^*-algebras. They developed from the Cuntz-Krieger algebras, which were defined in 1980. Graph C^*-algebras have been studied broadly for the last 40 years. The reason for this is that the graph provides a handy tool for characterizing and visualizing properties of the associated graph C^*-algebra.

The main goal of this thesis is to generalize said object in the hope to find interesting connections and differences to the well studied graph C^*-algebra. We do so by transferring the concept of graph C^*-algebras on hypergraphs. Hypergraphs are a generalization of graphs and therefore, a possible candidate to achieve our goal. The difference to graphs is that the range and source map of hypergraphs map into the power set of the vertices. Hence, an edge can join any number of vertices.

After introducing hypergraph C^*-algebras, we examine a collection of different examples. We were able to find an example that shows, that in contrast to the graph C^*-algebras, a hypergraph C^*-algebra does not need to be nuclear (see Proposition 3.12). Also we found a hypergraph C^*-algebra that is isomorphic to the well known Toeplitz algebra (see Proposition 3.10) and one that is isomorphic to the Cuntz algebra (see Proposition 3.11). This led us to the idea of *hyperization*. The line of reasoning is to find a "hyper version" of a given graph, that delivers us a *-homomorphism between the graph and the hypergraph C^*-algebra. One "hyper version" we found provides us with an injective *-homomorphism.
Keeping in mind, that by the Gelfand-Naimark Theorem every C^*-algebra has a representation on a Hilbert space, we went on and investigated some explicit representations. By representation we mean a *-homomorphism from the C^*-algebra into $\mathcal{B}(H)$ the space of bounded operators on a Hilbert space H.

We now give an outline of the thesis.
In Chapter 1 we provide mathematical basics that are indispensable for the construction of graph C^*-algebras. We will also present examples of C^*-algebras that will encounter us throughout this thesis.
Chapter 2 deals with graph C^*-algebras and their best known examples.
Finally, in Chapter 3, we pass to our own work: we introduce and study the case of hypergraph C^*-algebras. We give a definition and a collection of examples that we examined. Some examples are completely understood and some only partially. We

also present examples that are still quite mysterious to us. At the end of the thesis, we prove the non-triviality of the examples by presenting representations.

1 Preliminaries

Since we want to study hypergraph C^*-algebras, we need to introduce the concept of universal C^*-algebras first. To fully grasp the concept, we will show some beautiful examples of universal C^*-algebras. Before doing so, we will speak about projections in C^*-algebras, since they will play a major role in this thesis. A standard reference for the theory of C^*-algebras is Blackadar (2006).

1.1 Projections in C^*-algebras

In this chapter, we provide basic properties about projections.

Definition 1.1. Let A be a C^*-algebra. We call $p \in A$ a *projection* iff the equation $p = p^2 = p^*$ holds.

Definition 1.2. Let A be a C^*-algebra. We call $s \in A$ a *partial isometry* iff the equation $ss^*s = s$ holds.

Remark 1.3. *Notice that for every partial isometry s in a C^*-algebra A we have $s^* = s^*ss^*$.*

The following statements are about orthogonal projections on a closed subspace of a Hilbert space H. Due to the Gelfand-Naimark Theorem we know that for every C^*-algebra we have a faithful representation on a Hilbert space. Hence, we can use the upcoming statements for projections in some C^*-algebra. They are being proven in Raeburn (2005).

Proposition 1.4. *Let P and Q be orthogonal projections on a closed subspace of a Hilbert space H. The following statements are equivalent:*

(a) $PH \subset QH$

(b) $QP = P = PQ$

(c) $Q - P$ *is a projection.*

(d) $P \leq Q$ *(meaning* $\langle Ph, h\rangle \leq \langle Qh, h\rangle$ *for all* $h \in H$*)*

Proposition 1.5. *Let P and Q be orthogonal projections on a closed subspace of a Hilbert space H. The following statements are equivalent:*

(a) $PH \perp QH$

(b) $QP = 0 = PQ$

(c) *$P + Q$ is a projection.*

To show the main statement of this section we need some terminology and properties of positive elements.

Definition 1.6. In a C^*-algebra A we name an element $z \in A$ *positive* and write $0 \leq z$ iff an element $x \in A$ exists with $z = x^*x$. We write $a \leq z$ iff $0 \leq z - a$.

Remark 1.7. *One can show that a bounded operator T on a Hilbert space H is positive iff $\langle Th, h\rangle \geq 0$ for all $h \in H$.*

Proposition 1.8. *Let A be a C^*-algebra.*

(a) *Let $x \in A$. If $-x^*x \geq 0$, then $x = 0$.*

(b) *The relation in Definition 1.6 defines a partial order structure.*

(c) *The sum of positive elements is again positive.*

Proof. (a) see (Web)[Lemma 4.7].

(b) see (Web)[Corollary 4.9].

(c) Follows immediately from (b).

□

Let's show the main statement of this section.

Proposition 1.9. *Let $\{p_i \mid 1 \leq i \leq n\}$ be projections in a C^*-algebra A. Then we have that $p := \sum_{i=1}^n p_i$ is a projection iff $p_ip_j = 0$ for all $i \neq j$. In that case we say that the projections are* mutually orthogonal*.*

Proof. Let the projections $\{p_i \mid 1 \leq i \leq n\}$ be mutually orthogonal. We are going to prove that p is a projection. It is clear that we have $p^* = p$. Furthermore it holds that

$$p^2 = \sum_{j=1}^n \sum_{i=1}^n p_jp_i = \sum_{i=1}^n p_i^2 = \sum_{i=1}^n p_i = p.$$

Let's prove the converse. We are going to use an induction. The case of $n = 1$ is trivial. Assume that the converse holds true for n. Let $\sum_{i=1}^{n+1} p_i$ be a projection. Since $p_i = p_i^*p_i$ is positive for all $i \in \{1, \ldots, n\}$ it follows that $\sum_{i=1}^{n+1} p_i \geq p_{n+1}$. With Proposition 1.4 we know, that $\sum_{i=1}^n p_i = \sum_{i=1}^{n+1} p_i - p_{n+1}$ is a projection. The induction hypothesis implies that the projections $\{p_i \mid 1 \leq i \leq n\}$ are mutually orthogonal. Using Proposition 1.5 shows, that p_{n+1} is orthogonal to $\sum_{i=1}^n p_i$. For $i \leq n$ we have

$$0 \leq (p_ip_{n+1})^*(p_ip_{n+1}) = p_{n+1}p_ip_{n+1} \leq p_{n+1}(\sum_{j=1}^n p_j)p_{n+1} = 0$$

and hence

$$\|p_ip_{n+1}\|^2 = \|(p_ip_{n+1})^*(p_ip_{n+1})\| = 0.$$

We conclude $p_ip_{n+1} = 0$ for all $i \in \{1, \ldots, n\}$. □

1.2 Universal C^*-algebras

In this section we construct universal C^*-algebras. They are the fundamental mathematical object to define graph C^*-algebras.

Definition 1.10. Let $E = \{x_i \mid i \in I\}$ be a set of elements indexed by a set I.

(a) A *noncommutative monomial* in E is a word $x_{i_1} \cdots x_{i_m}$ with $i_1, ..., i_m \in I$ and $m \in \mathbb{N}\backslash\{0\}$.

(b) A *noncommutative polynomial* in E is a complex linear combination of noncommutative monomials: $\sum_{k=1}^{N} \alpha_k y_k$ with $N \in \mathbb{N}, \alpha_k \in \mathbb{C}$ and $y_1, ..., y_N$ being noncommutative monomials in E.

(c) The concatenation of two words is defined by the following

$$(x_{i_1} \cdots x_{i_m}) \cdot (x_{j_1} \cdots x_{j_n}) := x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_n}$$

where $x_{i_1} \cdots x_{i_m}$ and $x_{j_1} \cdots x_{j_n}$ are two words.

(d) The *free algebra on the generator set* E is the set of all noncommutative polynomials with the canonical addition and scalar multiplication. The multiplication of two elements from the free algebra is given by the concatenation.

Note that for every algebra B containing a set of elements $\{y_i \mid i \in I\}$ that is indexed by the same set I, we will find a homomorphism from the free algebra to the algebra B sending x_i to y_i, for all $i \in I$.
Let $E = \{x_i \mid i \in I\}$ be a set of elements. By adding another set $E^* = \{x_i^* \mid i \in I\}$ which is disjoint with E and by defining an involution on $E \cup E^*$ using the following

$$(\alpha x_{i_1}^{\epsilon_1} ... x_{i_m}^{\epsilon_m})^* := \overline{\alpha} x_{i_m}^{\overline{\epsilon}_m} ... x_{i_1}^{\overline{\epsilon}_1}$$

where $\alpha \in \mathbb{C}, \epsilon_k \in \{1, *\}$ and $\overline{\epsilon}_k := \begin{cases} 1 & \epsilon_k = * \\ * & \epsilon_k = 1 \end{cases}$

we obtain the *free *-algebra* $P(E)$ on the generator set E.

Definition 1.11. Let $E = \{x_i \mid i \in I\}$ be a set of elements indexed by a set I.

(a) Let $R \subset P(E)$ be a set of polynomials.

(b) Let $J(R) \subset P(E)$ be a two-sided ideal generated by R. This means that $J(R)$ is the smallest two-sided ideal that contains R. The *universal *-algebra* with *generator* E and *relations* R is defined as the quotient space $A(E|R) := P(E)/J(R)$.

Since $J(R)$ is an ideal, the structure of an algebra is kept. By abuse of notation we will write x_i for $\dot{x_i} \in A(E|R)$.
We still miss a C^*-norm. First, we will have a look at C^*-seminorms.

Definition 1.12. Let A be a *-algebra.
A C^*-*seminorm* on A is a mapping $p : A \to [0, \infty)$, such that

(a) $p(\lambda x) = |\lambda| p(x)$ and $p(x+y) \leq p(x) + p(y)$ for all $x, y \in A$ and $\lambda \in \mathbb{C}$

(b) $p(xy) \leq p(x)p(y)$ for all $x, y \in A$

(c) $p(x^*x) = p(x)^2$ for all $x \in A$

holds. We are now able to define universal C^*-algebras.

Definition 1.13. Let E be a set of generators and $R \subset P(E)$ relations. Define

$$\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E|R)\}.$$

If $\|x\| < \infty$ for all $x \in A(E|R)$ holds, then $\|\cdot\|$ is a C^*-seminorm. To convince oneself, one can easily check (a)-(c) from Definition 1.12. Furthermore, note that $\{x \in A(E|R) \mid \|x\| = 0\}$ is a two-sided ideal: let $z \in \{x \in A(E|R) \mid \|x\| = 0\}$ and $y \in A(E|R)$. By using Definition 1.12 (b) we see that

$$0 \leq \|yz\| \leq \|y\|\|z\| = 0$$
$$0 \leq \|zy\| \leq \|z\|\|y\| = 0$$

applies.
So if $\|x\| < \infty$ holds for all $x \in A(E|R)$, we define the *universal C^*-algebra $C^*(E|R)$* as the completion with respect to the norm $\|\dot{x}\| := \|x\|$:

$$C^*(E|R) := \overline{A(E|R)/\{x \in A(E|R) \mid \|x\| = 0\}}^{\|\cdot\|}$$

where $\dot{x} \in A(E|R)/\{x \in A(E|R) \mid \|x\| = 0\}$ is the equivalence class of x. Observe that by taking the quotient space the C^*-seminorm becomes a C^*-norm. The completion yields a C^*-algebra.

The following lemma will provide a useful tool for proving that a universal C^*-algebra actually exists.

Lemma 1.14. *Let $E = \{x_i \mid i \in I\}$ be a set of generators and $R \subset P(E)$ relations. If a constant C exists such that $p(x_i) < C$ holds for all $i \in I$ and all C^*-seminorms p on $A(E|R)$, then it follows that $\|x\| < \infty$ holds for all $x \in A(E|R)$. In that case, we say that the universal C^*-algebra exists in the sense as described above.*

Proof. The norm of a monomial of length N is bounded by C^N and hence every polynomial in $A(E|R)$ is bounded. □

Before we present some examples, we need some basic statements from functional calculus.

Definition 1.15. Let A be a C^*-algebra and $M \subset A$ a subset of A. By $C^*(M)$ we define the intersection of all *-subalgebras $B \subset A$ with $M \subset B$. Notice that $C^*(M)$ is the smallest *-subalgebra in A that contains M.
In the case that A is unital and that $x \in A$ we define in an analogous way $C^*(x, 1)$.

Perceive that *-homomorphisms are already uniquely defined on the generator set.

Lemma 1.16. *Let A and B be C^*-algebras and $M \subset A$ a subset of A. Let $\phi, \psi : A \to B$ be two *-homomorphisms. If $\phi(x) = \psi(x)$ for all $x \in M$, then it follows that $\phi(x) = \psi(x)$ for all $x \in C^*(M)$.*

Proof. Since ϕ and ψ are *-homomorphisms we obtain by $D := \{x \in A \mid \phi(x) = \psi(x)\} \subset A$ a C^*-algebra in A that contains M. Hence we have that $C^*(M) \subset D$. □

In the following, we will discuss the so called universal property. Note that our universal C^*-algebra may exist, but it could still be the case that the algebra is trivial. The universal property is a tool for obtaining *-homomorphisms between C^*-algebras and for proving the non-triviality of a universal C^*-algebra.
Let $E = \{x_i \mid i \in I\}$ be a set of generators and $R \subset P(E)$ relations. We say that elements $\{y_i \mid i \in I\}$ in some *-algebra B satisfy the relations R, if every polynomial $p \in R$ vanishes, when we replace x_i with y_i.

Proposition 1.17. *Let $E = \{x_i \mid i \in I\}$ be a generator set and $R \subset P(E)$ relations, such that the universal C^*-algebra $C^*(E|R)$ exists. Let $E' = \{y_i \mid i \in I\}$ be a subset of some C^*-algebra B. If the elements in E' satisfy the relations R, then there exists a* unique **-homomorphism $\phi : C^*(E|R) \to B$, where x_i is being sent to y_i for all $i \in I$.*

Proof. Recall the *-homomorphism $\phi : P(E) \to B$, sending x_i to y_i. Since the elements y_i satisfy the relations R and hence the two-sided ideal $P(R)$, generated by R, vanishes in B, the *-homomorphism ϕ induces another *-homomorphism $\phi_0 : A(E|R) \to B$. To prove this, one can consider following definition $\phi_0 : A(E|R) \to B, \phi_0(\dot{x}) := \phi(x)$, where $\dot{x} \in A(E|R)$. It is well defined. Let $\dot{x}, \dot{z} \in A(E|R)$ with $\dot{x} = \dot{z}$ and therefore $x - z \in R$. Then we have

$$\phi_0(\dot{x}) - \phi_0(\dot{z}) = \phi(x - z) = 0.$$

Keep in mind that ϕ_0 is sending $\dot{x}_i$ respectively x_i to y_i.
Define $p(\dot{x}) := \|\phi_0(\dot{x})\|_B$ for all $\dot{x} \in A(E|R)$. One can show that this a C^*-seminorm and hence it follows that $\|\phi_0(\dot{x})\| \leq \|\dot{x}\|$ holds. Therefore ϕ_0 is continuous. We may extend it to a *-homomorphism $\phi : C^*(E|R) \to B$, sending x_i to y_i for all $i \in I$. Uniqueness is by Lemma 1.16. □

The following lemma will show the existence of C^*-algebras, that we will consider throughout this thesis.

Lemma 1.18. *Let A be a universal C^*-algebra that is generated by a partial isometry x and/or projection y. Then the C^*-algebra A exists.*

Proof. Let p be a C^*-seminorm on A. We have

$$p(y)^2 = p(y^*y) = p(y^2) = p(y) \in \{0,1\} \text{ and}$$
$$p(x)^4 = p(x^*x)^2 = p(x^*xx^*x) = p(x^*x) = p(x)^2 \in \{0,1\}.$$

By Lemma 1.14 the C^*-algebra A exists. □

In the last step of this section, we will introduce how to form a product of universal C^*-algebras. We need this definition in a later part of the thesis.

Definition 1.19. Let $A = C^*(E_1|R_1)$ and $B = C^*(E_2|R_2)$ be unital universal C^*-algebras. We call

$$A *_{\mathbb{C}} B := C^*(E_1, E_2|R_1, R_2 \text{ and } 1_A = 1_B)$$

the *free product* of A and B.

1.3 Examples

In the previous chapter we have seen some tools to prove the existence and non-triviality of universal C^*-algebras. In this chapter we want to actually use them.
The first example we look at is the universal C^*-algebra generated by a unitary. An element u in a C^*-algebra is a unitary iff it fulfills the following equation $u^*u = 1 = uu^*$, where 1 is the unit element of the given C^*-algebra. To characterize this universal C^*-algebra we need two important theorems from functional calculus. The proofs can be found in Web[Kapitel 3].

Theorem 1.20 (Stone-Weierstraß)**.** *Let X be a compact Hausdorff space and $A \subset C(X)$ a closed, unital *-subalgebra. By $C(X)$ we define the space of all continuous functions from X to $\mathbb{C}$. If there exists a function $f \in C(X)$ for all $s, t \in X, s \neq t$ with $f(s) \neq f(t)$, then we have $A = C(X)$.*

Theorem 1.21 (Continuous functional calculus)**.** *Let A be a unital C^*-algebra and $x \in A$ normal, meaning $x^*x = xx^*$. Then there exists an isometric isomorphism $\phi : C(sp(x)) \to C^*(x, 1) \subset A$ with $\phi(\mathrm{id}) = x$ and $\phi(1) = 1$, where $sp(x)$ is the spectrum of x.*

Let's have a look at our first example. For our universal C^*-algebra generated by a unitary we write $C^*(u, 1|u^*u = 1 = uu^*)$. Notice that we will not explicitly write down the relation of 1 being the unit element. The existence is by Lemma 1.18 since every unitary is also a partial isometry.

