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Bachelor Thesis

Combinatorial Models of Frieze Patterns

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I hereby confirm that I have written the Bachelor Thesis with the title **Combinatorial Models of Frieze Patterns** (supervisor: **Prof. Dr. Moritz Weber**) on my own and that I have not used any other materials than the ones referred to in this thesis.
I further confirm that I did not submit this or a very similar thesis at another place.

Saarbrücken, April 23, 2018

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Contents

0 Introduction

Frieze patterns have been invented by H.S.M. Coxeter in 1971 [9]. In the 1970s, some fundamental and attractive properties like periodicity and glide reflection symmetry have been developed by Conway and Coxeter [7]. The initial approach to frieze patterns is over natural numbers or positive integers. However, the extensions of frieze patterns over real numbers or complex numbers are available nowadays.

Frieze pattern is a combinatorial model, which consists of rows of numbers, where the first 2 rows at top and bottom are 0's and 1's and the minor of every adjacent 2×2 entries is equal to 1. Moreover, if the minor of every adjacent 3×3 entries is equal to 0, then this frieze pattern is tame. For example, a tame frieze pattern with width 4 is as follows:

row 0	0		0		0		0		0		0		0		0		0		0		0		0
row 1		1		1		1		1		1		1		1		1		1		1		1	
row 2	...		3		1		2		4		1		2		2		3		1		2		...
row 3				2		1		7		3		1		3		5		2		1		...	
row 4	...		3		1		3		5		2		1		7		3		1		3		...
row 5				1		2		2		3		1		2		4		1		2		...	
row 6	...		1		1		1		1		1		1		1		1		1		1		...
row 7		0		0		0		0		0		0		0		0		0		0		0	

Notice that each tame frieze pattern satisfies periodicity. We define a finite sequence, which consists of elements in a period from row 2, as a quiddity cycle. Due to the close connection to cluster algebras of type A, which was introduced by Fomin and Zelevinsky in 2002 [12], the combinatorial model of frieze patterns has been endowed with plenty of interesting statements. For examples: The transformation from frieze patterns and its extension SL_2-tiling over positive integers to polygons with specific triangulations [4, 6], and in reverse [17, 15]; the generalized frieze patterns over natural numbers [5]; the combinatorial model of frieze patterns in cluster algebras of other types [13, 2, 1]. In particular, the elements of frieze patterns in these investigations were always constrained to integers or natural numbers. The results about integer-valued frieze patterns have been summarized by Sophie Morier-Genoud [17].

In recent years, the domain of elements in a frieze pattern has been widened to integers, real numbers and complex numbers by Michael Cuntz [11]. The most significant conclusions of recent research on frieze pattern lie in two aspects: on the one hand, the triangulation methods were improved in integer-valued frieze patterns. As a result of [7], each positive-integer-valued pattern is in one-to-one correspondence to triangulation of a convex polygon. However, this case is changed over integers: each triangulation determines just one frieze pattern, but conversely, a frieze pattern may have several different triangulations [11]. On the other hand, frieze patterns are constructed in another approach by Michael Cuntz using quiddity cycles. Since there is still no proper noun for this approach, we call it "Cuntz Frieze Pattern". For a fixed subset of a commutative ring, one can classify a quiddity cycle of a frieze pattern by reducibility, where a "reducible" quiddity cycle can be presented as "direct sum" of two other quiddity cycles, but an irreducible quiddity cycle can not [10].

It is natural to ask whether a quiddity cycle, which can be presented as direct sum of two other quiddity cycles, can be further presented as direct sum of three or even more frieze patterns. Moreover, we investigate whether there exists a method, which decomposes each integer-valued frieze pattern into different irreducible frieze patterns in a unique way.

The main results of my thesis are listed in the following 3 parts. Firstly, in most cases the operator "direct sum" fulfils neither commutativity nor associativity. But no matter whether we commute two quiddity cycles, the direct sum of them always builds the same frieze pattern. The case for associativity is similar.(See Proposition 3.18) Secondly, each quiddity cycle can be reduced to several irreducible quiddity cycles by using a fixed formula, which is called "Factorization". Because of non-commutativity, a quiddity cycle may have some different factorizations.(See Proposition 4.5) Thirdly, we explore an interesting class of quiddity cycles – simple quiddity cycles, whose subsequences are always not quiddity cycles any more. Moreover, some useful properties about simple quiddity cycles like the relationship to reducibility are introduced in Section 4.2.

This thesis is structured as follows: In Section 1, the basic definitions and properties of Coxeter frieze patterns and the extension SL_{k+1} frieze patterns are introduced, followed by Section 2 the process of triangulation for positive-integer-valued frieze patterns. In Section 3, we introduce the process invented by Michael Cuntz to build frieze patterns by using quiddity cycles. Actually, "tame Coxeter Frieze Patterns" and "Cuntz Frieze Patterns" can be shown to be equivalent (See Section 3.3). Furthermore, with the definition of a "direct sum" of two quiddity cycles, the reducibility is defined in reverse (See Section 3.2). In Section 4, a new concept "Factorization" is defined as a formula to decompose frieze patterns until all the factors are irreducible. Moreover, the definitions and properties about simple quiddity cycles are introduced in Section 4.2. Parts of Section 3 and Section

4 are my own results.

1 Definition of Frieze Patterns

In this section we introduce some fundamental definitions and properties of Coxeter frieze patterns (Section 1.1) as well as Coxeter frieze patterns in the ring of positive integers (Section 1.2). Moreover, we introduce SL_{k+1} frieze patterns, an extension of Coxeter frieze patterns (Section 1.3).

1.1 Coxeter Frieze Patterns

The following definition has been given by H.S.M. Coxeter in 1971 [9].

Definition 1.1. (Coxeter frieze patterns)[9] Let m be a natural number and $E = (e_{i,j})_{i,j\in\mathbb{Z}, i-2\leq j\leq i+m+1}$ be an array of numbers. If E satisfies the following conditions, it is called *Coxeter frieze patterns* with *width* m.

1. For all $i \in \mathbb{Z}$ we have $e_{i,i-2} = e_{i,i+m+1} = 0$.

2. For all $i \in \mathbb{Z}$ we have $e_{i,i-1} = e_{i,i+m} = 1$.

3. Every four adjacent entries a, b, c, d forming a diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

   satisfy the unimodular rule: $ad - bc = 1$.

Moreover, if the determinant of every 3×3 adjacent entries of E equals 0, E is *tame*.

Example 1.2. (1) Each Coxeter frieze pattern has a general form:

row 0	0	0	0	0	0	0	0	0	0	
row 1	1	1	1	1	1	1	1	1		
row 2	...	$e_{1,1}$	$e_{2,2}$	$e_{3,3}$	...	$e_{i,i}$	...			
$\vdots$	...	$\ddots$	$\ddots$	$\ddots$	$\ddots$					
row m	...	$e_{1,m-1}$	$e_{2,m}$	$e_{3,m+1}$	$e_{i,i+m-2}$	...				
row $m+1$	...	$e_{1,m}$	$e_{2,m+1}$	$e_{3,m+2}$	$e_{i,i+m-1}$	...				
row $m+2$	1	1	1	1	1	1	1	1		
row $m+3$	0	0	0	0	0	0	0	0		

(2) If $m = 0$, we obtain the minimal Coxeter frieze pattern:

row 0	...	0	0	0	0	0	0	0	0	0	0	0	...
row 1	...	1	1	1	1	1	1	1	1	1	1	...	
row 2	...	1	1	1	1	1	1	1	1	1	1	1	...
row 3	...	0	0	0	0	0	0	0	0	0	0	...	

(3) For $m = 4$, an example is as follows, it is tame:

row 0	0	0	0	0	0	0	0	0	0	0	0	0
row 1	1	1	1	1	1	1	1	1	1	1	1	
row 2	...	3	1	2	4	1	2	2	3	1	2	...
row 3	2	1	7	3	1	3	5	2	1	...		
row 4	...	3	1	3	5	2	1	7	3	1	3	...
row 5	1	2	2	3	1	2	4	1	2	...		
row 6	...	1	1	1	1	1	1	1	1	1	1	...
row 7	0	0	0	0	0	0	0	0	0	0	0	

Proposition 1.3. (Periodicity) [9] *If we have a tame Coxeter frieze pattern with width m, then $n := m + 3$ is the period of this frieze. i.e. $e_{i,j} = e_{i+n,j+n}$ for every $i, j \in \mathbb{N}$, with $i \leq j \leq i + m - 1$.*
For example, the frieze with width 4 in Example 1.2(3) has period 7.

Remark 1.4. If a Coxeter frieze is not tame, it may have no periodicity.
In the following example, the numbers in row 2 fulfil:

$$e_{i,i} = \begin{cases} 3 \times 2^{\frac{i}{2}-3}, & i \text{ even} \\ 2^{\frac{5-i}{2}}, & i \text{ odd} \end{cases}$$

when i even and $i \to \infty$, $e_{i,i} \to \infty$, so it has no periodicity.

row 0	0		0		0		0		0		0		0		0		0		0		0		0
row 1		1		1		1		1		1		1		1		1		1		1		1	
row 2	...		4		$\frac{3}{4}$		2		$\frac{3}{2}$		1		3		$\frac{1}{2}$		6		$\frac{1}{4}$		12		...
row 3	...	$\frac{1}{2}$		2		$\frac{1}{2}$		2		$\frac{1}{2}$		2		$\frac{1}{2}$		2		$\frac{1}{2}$		2		$\frac{1}{2}$	...
row 4	...		0		0		0		0		0		0		0		0		0		0		...
row 5	...	-2		$-\frac{1}{2}$		-2		$-\frac{1}{2}$		-2		$-\frac{1}{2}$		-2		$-\frac{1}{2}$		-2		$-\frac{1}{2}$		-2	...
row 6			1		1		1		1		1		1		1		1		1		1		
row 7		0		0		0		0		0		0		0		0		0		0		0	

The following result may be found in [9], but we give an alternative proof.