Proposition 1.22. *Let A be a unital C^*-algebra and $z \in A$ a unitary element with $sp(z) = S^1$, where $S^1 := \{x \in \mathbb{C} \mid |x| = 1\}$. Then we have $C^*(u, 1|u^*u = 1 = uu^*) \cong C^*(z) \subset A$, meaning that $C^*(u, 1|u^*u = 1 = uu^*)$ is isomorphic to $C^*(z)$. Notice that we write $C^*(z)$ instead of $C^*(z, 1)$ because z is a unitary.*

Proof. Define $C^*(u) := C^*(u, 1|u^*u = 1 = uu^*)$. Observe that by the universal property 1.17 we obtain a *-homomorphism $\phi : C^*(u) \to C^*(z)$ that sends u to z. Since $C^*(z)$ is generated by z we obtain that ϕ is surjective.
Notice that z is also a normal element. Therefore we can use the continuous functional calculus and we get an isomorphism $\Psi_1 : C(sp(z)) \to C^*(z)$. Using an argument by analogy, there exists another isomorphism $\Psi_2 : C(sp(u)) \to C^*(u)$. For our next step it is important to know that the spectrum of a unitary element is always a subset of S^1. Let $\Phi : C(S^1) \to C(sp(u))$ be the restriction mapping defined

by $f \mapsto f|_{sp(u)}, f \in C(S^1)$. By assumption we have $C(sp(z)) = C(S^1)$. We define $\psi : C^*(z) \to C^*(u)$ by $\psi := \Psi_2 \circ \Phi \circ \Psi_1^{-1}$. It follows that

$$\psi(z) = (\Psi_2 \circ \Phi \circ \Psi_1^{-1})(z) = \Psi_2(\Phi(\Psi_1^{-1}(z))) = \Psi_2(\Phi(\mathrm{id}|_{C(sp(z))})) = \Psi_2(\mathrm{id}|_{sp(u)}) = u.$$

Using Lemma 1.16 we see that $\psi \circ \phi$ is already the identity function on $C^*(u)$. One can conclude immediately, that ϕ is injective and therefore, the proof is finished. □

Corollary 1.23. *We have* $C^*(u, 1|u^*u = 1 = uu^*) \cong C(S^1)$.

Proof. Consider the identity function z on S^1 given by $z(t) = t$ for all $t \in S^1$. Notice that $z \in C(S^1)$ and $z^*z = 1 = zz^*$ holds, where 1 is the constant function $1(t) = t$. We have $sp(z) = S^1$. It follows from the fact that $\lambda - t, t \in S^1$ is not invertible iff $\lambda \in S^1$. Observe that $C(S^1)$ is a unital C^*-algebra. Using the previous proposition, we have $C^*(u, 1|u^*u = 1 = uu^*) \cong C^*(z)$. By construction we know that $C^*(z) \subset C(S^1)$ is a closed unital *-subalgebra. Let $s, t \in S^1, s \neq t$. We have $z \in C^*(z)$ with $z(s) \neq z(t)$. By Stone-Weierstraß Theorem we obtain $C^*(z) = C(S^1)$ and therefore, $C^*(u, 1|u^*u = 1 = uu^*) \cong C(S^1)$. □

For our next example we consider a universal C^*-algebra generated by an *isometry* instead by a unitary.

Definition 1.24 (Toeplitz algebra)**.** The universal C^*-algebra

$$\mathcal{T} := C^*(u, 1|u^*u = 1)$$

generated by an isometry u, meaning $u^*u = 1$, is the so called *Toeplitz* algebra.

Notice that the Toeplitz algebra exists for the same reasons as for the universal C^*-algebra generated by a unitary element. The Toeplitz algebra is non-trivial. One can consider the Hilbert space $\ell^2(\mathbb{N})$ of square-summable $\mathbb{C}$-valued sequences. Let $\{e_n \mid n \in \mathbb{N}\}$ be the standard orthonormal basis. The unilateral shift S given by $Se_n = e_{n+1}$ is an isometry. With the universal property we have found a representation of $\mathcal{T}$ on $\ell^2(\mathbb{N})$.
The existence of the next universal C^*-algebra follows by the same argument as before.

Definition 1.25 (Cuntz algebra)**.** Let $n \in \mathbb{N}$ and $n \geq 2$. We call the universal C^*-algebra

$$\mathcal{O}_n := C^*(S_1, \ldots, S_n \mid S_i^* S_i = 1 \text{ for all } \; i = 1, \ldots, n; \sum_{i=1}^{n} S_i S_i^* = 1)$$

the *Cuntz algebra.*

The Cuntz algebra is also non-trivial. We sketch the proof. Consider a separable Hilbert space H with orthonormal basis $(e_k)_{k \in \mathbb{N}}$. Notice that the basis is countable since we have a separable Hilbert space. Consider injective functions $f_1, \ldots, f_n : \mathbb{N} \to \mathbb{N}$ with $f_i(\mathbb{N}) \cap f_j(\mathbb{N}) = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} f_i(\mathbb{N}) = \mathbb{N}$. One could for example take $f_i(m) := n(m-1) + i$. By $T_i e_k := e_{f_i(k)}$ we obtain operators $T_1, \ldots, T_n$ that fulfill

the relations from the Cuntz algebra. Notice that since f_i is injective a left inverse g_i exists for all $i = 1, \dots, n$. We set $g_i(k) = 0$ for every $k \notin \mathrm{im} f_i$ and $i = 1, \dots, n$. We also set $e_0 := 0$. One can show that T_i^* is given by $T_i^* e_k = e_{g_i(k)}$.
Let $k \in \mathbb{N}$. Then there exists a $j \in \{1, \dots, n\}$ and $m \in \mathbb{N}$, such that $f_j(m) = k$ and $k \notin \mathrm{Im}(f_i)$ for all $i = 1, \dots, n$ with $i \neq j$. So we have

$$\sum_{i=1}^{n} T_i(T_i^*(e_k)) = \sum_{i=1}^{n} e_{f_i(g_i(k))} = e_{f_j(g_j(f_j(m)))} = e_{f_j(m)} = e_k$$

Since k was arbitrary, it follows $\sum_{i=1}^{n} T_i T_i^* = 1$.
We now come to a third example. Let $M_N(\mathbb{C})$ be the $N \times N$-matrices and $E_{ij} \in M_N(\mathbb{C})$ the matrix units, meaning that at the i-j-th place there is a 1 and elsewhere a 0.

Proposition 1.26. *Let $N \in \mathbb{N}$ with $N \geq 2$. The following C^*-algebras are isomorphic.*

(a) $M_N(\mathbb{C})$

(b) $C^*(e_{ij}; i, j = 1, \dots, N | e_{ij}^* = e_{ji}; e_{ij} e_{kl} = \delta_{jk} e_{il}$ *for all* $i, j, k, l)$

Proof. Let A be the universal C^*-algebra in (b). First of all, we prove the existence of A. Let p be a C^*-seminorm. Then we have $p(e_{jj})^2 = p(e_{jj}^* e_{jj}) = p(e_{jj}) \in \{0, 1\}$ and hence $p(e_{ij})^2 = p(e_{ji} e_{ij}) = p(e_{jj}) \in \{0, 1\}$. By Lemma 1.14 we conclude the existence of A. The matrix units satisfy the relations of A. Hence there exists a *-homomorphism $\phi : A \to M_N(\mathbb{C})$, sending e_{ij} to E_{ij}. Since the matrix units E_{ij} are generators of $M_N(\mathbb{C})$, we see that ϕ is surjective. Due to the second relation of A, we observe that $e_{ij}, i, j = 1 \dots, N$ are already all possible monomials and therefore A is N^2-dimensional. The fact that ϕ is surjective implies that ϕ is also injective. □

Definition 1.27. We name a C^*-algebra A *simple*, if it has no proper ideal, meaning for every closed ideal $I \subset A$ it immediately follows that either $I = \{0\}$ or $I = A$ holds.

A common example for a simple C^*-algebra is the space of $N \times N$-matrices $M_N(\mathbb{C})$. By $\mathcal{K}(H)$ we define the space of all compact operators on H, where H is a separable Hilbert space. What comes next is an infinite version of Proposition 1.26.

Proposition 1.28. *The following C^*-algebras are isomorphic.*

(a) $\mathcal{K}(H)$

(b) $C^*(e_{ij}; i, j \in \mathbb{N} | e_{ij}^* = e_{ji}; e_{ij} e_{kl} = \delta_{jk} e_{il}$ *for all* $i, j, k, l)$

Proof. We define $A := C^*(e_{ij}; i, j \in \mathbb{N} | e_{ij}^* = e_{ji}; e_{ij} e_{kl} = \delta_{jk} e_{il}$ for all $i, j, k, l)$. The existence follows in an analogous way as in the proof of Proposition 1.26.
Let $\langle \cdot, \cdot \rangle$ be the scalar product of H and $\| \cdot \|$ the induced norm. Since H is separable, we can choose a countable orthonormal basis $(e_n)_{n \in \mathbb{N}}$ in H. For $i, j \in \mathbb{N}$ we define the operator f_{ij} by $f_{ij} e_n := \delta_{jn} e_i$, where $n \in \mathbb{N}$. We have

$$\|f_{ij} e_n\| = \|\delta_{jn} e_i\| = \delta_{jn} \|e_i\| = \delta_{jn} \|e_n\|.$$

Therefore f_{ij} is a bounded and hence continuous operator. Furthermore, it follows that

$$f_{ij}e_n = \delta_{jn}e_i = \delta_{jn}\langle e_n, e_n\rangle e_i$$

applies. Hence f_{ij} is an operator with a finite image and that is why f_{ij} is a compact operator. In the next stept we want to prove that f_{ij} fulfills the relations of A. Let $k, l \in \mathbb{N}$ then we have

$$f_{ij}(f_{kl}e_n) = f_{ij}(\delta_{ln}e_k) = \delta_{ln}\delta_{jk}e_i = \delta_{jk}f_{il}e_n.$$

Let $x, y \in H$. Due to the orthonormal basis we can rewrite x and y as $x = \sum_{n\in\mathbb{N}}\langle x, e_n\rangle e_n$ and $y = \sum_{n\in\mathbb{N}}\langle y, e_n\rangle e_n$. Using the continuity of f_{ij} we conclude

$$f_{ij}x = f_{ij}\sum_{n\in\mathbb{N}}\langle x, e_n\rangle e_n = \sum_{n\in\mathbb{N}}\langle x, e_n\rangle\delta_{jn}e_i = \langle x, e_j\rangle e_i$$

and hence

$$\begin{aligned}
\langle f_{ij}x, y\rangle &= \langle\langle x, e_j\rangle e_i, y\rangle \\
&= \langle\langle x, e_j\rangle e_i, \sum_{n\in\mathbb{N}}\langle y, e_n\rangle e_n\rangle \\
&= \langle\langle x, e_j\rangle e_i, \langle y, e_i\rangle e_i\rangle \\
&= \langle x, e_j\rangle\langle e_i, e_i\rangle\langle y, e_i\rangle \\
&= \langle x, e_j\rangle\langle e_j, e_j\rangle\langle y, e_i\rangle \\
&= \langle\langle x, e_j\rangle e_j, \langle y, e_i\rangle e_j\rangle \\
&= \langle\sum_{n\in\mathbb{N}}\langle x, e_n\rangle e_n, \langle y, e_i\rangle e_j\rangle \\
&= \langle x, f_{ji}y\rangle.
\end{aligned}$$

We see that $f_{ij}^* = f_{ji}$ holds. The operators f_{ij} satsify the relations of the universal C^*-algebra A. Hence there exists a *-homomorphism $\phi : A \to \mathcal{K}(H)$, sending e_{ij} to f_{ij} for all i, j. In the following we will show that ϕ is an isomorphism.
The image of ϕ contains all linear combinations of the operators f_{ij}. Remember the fact that the image of a C^*-algebra under a *-homomorphism is again a C^*-algebra. Therefore, the image $\phi(A)$ contains even all limits of the linear combinations. One can show that every arbitrary compact operator can be approximated by linear combinations of f_{ij}. This shows that ϕ is surjective.
To show injectivity we regard $M_N := C^*(e_{ij}; i, j = 1, \ldots, N | e_{ij}^* = e_{ji}; e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i, j, k, l)$ for $N \in \mathbb{N}$. By Proposition 1.26 we have $M_N \cong M_N(\mathbb{C})$. Define ϕ_N as the restriction mapping from ϕ on M_N. Since the kernel of the *-homomorphism ϕ_N is an ideal in M_N and since M_N is isomorphic to $M_N(\mathbb{C})$, and hence simple, we have that ϕ_N is injective. Using the property, that a *-homomorphism between two C^*-algebras is injective iff it is isometric, provides us the fact that ϕ_N is also isometric. Therefore, ϕ is isometric on the dense subset $\bigcup_{n\in\mathbb{N}} M_N \subset A$. In turn, it follows that ϕ is injective on A. □

Let's come to our last example for this section.

Proposition 1.29. *Let $N \in \mathbb{N}$. The following C^*-algebras are isomorphic.*

(a) $\mathbb{C}^N$ *as C^*-algebra with pointwise operation*

(b) $C^*(p_1, ..., p_N, 1 \mid p_i = p_i^* = p_i^2, i = 1, ..., N; \sum_{i=1}^N p_i = 1)$

Proof. We define $A_N := C^*(p_1, ..., p_N, 1 \mid p_i = p_i^* = p_i^2, i = 1, ..., N; \sum_{i=1}^N p_i = 1)$. Notice that A_N is generated by projections. Therefore, the universal C^*-algebra A_N exists by Lemma 1.18. Observe that $\tilde{p}_i := e_i = (0, ..., 0, 1, 0, ..., 0)$, with an 1 at the i-th place, satisfies the relations of A_N. Hence we have a *-homomorphism $\phi : A_N \to \mathbb{C}^N$, sending p_i to $\tilde{p}_i = e_i$. Since the e_i are the basis of $\mathbb{C}^N$, we have that ϕ is surjective. If we show, that A_N is N-dimensional we are done. So we need to prove, that $p_i, i = 1, ..., N$ are already all monomials in A_N. We want to show, that $p_i p_j = 0$ for all $i \neq j$. Observe that $1 = \sum_{i=1}^N p_i$ is a projection. With Proposition 1.9 it follows that $p_i p_j = 0$ for all $i \neq j$. □

2 Graph C^*-algebras

This chapter deals with the construction of graph C^*-algebras and their most known examples.

2.1 Definition and properties

After learning about universal C^*-algebras, we are now able to use them to define graph C^*-algebras. First of all, we need to introduce graphs.

Definition 2.1. A *directed finite graph* $\Gamma = (V, E, r, s)$ consists of two finite sets V, E and functions $r : E \to V$, $s : E \to V$. The elements of V are called *vertices* and the elements of E are called *edges*. The map r is named *range* map and the map s is named *source* map. Throughout this thesis we will mostly look at directed finite graphs, therefore we will just say graphs. We say that $v \in V$ is a *sink* iff the set $s^{-1}(v)$ is empty and we call v a *source* iff $r^{-1}(v)$ is empty.

In the following we want to define graph C^*-algebras.

Definition 2.2. Let $\Gamma = (V, E, r, s)$ be a graph. The *graph C^*-algebra* $C^*(\Gamma)$ of the graph Γ is the universal C^*-algebra generated by mutually orthogonal projections p_v for all $v \in V$ and partial isometries s_e for all $e \in E$ such that the following relations hold

(R1) $s_e^* s_f = \delta_{ef} p_{r(e)}$ for all $e, f \in E$

(R2) $p_v = \sum_{\substack{e \in E \\ s(e)=v}} s_e s_e^*$ for all $v \in V$, in case that v is not a sink.

Remark 2.3. *Every graph C^*-algebra exists by Lemma 1.18.*

Remark 2.4. *Notice that $s_e^* s_f = 0$ iff $e \neq f$ and hence $s_e s_e^* s_f s_f^* = 0$ for all $e, f \in E$ with $e \neq f$. So the projections $\{s_e s_e^* \mid e \in E\}$ are mutually orthogonal.*

Remark 2.5. *If the graph is infinite, meaning that V and E are infinite but countable sets, we only consider vertices $v \in V$ for (R2) where $s^{-1}(v)$ is non-empty and finite. Because otherwise we need to consider the infinite sum $\sum_{\substack{e \in E \\ s(e)=v}} s_e s_e^*$ of mutually orthogonal projections which does not converge in norm. To see this let $0 < \epsilon < 1$ and assume that the sum converges. Notice that we replace E with $\mathbb{N}$ since E is countable. Since it converges it is also a Cauchy sequence. Hence there exists $N \in \mathbb{N}$ such that $\|\sum_{i=n}^{m} s_i s_i^*\| < \epsilon$ for $n, m \geq N$ which implies that $\|s_n s_n^*\| < \epsilon$ for $n \geq N$. Since ϵ was arbitrary we conclude $\|s_n s_n^*\| = 0$ for $n \geq N$. This would imply that $s_n s_n^* = 0$ for all $n \geq N$ which would be a contradiction. We also add Relation (R3) to our relations:*

(R3) $s_e s_e^* \leq p_{s(e)}$ *for all* $e \in E$.

We need this relation (R3) for vertices where $s^{-1}(v)$ *is infinite. If for all* $v \in V$ *the set* $s^{-1}(v)$ *is finite or empty one can show that (R2) implies (R3). In that case we call the graph* row-finite. *In this thesis we only consider row-finite graphs.*

Before we present some examples, we are going to prove a few properties of graph C^*-algebras. We will need them for the characterization of our examples.

Proposition 2.6. *Let* $\Gamma = (V, E, r, s)$ *be a graph and* $C^*(\Gamma)$ *the graph* C^**-algebra. Then the following equations hold*

$$s_e = s_e p_{r(e)} \tag{2.1}$$

and

$$s_e = p_{s(e)} s_e \tag{2.2}$$

for all $e \in E$.

Proof. Let $e \in E$. By (R1) and the fact that s_e is a partial isometry we have $s_e = s_e s_e^* s_e = s_e p_{r(e)}$. We are going to prove Equation 2.2.
Observe that we can write $p_{s(e)} = \sum_{s(f)=s(e)} s_f s_f^*$. Consequently it is

$$p_{s(e)} s_e = \sum_{s(f)=s(e)} s_f s_f^* s_e \overset{R1}{=} s_e s_e^* s_e = s_e.$$

□

Proposition 2.7. *Let* $\Gamma = (V, E, r, s)$ *be a graph and* $C^*(\Gamma)$ *the graph* C^**-algebra. Then we have that* $\sum_{v \in V} p_v$ *is the identity in* $C^*(\Gamma)$ *and hence* $\sum_{v \in V} p_v = 1$.

Proof. Let $w \in V$. Since the projections are mutually orthogonal we have

$$p_w \sum_{v \in V} p_v = \sum_{v \in V} p_w p_v = p_w^2 = p_w = (\sum_{v \in V} p_v) p_w.$$

Let $e \in E$. Since s_e is a partial isometry we have by Relation (R1)

$$s_e \sum_{v \in V} p_v = s_e s_e^* s_e \sum_{v \in V} p_v = s_e p_{r(e)} \sum_{v \in V} p_v = s_e p_{r(e)} = s_e.$$

Using Equation 2.2 it follows

$$(\sum_{v \in V} p_v) s_e = (\sum_{v \in V} p_v) p_{s(e)} s_e = p_{s(e)} s_e = s_e$$

and hence $\sum_{v \in V} p_v = 1$. □

Proposition 2.8. *Let* $\Gamma = (V, E, r, s)$ *be a graph and* $C^*(\Gamma)$ *the graph* C^**-algebra. Then we have that* *(a) the projections $\{s_e s_e^* \mid e \in E\}$ are mutually orthogonal*

(b) and $s_e s_f^* \neq 0 \Rightarrow r(e) = r(f)$.

Proof. (a) We have $(s_e s_e^*)^* = s_e s_e^*$ and $s_e s_e^* s_e s_e^* = s_e s_e^*$. Therefore $s_e s_e^*$ is a projection. To show orthogonality see Remark 2.4.

(b) It applies that $s_e s_f^* = s_e p_{r(e)} p_{r(f)} s_f^* = 0$, if $r(e) \neq r(f)$.

□

2.2 Examples

Finally, we will study some examples of graph C^*-algebras. The usage of the universal property 1.17 will be indispensable in the following section.

Proposition 2.9. *Let $C^*(\Gamma)$ be the graph C^*-algebra of the following graph Γ. We have $C^*(\Gamma) \cong C(S^1)$, meaning the graph C^*-algebra is isomorphic to the set of continuous functions on the unit circle.*

Figure 2.1: Graph 1

Proof. The relations (R1) and (R2) imply

$$s_e^* s_e = p_v = s_e s_e^*.$$

Using Proposition 2.7 we know that the projection p_v has to be the identity in $C^*(\Gamma)$. Using the equation from above, this implies that s_e is a unitary generator of $C^*(\Gamma)$. The universal property gives us a *-homomorphism $\phi_1 : C^*(u, 1|u^*u = 1 = uu^*) \to C^*(\Gamma)$, sending u to s_e and 1 to p_v.
By defining $\tilde{s}_e := u$ and $\tilde{p}_v := 1$ we see that $\tilde{s}_e$ and $\tilde{p}_v$ satisfy the relations (R1) and (R2). Note that $\tilde{s}_e$ is also a partial isometry and $\tilde{p}_v$ a projection. This is due to $uu^*u = u1 = u$ and $1^2 = 1 = 1^*$. So there exists another *-homomorphism $\phi_2 : C^*(\Gamma) \to C^*(u, 1|u^*u = 1 = uu^*)$, sending s_e to $\tilde{s}_e = u$ and p_v to $\tilde{p}_v = 1$. Those two homomorphism are inverse to each other. Using Corollary 1.23 we have $C(S^1) \cong C^*(u, 1|u^*u = 1 = uu^*) \cong C^*(\Gamma)$. □

Proposition 2.10. *Consider the following graph Γ and the associated graph C^*-algebra $C^*(\Gamma)$. We have $C^*(\Gamma) \cong \mathcal{T}$ where $\mathcal{T}$ is the Toeplitz algebra from Definition 1.24.*

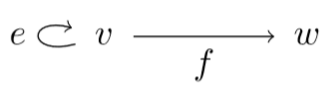

Figure 2.2: Graph 2

Proof. Step 1: There is a *-homomorphism $\phi_1 : C^*(u, 1|u^*u = 1) \to C^*(\Gamma)$, sending u to $(s_e + s_f)$ and 1 to $p_v + p_w$. Following from the relations (R1) and (R2) we have

$$\begin{aligned} p_v &= s_e^* s_e \\ p_w &= s_f^* s_f \\ p_v &= s_e s_e^* + s_f s_f^*. \end{aligned}$$

Using Proposition 2.7 we have that $p_v + p_w$ is the identity in $C^*(\Gamma)$. By using (R1) and Proposition 2.8 it follows that

$$\begin{aligned} &(s_e + s_f)^*(s_e + s_f) = s_e^* s_e + s_e^* s_f + s_f^* s_e + s_f^* s_f = p_v + 0 + 0 + p_w = p_v + p_w \\ &(s_e + s_f)(s_e + s_f)^* = s_e s_e^* + s_e s_f^* + s_f s_e^* + s_f s_f^* = p_v \\ &(s_e + s_f)^*(s_e + s_f) - (s_e + s_f)(s_e + s_f)^* = p_w \\ &(s_e + s_f)p_w = s_e p_w + s_f p_w = s_e s_f^* s_f + s_f = s_f \\ &(s_e + s_f)p_v = s_e p_v + s_f p_v = s_e + s_f s_e^* s_e = s_e \end{aligned}$$

holds. Meaning that $s_e + s_f$ is an isometry and it also generates $C^*(\Gamma)$. We get a *-homomorphism $\phi_1 : C^*(u, 1|u^*u = 1) \to C^*(\Gamma)$, sending u to $(s_e + s_f)$ and 1 to $p_v + p_w$.

Step 2: Consider the definition $\tilde{p}_v := uu^*, \tilde{p}_w := 1 - \tilde{p}_v, \tilde{s}_e := u\tilde{p}_v$ and $\tilde{s}_f := u\tilde{p}_w$. There exists a *-homomorphism $\phi_2 : C^*(\Gamma) \to C^*(u, 1|u^*u = 1)$, that sends p_v to $\tilde{p}_v = uu^*$, p_w to $\tilde{p}_w = 1 - uu^*$, s_e to $\tilde{s}_e = uuu^*$ and s_f to $\tilde{s}_f = u(1 - uu^*)$. It is

$$\begin{aligned} \tilde{s}_e \tilde{s}_e^* \tilde{s}_e &= uuu^*(uuu^*)^* uuu^* \\ &= uuu^*(uu^*)^* u^* uuu^* \\ &= uuu^* uu^* u^* uuu^* \\ &= uu(u^*u)u^*(u^*u)uu^* \\ &= uu(u^*u)u^* \\ &= uuu^* = \tilde{s}_e \end{aligned}$$

and

$$\begin{aligned} \tilde{s}_f \tilde{s}_f^* \tilde{s}_f &= u(1 - uu^*)(u(1 - uu^*))^* u(1 - uu^*) \\ &= u(1 - uu^*)(1 - uu^*)u^* u(1 - uu^*) \\ &= u(1 - uu^*)(1 - uu^*)(1 - uu^*) \\ &= u(1 - uu^*)(1 - 2uu^* + uu^* uu^*) \\ &= u(1 - uu^*)(1 - uu^*) \\ &= u(1 - uu^*) = \tilde{s}_f. \end{aligned}$$

Hence $\tilde{s}_e$ and $\tilde{s}_f$ are partial isometries. Further, it holds

$$\begin{aligned} &\tilde{p}_v^2 = uu^* uu^* = uu^* = \tilde{p}_v = (uu^*)^* = \tilde{p}_v^* \\ &\tilde{p}_w^2 = (1 - \tilde{p}_v)^2 = 1 - 2\tilde{p}_v + \tilde{p}_v^2 = 1 - \tilde{p}_v = \tilde{p}_w = (1 - \tilde{p}_v)^* = \tilde{p}_w^* \end{aligned}$$

and in addition

$$\tilde{p}_v \tilde{p}_w = \tilde{p}_v(1-\tilde{p}_v) = \tilde{p}_v - \tilde{p}_v^2 = 0.$$

Meaning that by $\tilde{p}_v$ and $\tilde{p}_w$ we defined mutually orthogonal projections. In the following, we want to check the relations (R1) and (R2). It applies

$$\tilde{s}_e^* \tilde{s}_e = \tilde{p}_v^* u^* u \tilde{p}_v = \tilde{p}_v^2 = \tilde{p}_v$$
$$\tilde{s}_f^* \tilde{s}_f = \tilde{p}_w^* u^* u \tilde{p}_w = \tilde{p}_w^2 = \tilde{p}_w$$
$$\tilde{s}_e^* \tilde{s}_f = \tilde{p}_v^* u^* u \tilde{p}_w = 0$$
$$\tilde{s}_e \tilde{s}_e^* + \tilde{s}_f \tilde{s}_f^* = u\tilde{p}_v \tilde{p}_v^* u^* + u\tilde{p}_w \tilde{p}_w^* u^* = u\tilde{p}_v u^* + u\tilde{p}_w u^* = u\tilde{p}_v u^* + u(1-\tilde{p}_v)u^* = uu^* = \tilde{p}_v.$$

We see that $\tilde{s}_e, \tilde{s}_f, \tilde{p}_v$ and $\tilde{p}_w$ satisfy the relations (R1) and (R2) and hence there exsits a *-homomorphism $\phi_2 : C^*(\Gamma) \to C^*(u, 1 | u^*u = 1)$, that sends p_v to $\tilde{p}_v = uu^*$, p_w to $\tilde{p}_w = 1 - uu^*$, s_e to $\tilde{s}_e = uuu^*$ and s_f to $\tilde{s}_f = u(1-uu^*)$.
Step 3: We have that $\phi_2 \circ \phi_1 = \mathrm{id}_{C^*(u,1|u^*u=1)}$ and $\phi_1 \circ \phi_2 = \mathrm{id}_{C^*(\Gamma)}$. One observe that

$$\phi_1(\phi_2(p_v)) = \phi_1(uu^*) = (s_e+s_f)(s_e+s_f)^* = p_v$$
$$\phi_1(\phi_2(p_w)) = \phi_1(1-uu^*) = (p_v+p_w) - p_v = p_w$$
$$\phi_1(\phi_2(s_e)) = \phi_1(uuu^*) = (s_e+s_f)(s_e+s_f)(s_e+s_f)^* = (s_e+s_f)p_v = s_e$$
$$\phi_1(\phi_2(s_f)) = \phi_1(u(1-uu^*)) = (s_e+s_f)p_w = s_f$$
$$\phi_2(\phi_1(u)) = \phi_2(s_e+s_f) = uuu^* + u(1-uu^*) = u$$
$$\phi_2(\phi_1(1)) = \phi_2(p_v+p_w) = 1$$

applies. We conclude by Definition 1.24 that we have $\mathcal{T} \cong C^*(\Gamma)$. □

Proposition 2.11. *Let $n \in \mathbb{N}$ and let $C^*(\Gamma)$ be the graph C^*-algebra of the following graph Γ. It follows $C^*(\Gamma) \cong M_n(\mathbb{C})$ where $M_n(\mathbb{C})$ is the set of $n \times n$-matrices.*

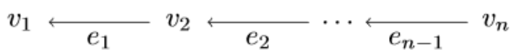

Figure 2.3: Graph 3

Proof. Step 1: In $C^*(\Gamma)$ it holds $s_{e_j} s_{e_{k-1}} = \delta_{jk} s_{e_j} s_{e_{j-1}}$ for $j = 1, \ldots, n-1; k = 2, \ldots, n$ and $s_{e_{j-1}}^* s_{e_{k-1}} = \delta_{jk} p_{v_{j-1}}$ for $j, k = 2, \ldots, n$. From our relations (R1) and (R2) we get

$$s_{e_i}^* s_{e_i} = p_{v_i} \text{ for all } i = 1, \ldots, n-1$$

and

$$s_{e_i} s_{e_i}^* = p_{v_{i+1}} \text{ for all } i = 1, \ldots, n-1.$$

Let $j = 1, ..., n-1$ and $k = 2, ..., n$. With Proposition 2.6 it follows that the relation

$$s_{e_j} s_{e_{k-1}} = s_{e_j} p_{v_j} p_{v_k} s_{e_{k-1}} = \delta_{jk} s_{e_j} s_{e_{j-1}}$$

holds and for $j, k = 2, \dots, n$

$$s^*_{e_{j-1}} s_{e_{k-1}} = (p_{e_j} s_{e_{j-1}})^* p_{e_k} s_{e_{k-1}} = s^*_{e_{j-1}} p_{e_j} p_{e_k} s_{e_{k-1}} = \delta_{jk} s^*_{e_{j-1}} s_{e_{j-1}} = \delta_{jk} p_{v_{j-1}}.$$

Step 2: Define $\tilde{E}_{i,i} := p_{v_i}$ for $i = 1, \dots, n$. For $i, j = 1, ..., n$ with $i > j$ we define $\tilde{E}_{i,j} := s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{j+1}} s_{e_j}$ and else $\tilde{E}_{i,j} := \tilde{E}^*_{j,i}$, where $\tilde{E}_{(j+1),j} := s_{e_j}$. There exists a *-homomorphism $\phi_1 : M_n(\mathbb{C}) \to C^*(\Gamma)$, sending $E_{i,j}$ to $\tilde{E}_{i,j} = s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{j+1}} s_{e_j}$ and $E_{i,i}$ to $\tilde{E}_{i,i} = p_{v_i}$. Recall Proposition 1.26, which told us that $M_n(\mathbb{C}) \cong C^*(e_{ij}; i, j = 1, \dots, n | e^*_{ij} = e_{ji}; e_{ij} e_{kl} = \delta_{jk} e_{il}$ for all $i, j, k, l)$. Obviously $\tilde{E}_{i,i}$ satisfy the relations of the matrix units. Let $i > j$ and $k > l$, then it is

$$\begin{aligned}\tilde{E}_{i,j} \tilde{E}_{k,l} &= s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{j+1}} s_{e_j} s_{e_{k-1}} s_{e_{k-2}} \dots s_{e_{l+1}} s_{e_l} \\ &= \delta_{j,k} s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{j+1}} s_{e_j} s_{e_{i-j}} s_{e_{j-2}} \dots s_{e_{l+1}} s_{e_l} = \delta_{j,k} \tilde{E}_{i,l}.\end{aligned}$$

For the next case let $j > i, k > l$ and with no loss of generality $i < l$. The remaining cases can be treated analogously. Then by using Equation 2.2

$$\begin{aligned}\tilde{E}_{i,j} \tilde{E}_{k,l} = \tilde{E}^*_{j,i} \tilde{E}_{k,l} &= s^*_{e_i} s^*_{e_{i+1}} \dots s^*_{e_{j-2}} s^*_{e_{j-1}} s_{e_{k-1}} s_{e_{k-2}} \dots s_{e_{l+1}} s_{e_l} \\ &= \delta_{j,k} s^*_{e_i} s^*_{e_{i+1}} \dots s^*_{e_{j-2}} s^*_{e_{j-1}} s_{e_{j-1}} s_{e_{j-2}} \dots s_{e_{l+1}} s_{e_l} \\ &= \delta_{j,k} s^*_{e_i} s^*_{e_{i+1}} \dots s^*_{e_{j-2}} p_{v_{j-1}} s_{e_{j-2}} \dots s_{e_{l+1}} s_{e_l} \\ &= \delta_{j,k} s^*_{e_i} s^*_{e_{i+1}} \dots s^*_{e_{j-2}} s_{e_{j-2}} \dots s_{e_{l+1}} s_{e_l} \\ &= \dots \\ &= \delta_{j,k} s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{l+1}} s_{e_l} \\ &= \delta_{j,k} \tilde{E}_{i,l}\end{aligned}$$

holds. In addition to that we have

$$\begin{aligned}\tilde{E}^*_{i,j} &= \tilde{E}_{j,i} \text{ for } i > j \\ \tilde{E}^*_{i,j} &= (\tilde{E}^*_{j,i})^* = \tilde{E}_{j,i} \text{ else.}\end{aligned}$$

Hence there exists a *-homomorphism $\phi_1 : M_n(\mathbb{C}) \to C^*(\Gamma)$, sending $E_{i,j}$ to $\tilde{E}_{i,j} = s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{j+1}} s_{e_j}$ and $E_{i,i}$ to $\tilde{E}_{i,i} = p_{v_i}$.
Step 3: Define $\tilde{p}_{v_i} := E_{i,i}$ and $\tilde{s}_{e_i} := E_{i+1,i}$. There is a *-homomorphism $\phi_2 : C^*(\Gamma) \to M_n(\mathbb{C})$, that sends p_{v_i} to $\tilde{p}_{v_i} = E_{i,i}$ and s_{e_i} to $\tilde{s}_{e_i} = E_{i+1,i}$. We see that $\tilde{p}_{v_i}$ is a projection, $\tilde{s}_{e_i}$ is a partial isometry and that they fulfill the relations (R1) and (R2), since we have

$$\begin{aligned}&E_{i+1,i} E^*_{i+1,i} E_{i+1,i} = E_{i+1,i+1} E_{i+1,i} = E_{i+1,i} \\ &E^2_{i,i} = E_{i,i} = E^*_{i,i} \\ &E^*_{i+1,i} E_{j+1,j} = \delta_{ij} E_{i,i} \\ &E_{i+1,i} E^*_{i+1,i} = E_{i+1,i+1}.\end{aligned}$$

We can conclude that there exists another *-homomorphism $\phi_2 : C^*(\Gamma) \to M_n(\mathbb{C})$, that sends p_{v_i} to $\tilde{p}_{v_i} = E_{i,i}$ and s_{e_i} to $\tilde{s}_{e_i} = E_{i+1,i}$.

Step 4: It is $\phi_2 \circ \phi_1 = \mathrm{id}_{M_n(\mathbb{C})}$ and $\phi_1 \circ \phi_2 = \mathrm{id}_{C^*(\Gamma)}$. We have

$$\begin{aligned}
&\phi_1(\phi_2(s_{e_i})) = \phi_1(E_{i+1,i}) = s_{e_i} \\
&\phi_1(\phi_2(p_{v_i})) = \phi_1(E_{i,i}) = p_{v_i} \\
&\phi_2(\phi_1(E_{i,j})) = \phi_2(s_{e_{i-1}} s_{e_{i-2}} \dots s_{e_{j+1}} s_{e_j}) = E_{i,i-1}E_{i-1,i-2}\dots E_{j+2,j+1}E_{j+1,j} = E_{i,j} \\
&\phi_2(\phi_1(E_{i,i})) = \phi_2(p_{v_i}) = E_{i,i}.
\end{aligned}$$

The *-homomorphisms are inverse to each other and one can conclude that $C^*(\Gamma) \cong M_n(\mathbb{C})$. □

For our next example we want to study an infinite version of Proposition 2.11.