Proposition 1.5. (Linear recurrence relations) [9] *Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a tame Coxeter frieze with width m. For any fixed i, if we denote $V_j := e_{i,j}$, and $a_j := e_{j,j}$ with $i, j \in \mathbb{Z}, i+1 \leq j \leq i+m+1$, then we obtain the <u>linear recurrence relations</u>:*

$$V_j = a_j V_{j-1} - V_{j-2}$$

Proof. Since E is a tame Coxeter frieze, for all $i, j \in \mathbb{Z}$ with $i+1 \leq j \leq i+m+1$ the determinant of every 3×3 adjacent entries equals 0.

$$\begin{array}{ccccc} & & e_{i+2,j-2} & & \\ & e_{i+1,j-2} & & e_{i+2,j-1} & \\ e_{i,j-2} & & e_{i+1,j-1} & & e_{i+2,j} \\ & e_{i,j-1} & & e_{i+1,j} & \\ & & e_{i,j} & & \end{array} \quad (*)$$

So, for every 3×3 entries having above form, we have:

$$\begin{aligned} 0 &= \begin{vmatrix} e_{i,j-2} & e_{i+1,j-2} & e_{i+2,j-2} \\ e_{i,j-1} & e_{i+1,j-1} & e_{i+2,j-1} \\ e_{i,j} & e_{i+1,j} & e_{i+2,j} \end{vmatrix} \\ &= e_{i,j-2} \begin{vmatrix} e_{i+1,j-1} & e_{i+2,j-1} \\ e_{i+1,j} & e_{i+2,j} \end{vmatrix} - e_{i,j-1} \begin{vmatrix} e_{i+1,j-2} & e_{i+2,j-2} \\ e_{i+1,j} & e_{i+2,j} \end{vmatrix} + e_{i,j} \begin{vmatrix} e_{i+1,j-2} & e_{i+2,j-2} \\ e_{i+1,j-1} & e_{i+2,j-1} \end{vmatrix} \\ &= e_{i,j-2} - e_{i,j-1} \begin{vmatrix} e_{i+1,j-2} & e_{i+2,j-2} \\ e_{i+1,j} & e_{i+2,j} \end{vmatrix} + e_{i,j} \end{aligned}$$

That means, for $t := \begin{vmatrix} e_{i+1,j-2} & e_{i+2,j-2} \\ e_{i+1,j} & e_{i+2,j} \end{vmatrix} = e_{i+1,j-2}e_{i+2,j} - e_{i+2,j-2}e_{i+1,j} \in \mathbb{C}$ we have $e_{i,j} = te_{i,j-1} - e_{i,j-2}$. Moreover:

$$1 = \begin{vmatrix} e_{i,j-2} & e_{i+1,j-2} \\ e_{i,j-1} & e_{i+1,j-1} \end{vmatrix} = e_{i,j-2}e_{i+1,j-1} - e_{i,j-1}e_{i+1,j-2} \tag{1.1}$$

$$1 = \begin{vmatrix} e_{i,j-1} & e_{i+1,j-1} \\ e_{i,j} & e_{i+1,j} \end{vmatrix} = e_{i,j-1}e_{i+1,j} - e_{i,j}e_{i+1,j-1} \tag{1.2}$$

That means:

$$\begin{aligned} & e_{i,j-2}e_{i+1,j-1} - e_{i,j-1}e_{i+1,j-2} = e_{i,j-1}e_{i+1,j} - e_{i,j}e_{i+1,j-1} \\ \Leftrightarrow \quad & e_{i+1,j-1}(e_{i,j-2} + e_{i,j}) = e_{i,j-1}(e_{i+1,j-2} + e_{i+1,j}) \end{aligned}$$

Case 1: If $e_{i,j-1} \neq 0$ and $e_{i+1,j-1} \neq 0$, we have $\frac{e_{i+1,j-2}+e_{i+1,j}}{e_{i+1,j-1}} = \frac{e_{i,j-2}+e_{i,j}}{e_{i,j-1}} = t$ and therefore $e_{i+1,j} = te_{i+1,j-1} - e_{i+1,j-2}$.

Case 2: If $e_{i,j-1} = 0$ and $e_{i+1,j-1} = 0$, similar to (1.1) we have $1 = e_{i+1,j-1}e_{i+2,j} - e_{i+1,j}e_{i+2,j-1} = 0 - 0 = 0$, which is contradiction.

Case 3: If $e_{i,j-1} \neq 0$ and $e_{i+1,j-1} = 0$, we have $e_{i+1,j-2} + e_{i+1,j} = 0$, which implies that $e_{i+1,j-2} + e_{i+1,j} = 0 = t \times 0 = te_{i+1,j-1}$.

Case 4: If $e_{i,j-1} = 0$ and $e_{i+1,j-1} \neq 0$, similar to (1.1) and (1.2) we have:

$$1 = \begin{vmatrix} e_{i+1,j-2} & e_{i+2,j-2} \\ e_{i+1,j-1} & e_{i+2,j-1} \end{vmatrix} = e_{i+1,j-2}e_{i+2,j-1} - e_{i+1,j-1}e_{i+2,j-2} \tag{1.3}$$

$$1 = \begin{vmatrix} e_{i+1,j-1} & e_{i+2,j-1} \\ e_{i+1,j} & e_{i+2,j} \end{vmatrix} = e_{i+1,j-1}e_{i+2,j} - e_{i+1,j}e_{i+2,j-1} \tag{1.4}$$

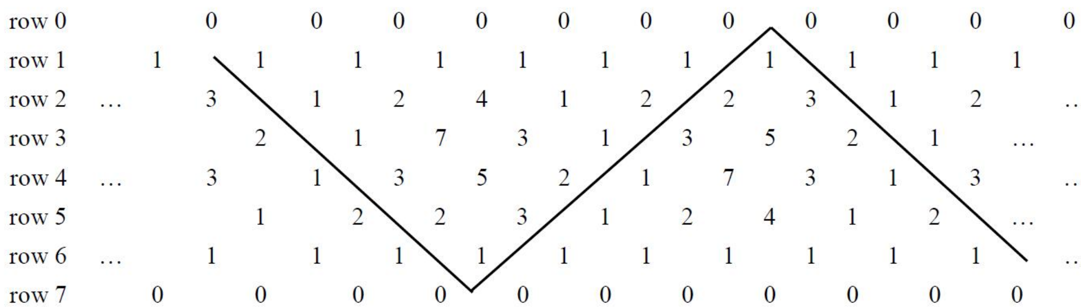
which implies:

$$\begin{aligned} te_{i+1,j-1} &= (e_{i+1,j-2}e_{i+2,j} - e_{i+1,j}e_{i+2,j-2})e_{i+1,j-1} \\ &= (e_{i+1,j-2}\frac{e_{i+1,j}e_{i+2,j-1}+1}{e_{i+1,j-1}} - e_{i+1,j}\frac{e_{i+1,j-2}e_{i+2,j-1}-1}{e_{i+1,j-1}})e_{i+1,j-1} \\ &= e_{i+1,j-2} + e_{i+1,j} \end{aligned}$$

In summary, this t satisfies that for all $i \in \mathbb{Z}, e_{i,j} = te_{i,j-1} - e_{i,j-2}$.
Especially for $i = j$ we have $t = \frac{e_{j,j-2}+e_{j,j}}{e_{i,j-1}} = \frac{0+e_{j,j}}{1} = e_{j,j}$, which implies $V_j = a_jV_{j-1} - V_{j-2}$. □

A proof for the following well-known fact may be found in [17].

Proposition 1.6. (Glide symmetry) *Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a tame Coxeter frieze with width m. If m is even, then E is invariant under a glide reflection with respect to the horizontal median line between row $\frac{m}{2}+1$ and row $\frac{m}{2}+2$. If m is odd, then E is invariant under a glide reflection with respect to row $\frac{m+3}{2}$.*



1.2 Integer-valued Frieze Patterns

The following three propositions 1.7, 1.8, 1.9 were already shown by Coxeter in [9] for frieze patterns over positive integers, and were completed by Morier-Genoud in [17] for frieze patterns over integers. Both proofs employed Laurent polynomials. We give alternative proofs for these three propositions in another approach.

Proposition 1.7. (Integer-valued frieze patterns)[17] *Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a Coxeter frieze. If for all $i \in \mathbb{Z}$, $e_{i,i} \in \mathbb{Z}$, then all the entries in E are integers.*

Proof. It is trivial that for all $i \in \mathbb{Z}$, $e_{i,i-2} = e_{i,i+m+1} = 0 \in \mathbb{Z}$ and $e_{i,i-1} = e_{i,i+m} = 1 \in \mathbb{Z}$. Then we still need to show: $e_{i,j} \in \mathbb{Z}$ for all $i, j \in \mathbb{Z}$ with $i+1 \le j \le i+m-1$. We do induction over j.