Proposition 2.12. *We consider the next graph Γ and the associated graph C^*-algebra $C^*(\Gamma)$. It follows $C^*(\Gamma) \cong \mathcal{K}(H)$ where $\mathcal{K}(H)$ is the set of compact operators on a Hilbert space H.*

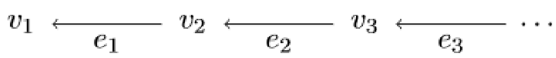

Figure 2.4: Graph 4

Proof. Keep in mind that we need to use the relations from Remark 2.5. Observe that Γ is a row-finite graph. Hence, it follows for all $i \in \mathbb{N}$

$$\begin{aligned}
s_{e_i}^* s_{e_i} &= p_{v_i} \\
s_{e_i} s_{e_i}^* &= p_{v_{i+1}}.
\end{aligned}$$

By taking an analogue approach as in Example 2.11 one can find an isomorphism to $C^*(e_{ij}; i, j \in \mathbb{N} | e_{ij}^* = e_{ji}; e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all $i, j, k, l)$. With Proposition 1.28 we have $C^*(\Gamma) \cong \mathcal{K}(H)$. □

Proposition 2.13. *Let $n \in \mathbb{N}$ and $n \geq 2$. Consider to the next graph Γ the associated graph C^*-algebra $C^*(\Gamma)$. It follows $C^*(\Gamma) \cong \mathcal{O}_n$ where $\mathcal{O}_n$ is the Cuntz algebra from Definition 1.25. The figure shown below is taken from* $(Eifler,\ 2016)[p.30]$.

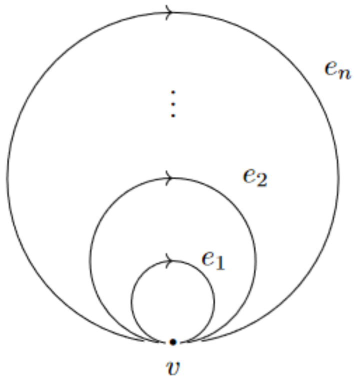

Figure 2.5: Graph 5

Proof. It holds $s_{e_i}^* s_{e_i} = p_v$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n s_{e_i} s_{e_i}^* = p_v$. With Proposition 2.7 the projection p_v is the identity in $C^*(\Gamma)$ and therefore, s_{e_i} is an isometry for all $i = 1, \ldots, n$. Also $\{s_{e_i} | i = 1, ..., n\}$ is the generator of $C^*(\Gamma)$. Recall Definition 1.25 and the universal C^*-algebra $\mathcal{O}_n = C^*(S_1, \ldots, S_n \mid S_i^* S_i = 1 \text{ for all } i = 1, \ldots, n;$ $\sum_{i=1}^n S_i S_i^* = 1)$. We obtain a surjective *-homomorphism $\phi_1 : \mathcal{O}_n \to C^*(\Gamma)$, sending S_i to s_{e_i}.
On the other hand with $\tilde{s}_{e_i} := S_i$ and $\tilde{p}_v := \sum_{i=1}^n S_i S_i^*$ we see that $\tilde{s}_{e_i}$ and $\tilde{p}_v$ satisfy the relations of the graph C^*-algebra $C^*(\Gamma)$. Nevertheless, we check that $\tilde{s}_{e_i}^* \tilde{s}_{e_j} = 0$ for $i \neq j$. By Proposition 1.9 we have that $\{S_i S_i^* \mid i = 1, ..., n\}$ are mutually orthogonal and hence for $i \neq j$

$$\begin{aligned} \tilde{s}_{e_i}^* \tilde{s}_{e_j} &= S_i^* S_j \\ &= S_i^* S_i S_i^* S_j S_j^* S_j \\ &= S_i^* 0 S_j = 0. \end{aligned}$$

The resulting *-homomorphism $\phi_2 : C^*(\Gamma) \to \mathcal{O}_n$ sends s_{e_i} to $\tilde{s}_{e_i} = S_i$. Observe that ϕ_1 and ϕ_2 are inverse to each other. We have $C^*(\Gamma) \cong \mathcal{O}_n$. □

3 Hypergraph C^*-algebras

We now come to the main section of this thesis, where we introduce the so called hypergraph C^*-algebras. They are new mathematical objects. The definition of hypergraph C^*-algebras was conveyed to us by Simon Schmidt and Moritz Weber. We then enriched the theory with the remaining results of this section. First, we will define directed hypergraphs.

3.1 Definition and properties

Definition 3.1. A *directed finite hypergraph* $H\Gamma = (V, E, r, s)$ consists of two finite sets V, E and two mappings $r, s : E \to \mathcal{P}(V)\backslash\{\emptyset\}$. The set V contains vertices, while the set E contains *hyperedges*. The difference to a directed finite graph, as in Definition 2.1, is that the hyperedges can join any number of vertices whereas for graphs, we always have $|r(e)| = 1 = |s(e)|$ for all $e \in E$. Therefore, the range map r and the source map s map to the power set $\mathcal{P}(V)$ of V rather than to V. We only study directed finite hypergraphs. We write hypergraph instead of directed finite hypergraph.

For a better understanding, we take a look at an example of a hypergraph.

Example 3.2. Consider the following hypergraph $H\Gamma$.

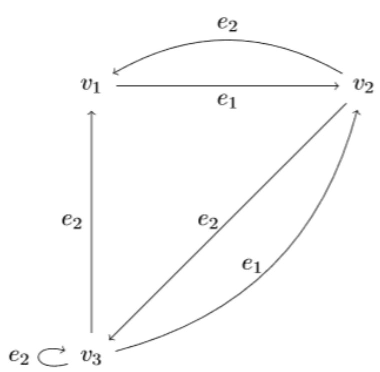

Figure 3.1: Hypergraph 1

We have $V = \{v_1, v_2, v_3\}$ and $E = \{e_1, e_2\}$. For the image of our range and source map we have

$$r(e_1) = \{v_2\}, \quad s(e_1) = \{v_1, v_3\}$$
$$r(e_2) = \{v_1, v_3\}, \quad s(e_2) = \{v_2, v_3\}.$$

Throughout this thesis we display hypergraphs like the example above. There are several different ways to display hypergraphs but for our following examples this way of representing them is just fine. For more complicated cases one might consider to draw circles around the vertices instead of connecting them with arrows. I should notice that we did all the drawings of graphs by ourselves except the drawing from Proposition 2.13.

Remark 3.3. *Notice that every graph* $\Gamma = (V, E, r, s)$ *is also a hypergraph* $H\Gamma = (V, E, r', s')$ *by defining* $r' : E \to \mathcal{P}(V), e \mapsto \{r(e)\}$ *and* $s' : E \to \mathcal{P}(V), e \mapsto \{s(e)\}$*.*

Similar as in the Definition 2.2 for constructing graph C^*-algebras, we will define hypergraph C^*-algebras.

Definition 3.4. Let $H\Gamma = (V, E, r, s)$ be a hypergraph. The hypergraph C^*-algebra $C^*(H\Gamma)$ of the hypergraph $H\Gamma$ is the universal C^*-algebra generated by mutually orthogonal projections p_v for all $v \in V$ and partial isometries s_e for all $e \in E$ such that the following relations hold

(HR1) $s_e^* s_f = \delta_{ef} \sum_{v \in r(e)} p_v$ for all $e, f \in E$

(HR2) $s_e s_e^* \leq \sum_{v \in s(e)} p_v$ for all $e \in E$

(HR3) $p_w \leq \sum_{\substack{e \in E \\ w \in s(e)}} s_e s_e^*$ if $s^{-1}(w) \neq \emptyset$ for $w \in V$.

Remark 3.5. *Every hypergraph* C^**-algebra exists by Lemma 1.18.*

Keep in mind that with Proposition 1.4, we have for every projection p and q in some C^*-algebra A the equivalence $p \leq q \Leftrightarrow pq = p = qp$.

Lemma 3.6. *Let* p *and* q *be projections in some* C^**-algebra* A*. If* $p \leq q$ *and* $q \leq p$ *applies, then we have* $p = q$*.*

Proof. It follows

$$p = pq = q.$$

□

Lemma 3.7. *Let* p_1, p_2, q *be projections in some* C^**-algebra* A*. If* $p_i \leq q$ *for* $i = 1, 2$ *and* $q \leq p_1 + p_2$ *applies, then we have* $q = p_1 + p_2$*.*

Proof. It follows

$$q = q(p_1 + p_2) = qp_1 + qp_2 = p_1 + p_2.$$

□

We are still left with the question whether the hypergraph C^*-algebra $C^*(H\Gamma)$ of an arbitrary hypergraph $H\Gamma$ is trivial or not. The following statement shows that the class of hypergraph C^*-algebras contains the class of graph C^*-algebras and therefore, our definition of hypergraph C^*-algebras is a generalization as wished. We know that every graph C^*-algebra is non-trivial (see (Raeburn, 2005)) and therefore, we found a class of non-trivial hypergraph C^*-algebras. The proposition also shows that the Relations (HR2) and (HR3) are the corresponding relations to Relation (R2).

Proposition 3.8. *Consider a graph $\Gamma = (V, E, r, s)$ and interpret it as a hypergraph $H\Gamma = (V, E, r', s')$ in the sense of Remark 3.3. For our graph C^*-algebra we write*

$$C^*(\Gamma) = C^*(\tilde{s}_e,\ e \in E; \tilde{p}_v,\ v \in V \mid \tilde{p}_v\tilde{p}_w = 0, v \neq w; \tilde{s}_e^*\tilde{s}_f = \delta_{ef}\tilde{p}_{r(e)}; \sum_{\substack{e\in E\\ s(e)=w}} \tilde{s}_e\tilde{s}_e^* = \tilde{p}_w)$$

where $\tilde{s}_e$ is a partial isometry for all $e \in E$ and $\tilde{p}_v$ is a projection for all $v \in V$. Then we have $C^*(\Gamma) \cong C^*(H\Gamma)$.

Proof. First, we check that the generators of $C^*(\Gamma)$ fulfill the relations of $C^*(H\Gamma)$. Since the only element in the set $r'(e)$ is the vertex $r(e)$ we have

$$\tilde{s}_e^*\tilde{s}_f = \delta_{ef}\tilde{p}_{r(e)} = \delta_{ef} \sum_{\substack{v\in V\\ v\in r'(e)}} \tilde{p}_v.$$

We see that Relation (HR1) is fulfilled. For the same reasons it follows for $w \in V$ with $s^{-1}(w) \neq \emptyset$ that

$$\tilde{p}_w = \sum_{\substack{e\in E\\ w=s(e)}} \tilde{s}_e\tilde{s}_e^* = \sum_{\substack{e\in E\\ w\in s'(e)}} \tilde{s}_e\tilde{s}_e^*$$

applies and hence

$$\tilde{p}_w = \tilde{p}_w\tilde{p}_w = \sum_{\substack{e\in E\\ w\in s'(e)}} \tilde{s}_e\tilde{s}_e^*\tilde{p}_w = \tilde{p}_w \sum_{\substack{e\in E\\ w\in s'(e)}} \tilde{s}_e\tilde{s}_e^*.$$

Therefore, Relation (HR3) $\tilde{p}_w \leq \sum_{\substack{e\in E\\ w\in s'(e)}} \tilde{s}_e\tilde{s}_e^*$ is fulfilled. Let's check Relation (HR2). By Equation 2.2 we have $\tilde{s}_e = \tilde{p}_{s(e)}\tilde{s}_e$. So it follows

$$\tilde{s}_e\tilde{s}_e^* = \tilde{p}_{s(e)}\tilde{s}_e\tilde{s}_e^* = \tilde{s}_e\tilde{s}_e^*\tilde{p}_{s(e)}^* = \tilde{s}_e\tilde{s}_e^*\tilde{p}_{s(e)}$$

and hence

$$\tilde{s}_e\tilde{s}_e^* \leq \tilde{p}_{s(e)} = \sum_{\substack{v\in V\\ v=s(e)}} \tilde{p}_v = \sum_{\substack{v\in V\\ v\in s'(e)}} \tilde{p}_v.$$

By the universal property we obtain a *-homomorphism $\phi_1 : C^*(H\Gamma) \to C^*(\Gamma)$, sending s_e to $\tilde{s}_e$ and p_v to $\tilde{p}_v$. To obtain a *-homomorphism that is inverse to ϕ_1, we are going to prove that the generators of $C^*(H\Gamma)$ satisfy the Relations (R1) and (R2) of $C^*(\Gamma)$. Using the same argument as in the above direction, we have

$$s_e^*s_f = \delta_{ef} \sum_{\substack{v\in V\\ v\in r'(e)}} p_v = \delta_{ef}p_{r(e)}.$$

We see that Relation (R1) is satisfied. To show that (R2) is fulfilled we need Relations (HR2) and (HR3). Let $w \in V$ with $s^{-1}(w) \neq \emptyset$ and hence there exists at least one $f \in E$ with $s(f) = w$. With (HR2) it follows

$$s_e s_e^* \leq \sum_{\substack{v \in V \\ v \in s'(e)}} p_v = p_{s(e)}$$

and using Relation (HR3) we have

$$p_w = p_{s(f)} \leq \sum_{\substack{e \in E \\ s(f) \in s'(e)}} s_e s_e^* = \sum_{\substack{e \in E \\ s(f) = s(e)}} s_e s_e^*.$$

By Lemma 3.7 we conclude

$$p_w = \sum_{\substack{e \in E \\ w = s(e)}} s_e s_e^*.$$

Therefore, a *-homomorphism $\phi_2 : C^*(\Gamma) \to C^*(H\Gamma)$, that sends $\tilde{s}_e$ to s_e and $\tilde{p}_v$ to p_v, exists. Notice that ϕ_1 and ϕ_2 are inverse to each other. □

In the following, we will present a useful statement that will accompany us through the remaining parts of this thesis. It says that the projections sum up to the identity. We know that it holds of every graph C^*-algebra (see Proposition 2.7). It's nice to notice that although we generalize graph C^*-algebras we are able to show that this statement still holds.

Theorem 3.9. *For every hypergraph $H\Gamma = (V, E, r, s)$ and hypergraph C^*-algebra $C^*(H\Gamma)$ we have that $\sum_{v \in V} p_v$ is the unit element in $C^*(H\Gamma)$ and therefore, $\sum_{v \in V} p_v = 1$.*

Proof. Using Relation (HR1), we have

$$\begin{aligned} s_e \sum_{v \in V} p_v &= s_e s_e^* s_e \sum_{v \in V} p_v \\ &= s_e \sum_{v \in r(e)} p_v \sum_{v \in V} p_v \\ &= s_e \sum_{v \in r(e)} p_v \\ &= s_e. \end{aligned}$$

It follows with Relation (HR2)

$$\begin{aligned} (\sum_{v \in V} p_v) s_e &= (\sum_{v \in V} p_v) s_e s_e^* s_e \\ &= (\sum_{v \in V} p_v)(\sum_{v \in s(e)} p_v) s_e s_e^* s_e \\ &= (\sum_{v \in s(e)} p_v) s_e s_e^* s_e \\ &= s_e. \end{aligned}$$

Notice that we have

$$\sum_{v\in V} p_v p_w = p_w = p_w \sum_{v\in V} p_v \text{ for all } w \in V \text{ and}$$
$$(\sum_{v\in V} p_v)^2 = \sum_{v\in V} p_v = (\sum_{v\in V} p_v)^*.$$

We conclude that $\sum_{v\in V} p_v$ is the unit element in $C^*(H\Gamma)$. □

3.2 Examples

In this chapter we will present some interesting hypergraphs. One should recall Theorem 3.9, since it will be of great benefit.

3.2.1 Toeplitz algebra

The first examples we present are interesting cases since we found isomorphisms from the hypergraph C^*-algebras to some well known universal C^*-algebras. Therefore, we fully understand these examples.

Proposition 3.10. *Consider the hypergraph $H\Gamma$ with vertices $\{v_1, v_2\}$ and edges $\{e_1\}$. The image of the range and source map looks as follows $r(e_1) = \{v_1, v_2\}$, $s(e_1) = \{v_1\}$. We have $C^*(H\Gamma) \cong \mathcal{T}$ where $\mathcal{T}$ is the Toeplitz algebra from Definition 1.24.*

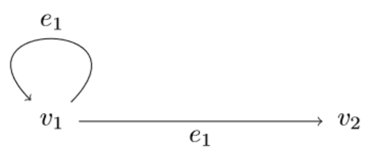

Figure 3.2: Hypergraph 2

Proof. Step 1: There exists a *-homomorphism $\phi_1 : \mathcal{T} = C^*(u, 1 | u^*u = 1) \to C^*(H\Gamma)$, sending u to s_{e_1} and 1 to $p_{v_1} + p_{v_2} = s_{e_1}^* s_{e_1}$. We are going to examine the associated hypergraph C^*-algebra $C^*(H\Gamma)$. Using the Relation (HR1) we get $s_{e_1}^* s_{e_1} = p_{v_1} + p_{v_2}$. Therefore, we have $s_{e_1}^* s_{e_1} = 1$ by Theorem 3.9. The remaining Relations (HR2) and (HR3) imply $s_{e_1} s_{e_1}^* \leq p_{v_1}$ and $s_{e_1} s_{e_1}^* \geq p_{v_1}$. Hence $s_{e_1} s_{e_1}^* = p_{v_1}$ by Lemma 3.6. Since $s_{e_1}^* s_{e_1}$ is the unit, we know that s_{e_1} is an isometry. We obtain a *-homomorphism $\phi_1 : \mathcal{T} = C^*(u, 1 | u^*u = 1) \to C^*(H\Gamma)$, sending u to s_{e_1} and 1 to $p_{v_1} + p_{v_2} = s_{e_1}^* s_{e_1}$.
Step 2: Define $\tilde{s}_{e_1} := u, \tilde{p}_{v_1} := uu^*$ and $\tilde{p}_{v_2} := 1 - uu^*$. There is a *-homomorphism $\phi_2 : C^*(H\Gamma) \to \mathcal{T}$, that sends s_{e_1} to $\tilde{s}_{e_1} = u$, p_{v_1} to $\tilde{p}_{v_1} = uu^*$ and p_{v_2} to $\tilde{p}_{v_2} = (1 - uu^*)$.

We see that $\tilde{s}_{e_1}$ is a partial isometry and that $\tilde{p}_{v_1}$ and $\tilde{p}_{v_2}$ define projections:

$$\begin{aligned}
&\tilde{s}_{e_1}\tilde{s}_{e_1}^*\tilde{s}_{e_1} = uu^*u = u1 = u = \tilde{s}_{e_1}\\
&\tilde{p}_{v_1}^* = (uu^*)^* = uu^* = \tilde{p}_{v_1}\\
&\tilde{p}_{v_1}^2 = uu^*uu^* = u1u^* = uu^* = \tilde{p}_{v_1}\\
&\tilde{p}_{v_2}^* = (1-uu^*)^* = 1^* - uu^* = 1 - uu^* = \tilde{p}_{v_2}\\
&\tilde{p}_{v_2}^2 = (1-uu^*)^2 = 1 - 2uu^* + uu^*uu^* = 1 - uu^* = \tilde{p}_{v_2}.
\end{aligned}$$

We check the relations of $C^*(H\Gamma)$ and we start with Relation (HR1). We have

$$\tilde{s}_{e_1}^*\tilde{s}_{e_1} = u^*u = 1 = uu^* + (1-uu^*) = \tilde{p}_{v_1} + \tilde{p}_{v_2}.$$

Therefore, Relation (HR1) is fulfilled. Furthermore, it is

$$\tilde{s}_{e_1}\tilde{s}_{e_1}^* = uu^* = \tilde{p}_{v_1}$$

and we see that Relations (HR2) and (HR3) are satisfied. Hence $\tilde{s}_{e_1}, \tilde{p}_{v_1}$ and $\tilde{p}_{v_2}$ fulfill the relations of $C^*(H\Gamma)$. Hence, there exists a *-homomorphism $\phi_2 : C^*(H\Gamma) \to \mathcal{T}$, that sends s_{e_1} to $\tilde{s}_{e_1} = u$, p_{v_1} to $\tilde{p}_{v_1} = uu^*$ and p_{v_2} to $\tilde{p}_{v_2} = (1-uu^*)$.
Step 3: It is $\phi_2 \circ \phi_1 = \mathrm{id}_{C^*(H\Gamma)}$ and $\phi_1 \circ \phi_2 = \mathrm{id}_{\mathcal{T}}$. We have

$$\begin{aligned}
&\phi_1(\phi_2(s_{e_1})) = \phi_1(u) = s_{e_1}\\
&\phi_1(\phi_2(p_{v_1})) = \phi_1(uu^*) = s_{e_1}s_{e_1}^* = p_{v_1}\\
&\phi_1(\phi_2(p_{v_2})) = \phi_1(1-uu^*) = p_{v_1} + p_{v_2} - p_{v_1} = p_{v_2}\\
&\phi_2(\phi_1(u)) = \phi_2(s_{e_1}) = u\\
&\phi_2(\phi_1(1)) = \phi_2(p_{v_1} + p_{v_2}) = uu^* + 1 - uu^* = 1.
\end{aligned}$$

The *-homomorphisms are inverse to each other and therefore, we have $\mathcal{T} \cong C^*(H\Gamma)$. □

3.2.2 Cuntz algebra

Proposition 3.11. *Let $n \in \mathbb{N}$ and $n \geq 2$. Consider the hypergraph $H\Gamma$ with vertices $\{v_1, ..., v_n\}$ and edges $\{e_1, ..., e_n\}$. The range and source map are defined as follows $r(e_i) = \{v_1, ..., v_n\}$, $s(e_i) = \{v_i\}$ for all $i = 1, ..., n$. It is $C^*(H\Gamma) \cong \mathcal{O}_n$ where $\mathcal{O}_n$ is the Cuntz algebra from Definition 1.25.*

Proof. By the relations of the hypergraph C^*-algebra $C^*(H\Gamma)$ we have

$$\begin{aligned}
&s_{e_i}^*s_{e_j} = \delta_{i,j}\sum_{j=1}^{n} p_{v_j} \text{ for all } i,j = 1, ..., n\\
&s_{e_i}s_{e_i}^* \leq p_{v_i} \text{ for all } i = 1, ..., n\\
&s_{e_i}s_{e_i}^* \geq p_{v_i} \text{ for all } i = 1, ..., n.
\end{aligned}$$

Notice that $s_{e_i}^* s_{e_i} = 1$ for all $i = 1, \dots, n$ by Theorem 3.9. With the last two relations we obtain $s_{e_i} s_{e_i}^* = p_{v_i}$ for all $i = 1, ..., n$ by Lemma 3.6. It follows that $1 = s_{e_i}^* s_{e_i} = (\sum_{j=1}^n p_{v_j}) = \sum_{j=1}^n s_{e_j} s_{e_j}^*$ holds and that $\{s_{e_i} \mid i = 1, \dots, n\}$ are generators of $C^*(H\Gamma)$. We obtain a *-homomorphism $\phi_1 : \mathcal{O}_n = C^*(S_1, \dots, S_n \mid S_i^* S_i = 1$ for all $i = 1, \dots, n; \sum_{i=1}^n S_i S_i^* = 1) \to C^*(H\Gamma)$, that sends S_i to s_{e_i}.
By $\tilde{s}_{e_i} := S_i$ we define a family $\{\tilde{s}_{e_i} \mid i = 1, \dots, n\}$ that satisfies the relations of the hypergraph C^*-algebra $C^*(H\Gamma)$. To see that $\tilde{s}_{e_i}^* \tilde{s}_{e_j} = 0$ for $i \neq j$ one can use the same argument as in Proposition 2.13. Therefore, we obtain a *-homomorphism $\phi_2 : C^*(H\Gamma) \to \mathcal{O}_n$, that is inverse to ϕ_1 and hence $\mathcal{O}_n \cong C^*(H\Gamma)$. □

3.2.3 Free products of $C(S^1)$ resp. $\mathcal{O}_m$ with $\mathbb{C}^n$

The next example will show that the class of hypergraph C^*-algebras is strictly larger than the class of graph C^*-algebras. One should recall the Definition 1.19 of free products: When we have two unital universal C^*-algebras $A = C^*(E_1|R_1)$ and $B = C^*(E_2|R_2)$, we call

$$A *_{\mathbb{C}} B := C^*(E_1, E_2 | R_1, R_2 \text{ and } 1_A = 1_B)$$

the free product of A and B.

Proposition 3.12. *Let $n \in \mathbb{N}$ and consider the hypergraph $H\Gamma$ with vertices $\{v_1, \dots, v_n\}$, edges $\{e_1\}$ and the mappings defined as $r(e_1) = \{v_1, \dots, v_n\}$, $s(e_1) = \{v_1, \dots, v_n\}$. We have $C^*(H\Gamma) \cong C(S^1) *_{\mathbb{C}} \mathbb{C}^n$.*

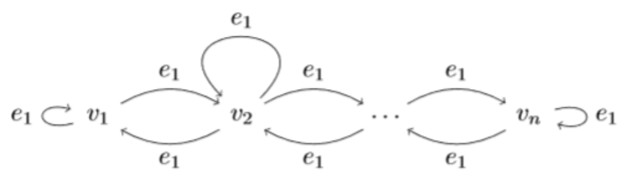

Figure 3.3: Hypergraph 3

Proof. For the hypergraph C^*-algebra $C^*(H\Gamma)$ we obtain $s_{e_1}^* s_{e_1} = \sum_{i=1}^n p_{v_i}$ by Relation (HR1). Once again $s_{e_1}^* s_{e_1}$ is the unit by Theorem 3.9. Further, by Relations (HR2) and (HR3) we have $s_{e_1} s_{e_1}^* \leq \sum_{i=1}^n p_{v_i}$ and $s_{e_1} s_{e_1}^* \geq p_{v_i}$ for all $i = 1, ..., n$. This implies $s_{e_1} s_{e_1}^* = \sum_{i=1}^n p_{v_i}$ by Lemma 3.7 and also $s_{e_1} s_{e_1}^* = 1$ by Theorem 3.9. Hence we have $s_{e_1} s_{e_1}^* = 1 = s_{e_1}^* s_{e_1}$.
Consider the universal C*-algebras $C(S^1) \cong C^*(u, 1 | uu^* = 1 = u^* u)$ and $\mathbb{C}^n \cong C^*(p_i, i = 1, ..., n | p_i^* = p_i = p_i^2; \sum_{i=1}^n p_i = 1) \cong \mathbb{C}^n$. We will show that we have $C(S^1) *_{\mathbb{C}} \mathbb{C}^n \cong C^*(H\Gamma)$. With the considerations above, we know that s_{e_1} satisfies the relations of $C(S^1)$ and that $\{p_{v_i} \mid i = 1, \dots, n\}$ satisfy the relations of $\mathbb{C}^n$. With $s_{e_1}^* s_{e_1} = 1 = \sum_{i=1}^n p_{v_i}$ we obtain a *-homomorphism $\phi_1 : C(S^1) *_{\mathbb{C}} \mathbb{C}^n \to C^*(H\Gamma)$, sending u to s_{e_1} and p_i to p_{v_i} for all $i = 1, \dots, n$.

Conversely, with $\tilde{s}_{e_1} := u$ and $\tilde{p}_{v_i} := p_i$, we have that $\tilde{s}_{e_1}$ is a partial isometry, $\tilde{s}_{e_1}\tilde{s}_{e_1}^*$ and $\tilde{s}_{e_1}^*\tilde{s}_{e_1}$ are the unit element, $\tilde{p}_{v_i}$ are projections and especially we have $\tilde{s}_{e_1}^*\tilde{s}_{e_1} = 1_{C(S^1)} = 1_{\mathbb{C}^n} = \sum_{i=1}^n \tilde{p}_{v_i}$ in $C(S^1) *_{\mathbb{C}} \mathbb{C}^n$.
The last two relations of our hypergraph C*-algebra $\tilde{s}_{e_1}\tilde{s}_{e_1}^* \leq \sum_{i=1}^n \tilde{p}_{v_i}$ and $\tilde{s}_{e_1}\tilde{s}_{e_1}^* \geq \tilde{p}_{v_i}$ for all $i = 1, ..., n$ are implied by the following

$$\tilde{s}_{e_1}\tilde{s}_{e_1}^* \sum_{i=1}^n \tilde{p}_{v_i} = 1_{C(S^1)}1_{\mathbb{C}^n} = 1_{C(S^1)} = \tilde{s}_{e_1}\tilde{s}_{e_1}^* = (\sum_{i=1}^n \tilde{p}_{v_i})\tilde{s}_{e_1}\tilde{s}_{e_1}^*$$
$$\tilde{s}_{e_1}\tilde{s}_{e_1}^*\tilde{p}_{v_i} = 1_{C(S^1)}\tilde{p}_{v_i} = \tilde{p}_{v_i} = \tilde{p}_{v_i}1_{C(S^1)} = \tilde{p}_{v_i}\tilde{s}_{e_1}\tilde{s}_{e_1}^*.$$

We obtain a *-homomorphism ϕ_2 that is inverse to ϕ_1. Hence we have $C(S^1) *_{\mathbb{C}} \mathbb{C}^n \cong C^*(H\Gamma)$. □

Remark 3.13. *One can define the term "nuclear" C^*-algebra. In (Raeburn, 2005)[p.34 Remark 4.3] it is stated that every graph C^*-algebra is nuclear. This is interesting due to the fact that one can show that the hypergraph C^*-algebra $C^*(H\Gamma)$ from Theorem 3.12 is not nuclear. In Proposition 3.8 it is shown that every graph C^*-algebra is a hypergraph C^*-algebra. Now we see that the class of hypergraph C^*-algebras is in fact strictly larger than the class of graph C^*-algebras. Since the theory of the term "nuclear" goes beyond the scope of this thesis, we present a sketch of the proof which was provided to us by Moritz Weber.*

Proof. In Proposition 3.12 we proved that $C^*(H\Gamma)$ is isomorphic to the free product $C(S^1) *_{\mathbb{C}} \mathbb{C}^n$. This free product is not nuclear and therefore, $C^*(H\Gamma)$ is not nuclear. To see this, notice that the set of integers $\mathbb{Z}$ and the cyclic group of order n, $\mathbb{Z}/n\mathbb{Z}$ form discrete groups and therefore, we are able to define the group C^*-algebras $C^*(\mathbb{Z})$ and $C^*(\mathbb{Z}/n\mathbb{Z})$. It is $C(S^1) \cong C^*(\mathbb{Z})$ and $\mathbb{C}^n \cong C^*(\mathbb{Z}/n\mathbb{Z})$. In group C^*-algebra theory one can show that for two groups G_1,G_2 and their free product $G_1 * G_2$ we have following isomorphism $C^*(G_1) *_{\mathbb{C}} C^*(G_2) \cong C^*(G_1 * G_2)$. Hence we have $C(S^1) *_{\mathbb{C}} \mathbb{C}^n \cong C^*(\mathbb{Z} * (\mathbb{Z}/n\mathbb{Z}))$. Since $\mathbb{F}_2 \subset \mathbb{Z} * (\mathbb{Z}/n\mathbb{Z})$ holds, it follows that $\mathbb{Z} * (\mathbb{Z}/n\mathbb{Z})$ is not *amenable* which is equivalent to $C^*(\mathbb{Z} * (\mathbb{Z}/n\mathbb{Z}))$ not being nuclear. Hence we are done. □

What happens to Proposition 3.12, when we take $m \geq 2$ edges instead of 1 edge? Looking back at the proposition before, one would expect to meet the Cuntz algebra. Let's show that we can confirm our expectations.

Proposition 3.14. *Let $n, m \in \mathbb{N}$ with $m \geq 2$. Consider the hypergraph $H\Gamma$ with the following properties: vertices $\{v_1, ..., v_n\}$, edges $\{e_1, ..., e_m\}$ and $r(e_i) = \{v_1, ..., v_n\}$, $s(e_i) = \{v_1, ..., v_n\}$ for all $i = 1, ..., m$. We have $C^*(H\Gamma) \cong \mathcal{O}_m *_{\mathbb{C}} \mathbb{C}^n$.*

Proof. Using the relations of the associated hypergraph C^*-algebra $C^*(H\Gamma)$, we obtain

$$s_{e_i}^* s_{e_j} = \delta_{i,j} \sum_{k=1}^{n} p_{v_k} \text{ for all } i, j = 1, ..., m$$

$$s_{e_i} s_{e_i}^* \leq \sum_{j=1}^{n} p_{v_j} \text{ for all } i = 1, ..., m$$

$$p_{v_i} \leq \sum_{j=1}^{m} s_{e_j} s_{e_j}^* \text{ for all } i = 1, ..., n.$$

With Theorem 3.9 we know that $s_{e_i}^* s_{e_i} = \sum_{j=1}^{n} p_{v_j} = 1$ for all $i = 1, ..., m$. Furthermore, we have with Relation (HR2) and (HR3)

$$\begin{aligned} 1 = \sum_{i=1}^{n} p_{v_i} &= \sum_{i=1}^{n} (p_{v_i} \sum_{j=1}^{m} s_{e_j} s_{e_j}^*) \\ &= \sum_{j=1}^{m} (\sum_{i=1}^{n} p_{v_i}) s_{e_j} s_{e_j}^* \\ &= \sum_{j=1}^{m} s_{e_j} s_{e_j}^*. \end{aligned}$$

Recall $\mathcal{O}_m = C^*(S_1, \ldots, S_m \mid S_i^* S_i = 1 \text{ for all } i = 1, \ldots, m; \sum_{i=1}^{m} S_i S_i^* = 1)$ and $\mathbb{C}^n \cong C^*(p_i, i = 1, ..., n | p_i^* = p_i = p_i^2; \sum_{i=1}^{n} p_i = 1)$. Hence we have a *-homomorphism $\phi_1 : \mathcal{O}_m *_{\mathbb{C}} \mathbb{C}^n \to C^*(H\Gamma)$ that sends S_i to s_{e_i} for all $i = 1, ..., m$ and p_i to p_{v_i} for all $i = 1, ..., n$.
Let $i = 1, ..., m$. Conversely, we have $S_i^* S_i = \sum_{j=1}^{m} S_j S_j^* = 1 = \sum_{j=1}^{n} p_j$ in the free product. With S_i we also defined partial isometries for all $i = 1, ..., m$. Hence, the Relations (HR1)-(HR3) are satisfied in $\mathcal{O}_m *_{\mathbb{C}} \mathbb{C}^n$. We obtain a *-homomorphism ϕ_2 that is inverse to ϕ_1. □

3.2.4 Two mysterious examples

In the following example we were able to show that the partial isometry and their conjugated form add up to a unitary.

Proposition 3.15. *Define the hypergraph $H\Gamma$ with vertices $\{v_1, v_2, v_3, v_4\}$ and edges $\{e_1\}$. The image of the range and source map is defined like this*
$r(e_1) = \{v_1, v_2\}$ *and* $s(e_1) = \{v_3, v_4\}$.
*There exists a *-homomorphism $\phi : C(S^1) \to C^*(H\Gamma)$, sending the identity function z on S^1 to $s_{e_1} + s_{e_1}^*$.*

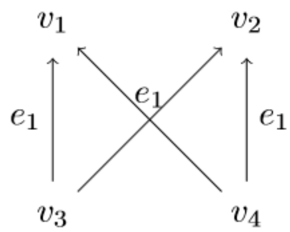

Figure 3.4: Hypergraph 6

Proof. Step 1: We have $s_{e_1}^* s_{e_1} = p_{v_1} + p_{v_2}$ and $s_{e_1} s_{e_1}^* = p_{v_3} + p_{v_4}$. By the relations of $C^*(H\Gamma)$ we have

$$\begin{aligned} &s_{e_1}^* s_{e_1} = p_{v_1} + p_{v_2} \\ &s_{e_1} s_{e_1}^* \leq p_{v_3} + p_{v_4} \\ &p_{v_3} \text{ respectively } p_{v_4} \leq s_{e_1} s_{e_1}^*. \end{aligned}$$

Using Lemma 3.7 implies

$$s_{e_1} s_{e_1}^* = p_{v_3} + p_{v_4}.$$

Step 2: We have $s_{e_1}^* s_{e_1} + s_{e_1} s_{e_1}^* = 1$. Using Theorem 3.9 we conclude from Step 1 that

$$s_{e_1}^* s_{e_1} + s_{e_1} s_{e_1}^* = \sum_{j=1}^{4} p_{v_j} = 1.$$

Step 3: We have that $s_{e_1} + s_{e_1}^*$ is a unitary. First, we show that $s_{e_1}^2 = 0$. It is

$$s_{e_1}^2 = s_{e_1} s_{e_1}^* s_{e_1} s_{e_1} s_{e_1}^* s_{e_1} = s_{e_1}(p_{v_1} + p_{v_2})(p_{v_3} + p_{v_4}) s_{e_1} = 0$$

and therefore,

$$(s_{e_1} + s_{e_1}^*)(s_{e_1} + s_{e_1}^*)^* = s_{e_1} s_{e_1}^* + s_{e_1}^2 + (s_{e_1}^*)^2 + s_{e_1}^* s_{e_1} = 1 = (s_{e_1} + s_{e_1}^*)^*(s_{e_1} + s_{e_1}^*).$$

Step 4: We have a *-homomorphism $\phi : C(S^1) \to C^*(H\Gamma)$. By the universal property we have a *-homomorphism $\phi_1 : C^*(u, 1 \mid uu^* = 1 = u^*u) \to C^*(H\Gamma)$ sending u to $s_{e_1} + s_{e_1}^*$. From Corollary 1.23 we have a *-isomomorphism $\Phi : C(S^1) \to C^*(u, 1 \mid uu^* = 1 = u^*u)$ that sends the identity function z on S^1 to u. Hence we have *-homomorphism $\phi : C(S^1) \to C^*(H\Gamma)$ sending z to $s_{e_1} + s_{e_1}^*$. □

The next example is of great interest since we do not fully understand it. We found a non-surjective *-homomorphism from a universal C^*-algebra into the hypergraph C^*-algebra. The said universal C^*-algebra is an interesting mathematical object which makes this example worth for further studies. To see this, we need a lemma beforehand.