Induction base: $j = i+1$: Since $1 = \begin{vmatrix} e_{i,i} & e_{i+1,i} \\ e_{i,i+1} & e_{i+1,i+1} \end{vmatrix} = e_{i,i}e_{i+1,i+1} - e_{i+1,i}e_{i,i+1} = e_{i,i}e_{i+1,i+1} - e_{i,i+1}$, so we have $e_{i,i+1} = e_{i,i}e_{i+1,i+1} - 1 \in \mathbb{Z}$.

Induction hypothesis: assume for j we have $e_{i,j} \in \mathbb{Z}$.

Induction step: for $j+1$

$$\begin{array}{ccccccc} & & & e_{i+2,j-1} & & & \\ & & e_{i+1,j-1} & & e_{i+2,j} & & \\ & e_{i,j-1} & & e_{i+1,j} & & e_{i+2,j+1} & \\ & & e_{i,j} & & e_{i+1,j+1} & & \\ & & & e_{i,j+1} & & & \end{array}$$

For each adjacent 3×3 diamond having the form above, we obtain:

$$1 = \begin{vmatrix} e_{i+1,j-1} & e_{i+2,j-1} \\ e_{i+1,j} & e_{i+2,j} \end{vmatrix} = e_{i+1,j-1}e_{i+2,j} - e_{i+1,j}e_{i+2,j-1} \tag{1.5}$$

$$1 = \begin{vmatrix} e_{i,j-1} & e_{i+1,j-1} \\ e_{i,j} & e_{i+1,j} \end{vmatrix} = e_{i,j-1}e_{i+1,j} - e_{i,j}e_{i+1,j-1} \tag{1.6}$$

$$1 = \begin{vmatrix} e_{i+1,j} & e_{i+2,j} \\ e_{i+1,j+1} & e_{i+2,j+1} \end{vmatrix} = e_{i+1,j}e_{i+2,j+1} - e_{i+1,j+1}e_{i+2,j} \tag{1.7}$$

$$1 = \begin{vmatrix} e_{i,j} & e_{i+1,j} \\ e_{i,j+1} & e_{i+1,j+1} \end{vmatrix} = e_{i,j}e_{i+1,j+1} - e_{i,j+1}e_{i+1,j} \tag{1.8}$$

(1.5) implies that $e_{i+1,j}$ is coprime to $e_{i+1,j-1}$, and $e_{i+1,j}$ is coprime to $e_{i+2,j}$. Moreover, we have:

$$\mathbb{Z} \ni e_{i,j}e_{i+1,j+1} - 1 \overset{(1.6)(1.7)}{=} \frac{e_{i,j-1}e_{i+1,j} - 1}{e_{i+1,j-1}} \times \frac{e_{i+2,j+1}e_{i+1,j} - 1}{e_{i+2,j}} - 1$$
$$= \frac{e_{i+1,j}(e_{i,j-1}e_{i+1,j}e_{i+2,j+1} - e_{i+2,j+1} - e_{i,j-1}) + 1 - e_{i+1,j-1}e_{i+2,j}}{e_{i+1,j-1}e_{i+2,j}}$$

Therefore $e_{i,j+1} \overset{(1.8)}{=} \frac{e_{i,j}e_{i+1,j+1}-1}{e_{i+1,j}} \in \mathbb{Z} \Leftrightarrow e_{i+1,j} \mid 1 - e_{i+1,j-1}e_{i+2,j}$, which is indicated by (1.5). □

Proposition 1.8. [17] *If a Coxeter frieze $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ consists of integer numbers, then E is tame.*

Proof. On the one hand, with (1.6) in Proposition 1.7 we have $1 = e_{i,j-1}e_{i+1,j} - e_{i,j}e_{i+1,j-1}$. Similarly, we obtain $1 = e_{i+1,j-2}e_{i+2,j-1} - e_{i+1,j-1}e_{i+2,j-2}$. These two equations imply:

$$e_{i,j-1}e_{i+1,j}e_{i+1,j-2}e_{i+2,j-1} = e_{i+1,j-1}(e_{i,j}e_{i+1,j-2}e_{i+2,j-1} + e_{i+2,j-2}) + 1$$

On the other hand we have:

$$e_{i,j-1}e_{i+1,j}e_{i+1,j-2}e_{i+2,j-1} = (e_{i,j-2}e_{i+1,j-1} - 1) \times (e_{i+2,j}e_{i+1,j-1} - 1)$$
$$= e_{i+1,j-1}(e_{i,j-2}e_{i+1,j-1}e_{i+2,j} - e_{i+2,j} - e_{i,j-2}) + 1$$

Then we obtain:

$$e_{i,j}e_{i+1,j-2}e_{i+2,j-1} = e_{i,j-2}e_{i+1,j-1}e_{i+2,j} - e_{i,j-2} - e_{i+2,j} - e_{i+2,j-2}$$

According to adjacent 3×3 form in $(*)$ Proposition 1.5 we have:

$$\begin{vmatrix} e_{i,j-2} & e_{i+1,j-2} & e_{i+2,j-2} \\ e_{i,j-1} & e_{i+1,j-1} & e_{i+2,j-1} \\ e_{i,j} & e_{i+1,j} & e_{i+2,j} \end{vmatrix}$$

$$= e_{i,j-2} - e_{i+1,j-2}\begin{vmatrix} e_{i,j-1} & e_{i+2,j-1} \\ e_{i,j} & e_{i+2,j} \end{vmatrix} + e_{i+2,j-2}$$

$$= e_{i,j-2} + e_{i+2,j-2} - e_{i+1,j-2}e_{i,j-1}e_{i+2,j} + e_{i+1,j-2}e_{i+2,j-1}e_{i,j}$$

$$= e_{i,j-2} + e_{i+2,j-2} - e_{i+1,j-2}e_{i,j-1}e_{i+2,j} + e_{i,j-2}e_{i+1,j-1}e_{i+2,j} - e_{i,j-2} - e_{i+2,j} - e_{i+2,j-2}$$

$$= e_{i,j-2}e_{i+1,j-1}e_{i+2,j} - e_{i+1,j-2}e_{i,j-1}e_{i+2,j} - e_{i+2,j}$$

$$= e_{i+2,j}(e_{i,j-2}e_{i+1,j-1} - e_{i+1,j-2}e_{i,j-1} - 1) = 0$$

□

Proposition 1.9. (Positive-integer-valued frieze patterns) *Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a Coxeter frieze. E consists of positive integers (expect the bound 0's at top and bottom) if and only if there is a $i \in \mathbb{Z}$, for all $j \in \mathbb{Z}$ with $i \leq j \leq i+m-1, V_j := e_{i,j}$ is a positive integer and V_{j-1} divides $V_j + V_{j-2}$.*

Proof. " $\Rightarrow$ " see Proposition 1.5.
" $\Leftarrow$ " We just need to show for $i+1$ and $\forall j \in \mathbb{Z}$ with $i+1 \leq j \leq i+m, e_{i+1,j}$ is also a positive integer and $e_{i+1,j-1}$ divides $e_{i+1,j} + e_{i+1,j-2}$.
Induction base: $j = i+1$

$$1 = \begin{vmatrix} e_{i,i} & e_{i+1,i} \\ e_{i,i+1} & e_{i+1,i+1} \end{vmatrix} = e_{i,i}e_{i+1,i+1} - e_{i,i+1}e_{i+1,i} = e_{i,i}e_{i+1,i+1} - e_{i,i+1}$$

So we get $e_{i+1,i+1} = \frac{e_{i,i+1}+1}{e_{i,i}} = \frac{e_{i,i+1}+e_{i,i-1}}{e_{i,i}} \in \mathbb{N}_+$.
And $e_{i+1,i}$ divides $e_{i+1,i+1} + e_{i+1,i-1}$, since $\frac{e_{i+1,i+1}+e_{i+1,i-1}}{e_{i+1,i}} = \frac{e_{i+1,i+1}+0}{1} = e_{i+1,i+1} \in \mathbb{N}_+$.
Induction hypothesis: assume for j we have that $e_{i+1,j}$ is a positive integer and $e_{i+1,j-1}$ divides $e_{i+1,j} + e_{i+1,j-2}$.

Induction step: for $j+1$ we have:

$$1 = \begin{vmatrix} e_{i,j-1} & e_{i+1,j-1} \\ e_{i,j} & e_{i+1,j} \end{vmatrix} = e_{i,j-1}e_{i+1,j} - e_{i,j}e_{i+1,j-1}$$

$$1 = \begin{vmatrix} e_{i,j} & e_{i+1,j} \\ e_{i,j+1} & e_{i+1,j+1} \end{vmatrix} = e_{i,j}e_{i+1,j+1} - e_{i,j+1}e_{i+1,j}$$

we obtain $e_{i+1,j}(e_{i,j+1} + e_{i,j-1}) = e_{i,j}(e_{i+1,j+1} + e_{i+1,j-1})$.