Lemma 3.16. *The following universal C^*-algebras are isomorphic.*

(a) $A := C^*(s_1, s_2, 1 \mid s_1 s_1^* = s_2^* s_2; s_2 s_2^* = s_1^* s_1; s_1^* s_1 + s_2^* s_2 = 1)$ *where* s_1 *and* s_2 *are partial isometries and*

(b) $B := C^*(u, p \mid up = (1-p)u)$ *where* u *is a unitary and* p *a projection*

Proof. Step 1: Define $\tilde{s}_1 := up$ and $\tilde{s}_2 := u(1-p)$. There exists a *-homomorphism $\phi_1 : A \to B$ sending s_1 to $\tilde{s}_1 = up$ and s_2 to $\tilde{s}_2 = u(1-p)$. Let's check that $\tilde{s}_1 = up$ and $\tilde{s}_2 = u(1-p)$ are partial isometries. It is

$$(up)(up)^*(up) = upp^*u^*up = up^2 1p = upp = up$$

and

$$\begin{aligned} u(1-p)(u(1-p))^*u(1-p) &= u(1-p)(1-p)^*u^*u(1-p) \\ &= u(1-p)(1-p)1(1-p) \\ &= u(1-p+p-p)(1-p) \\ &= u(1-p). \end{aligned}$$

We prove that $\tilde{s}_1^*\tilde{s}_1 + \tilde{s}_2^*\tilde{s}_2 = 1$ holds:

$$(up)^*up + (u(1-p))^*u(1-p) = p^*u^*up + (1-p)u^*u(1-p) = p + (1-p) = 1.$$

Let's show that $\tilde{s}_1\tilde{s}_1^* = \tilde{s}_2^*\tilde{s}_2$ and $\tilde{s}_2\tilde{s}_2^* = \tilde{s}_1^*\tilde{s}_1$ holds. The relation $up = (1-p)u$ from B implies $upu^* = (1-p)$. Hence

$$\tilde{s}_1\tilde{s}_1^* = (up)(up)^* = upu^* = 1-p = (1-p)u^*u(1-p) = (u(1-p))^*u(1-p) = \tilde{s}_2^*\tilde{s}_2$$

and

$$\tilde{s}_2\tilde{s}_2^* = u(1-p)(u(1-p))^* = u(1-p)u^* = 1 - upu^* = 1-(1-p) = p = (up)^*up = \tilde{s}_1^*\tilde{s}_1.$$

We obtain a *-homomorphism $\phi_1 : A \to B$ sending s_1 to $\tilde{s}_1 = up$ and s_2 to $\tilde{s}_2 = u(1-p)$.
Step 2: Define $\tilde{u} := s_1 + s_2, \tilde{p} := s_1^* s_1$. There is a *-homomorphism $\phi_2 : B \to A$, sending u to $\tilde{u} = s_1 + s_2$ and p to $\tilde{p} = s_1^* s_1$. Notice that by Proposition 1.9 the projections $s_1^* s_1$ and $s_2^* s_2$ are mutually orthogonal and hence

$$s_1 s_2^* = s_1 s_1^* s_1 s_2^* s_2 s_2^* = 0 = s_2 s_1^*.$$

We see that that $\tilde{u}$ is an isometry and $\tilde{p}$ a projection:

$$(s_1+s_2)(s_1+s_2)^* = s_1 s_1^* + s_1 s_2^* + s_2 s_1^* + s_2 s_2^* = 1 = (s_1+s_2)^*(s_1+s_2)$$

and

$$\tilde{p}^2 = s_1^* s_1 s_1^* s_1 = s_1^* s_1 = \tilde{p} = (s_1^* s_1)^* = \tilde{p}^*.$$

By using the relations from A we have

$$\begin{aligned}(1-\tilde{p})\tilde{u} &= (1-s_1^*s_1)(s_1+s_2)\\ &= s_1+s_2-s_1^*s_1s_1-s_1^*s_1s_2\\ &= s_1+s_2-s_1^*s_1s_1-s_2s_2^*s_2\\ &= s_1-s_1^*s_1s_1\\ &= s_1-s_1^*s_1s_1s_1^*s_1\\ &= s_1-s_1^*s_1s_2^*s_2s_1\\ &= s_1-0s_1\\ &= s_1\\ &= (s_1+s_2)s_1^*s_1=\tilde{u}\tilde{p}.\end{aligned}$$

Hence, we obtain a *-homomorphism $\phi_2 : B \to A$, sending u to $\tilde{u} = s_1 + s_2$ and p to $\tilde{p} = s_1^*s_1$.
Step 3: We show that ϕ_1 and ϕ_2 are inverse to each other:

$$\begin{aligned}&\phi_1(\phi_2(u)) = \phi_1(s_1+s_2) = up + u(1-p) = u\\ &\phi_1(\phi_2(p)) = \phi_1(s_1^*s_1) = (up)^*up = p\\ &\phi_2(\phi_1(s_1)) = \phi_2(up) = (s_1+s_2)s_1^*s_1 = s_1\\ &\phi_2(\phi_1(s_2)) = \phi_2(u(1-p)) = (s_1+s_2)(1-s_1^*s_1) = s_1+s_2-s_1s_1^*s_1-s_2s_1^*s_1 = s_2.\end{aligned}$$

Therefore, we have $A \cong B$. □

Proposition 3.17. *Let $B := C^*(u,p \mid up = (1-p)u)$ be the universal C^*-algebra from Lemma 3.16 where u is a unitary and p a projection. We define the hypergraph $H\Gamma$ with vertices $\{v_1, v_2, v_3, v_4\}$, edges $\{e_1, e_2\}$ and range and source map $r(e_1) = \{v_3, v_4\}$, $r(e_2) = \{v_1, v_2\}$, $s(e_1) = \{v_1, v_2\}$, $s(e_2) = \{v_3, v_4\}$. We have a *-homomorphism $B \to C^*(H\Gamma)$,sending $u \mapsto s_{e_1} + s_{e_2}$ and $p \mapsto s_{e_1}^* s_{e_1}$.*

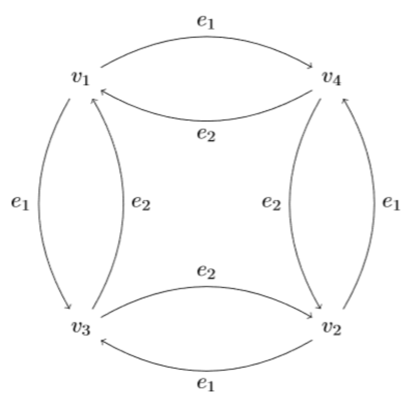

Figure 3.5: Hypergraph 4

Proof. We show that the partial isometries fulfill the relations from the universal C^*-algebra $A := C^*(s_1, s_2, 1 \mid s_1 s_1^* = s_2^* s_2; s_2 s_2^* = s_1^* s_1; s_1^* s_1 + s_2^* s_2 = 1)$ which we defined in Lemma 3.16. From the relations of the hypergraph C^*-algebra $C^*(H\Gamma)$ we have

$$s_{e_1}^* s_{e_1} = p_{v_3} + p_{v_4}$$
$$s_{e_2}^* s_{e_2} = p_{v_1} + p_{v_2}$$

and

$$s_{e_1} s_{e_1}^* \leq p_{v_1} + p_{v_2}$$
$$s_{e_2} s_{e_2}^* \leq p_{v_3} + p_{v_4}$$
$$s_{e_1} s_{e_1}^* \geq p_{v_1} \text{ respectively } p_{v_2}$$
$$s_{e_2} s_{e_2}^* \geq p_{v_3} \text{ respectively } p_{v_4}.$$

Using Lemma 3.7 it follows

$$s_{e_1} s_{e_1}^* = (p_{v_1} + p_{v_2}) = s_{e_2}^* s_{e_2}$$

and in an analogous way $s_{e_2} s_{e_2}^* = s_{e_1}^* s_{e_1}$. With Theorem 3.9 we conclude

$$s_{e_2} s_{e_2}^* + s_{e_1} s_{e_1}^* = s_{e_1}^* s_{e_1} + s_{e_2}^* s_{e_2} = \sum_{i=1}^{4} p_{v_i} = 1.$$

We see that s_{e_1} and s_{e_2} satisfy the relations from the universal C^*-algebra A from Lemma 3.16 which is isomorphic to B. The *-isomorphism $\phi_2 : B \to A$ from Lemma 3.16 sends u to $\tilde{u} = s_1 + s_2$ and p to $\tilde{p} = s_1^* s_1$. By the universal property we obtain a *-homomorphism $\phi : B \to C^*(H\Gamma)$ that maps u to $s_{e_1} + s_{e_2}$ and p to $s_{e_1}^* s_{e_1}$. □

Remark 3.18. *It is possible to show that the universal C^*-algebra B from Lemma* 3.16 *is isomorphic to the "cross-product" $C^*(\mathbb{Z}/2\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ where α is an isomorphism $\alpha : C^*(\mathbb{Z}/2\mathbb{Z}) \to C^*(\mathbb{Z}/2\mathbb{Z})$ sending p to $(1-p)$ (see (Weber, 2007)). This makes the example above interesting. Sadly, we only found a non-surjective *-homomorphism since we only mapped to $s_{e_1} + s_{e_2}$ and $s_{e_1}^* s_{e_1}$ of $C^*(H\Gamma)$. To fully understand this hypergraph C^*-algebra we need to find a way how to deal with the projections in $C^*(H\Gamma)$. A possible approach would be to look at similar hypergraphs. One defines the vertices $\{v_1, v_2, v_3, v_4\}$, edges $\{e_1, e_2\}$ and the source map $s(e_1) = \{v_1, v_2\}, s(e_2) = \{v_3, v_4\}$. Then one considers all possible ways of defining the range map r. In Proposition 3.21 we investigated such an example but with a less interesting outcome.*

3.2.5 Further Examples

What comes next is a collection of examples that one might title as uninteresting. We were not able to find any outstanding relations in them. This should not stop us from presenting them. As a matter of fact, we show some representations on those examples in the next section. Therefore, one might have a look at them.

Lemma 3.19. *Consider the hypergraph $C^*(H\Gamma)$ consisting of vertices $\{v_1, v_2, v_3, v_4\}$, edges $\{e_1, e_2, e_3, e_4\}$ and the following images of the range and source map*
$r(e_1) = \{v_1, v_2\}$ *and* $s(e_1) = \{v_1\}$
$r(e_2) = \{v_2, v_3\}$ *and* $s(e_2) = \{v_2\}$
$r(e_3) = \{v_3, v_4\}$ *and* $s(e_3) = \{v_3\}$
$r(e_4) = \{v_4, v_1\}$ *and* $s(e_4) = \{v_4\}$.
We have

(a) $s_{e_i}^* s_{e_i} = p_{v_i} + p_{v_{i+1}}$ *for all* $i = 1, ..., 3$

(b) $s_{e_4}^* s_{e_4} = p_{v_4} + p_{v_1}$

(c) $s_{e_i} s_{e_i}^* = p_{v_i}$ *for all* $i = 1, ..., 4$

(d) $1 = \sum_{j=1}^4 s_{e_j} s_{e_j}^* = s_{e_1}^* s_{e_1} + s_{e_3}^* s_{e_3} = s_{e_2}^* s_{e_2} + s_{e_4}^* s_{e_4}$

(e) $C^*(H\Gamma)$ *is generated by* $\{s_{e_i} | i = 1, ..., 4\}$.

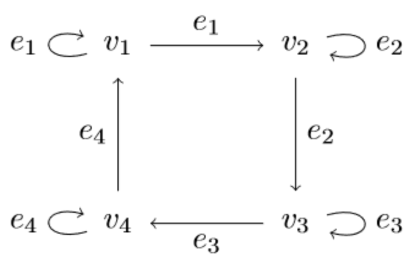

Figure 3.6: Hypergraph 5

Proof. By the relations of the associated hypergraph C^*-algebra and Lemma 3.6 we have

$$s_{e_i}^* s_{e_i} = p_{v_i} + p_{v_{i+1}} \text{ for all } i = 1, ..., 3 \text{ and}$$
$$s_{e_4}^* s_{e_4} = p_{v_4} + p_{v_1} \text{ and}$$
$$s_{e_i} s_{e_i}^* = p_{v_i} \text{ for all } i = 1, ..., 4.$$

Using again Theorem 3.9 we have

$$1 = \sum_{j=1}^{4} s_{e_j} s_{e_j}^* = \sum_{j=1}^{4} p_{v_j} = s_{e_1}^* s_{e_1} + s_{e_3}^* s_{e_3} = s_{e_2}^* s_{e_2} + s_{e_4}^* s_{e_4}.$$

We also see that the partial isometries s_{e_i} already generate the hypergraph C^*-algebra $C^*(H\Gamma)$, since $s_{e_i} s_{e_i}^* = p_{v_i}$ holds for all $i = 1, ..., 4$. □

Lemma 3.20. *Consider the hypergraph $H\Gamma$ with vertices $\{v_1, v_2, v_3\}$, edges $\{e_1\}$ and range and source map defined like this*
$r(e_1) = \{v_1, v_2\}$ *and* $s(e_1) = \{v_1, v_3\}$.
We have

(a) $s_{e_1}^* s_{e_1} = p_{v_1} + p_{v_2}$

(b) $s_{e_1} s_{e_1}^* = p_{v_1} + p_{v_3}$

(c) $C^*(H\Gamma)$ *is generated by* s_{e_1}.

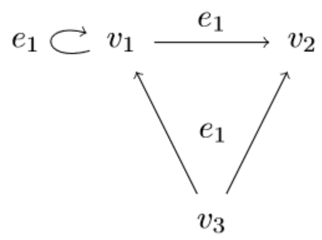

Figure 3.7: Hypergraph 7

Proof. The relations of the hypergraph C^*-algebra and Lemma 3.7 immediately imply

$$\begin{aligned} s_{e_1}^* s_{e_1} &= p_{v_1} + p_{v_2} \\ s_{e_1} s_{e_1}^* &= p_{v_1} + p_{v_3}. \end{aligned}$$

We have

$$\begin{aligned} s_{e_1}^* s_{e_1} s_{e_1} s_{e_1}^* &= p_{v_1} \\ s_{e_1}^* s_{e_1} - s_{e_1}^* s_{e_1} s_{e_1} s_{e_1}^* &= p_{v_2} \\ s_{e_1} s_{e_1}^* - s_{e_1}^* s_{e_1} s_{e_1} s_{e_1}^* &= p_{v_3} \end{aligned}$$

and therefore, we see that $C^*(H\Gamma)$ is already generated by the partial isometry s_{e_1}. □

Lemma 3.21. *Consider the hypergraph* $H\Gamma$ *with vertices* $\{v_1, v_2, v_3, v_4\}$ *and edges* $\{e_1, e_2\}$. *The image of the range and source map are defined like this*
$r(e_1) = \{v_1, v_3\}$ *and* $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_2, v_4\}$ *and* $s(e_2) = \{v_3, v_4\}$.
It follows

(a) $s_{e_1}^* s_{e_1} = p_{v_1} + p_{v_3}$

(b) $s_{e_2}^* s_{e_2} = p_{v_2} + p_{v_4}$

(c) $s_{e_1} s_{e_1}^* = p_{v_1} + p_{v_2}$

(d) $s_{e_2} s_{e_2}^* = p_{v_3} + p_{v_4}$

(e) $C^*(H\Gamma)$ *is generated by* s_{e_1} *and* s_{e_2}.

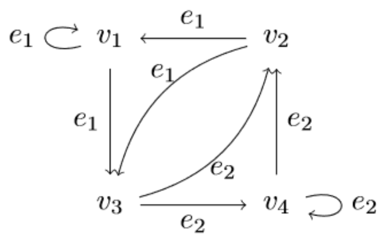

Figure 3.8: Hypergraph 8

Proof. Using the relations from $C^*(H\Gamma)$ and Lemma 3.7, we have

$$\begin{aligned} s_{e_1}^* s_{e_1} &= p_{v_1} + p_{v_3} \\ s_{e_2}^* s_{e_2} &= p_{v_2} + p_{v_4} \\ s_{e_1} s_{e_1}^* &= p_{v_1} + p_{v_2} \\ s_{e_2} s_{e_2}^* &= p_{v_3} + p_{v_4}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} s_{e_1}^* s_{e_1} s_{e_2} s_{e_2}^* &= p_{v_3} \\ s_{e_2}^* s_{e_2} s_{e_1} s_{e_1}^* &= p_{v_2} \\ s_{e_2}^* s_{e_2} s_{e_2} s_{e_2}^* &= p_{v_4} \\ s_{e_1}^* s_{e_1} s_{e_1} s_{e_1}^* &= p_{v_1}. \end{aligned}$$

Hence the hypergraph C^*-algebra is generated by the partial isometries. □

Lemma 3.22. *Consider the hypergraph $H\Gamma$ with vertices $\{v_1, v_2, v_3, v_4\}$, edges $\{e_1, e_2\}$ and range and source map defined like the following*
$r(e_1) = \{v_1, v_2, v_3\}$ *and* $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_3, v_4, v_1\}$ *and* $s(e_2) = \{v_3, v_4\}$.
We have

(a) $s_{e_1}^* s_{e_1} = p_{v_1} + p_{v_2} + p_{v_3}$

(b) $s_{e_2}^* s_{e_2} = p_{v_3} + p_{v_4} + p_{v_1}$

(c) $s_{e_1} s_{e_1}^* = p_{v_1} + p_{v_2}$

(d) $s_{e_2} s_{e_2}^* = p_{v_3} + p_{v_4}$

(e) $C^*(H\Gamma)$ *is generated by* s_{e_1} *and* s_{e_2}.