Case 1: If $e_{i+1,j} = 0$, then we have $1 = e_{i,j-1}e_{i+1,j} - e_{i,j}e_{i+1,j-1} = -e_{i,j}e_{i+1,j-1} < 0$ which is a contradiction.

Case 2: If $e_{i+1,j} \neq 0$, then we have $\exists p \in \mathbb{N}_+$ such that $\frac{e_{i+1,j+1}+e_{i+1,j-1}}{e_{i+1,j}} = \frac{e_{i,j+1}+e_{i,j-1}}{e_{i,j}} = p$. That means, $e_{i+1,j}$ divides $e_{i+1,j+1} + e_{i+1,j-1}$.

Moreover, $\left.\begin{array}{l} e_{i+1,j+1} = pe_{i+1,j-1} - e_{i+1,j-1} \in \mathbb{Z} \\ e_{i+1,j+1} = \frac{e_{i,j+1}e_{i+1,j}+1}{e_{i,j}} > 0 \end{array}\right\} \Rightarrow e_{i+1,j+1} \in \mathbb{N}_+$

□

Proposition 1.10. *Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a positive-integer-valued Coxeter frieze (expect bound 0's at top and bottom) with width $m \geq 1$, if there exists $i \in \mathbb{Z}$ with $e_{i,i} = 1$, then $e_{i+1,i+1} \neq 1$.*

Proof. Suppose $\exists i \in \mathbb{Z}, e_{i,i} = e_{i+1,i+1} = 1$. Then $1 = \begin{vmatrix} e_{i,i} & e_{i+1,i} \\ e_{i,i+1} & e_{i+1,i+1} \end{vmatrix} = e_{i,i}e_{i+1,i+1} - e_{i+1,i}e_{i,i+1} = 1 - e_{i,i+1}$, which implies $e_{i,i+1} = 0$. But E is a positive-integer-valued frieze, which implies $e_{i,j} > 0$ for all $i,j \in \mathbb{Z}, i-1 \leq j \leq i+m,$. Because of $m \geq 1$ we have $e_{i,i+1} > 0$, which is a contradiction. □

This following proposition and its basic idea for proof were given by Coxeter in 1971 [9], but we prove it completely.

Proposition 1.11. (Expression of each entry in frieze patterns) [9] *If we denote $a_i := e_{i,i}$, the row 2 in a tame Coxeter frieze (See Example 1.2(1)), then for every $i,j \in \mathbb{N}, i \leq j \leq i+m-1$, we have:*

$$e_{i,j} = \begin{vmatrix} a_i & 1 & & & & \\ 1 & a_{i+1} & 1 & & & \\ & 1 & a_{i+2} & 1 & & \\ & & \vdots & \vdots & \vdots & \\ & & & 1 & a_{i-1} & 1 \\ & & & & 1 & a_i \end{vmatrix}$$

Proof. Induction base: $j = i$, $e_{i,i} = a_i = |a_i|$ trivial.

Induction hypothesis: assume for j we have:

$$e_{i,t} = \begin{vmatrix} a_i & 1 & & & & \\ 1 & a_{i+1} & 1 & & & \\ & 1 & a_{i+2} & 1 & & \\ & & \vdots & \vdots & \vdots & \\ & & & 1 & a_{j-1} & 1 \\ & & & & 1 & a_j \end{vmatrix} \text{ for all } i \in \mathbb{Z}$$

Induction step: for $j+1$ we have:

$$\begin{vmatrix} a_i & 1 & & & & \\ 1 & a_{i+1} & 1 & & & \\ & 1 & a_{i+2} & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & a_j & 1 \\ & & & & 1 & a_{j+1} \end{vmatrix}$$

$$= a_{j+1} \begin{vmatrix} a_i & 1 & & & & \\ 1 & a_{i+1} & 1 & & & \\ & 1 & a_{i+2} & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & a_{j-1} & 1 \\ & & & & 1 & a_j \end{vmatrix} - \begin{vmatrix} a_i & 1 & & & & \\ 1 & a_{i+1} & 1 & & & \\ & 1 & a_{i+2} & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & a_{j-2} & 1 \\ & & & & 1 & a_{j-1} \end{vmatrix}$$

$$= a_{j+1}e_{i,j} - e_{i,j-1} \overset{\text{Prop1.5}}{=} e_{i,j+1}$$ □

1.3 SL_{k+1} Frieze Patterns

SL_{k+1} frieze pattern is a well-known extension of Coxeter frieze pattern. The following definitions and propositions can be consulted in [1, 8, 3, 16, 17].

Definition 1.12. **(SL_{k+1}–tiling and SL_{k+1}-frieze)**[8]

(1) An <u>SL_{k+1}*-tiling*</u> is a bi-infinite matrix $M = (m_{i,j})_{i,j\in\mathbb{Z}}$, where all adjacent minors of order $k+1$ equals 1. i.e. for all $i, j \in \mathbb{Z}$

$$M_{i,j}^{(k+1)} := \begin{vmatrix} m_{i,j} & m_{i,j+1} & \cdots & m_{i,j+k-1} & m_{i,j+k} \\ m_{i+1,j} & m_{i+1,j+1} & \cdots & m_{i+1,j+k-1} & m_{i+1,j+k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{i+k-1,j} & m_{i+k-1,j+1} & \cdots & m_{i+k-1,j+k-1} & m_{i+k-1,j+k} \\ m_{i+k,j} & m_{i+k,j+1} & \cdots & m_{i+k,j+k-1} & m_{i+k,j+k} \end{vmatrix} = 1$$

(2) An SL_{k+1}-tiling is called <u>*tame*</u> if in addition all adjacent minors of order $k+2$ vanish. i.e. $M_{i,j}^{(k+2)} = 0$ for all $i, j \in \mathbb{Z}$.

(3) Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be an infinite matrix, if E satisfies the following conditions, then E is called a <u>SL_{k+1}*-frieze*</u> with <u>*width*</u> m:

1. E is SL_{k+1}-tiling.
2. $\exists m \in \mathbb{N}$, such that

$$\begin{aligned} e_{i,i-1} = e_{i,i+m} = 1, &\qquad \forall i \in \mathbb{Z} \\ e_{i,i-1-l} = e_{i,i+m+l} = 0, &\qquad \forall i, l \in \mathbb{Z}, 1 \le l \le k \end{aligned}$$

Example 1.13. (1) Let $k = 1$, then SL_2-frieze is equivalent to Coxeter frieze.

(2) Let $k = 2$, an example of SL_3-frieze with width 4 is as following:

row 0	0		0		0		0		0		0		0		0		0		0		0
row 1		1		1		1		1		1		1		1		1		1		1	
row 2	4		3		1		5		4		3		1		6		4		3		1
row 3		8		1		4		9		8		2		2		21		8		1	
row 4	40		1		3		6		8		5		1		6		40		1		3
row 5		4		1		4		3		4		2		1		11		4		1	
row 6	1		1		1		1		1		1		1		1		1		1		1
row 7		0		0		0		0		0		0		0		0		0		0	

Definition 1.14. **(Projective dual)**[17]

(1) A <u>*r-derived array*</u> $\partial_r M$ of a SL_{k+1}-tiling $M = (m_{i,j})_{i,j\in\mathbb{Z}}$ is a bi-infinite matrix, whose elements are the adjacent minors of order r in M. i.e.

$$\partial_r M := (M_{i,j}^{(r)})_{i,j\in\mathbb{Z}}$$

(2) Let $M = (m_{i,j})_{i,j\in\mathbb{Z}}$ be a SL_{k+1}-tiling. The projective dual M^* is the k-derived array of M.

Proposition 1.15. (Projective dual of a SL_{k+1}-tiling)[3] *Let M be a SL_{k+1}-tiling and M^* be its projective dual.*

(1) *M^* is also a SL_{k+1}-tiling.*

(2) *If M is tame, then M^* is also tame.*

(3) *For all $r \in \mathbb{N}, 1 \le r \le k, (\partial_r M^*)_{i,j} = (\partial_{k+1-r} M)_{i+r-1,j+r-1}$.*

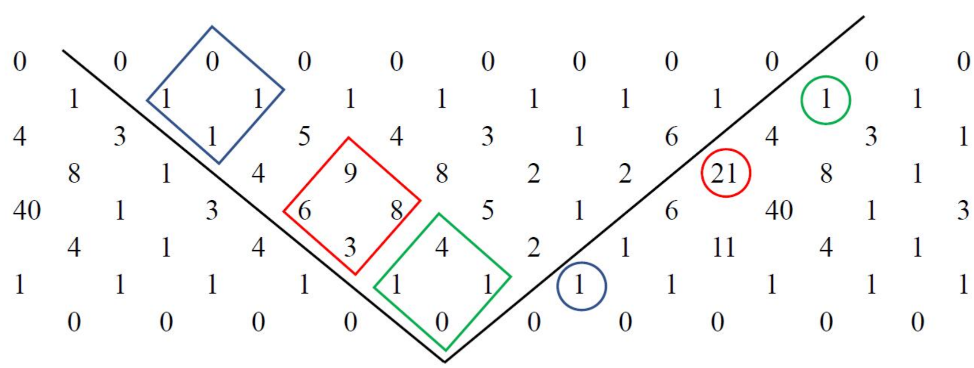
(4) *If we left-shift M $k-1$ elements, then we get $(M^*)^*$.*

Proof. The proofs of (1)-(3) can be consulted in [3]. We just give the proof of (4).
Proof of (4): Choose $r = k$ in (3), then we have:

$$((M^*)^*)_{i,j} = (\partial_k M^*)_{i,j} = (\partial_1 M)_{i+k-1,j+k-1} = (M)_{i+k-1,j+k-1}$$

$\square$

Proposition 1.16. (Glide symmetry of projective dual)[17]



Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a tame SL_{k+1}-frieze with width m. If m is even, then E has glide reflection to E^* with respect to the horizontal median line between row $\frac{m}{2}+1$ and row $\frac{m}{2}+2$. If m is odd, then E has glide reflection to E^* with respect to row $\frac{m+3}{2}$.