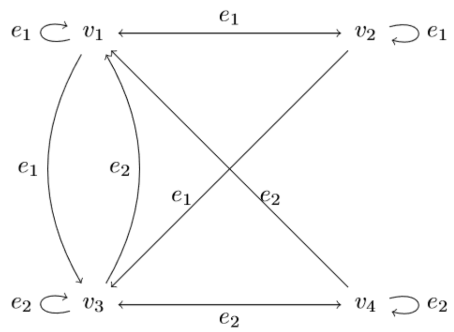

Figure 3.9: Hypergraph 9

Proof. The relations of $C^*(H\Gamma)$ and Lemma 3.7 imply

$$\begin{aligned} s_{e_1}^* s_{e_1} &= p_{v_1} + p_{v_2} + p_{v_3} \\ s_{e_2}^* s_{e_2} &= p_{v_3} + p_{v_4} + p_{v_1} \\ s_{e_1} s_{e_1}^* &= p_{v_1} + p_{v_2} \\ s_{e_2} s_{e_2}^* &= p_{v_3} + p_{v_4}. \end{aligned}$$

It holds that

$$\begin{aligned} & s_{e_1}^* s_{e_1} - s_{e_1} s_{e_1}^* = p_{v_3} \\ & s_{e_2}^* s_{e_2} - s_{e_2} s_{e_2}^* = p_{v_1} \\ & s_{e_1}^* s_{e_1} - (s_{e_1}^* s_{e_1} - s_{e_1} s_{e_1}^*) - (s_{e_2}^* s_{e_2} - s_{e_2} s_{e_2}^*) = p_{v_2} \\ & s_{e_2}^* s_{e_2} - (s_{e_1}^* s_{e_1} - s_{e_1} s_{e_1}^*) - (s_{e_2}^* s_{e_2} - s_{e_2} s_{e_2}^*) = p_{v_4}. \end{aligned}$$

Hence the hypergraph C^*-algebra $C^*(H\Gamma)$ is generated by the partial isometry s_{e_1} and s_{e_2}. □

Lemma 3.23. *Consider the hypergraph $H\Gamma$ with vertices $\{v_1, ..., v_6\}$ and edges $\{e_1, e_2, e_3\}$. We define the range and source map:*
$r(e_1) = \{v_1, v_2, v_3\}$ *and* $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_3, v_4, v_5\}$ *and* $s(e_2) = \{v_3, v_4\}$
$r(e_3) = \{v_5, v_6, v_1\}$ *and* $s(e_1) = \{v_5, v_6\}$.
It follows

(a) $s_{e_1}^* s_{e_1} = p_{v_1} + p_{v_2} + p_{v_3}$

(b) $s_{e_2}^* s_{e_2} = p_{v_3} + p_{v_4} + p_{v_5}$

(c) $s_{e_3}^* s_{e_3} = p_{v_5} + p_{v_6} + p_{v_1}$

(d) $s_{e_1} s_{e_1}^* = p_{v_1} + p_{v_2}$

(e) $s_{e_2} s_{e_2}^* = p_{v_3} + p_{v_4}$

(f) $s_{e_3}s_{e_3}^* = p_{v_5} + p_{v_6}$

(g) $C^*(H\Gamma)$ *is generated by the partial isometries.*

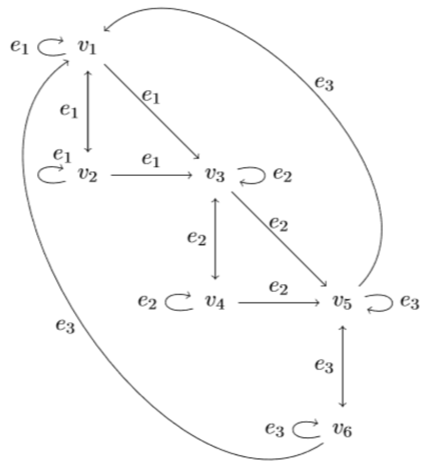

Figure 3.10: Hypergraph 10

Proof. By the relations of the hypergraph C^*-algebra and Lemma 3.7 we obtain

$$\begin{aligned}
s_{e_1}^* s_{e_1} &= p_{v_1} + p_{v_2} + p_{v_3} \\
s_{e_2}^* s_{e_2} &= p_{v_3} + p_{v_4} + p_{v_5} \\
s_{e_3}^* s_{e_3} &= p_{v_5} + p_{v_6} + p_{v_1} \\
s_{e_1} s_{e_1}^* &= p_{v_1} + p_{v_2} \\
s_{e_2} s_{e_2}^* &= p_{v_3} + p_{v_4} \\
s_{e_3} s_{e_3}^* &= p_{v_5} + p_{v_6}.
\end{aligned}$$

We have

$$\begin{aligned}
& s_{e_1}^* s_{e_1} s_{e_2}^* s_{e_2} = p_{v_3} \\
& s_{e_1}^* s_{e_1} s_{e_3}^* s_{e_3} = p_{v_1} \\
& s_{e_1}^* s_{e_1} - (s_{e_1}^* s_{e_1} s_{e_2}^* s_{e_2} + s_{e_1}^* s_{e_1} s_{e_3}^* s_{e_3}) = p_{v_2} \\
& s_{e_2}^* s_{e_2} s_{e_3}^* s_{e_3} = p_{v_5} \\
& s_{e_3} s_{e_3}^* - s_{e_2}^* s_{e_2} s_{e_3}^* s_{e_3} = p_{v_6} \\
& s_{e_2} s_{e_2}^* - s_{e_1}^* s_{e_1} s_{e_2}^* s_{e_2} = p_{v_4}.
\end{aligned}$$

So all projections, and therefore $C^*(H\Gamma)$, are already generated by the partial isometries. □

Lemma 3.24. *Consider the hypergraph $H\Gamma$ given by vertices $\{v_1, ..., v_6\}$, edges $\{e_1, e_2, e_3\}$ and following images of the range and source map*
$r(e_1) = \{v_1, v_2, v_3\}$ *and* $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_4, v_5, v_6\}$ *and* $s(e_2) = \{v_5, v_6\}$
$r(e_3) = \{v_3, v_4\}$ *and* $s(e_3) = \{v_3, v_4\}$.
We have

(a) $s_{e_1}^* s_{e_1} + s_{e_2}^* s_{e_2} = 1$

(b) $s_{e_1} s_{e_1}^* + s_{e_2} s_{e_2}^* + s_{e_3} s_{e_3}^* = 1$

(c) $s_{e_3} s_{e_3}^* = s_{e_3}^* s_{e_3}$.

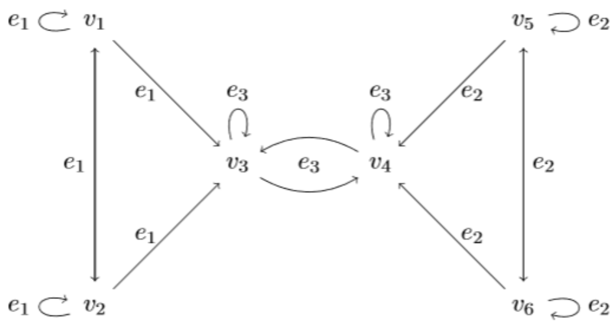

Figure 3.11: Hypergraph 11

Proof. Using the relations of the hypergraph C^*-algebra $C^*(H\Gamma)$ and Lemma 3.7 yields

$$\begin{aligned}
s_{e_1}^* s_{e_1} &= p_{v_1} + p_{v_2} + p_{v_3} \\
s_{e_2}^* s_{e_2} &= p_{v_4} + p_{v_5} + p_{v_6} \\
s_{e_3}^* s_{e_3} &= p_{v_3} + p_{v_4} \\
s_{e_1} s_{e_1}^* &= p_{v_1} + p_{v_2} \\
s_{e_2} s_{e_2}^* &= p_{v_5} + p_{v_6} \\
s_{e_3} s_{e_3}^* &= p_{v_3} + p_{v_4}.
\end{aligned}$$

Theorem 3.9 shows $\sum_{i=1}^{6} p_{v_i} = 1$. □

Lemma 3.25. *With vertices $\{v_1, ..., v_5\}$, edges $\{e_1, e_2\}$ and the following image of the range and source map*
$r(e_1) = \{v_1, v_3, v_5\}$ *and* $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_2, v_4\}$ *and* $s(e_2) = \{v_3\}$
we define the hypergraph $H\Gamma$. We have

(a) $s_{e_1}^* s_{e_1} = p_{v_1} + p_{v_3} + p_{v_5}$

(b) $s_{e_2}^* s_{e_2} = p_{v_2} + p_{v_4}$

(c) $s_{e_1}s_{e_1}^* = p_{v_1} + p_{v_2}$

(d) $s_{e_2}s_{e_2}^* = p_{v_3}$

(e) the partial isometries generate $C^*(H\Gamma)$.

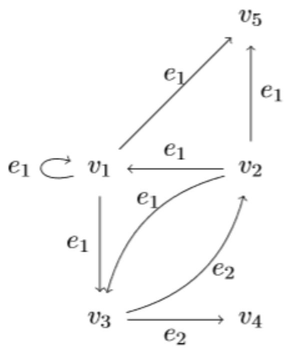

Figure 3.12: Hypergraph 12

Proof. The relations of the hypergraph C^*-algebra $C^*(H\Gamma)$ and Lemma 3.6 and Lemma 3.7 imply following equations

$$\begin{aligned} s_{e_1}^*s_{e_1} &= p_{v_1} + p_{v_3} + p_{v_5} \\ s_{e_2}^*s_{e_2} &= p_{v_2} + p_{v_4} \\ s_{e_1}s_{e_1}^* &= p_{v_1} + p_{v_2} \\ s_{e_2}s_{e_2}^* &= p_{v_3}. \end{aligned}$$

It applies

$$\begin{aligned} s_{e_1}^*s_{e_1}s_{e_1}s_{e_1}^* &= p_{v_1} \\ s_{e_2}^*s_{e_2}s_{e_1}s_{e_1}^* &= p_{v_2} \\ s_{e_1}^*s_{e_1} - (s_{e_1}^*s_{e_1}s_{e_1}s_{e_1}^* + s_{e_2}s_{e_2}^*) &= p_{v_5} \\ s_{e_2}^*s_{e_2} - s_{e_2}^*s_{e_2}s_{e_1}s_{e_1}^* &= p_{v_4}. \end{aligned}$$

The projections and therefore, the hypergraph C^*-algebra $C^*(H\Gamma)$ are generated by the partial isometries. □

3.2.6 Hyperization

In the subsection before we found isomomorphisms from hypergraph C^*-algebras to some well known universal C^*-algebras. From Section 2 we know graphs that generate a graph C^*-algebra that is isomorphic to these universal C^*-algebra. We asked ourselves whether it is possible or not to define a "hyper" version of a given graph such that the hypergraph is not actually a graph, like we did in Remark 3.3, and that the generated graph and hypergraph C^*-algebras are isomorphic to each other. We were able to find a "hyper" version that produces an injective homomorphism to the given graph C^*-algebra.

Proposition 3.26. *Let $\Gamma = (V, E, r, s)$ be a graph. For our graph C^*-algebra we write*

$$C^*(\Gamma) = C^*(\tilde{s}_e,\ e \in E; \tilde{p}_v,\ v \in V \mid \tilde{p}_v\tilde{p}_w = 0, v \neq w; \tilde{s}_e^*\tilde{s}_f = \delta_{ef}\tilde{p}_{r(e)}; \sum_{\substack{e\in E\\ s(e)=w}} \tilde{s}_e\tilde{s}_e^* = \tilde{p}_w)$$

where $\tilde{s}_e$ is a partial isometry for all $e \in E$ and $\tilde{p}_v$ is a projection for all $v \in V$. Consider the hypergraph $H\Gamma = (\dot{V}, E, \dot{r}, \dot{s})$ consisting of

$$\dot{V} = \{v, v'\} \text{ for all } v \in V$$
$$\dot{r}(e) = \{r(e), r(e)'\} \text{ for all } e \in E$$
$$\dot{s}(e) = \{s(e), s(e)'\} \text{ for all } e \in E.$$

*We have an injective *-homomorphism $\phi : C^*(\Gamma) \to C^*(H\Gamma)$, sending $\tilde{s}_e$ to s_e and $\tilde{p}_v$ to $p_v + p_{v'}$. We also have a surjective *-homomorphism $\alpha : C^*(H\Gamma) \to C^*(\Gamma)$ that sends s_e to $\tilde{s}_e$, p_v to $\tilde{p}_v$ and $p_{v'}$ to 0.*

Proof. Step 1: There exists a *-homomorphism $\phi : C^*(\Gamma) \to C^*(H\Gamma)$, sending $\tilde{s}_e$ to s_e and $\tilde{p}_v$ to $p_v + p_{v'}$. We check that the partial isometries $\{s_e | e \in E\}$ and projections $\{p_v + p_{v'} | v \in V\}$ from the hypergraph C^*-algebra $C^*(H\Gamma)$ satisfy the relations of the graph C^*-algebra $C^*(\Gamma)$. Using Relations (HR1)-(HR3) we obtain

$$s_e^*s_f = \delta_{ef} \sum_{\substack{v\in\dot{V}\\ v\in\dot{r}(e)}} p_v = \delta_{ef}(p_{r(e)} + p_{r(e)'})$$
$$s_es_e^* \leq \sum_{\substack{v\in\dot{V}\\ v\in\dot{s}(e)}} p_v = p_{s(e)} + p_{s(e)'}$$
$$p_v \leq \sum_{\substack{f\in E\\ v\in\dot{s}(f)}} s_fs_f^* = \sum_{\substack{f\in E\\ v=s(f)}} s_fs_f^*$$
$$p_{v'} \leq \sum_{\substack{f\in E\\ v'\in\dot{s}(f)}} s_fs_f^* = \sum_{\substack{f\in E\\ v\in\dot{s}(f)}} s_fs_f^* = \sum_{\substack{f\in E\\ v=s(f)}} s_fs_f^*.$$

Notice that the first Relation (R1) of our graph C^*-algebra $C^*(H\Gamma)$ is fulfilled. For the second and last Relation (R2) we let be $v \in \dot{V}$ that is not a sink. Then there exists at least one $e \in E$ such that $s(e) = v$ and hence $v \in \dot{s}(e)$. Observe that $v' \in \dot{s}(e)$. By the last 3 relations from above we have for that $v \in \dot{V}$

$$\begin{aligned}
p_v + p_{v'} &= p_{s(e)} + p_{s(e)'} \\
&= (p_{s(e)} + p_{s(e)'}) \sum_{\substack{f \in E \\ s(e) \in \dot{s}(f)}} s_f {s_f}^* \\
&= \sum_{\substack{f \in E \\ s(e) \in \dot{s}(f)}} (p_{s(e)} + p_{s(e)'}) s_f {s_f}^* \\
&= \sum_{\substack{f \in E \\ s(e) = s(f)}} (p_{s(e)} + p_{s(e)'}) s_f s_f^* \\
&\overset{(HR2)}{=} \sum_{\substack{f \in E \\ s(e) \in \dot{s}(f)}} s_f {s_f}^* .
\end{aligned}$$

Hence we obtain a *-homomorphism $\phi : C^*(\Gamma) \to C^*(H\Gamma)$, sending $\tilde{s}_e$ to s_e and $\tilde{p}_v$ to $p_v + p_{v'}$.
Step 2: We show that ϕ is injective. Define $\alpha : C^*(H\Gamma) \to C^*(\Gamma)$ with

$$\begin{aligned}
\alpha(s_e) &= \tilde{s}_e \\
\alpha(p_v) &= \tilde{p}_v \\
\alpha(p_{v'}) &= 0
\end{aligned}$$

and observe that

$$\begin{aligned}
\alpha(\phi(\tilde{s}_e)) &= \alpha(s_e) = \tilde{s}_e \\
\alpha(\phi(\tilde{p}_v)) &= \alpha(p_v + p_{v'}) = \tilde{p}_v .
\end{aligned}$$

So α is the left inverse to ϕ and therefore, the *-homomorphism ϕ is injective. Notice that α is by definition a surjective *-homomorphism. □

Remark 3.27. *Notice that there are other ways to define a "hyper" version of a given graph. One could also consider taking two 2 edges on top of taking 2 vertices as in our version. Since we found an injective *-homomorphism for the version above, this is the one we present.*

3.3 Representations

Recall that we showed non-triviality of hypergraph C^*-algebras for a certain class of hypergraphs (see Proposition 3.8). Since we did not find a general way of proving non-triviality we investigated the graphs from Section 3.2 and found representations of them. Notice that we do not explicitly write down the representations. We present the operators that are needed to obtain the representations.

3.3.1 Representations on $\ell^2(\mathbb{Z}^2)$

Remark 3.28. *Notice that the cardinal number of the image of the range map r is 2 throughout the sections 3.3.1 and 3.3.2. As soon as we use $|r(e)| = 3$ for $e \in E$ we need to switch from representations on $\ell^2(\mathbb{Z}^2)$ to representations on $\ell^2(\mathbb{Z}^3)$. One might investigate this observation.*

One consider the Hilbert space $H := \ell^2(\mathbb{Z}^2)$ with orthonormal basis $e_{(x,y)}$ where $x, y \in \mathbb{Z}$. We define the following closed subspaces of H with decomposition $H := H_1 \oplus H_2 \oplus H_3 \oplus H_4$

$$\begin{aligned} H_1 &:= \langle e_{(x,y)} \mid x \geq 0, y \geq 0 \rangle \\ H_2 &:= \langle e_{(x,y)} \mid x < 0, y \geq 0 \rangle \\ H_3 &:= \langle e_{(x,y)} \mid x < 0, y < 0 \rangle \\ H_4 &:= \langle e_{(x,y)} \mid x \geq 0, y < 0 \rangle. \end{aligned}$$

Let $P_i \in \mathcal{B}(H)$ be the corresponding projection on H_i for all $i = 1, ..., 4$. Furthermore, let

$$\begin{aligned} f &: \mathbb{Z} \to \mathbb{N}_0 \\ g &: \mathbb{Z} \to \mathbb{Z} \backslash \mathbb{N}_0 \\ h &: \mathbb{N} \to \mathbb{Z} \backslash \mathbb{N}_0 \end{aligned}$$

be bijections.

(a) Recall the hypergraph $H\Gamma$ from Proposition 3.15. It is given by vertices $\{v_1, v_2, v_3, v_4\}$, edges $\{e_1\}$ and the following images of the range and source map $r(e_1) = \{v_1, v_2\}$ and $s(e_1) = \{v_3, v_4\}$.

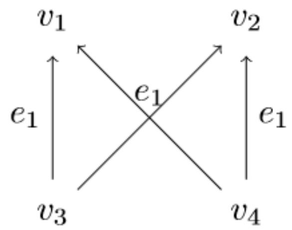

Figure 3.13: Hypergraph 6

We define by S_1 a partial isometry that fulfills the relations of the hypergraph C^*-algebra

$$S_1(e_{(x,y)}) := \begin{cases} e_{(x,h(y))} & y \geq 0 \\ e_{(x,h(-y))} & \text{sonst} \end{cases}; \qquad S_1^* S_1 = P_1 + P_2; \qquad S_1 S_1^* = P_3 + P_4.$$

(b) Recall the hypergraph $H\Gamma$ from Lemma 3.19. It is given by vertices $\{v_1, v_2, v_3, v_4\}$, edges $\{e_1, e_2, e_3, e_4\}$ and the following images of the range and source map
$r(e_1) = \{v_1, v_2\}$ and $s(e_1) = \{v_1\}$
$r(e_2) = \{v_2, v_3\}$ and $s(e_2) = \{v_2\}$
$r(e_3) = \{v_3, v_4\}$ and $s(e_3) = \{v_3\}$
$r(e_4) = \{v_4, v_1\}$ and $s(e_4) = \{v_4\}$.

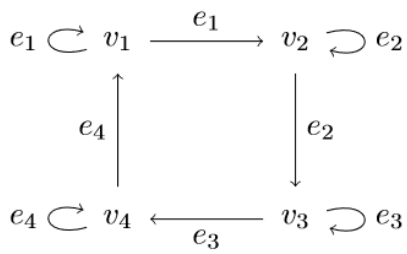

Figure 3.14: Hypergraph 5

We define by S_i partial isometries for all $i = 1, ..., 4$ that satisfy the relations of the hypergraph C^*-algebra $C^*(H\Gamma)$

$$\begin{aligned}
S_1(e_{(x,y)}) &:= \delta_{y\geq 0} e_{(f(x),y)} & S_1^*S_1 &= P_1 + P_2; & S_1S_1^* &= P_1\\
S_2(e_{(x,y)}) &:= \delta_{x<0} e_{(x,f(y))} & S_2^*S_2 &= P_2 + P_3; & S_2S_2^* &= P_2\\
S_3(e_{(x,y)}) &:= \delta_{y<0} e_{(g(x),y)} & S_3^*S_3 &= P_3 + P_4; & S_3S_3^* &= P_3\\
S_4(e_{(x,y)}) &:= \delta_{x\geq 0} e_{(x,g(y))} & S_4^*S_4 &= P_4 + P_1; & S_4S_4^* &= P_4.
\end{aligned}$$

3.3.2 Further representations on $\ell^2(\mathbb{Z}^2)$

It was not possible to use the decomposition from Section 3.3.1 for the next examples. Hence we used a different one.
One consider the Hilbert space $H := \ell^2(\mathbb{Z}^2)$ with orthonormal basis $e_{(x,y)}$ where $x, y \in \mathbb{Z}$. We define the following closed subspaces of H with decomposition $H := H_1 \oplus H_2 \oplus H_3 \oplus H_4$

$$\begin{aligned}
H_1 &:= \langle e_{(x,y)} \mid x \geq 0, y \geq 0\rangle\\
H_2 &:= \langle e_{(x,y)} \mid x < 0, y \geq 0\rangle\\
H_3 &:= \langle e_{(x,y)} \mid x \geq 0, y < 0\rangle\\
H_4 &:= \langle e_{(x,y)} \mid x < 0, y < 0\rangle.
\end{aligned}$$

Let $P_i \in \mathcal{B}(H)$ be the corresponding projection on H_i for all $i = 1, ..., 4$. Furthermore, let

$$\begin{aligned}
f &: \mathbb{Z} \to \mathbb{N}_0\\
g &: \mathbb{N}_0 \to \mathbb{Z}\\
h &: \mathbb{Z}\backslash\mathbb{N}_0 \to \mathbb{Z}\\
j &: \mathbb{Z} \to \mathbb{Z}\backslash\mathbb{N}_0
\end{aligned}$$

be bijections.

(a) Recall the hypergraph $H\Gamma$ from Lemma 3.20. It is given by vertices $\{v_1, v_2, v_3\}$, edges $\{e_1\}$ and the following images of the range and source map
$r(e_1) = \{v_1, v_2\}$ and $s(e_1) = \{v_1, v_3\}$.

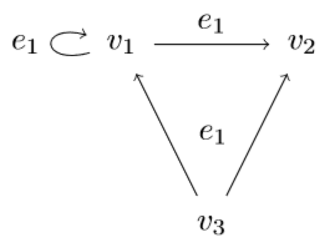

Figure 3.15: Hypergraph 7

We define by S_1 a partial isometry that satisfies the relations of the hypergraph C^*-algebra $C^*(H\Gamma)$

$$S_1(e_{(x,y)}) := \begin{cases} e_{(f(x),g(y))} & y \geq 0 \\ e_{(f(x),g(-y))} & \text{sonst} \end{cases}; \qquad S_1^*S_1 = P_1 + P_2; \qquad S_1S_1^* = P_1 + P_3.$$

(b) Recall the hypergraph $H\Gamma$ from Lemma 3.21. It is given by vertices $\{v_1, v_2, v_3, v_4\}$, edges $\{e_1, e_2\}$ and the following images of the range and source map
$r(e_1) = \{v_1, v_3\}$ and $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_2, v_4\}$ and $s(e_2) = \{v_3, v_4\}$.

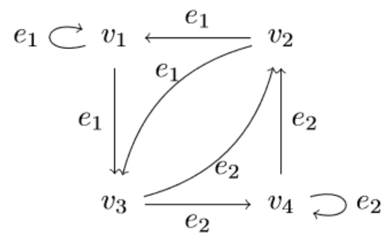

Figure 3.16: Hypergraph 8

We define by S_1 and S_2 partial isometries that fulfill the relations of the hypergraph C*-algebra $C^*(H\Gamma)$

$$S_1(e_{(x,y)}) := \begin{cases} e_{(g(x),f(y))} & x \geq 0 \\ e_{(g(-x),f(y))} & \text{sonst} \end{cases}; \qquad S_1^*S_1 = P_1 + P_3; \qquad S_1S_1^* = P_1 + P_2$$

$$S_2(e_{(x,y)}) := \begin{cases} e_{(h(x),j(y))} & x < 0 \\ e_{(h(-x),j(y))} & \text{sonst} \end{cases}; \qquad S_2^*S_2 = P_2 + P_4 \qquad S_2S_2^* = P_3 + P_4.$$

3.3.3 Representations on $\ell^2(\mathbb{Z}^3)$

As we discussed earlier we need to switch to representations on $\ell^2(\mathbb{Z}^3)$ since we start to investigate hypergraphs with $|r(e)| = 3$ for $e \in E$.
One consider the Hilbert space $\ell^2(\mathbb{Z}^3)$ with orthonormal basis $e_{(x,y,z)}$ where $x, y, z \in \mathbb{Z}$. We define the following closed subspaces of H with decomposition $H := H_1 \oplus H_2 \oplus \cdots \oplus H_8$

$$\begin{aligned} H_1 &:= \langle e_{(x,y,z)} \mid x \geq 0, y \geq 0, z \geq 0 \rangle \\ H_2 &:= \langle e_{(x,y,z)} \mid x \geq 0, y \geq 0, z < 0 \rangle \\ H_3 &:= \langle e_{(x,y,z)} \mid x \geq 0, y < 0, z \geq 0 \rangle \\ H_4 &:= \langle e_{(x,y,z)} \mid x \geq 0, y < 0, z < 0 \rangle \\ H_5 &:= \langle e_{(x,y,z)} \mid x < 0, y \geq 0, z \geq 0 \rangle \\ H_6 &:= \langle e_{(x,y,z)} \mid x < 0, y \geq 0, z < 0 \rangle \\ H_7 &:= \langle e_{(x,y,z)} \mid x < 0, y < 0, z \geq 0 \rangle \\ H_8 &:= \langle e_{(x,y,z)} \mid x < 0, y < 0, z < 0 \rangle. \end{aligned}$$

Let $P_i \in \mathcal{B}(H)$ be the corresponding projections on H_i for all $i = 1, ..., 8$. Furthermore, let

$$\begin{aligned} f &: \mathbb{Z} \to \mathbb{N}_0 \\ g &: \mathbb{Z} \to \mathbb{Z} \backslash \mathbb{N}_0 \\ h &: \mathbb{N}_0 \to \mathbb{Z} \\ i &: \mathbb{Z} \backslash \mathbb{N}_0 \to \mathbb{N}_0 \end{aligned}$$

be bijections.

(a) Recall the hypergraph $H\Gamma$ from Lemma 3.22. It is given by vertices $\{v_1, v_2, v_3, v_4\}$, edges $\{e_1, e_2\}$ and the following images of the range and source map
$r(e_1) = \{v_1, v_2, v_3\}$ and $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_3, v_4, v_1\}$ and $s(e_2) = \{v_3, v_4\}$.

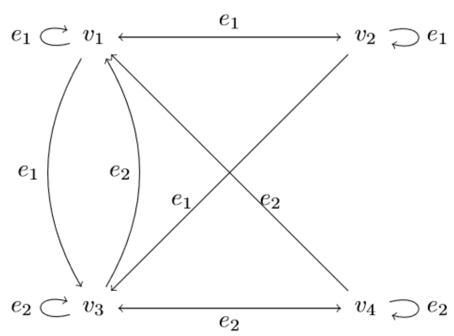

Figure 3.17: Hypergraph 9

We define by S_1 and S_2 partial isometries that satisfy the relations of the hypergraph C^*-algebra $C^*(H\Gamma)$

$$S_1(e_{(x,y,z)}) := \delta_{\substack{x\geq 0\\ y\geq 0;z\geq 0\\ y<0;z\geq 0\\ y\geq 0;z<0}} e_{(x,f(y),z)} \qquad S_1^*S_1 = P_1 + P_2 + P_3; \qquad S_1S_1^* = P_1 + P_2$$

$$S_2(e_{(x,y,z)}) := \delta_{\substack{x\geq 0\\ y\geq 0;z\geq 0\\ y<0;z\geq 0\\ y<0;z<0}} e_{(x,g(y),z)} \qquad S_2^*S_2 = P_3 + P_4 + P_1 \qquad S_2S_2^* = P_3 + P_4.$$

(b) Recall the hypergraph $H\Gamma$ from Lemma 3.23. It is given by vertices $\{v_1, ..., v_6\}$, edges $\{e_1, e_2, e_3\}$ and the following images of the range and source map
$r(e_1) = \{v_1, v_2, v_3\}$ and $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_3, v_4, v_5\}$ and $s(e_2) = \{v_3, v_4\}$
$r(e_3) = \{v_5, v_6, v_1\}$ and $s(e_1) = \{v_5, v_6\}$.

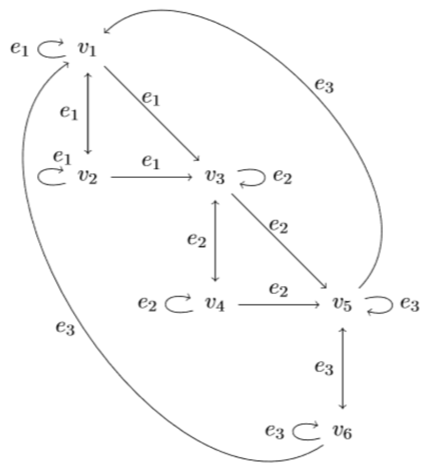

Figure 3.18: Hypergraph 10

We define by S_i partial isometries that satisfy the relations of the hypergraph C^*-algebra $H\Gamma$

$$S_1(e_{(x,y,z)}) = \delta_{\substack{x\geq 0 \\ y\geq 0; z\geq 0 \\ y<0; z\geq 0 \\ y\geq 0; z<0}} e_{(x,f(y),z)} \qquad S_1^*S_1 = P_1 + P_2 + P_3; \qquad S_1S_1^* = P_1 + P_2$$

$$S_2(e_{(x,y,z)}) := \delta_{\substack{x\geq 0; y<0; z\geq 0 \\ x\geq 0; y<0; z<0 \\ x<0; y\geq 0; z\geq 0}} e_{(f(x),y,z)} \qquad S_2^*S_2 = P_3 + P_4 + P_5; \qquad S_2S_2^* = P_3 + P_4$$

$$S_3(e_{(x,y,z)}) := \delta_{\substack{y\geq 0 \\ x\geq 0; z\geq 0 \\ x<0; z\geq 0 \\ x<0; z<0}} e_{(g(x),y,z)} \qquad S_3^*S_3 = P_5 + P_6 + P_1; \qquad S_3S_3^* = P_5 + P_6.$$

(c) Recall the hypergraph $H\Gamma$ from Lemma 3.24. It is given by vertices $\{v_1, ..., v_6\}$, edges $\{e_1, e_2, e_3\}$ and the following images of the range and source map
$r(e_1) = \{v_1, v_2, v_3\}$ and $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_4, v_5, v_6\}$ and $s(e_2) = \{v_5, v_6\}$
$r(e_3) = \{v_3, v_4\}$ and $s(e_3) = \{v_3, v_4\}$.

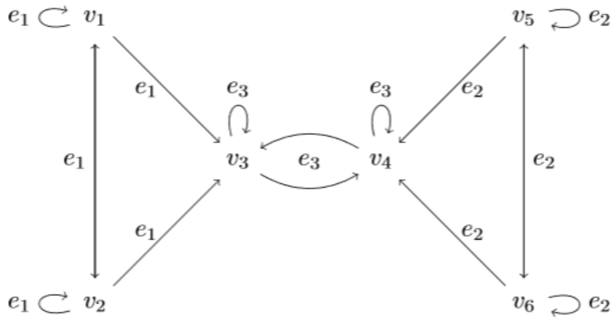

Figure 3.19: Hypergraph 11

We define by S_i partial isometries that fulfill the relations of the hypergraph C^*-algebra $H\Gamma$

$$S_1(e_{(x,y,z)}) := \delta_{\substack{x\geq 0\\ y\geq 0; z\geq 0\\ y<0; z\geq 0\\ y\geq 0; z<0}} e_{(x,f(y),z)} \qquad S_1^*S_1 = P_1 + P_2 + P_3; \qquad S_1S_1^* = P_1 + P_2$$

$$S_2(e_{(x,y,z)}) := \delta_{\substack{x\geq 0; y<0; z<0\\ x<0; y\geq 0; z\geq 0\\ x<0; y\geq 0; z<0}} e_{(g(x),y,z)} \qquad S_2^*S_2 = P_4 + P_5 + P_6; \qquad S_2S_2^* = P_5 + P_6$$

$$S_3(e_{(x,y,z)}) := \delta_{x\geq 0; y<0}\, e_{(x,y,z)} \qquad S_3^*S_3 = P_3 + P_4; \qquad S_3S_3^* = P_3 + P_4.$$

(d) Recall the hypergraph $H\Gamma$ from Lemma 3.25. It is given by vertices $\{v_1, ..., v_5\}$, edges $\{e_1, e_2\}$ and the following images of the range and source map
$r(e_1) = \{v_1, v_3, v_5\}$ and $s(e_1) = \{v_1, v_2\}$
$r(e_2) = \{v_2, v_4\}$ and $s(e_2) = \{v_3\}$.

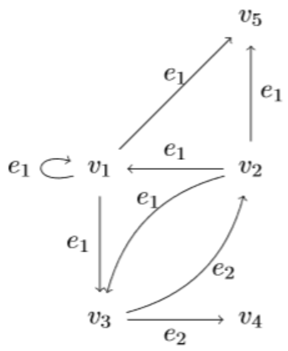

Figure 3.20: Hypergraph 12

We define by S_1 and S_2 partial isometries that fulfill the relations of the hypergraph C^*-algebra $C^*(H\Gamma)$

$$S_1(e_{(x,y,z)}) := \delta_{\substack{x\geq 0;y\geq 0;z\geq 0\\ x\geq 0;y\geq 0;z<0\\ x\geq 0;y<0;z\geq 0\\ x<0;y\geq 0;z\geq 0}} \begin{cases} e_{(f(x),f(y),h(z))} & z\geq 0\\ e_{(f(x),f(y),h(-z))} & \text{sonst} \end{cases}; \qquad S_1^*S_1 = P_1 + P_3 + P_5;$$

$$S_1S_1^* = P_1 + P_2$$

$$S_2(e_{(x,y,z)}) := \delta_{x\geq 0} \begin{cases} e_{(x,g(y),i(z))} & z<0\\ e_{(x,g(y),i(-z))} & \text{sonst} \end{cases}; \qquad S_2^*S_2 = P_2 + P_4; \qquad S_2S_2^* = P_3.$$

List of Figures

Bibliography

[Web] *C*-ALGEBRAS AND DYNAMICS, Lecture Notes des ISem24, Moritz Weber, 2020*

[Blackadar 2006] BLACKADAR, Bruce: *Operator Algebras.* 1. Springer, Berlin, Heidelberg, 2006. – ISBN 978–3–540–28486–4

[Eifler 2016] EIFLER, Kari: *Graph C^*-algebras and the Abelian core*, University of Waterloo, Diplomarbeit, 2016

[Raeburn 2005] RAEBURN, Iain: *Graph Algebras.* volume 103 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, 2005

[Weber 2007] WEBER, Moritz: *C*-Algebren, die von einer partiellen Isometrie erzeugt werden*, Westfälische Wilhelms-Universität Münster, Diplomarbeit, 2007