For example, in above figure, the determinant of each square equals to the value in the circle with same colour. This implies, if we define the reversal version of E as $E^{\perp}$, then E^* and $E^{\perp}$ concide up to a shift of indices.

Proposition 1.17. (Periodicity of tame SL_{k+1}-frieze)[17] *If we denote a SL_{k+1}-frieze $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ with width m. Then E fulfils:*

$$e_{i,j} = (-1)^k e_{i+k+m+2,j} \text{ and } e_{i,j} = (-1)^k e_{i,j+k+m+2}$$

In particular, $e_{i,j} = (-1)^k e_{i+k+m+2,j} = (-1)^{2k} e_{i+k+m+2,j+k+m+2} = e_{i+k+m+2,j+k+m+2}$. Therefore, E has period $k+m+2$.

The following proposition and its basic idea can be consulted in [3], but we prove it completely.

Proposition 1.18. (Linear recurrence relations)[3] *Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a tame SL_{k+1}-frieze. For all $i \in \mathbb{Z}$ fixed, $V_j := e_{i,j}$. Then it satisfies the linear difference equation:*

$$V_j = a_{j,1}V_{j-1} - a_{j,2}V_{j-2} + ... + (-1)^{k-1}a_{j,k}V_{j-k} + (-1)^k V_{j-k-1}$$

Proof. Since E is tame SL_{k+1}-frieze, each adjacent minor of order $k+2$ equals 0. By using Laplace expansion, $\forall i, j \in \mathbb{Z}$:

$$\begin{aligned}
0 &= \begin{vmatrix}
e_{i,j-k-1} & e_{i+1,j-k-1} & \cdots & e_{i+k,j-k-1} & e_{i+k+1,j-k-1} \\
e_{i,j-k} & e_{i+1,j-k} & \cdots & e_{i+k,j-k} & e_{i+k+1,j-k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
e_{i,j-1} & e_{i+1,j-1} & \cdots & e_{i+k,j-1} & e_{i+k+1,j-1} \\
e_{i,j} & e_{i+1,j} & \cdots & e_{i+k,j} & e_{i+k+1,j}
\end{vmatrix} \\
&= e_{i,j-k-1}M_{i,j-k-1} - e_{i,j-k}M_{i,j-k} + ... + (-1)^{k+2}e_{i,j-1}M_{i,j-1} + (-1)^{k+3}e_{i,j}M_{i,j} \\
\Leftrightarrow & e_{i,j}M_{i,j} = e_{i,j-1}M_{i,j-1} + ... + (-1)^{k-1}e_{i,j-k}M_{i,j-k} + (-1)^k e_{i,j-k-1}M_{i,j-k-1}
\end{aligned}$$

where $M_{i,j}$ is (i,j)-minor of E (the minor of the submatrix formed by deleting the i-th row and j-th column). Notice that $M_{i,j-k-1} = M_{i,j} = 1$. If we define $a_{j,t} := M_{i,j-t}$, then we obtain the linear difference equation. □

2 Triangulation for positive-integer-valued Coxeter Frieze

In this section we introduce two important definitions, namely quiddity cycles and triangulations. With a quiddity cycle of a positive-integer-valued Coxeter frieze pattern we can find a polygon with specific triangulation. In reverse, with this polygon we can recover the original Coxeter frieze pattern.

Definition 2.1. (Quiddity cycle in Coxeter frieze)[9] Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a Coxeter frieze with width m. $c = (c_1, c_2, ..., c_n)$ is a *quiddity cycle* of E if and only if $n = m+3$ is the period of E and c is a cycle in row 2. i.e. $\exists t \in \{0, 1, ..., n-1\}$ fixed, $\forall i \in \{1, 2, ..., n\}$ and $\forall k \in \mathbb{Z}$, we have $c_i = e_{kn+i+t,kn+i+t}$.

Remark 2.2. Let E be a Coxeter frieze with period n. Then E has n different quiddity cycles. But all these quiddity cycles are unique up to cyclic permutation.

Proposition 2.3. *Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a tame Coxeter frieze and $c = (c_1, c_2, ..., c_n) = (e_{t,t}, e_{t+1,t+1}, ..., e_{t+m+2,t+m+2})$ be a quiddity cycle of E for a fixed $t \in \mathbb{Z}$. Then for all $j \in \mathbb{Z}$ with $t \le j \le t+m+1$, we have:*

$$\begin{pmatrix} e_{t,j} & -e_{t,j-1} \\ e_{t+1,j} & -e_{t+1,j-1} \end{pmatrix} = \prod_{k=t}^{j} \begin{pmatrix} e_{k,k} & -1 \\ 1 & 0 \end{pmatrix}$$

Proof. We show this proposition by using induction over j.
Induction base: $j = t$,

$$\prod_{k=t}^{t} \begin{pmatrix} e_{k,k} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e_{t,t} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e_{t,t} & -e_{t,t-1} \\ e_{t+1,t} & -e_{t+1,t-1} \end{pmatrix}$$

Induction hypothesis: for j we have:

$$\prod_{k=t}^{j} \begin{pmatrix} e_{j,j} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e_{t,j} & -e_{t,j-1} \\ e_{t+1,j} & -e_{t+1,j-1} \end{pmatrix}$$

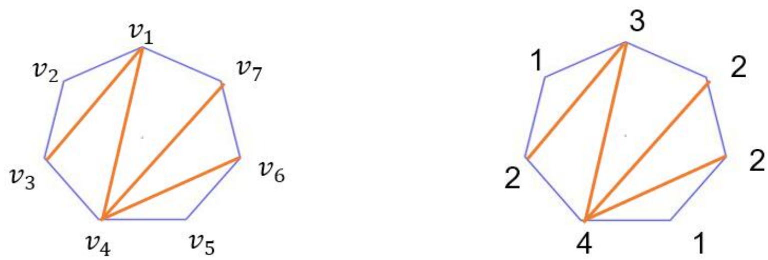
Induction step: for $j+1$ we have:

$$\begin{aligned}
\prod_{k=t}^{j+1} \begin{pmatrix} e_{k,k} & -1 \\ 1 & 0 \end{pmatrix} &= \prod_{k=t}^{j} \begin{pmatrix} e_{k,k} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_{j+1,j+1} & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} e_{t,j} & -e_{t,j-1} \\ e_{t+1,j} & -e_{t+1,j-1} \end{pmatrix} \begin{pmatrix} e_{j+1,j+1} & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} e_{t,j}e_{j+1,j+1} - e_{t,j-1} & -e_{t,j} \\ e_{t+1,j}e_{j+1,j+1} - e_{t+1,j-1} & -e_{t+1,j} \end{pmatrix} \\
&\overset{\text{Prop 1.5}}{=} \begin{pmatrix} e_{t,j+1} & -e_{t,j} \\ e_{t+1,j+1} & -e_{t+1,j} \end{pmatrix}
\end{aligned}$$

$\square$

Proposition 2.4. (Triangulation)[17] *Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a positive-integer-valued Coxeter frieze with width m and $c = (c_1, c_2, ..., c_n)$ be a quiddity cycle of E. Then c determines a triangulation of a convex n-polygon, where c_i counts the number of triangles, which contain the i-th vertex of this n-polygon.*

Example 2.5. Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be the Coxeter frieze in Example 1.2(3). Then tuple $c = (3, 1, 2, 4, 1, 2, 2)$ is a quiddity cycle of E. Moreover, the following figure is a triangulation of c.



Proposition 2.6. (Process to calculate frieze by triangulation)[17] *Let $c = (c_1,$ $c_2,...,c_n)$ be a quiddity cycle. Then we can calculate all entries of a Coxeter frieze E, where c is quiddity cycle of E, by using the following process.*

(1) *Choose one vertex v_i and tag it 0.*

(2) *Tag 1 all vertex, which is connected with v_i.*

(3) *If there is a triangle, whose two vertex are already tagged as a and b, then tag $a + b$ the third vertex.*

(4) *Repeat (3) until all vertex are tagged, $e_{i+1,j-1}$ = the tag at vertex v_j.*

(5) *Repeat (1)-(4) for $i \in \{1, 2, ..., n\}$, then we obtain the entries of a Coxeter frieze in a period, with periodicity we can extent these entries to a Coxeter frieze E.*

Example 2.7. Let $c = (3, 1, 2, 4, 1, 2, 2)$ be the quiddity cycle as in Example 2.5.

(1) Choose vertex v_2 and tag it 0. (figure 1)

(2) Tag v_1 and v_3 with 1, which is connected with v_2. (figure 2)

(3) If there is a triangle, whose two vertex are already tagged as a and b, then tag $a+b$ the third vertex. e.g. v_4. (figure 3)

(4) Repeat (3) until all vertex are tagged. (figure 4)

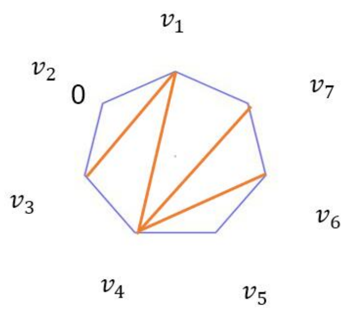


figure 1

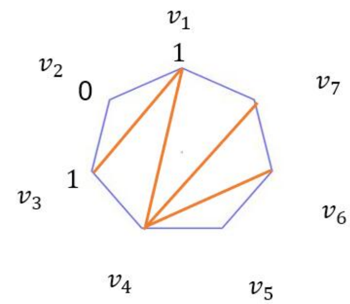


figure 2

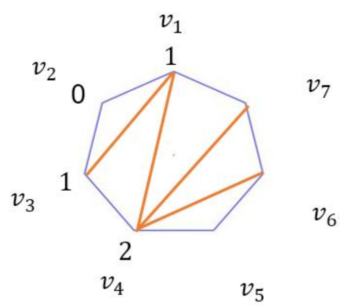


figure 3

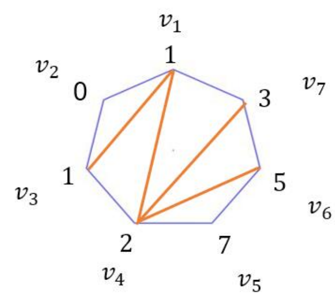


figure 4

Then $e_{3,j-1} :=$ the tag at vertex v_j. In particular, if $j-1<0$, we denote $e_{3,j-1+n} = e_{3,j-1}$. After that we obtain the third column in Example 1.2(3). If we repeat step (1)-(4) by choosing vertex $v_1, v_3, .., v_7$, then we obtain the other entries of a period of the frieze in Example 1.2(3).

Proposition 2.8. (Quantity of positive-integer Coxeter frieze) [14] *The quantity of positive-integer-valued Coxeter frieze with width m is finite, and is equal to the $(m+1)$-th Catalan number.*

$$C_{m+1} = \frac{1}{m+2}\begin{pmatrix} 2(m+1) \\ m+1 \end{pmatrix}$$

3 Cuntz Frieze Patterns

By Section 2 we know that we can extract a quiddity cycle of a Coxeter frieze pattern, and then with triangulation and this quiddity cycle we can recover the original Coxeter frieze pattern. That means, a finite sequence with some specific properties can build a frieze pattern. That is the basic idea of another approach to frieze by Cuntz [10]. Since there is still no proper noun for the result of this approach. We call it "Cuntz Frieze Patterns". In this section we will introduce the basic definitions and properties of Cuntz frieze patterns (Section 3.1). Moreover, we will give an introduction about the decomposition for a quiddity cycle (Section 3.2), and show the equivalence between Coxeter frieze patterns and Cuntz frieze patterns (Section 3.3). Finally, we give an intuition about the combinatorial models for integer-valued frieze patterns (Section 3.3).

3.1 Quiddity Cycles and Cuntz Frieze

The following definition has been given by Cuntz and Holm in 2017 [11].

Definition 3.1. (λ-quiddity cycle and Cuntz frieze)[10] Let R be a subset of a commutative ring and $\lambda \in \{\pm 1\}$, A *λ-quiddity cycle* over R is a tuple $c = (c_1, c_2, ..., c_n) \in R^n$ satisfying:

$$\prod_{k=1}^{n} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda \,\mathrm{id}$$

In particular, we also call a (-1)-quiddity cycle simply a *quiddity cycle*.

A *Cuntz frieze* $F = (f_{i,j})_{i,j \in \mathbb{Z}, i-2 \leq j \leq i+n-2}$ produced by a quiddity cycle $c = (c_1, c_2, ..., c_n)$ is defined by:

1. $c_{i+tn,i+tn} = c_i$ for all $t \in \mathbb{Z}$
2. $f_{i,j} := \left(\prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \right)_{1,1}$

where $M_{1,1}$ means the entry in the first row and first column of M. Especially, we denote $f_{i,i-2} = f_{i,i+n-2} = 0, f_{i,i-1} = f_{i,i+n-2} = 1$. Then n is the *period* of F and $m := n - 3$ is the *width* of F. Obviously, $f_{i,i} = c_i$ for all $i \in \mathbb{Z}$.

Remark 3.2. If we consider a number as a 1×1 matrix, then we have:

$$f_{i,j} = \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example 3.3. [10] Let $R = \mathbb{C}$, then we have the following results:

(1) $(0,0)$ is the only λ-quiddity cycle of length 2.

(2) $(-1,-1,-1)$ is the only 1-quiddity of length 3.

(3) $(1,1,1)$ is the only (-1)-quiddity of length 3.

(4) $(t, \frac{2}{t}, t, \frac{2}{t})$, t a unit and $(a, 0, -a, 0)$, a arbitrary, are the only λ-quiddity cycles of length 4.

Proposition 3.4. [10] *Let D_n be the dihedral group with $2n$ elements acting on $1, ..., n$ and $c = (c_1, c_2, ..., c_n)$ be a λ-quiddity cycle. Then $c^\sigma := (c_1, c_2, ..., c_n)^\sigma := (c_{\sigma(1)}, c_{\sigma(2)}, ..., c_{\sigma(n)})$ is a λ-quiddity cycle as well.*

In most articles, the authors show the equivalence between "Cuntz frieze patterns" and "tame Coxeter frieze patterns" at first and then obtain the linear recurrence relation for Cuntz frieze patterns directly. But we prove these claims in another easy approach: We firstly show the linear recurrence relation for Cuntz frieze patterns, which is helpful for the straightforward proof for the equivalence between "Cuntz frieze patterns" and "tame Coxeter frieze patterns" later.

Proposition 3.5. (Linear recurrence relation) [11] *Let $c = (c_1, c_2, ..., c_n)$ be a quiddity cycle and $F = (f_{i,j})_{i,j\in\mathbb{Z}}$ be the Cuntz frieze produced by c. Then for all $i, j \in \mathbb{Z}, i-2 \leq j \leq i+n-1$, $f_{i,j} = f_{i,j-1}f_{j,j} - f_{i,j-2}$.*

Proof.

$$\begin{aligned}
&f_{i,j-1}f_{j,j} - f_{i,j-2}\\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j-1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} c_j & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\\
&\quad - \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j-2} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\prod_{k=i}^{j-1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_j & -1 \\ 0 & 0 \end{pmatrix} - \prod_{k=i}^{j-2} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j-1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} c_j & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & c_{j-1} \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j-1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_j & -2 \\ 1 & -c_{j-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j-1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_j & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = f_{i,j}$$

□

Proposition 3.6. *Let $c = (c_1, c_2, ..., c_n)$ be a quiddity cycle and $F = (f_{i,j})_{i,j\in\mathbb{Z}}$ be the Cuntz frieze produced by c. Then for all $i, j \in \mathbb{Z}, i \leq j \leq i + n - 2$ we have:*

$$\prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{i,j} & -f_{i,j-1} \\ f_{i+1,j} & -f_{i+1,j-1} \end{pmatrix}$$

Proof. We prove this proposition by using induction over j.
Induction base: $j = i$,

$$\prod_{k=i}^{i} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_i & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{i,i} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{i,i} & -f_{i,i-1} \\ f_{i+1,i} & -f_{i+1,i-1} \end{pmatrix}$$

Induction hypothesis: for j we have:

$$\prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{i,j} & -f_{i,j-1} \\ f_{i+1,j} & -f_{i+1,j-1} \end{pmatrix}$$

Induction step: for $j + 1$ we have:

$$\prod_{k=i}^{j+1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} = \prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{j+1} & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} f_{i,j} & -f_{i,j-1} \\ f_{i+1,j} & -f_{i+1,j-1} \end{pmatrix} \begin{pmatrix} f_{j+1,j+1} & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} f_{i,j}f_{j+1,j+1} - f_{i,j-1} & -f_{i,j} \\ f_{i+1,j}f_{j+1,j+1} - f_{i+1,j-1} & -f_{i+1,j} \end{pmatrix}$$
$$\overset{\text{Prop 3.5}}{=} \begin{pmatrix} f_{i,j+1} & -f_{i,j} \\ f_{i+1,j+1} & -f_{i+1,j} \end{pmatrix}$$

□

Remark 3.7. (1) Let $c = (c_1, c_2, ..., c_n)$ be a quiddity cycle and $F = (f_{i,j})_{i,j\in\mathbb{Z}}$ be the Cuntz frieze produced by c. Then for all $i, j \in \mathbb{Z}, i-2 \le j \le i+n-2$:

$$\begin{aligned} f_{i,j} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j+1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \prod_{k=i-1}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \end{pmatrix} \prod_{k=i-1}^{j+1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

(2) As $AB \neq BA$ for $A, B \in M_2(R)$, if $c = (c_1, ..., c_{i-1}, c_i, c_{i+1}, c_{i+2}, ..., c_n)$ is a quiddity cycle, then in general $(c_1, ..., c_{i-1}, c_{i+1}, c_i, c_{i+2}, ..., c_n)$ is not a quiddity cycle for the case $c_i \neq c_{i+1}$.

Proposition 3.8. (Equivalence relation) *Let R be a subset of a commutative ring, then for a fixed $n \in \mathbb{N}_+$, "$a \sim b \Leftrightarrow \exists \sigma \in D_n, a = b^\sigma$" defines an equivalence relation on R.*

Proof. Reflexivity: Choose $\sigma = \mathrm{id} \in D_n$. Then $a = a^\sigma$.
Symmetry: Let $a, b \in R^n$, then we have:

$$a \sim b \Leftrightarrow \exists \sigma \in D_n, a = b^\sigma \Leftrightarrow \exists \pi = -\sigma \in D_n, b = b^{\sigma(-\sigma)} = a^{-\sigma} = a^\pi \Leftrightarrow b \sim a$$

Transitivity: If $a \sim b$ and $b \sim c$, then $\exists \sigma_1, \sigma_2 \in D_n, a = b^{\sigma_1}, b = c^{\sigma_2}$. Choose $\pi = \sigma_1\sigma_2 \in D_n$, then we have:

$$a = b^{\sigma_1} = c^{\sigma_1\sigma_2} = c^\pi \Rightarrow a \sim c$$

□

Proposition 3.9. *Let R be a subset of a commutative ring, $c = (c_1, c_2, ..., c_n) \in R^n$ be a finite sequence. If there exists $x, y \in R$ such that $d = (x, c_1, c_2, ..., c_n, y) \in R^{n+2}$ is a λ-quiddity cycle, then x, y are uniquely determined.*

Proof. Define $M := \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix} := \prod_{k=1}^n \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix}$. Then, we have:

$$\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \prod_{k=1}^{n} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix} \begin{pmatrix} y & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} m_{1,1}xy - m_{2,1}y + m_{1,2}x - m_{22} & m_{2,1} - m_{1,1}x \\ m_{1,1}y + m_{1,2} & -m_{1,1} \end{pmatrix}$$

Therefore x, y are uniquely determined, since

$$\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \prod_{k=1}^{n} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \Leftrightarrow \begin{cases} x = -\lambda m_{2,1} \\ y = \lambda m_{1,2} \end{cases}.$$

□

Proposition 3.10. *Let R be a subset of a commutative ring and $c = (c_1, c_2, ..., c_n) \in R^n$ be a λ-quiddity cycle. Then $-c = (-c_1, -c_2, ..., -c_n)$ is a $(-1)^n\lambda$-quiddity cycle.*

Proof. Notice that $\forall x \in R$, for the matrix $A := \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$ we have $|A| = \left|A^\top\right| = 1$, which implies that A is invertible. Moreover, we have:

$$\begin{aligned} \prod_{k=1}^{n} \begin{pmatrix} -c_k & -1 \\ 1 & 0 \end{pmatrix} &= \prod_{k=1}^{n} \left((-1) \begin{pmatrix} c_k & 1 \\ -1 & 0 \end{pmatrix} \right) = (-1)^n \prod_{k=1}^{n} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix}^\top \\ &= (-1)^n \left(\prod_{k=n}^{1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \right)^\top \overset{\text{Prop 3.7}}{=} (-1)^n \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}^\top = (-1)^n \lambda \,\mathrm{id} \end{aligned}$$

□

Proposition 3.11. [11] *Let $R = \mathbb{C}$ and $c = (c_1, c_2, ..., c_n) \in R^n$ be a λ-quiddity cycle. Then there are two different indices $j, k \in \{1, ..., n\}$ with $|c_j| < 2$ and $|c_k| < 2$.*

3.2 Reducibility

Definition 3.12. (Reducibility) [10] Let R be a subset of a commutative ring and $c = (c_1, c_2, ..., c_m) \in R^m$ $(m > 2)$ be a λ-quiddity cycle. Then, c is called *reducible* over R if there exists a λ'-quiddity cycle $a = (a_1, a_2, ..., a_k)$ and a λ''-quiddity cycle $b = (b_1, b_2, ..., b_l)$, such that:

1. $\lambda = -\lambda'\lambda''$
2. $k, l > 2$
3. there exists a permutation $\sigma \in D_n$, such that
   $c^\sigma = (a_1 + b_l, a_2, ..., a_{k-1}, a_k + b_1, b_2, ..., b_{l-1}) =: a \oplus b$

If c is not reducible then c is called *irreducible*.

Example 3.13. [10]

(1) The set of irreducible λ-quiddity cycles over $\mathbb{N}$ is $\{(1,1,1)\}$.

(2) The set of irreducible λ-quiddity cycles over $\mathbb{Z}$ is

$$\{(1,1,1),(-1,-1,-1),(a,0,-a,0),(0,a,0,-a)|a\in\mathbb{Z}\backslash\{\pm1\}\}.$$

(3) $(1,0,-1,0)$ is reducible over $\mathbb{Z}$, since $(0,1,0,-1)=(1,1,1)\oplus(-1,-1,-1)$.

Proposition 3.14. [10] *Let $a=(a_1,a_2,...,a_k)$ be a λ'-quiddity cycle and $b=(b_1,b_2,...,b_l)$ be a λ''-quiddity cycle. Then*

$$a\oplus b:=(a_1+b_l,a_2,...,a_{k-1},a_k+b_1,b_2,...,b_{l-1})$$

is a $(-\lambda'\lambda'')$-quiddity cycle.

Remark 3.15. (1) We denote by l_c the length of a quiddity cycle c. Let a,b be $\{\pm1\}$-quiddity cycles, then $l_{a\oplus b}=l_a+l_b-2$.

(2) Let $R=\mathbb{Z}$ and $c=(c_1,...,c_n)\in R^n$ be a quiddity cycle. If c can be decomposed into two irreducible $\{\pm1\}$-quiddity cycles a,b, such that $c=a\oplus b$, then $l_a,l_b\in\{3,4\}$.

(3) Let R be a subset of a commutative ring and $c=(c_1,...,c_n)\in R^n$ be an irreducible quiddity cycle. Then for all $c'\in R^n$ with $c\sim c'$, c' is irreducible. (In other words, c^σ is irreducible, for all $\sigma\in D_n$.)

Proof. Suppose $\exists\sigma\in D_n$, such that c^σ is reducible. That means $\exists a,b$ two $\{\pm1\}$-quiddity cycles over R and $\exists\tau\in D_n$, such that $c^\sigma=(a\oplus b)^\tau$. If we define $\pi=-\sigma\tau\in D_n$, then we have $c=(c^\sigma)^{-\sigma}=((a\oplus b)^\tau)^{-\sigma}=(a\oplus b)^\pi$, which implies that c is reducible. It is a contradiction to c irreducible. □

Proposition 3.16. *Let $a=(a_1,a_2,...,a_k)$ be a λ'-quiddity cycle and $b=(b_1,b_2,...,b_l)$ be a finite sequence. Then b is a λ''-quiddity cycle if and only if $a\oplus b$ is a $(-\lambda'\lambda'')$-quiddity cycle.*

Proof. " $\Rightarrow$ " see Proposition 3.14.

" $\Leftarrow$ " $(-\lambda'\lambda'')\,\mathrm{id}$

$$=\begin{pmatrix} a_1+b_l & -1 \\ 1 & 0\end{pmatrix}\prod_{t=2}^{k-1}\begin{pmatrix} a_t & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} a_k+b_1 & -1 \\ 1 & 0\end{pmatrix}\prod_{t=2}^{l-1}\begin{pmatrix} b_t & -1 \\ 1 & 0\end{pmatrix}$$

$$= \begin{pmatrix} b_l & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix} \prod_{t=2}^{k-1} \begin{pmatrix} a_t & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_k & -1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & -1 \\ 1 & 0 \end{pmatrix} \prod_{t=2}^{l-1} \begin{pmatrix} b_t & -1 \\ 1 & 0 \end{pmatrix}$$
$$= (-\lambda') \begin{pmatrix} b_l & -1 \\ 1 & 0 \end{pmatrix} \prod_{t=1}^{l-1} \begin{pmatrix} b_t & -1 \\ 1 & 0 \end{pmatrix}$$

Therefore, $(b_l, b_1, ..., b_{l-1})$ is a λ''-quiddity cycle. With Proposition 3.4 we know that b is also a λ''-quiddity cycle. □

Proposition 3.17. (Existence of reducibility) [10] *Let R be a commutative ring, $c = (c_1, c_2, ..., c_m) \in R^m$ be a λ-quiddity cycle, $F = (f_{i,j})_{i,j \in \mathbb{Z}}$, be the Cuntz frieze produced by c. Then, c is reducible over R if and only if $\exists i, j \in \mathbb{Z}, i \le j \le i + n - 4, f_{i,j} = 1$ or $f_{i,j} = -1$.*

Proposition 3.18. (Commutativity and associativity) *Let R be a subset of a commutative ring, $a = (a_1, a_2, ..., a_k) \in R^k$ be a λ-quiddity cycle, $b = (b_1, b_2, ..., b_l) \in R^l$ be a λ'-quiddity cycle $c = (c_1, c_2, ..., c_m) \in R^m$ be a λ''-quiddity cycle, $k, l, m > 2$.*

(1) $\exists \sigma \in D_{k+l}, a \oplus b = (b \oplus a)^\sigma$

(2) $\exists \sigma \in D_{l+m}, \tau \in D_l, (a \oplus b) \oplus c = a \oplus (b^\tau \oplus c)^\sigma$

(3) *If $k = l$, $\lambda = \lambda'$ and $a \oplus b = b \oplus a$, then $a = b$*

Proof. (1)

$$\begin{aligned} a \oplus b &= (a_1 + b_l, a_2, ..., a_{k-1}, a_k + b_1, b_2, ..., b_{l-1}) \\ &= (a_k + b_1, b_2, ..., b_{l-1}, a_1 + b_l, a_2, ..., a_{k-1})^\sigma \\ &= (b \oplus a)^\sigma \end{aligned}$$

(2)

$$\begin{aligned} (a \oplus b) \oplus c &= (a_1 + b_l, a_2, ..., a_{k-1}, a_k + b_1, b_2, ..., b_{l-1}) \oplus c \\ &= (a_1 + b_l + c_m, a_2, ..., a_{k-1}, a_k + b_1, b_2, ..., b_{l-2}, b_{l-1} + c_1, c_2, ..., c_{m-1}) \\ &= a \oplus (b_1, b_2, ..., b_{l-2}, b_{l-1} + c_1, c_2, ..., c_{m-1}, b_l + c_m) \\ &= a \oplus (b_l + c_m, b_1, b_2, ..., b_{l-2}, b_{l-1} + c_1, c_2, ..., c_{m-1})^\sigma \\ &= a \oplus ((b_l, b_1, b_2, ..., b_{l-1}) \oplus c)^\sigma \\ &= a \oplus (b^\tau \oplus c)^\sigma \end{aligned}$$

(3) $a \oplus b = (a_1 + b_l, a_2, ..., a_{k-1}, a_k + b_1, b_2, ..., b_{l-1})$, $b \oplus a = (b_1 + a_k, b_2, ..., b_{l-1}, b_l + a_1, a_2, ..., a_{k-1})$. With $a \oplus b = b \oplus a$ and $k = l$, we obtain that $a_2 = b_2, a_3 = b_3, ..., a_{k-1} = b_{k-1}$. With $\lambda = \lambda'$ and Proposition 3.9 we have $a_1 = b_1, a_k = b_k$. Therefore, $a = b$.

□

Remark 3.19. (1) Proposition 3.18 (1) just showed that $a \oplus b \sim b \oplus a$. In most cases we can not have $a \oplus b = b \oplus a$. For example, $a = (1, 1, 1)$ and $b = (2, 1, 2, 1)$, then $a \oplus b = (2, 1, 3, 1, 2) \neq (3, 1, 2, 2, 1) = b \oplus a$.

(2) Similarly, Proposition 3.18 (2) does not imply $(a \oplus b) \oplus c = a \oplus (b \oplus c)$. Actually, in most cases it is wrong. For example, $a = (1, 1, 1)$, $b = (2, 1, 2, 1)$ and $c = (1, 1, 1)$, then $(a \oplus b) \oplus c = (2, 1, 3, 1, 2) \oplus (1, 1, 1) = (3, 1, 3, 1, 3, 1) \neq (2, 1, 4, 1, 2, 2) = (1, 1, 1) \oplus (3, 1, 2, 2, 1) = a \oplus (b \oplus c)$.

(3) If $a \oplus b = (a \oplus c)^\sigma$ and $\sigma \in D_{k+l-2} = D_{k+m-2}$, then we may have $b \neq c$. For example,

$$(1, 1, 1) \oplus (0, 6, 0, -6) = (-5, 1, 1, 6, 0)$$

$$(1, 1, 1) \oplus (5, 0, -5, 0) = (1, 1, 6, 0, -5) = (-5, 1, 1, 6, 0)^\sigma$$

We can notice that $(5, 0, -5, 0) \neq (0, 6, 0, -6)$.

Proposition 3.20. *Let R be a subset of a commutative ring. Let $a = (a_1, a_2, ..., a_k) \in R^k$ be a λ-quiddity cycle, $b = (b_1, b_2, ..., b_l) \in R^l$ be a λ'-quiddity cycle and $c = (c_1, c_2, ..., c_m) \in R^m$ be a λ''-quiddity cycle, $k, l, m > 2$. If we define $d := a \oplus (b \oplus c)^\sigma$ as a $(\lambda\lambda'\lambda'')$-quiddity cycle for all $\sigma \in D_{l+m}$, then one of the following properties will be satisfied:*

(i) There exists $c' \in R^m$ and $\pi_1 \in D_{k+m}, \pi_2 \in D_{k+l+m}$, such that $c \sim c'$ and $d = ((a \oplus c')^{\pi_1} \oplus b)^{\pi_2}$.

(ii) There exists $b' \in R^l$ and $\pi_1 \in D_{k+l}, \pi_2 \in D_{k+l+m}$, such that $b \sim b'$ and $d = ((a \oplus b')^{\pi_1} \oplus c)^{\pi_2}$.

Proof. We consider $(b \oplus c)^\sigma$ in the following 4 cases.

Case 1: $(b \oplus c)^\sigma = (b \oplus c) = (b_1 + c_m, b_2, ..., b_{l-1}, b_l + c_1, c_2, ..., c_{m-1})$, then we have:

$$\begin{aligned}
a \oplus (b \oplus c)^{\sigma} &= a \oplus (b \oplus c) \\
&\overset{\text{Prop3.18(1)}}{=} ((b \oplus c) \oplus a)^{\sigma_1} \\
&\overset{\text{Prop3.18(2)}}{=} (b \oplus (c^{\sigma_2} \oplus a)^{\sigma_3})^{\sigma_1} \\
&\overset{\text{Prop3.18(1)}}{=} ((c^{\sigma_2} \oplus a)^{\sigma_3} \oplus b)^{\sigma_1\sigma_4} \\
&\overset{\text{Prop3.18(1)}}{=} ((a \oplus c^{\sigma_2})^{\sigma_3\sigma_5} \oplus b)^{\sigma_1\sigma_4} \\
&= ((a \oplus c')^{\pi_1} \oplus b)^{\pi_2}
\end{aligned}$$

where $\sigma_1, \sigma_4 \in D_{k+l+m}, \sigma_2 \in D_m, \sigma_3, \sigma_5 \in D_{k+m}$, and $\pi_1 = \sigma_3\sigma_5, \pi_2 = \sigma_1\sigma_4, c' = c^{\sigma_2}$. This case fulfils (i).

Case 2: $(b \oplus c)^{\sigma} = (b_i, b_{i+1}..., b_{l-1}, b_l + c_1, c_2, ..., c_{m-1}, b_1 + c_m, b_2, ..., b_{i-1})$ for all $i \in \{2, ..., l-1\}$, then we have:

$$\begin{aligned}
& a \oplus (b \oplus c)^{\sigma} \\
&= (a_1, ..., a_k) \oplus (b_i, b_{i+1}..., b_{l-1}, b_l + c_1, c_2, ..., c_{m-1}, b_1 + c_m, b_2, ..., b_{i-1}) \\
&= (a_1 + b_{i-1}, a_2, ..., a_{k-1}, a_k + b_i, b_{i+1}..., b_{l-1}, b_l + c_1, c_2, ..., c_{m-1}, b_1 + c_m, b_2, ..., \\
&\quad b_{i-2}) \\
&= (b_l + c_1, c_2, ..., c_{m-1}, b_1 + c_m, b_2, ..., b_{i-2}, a_1 + b_{i-1}, a_2, ..., a_{k-1}, a_k + b_i, b_{i+1}..., \\
&\quad b_{l-1})^{\sigma_1} \\
&= (c \oplus (b_1, b_2, ..., b_{i-2}, a_1 + b_{i-1}, a_2, ..., a_{k-1}, a_k + b_i, b_{i+1}..., b_{l-1}, b_l))^{\sigma_1} \\
&= (c \oplus (a \oplus b^{\sigma_2})^{\sigma_3})^{\sigma_1} \\
&\overset{\text{Prop3.18(1)}}{=} ((a \oplus b^{\sigma_2})^{\sigma_3} \oplus c)^{\sigma_1\sigma_4} \\
&= ((a \oplus b')^{\pi_1} \oplus c)^{\pi_2}
\end{aligned}$$

where $\sigma_1, \sigma_4 \in D_{k+l+m}, \sigma_2 \in D_l, \sigma_3 \in D_{k+l}$, and $\pi_1 = \sigma_3, \pi_2 = \sigma_1\sigma_4, b' = b^{\sigma_2}$. This case fulfils (ii).

Case 3: $(b \oplus c)^{\sigma} = (b_l + c_1, c_2, ..., c_{m-1}, b_1 + c_m, b_2, ..., b_{l-1}) = c \oplus b$, then similar to Case 1, we have:

$$\begin{aligned}
a \oplus (b \oplus c)^{\sigma} &= a \oplus (c \oplus b) \\
&\overset{\text{Prop3.18(1)}}{=} ((c \oplus b) \oplus a)^{\sigma_1} \\
&\overset{\text{Prop3.18(2)}}{=} (c \oplus (b^{\sigma_2} \oplus a)^{\sigma_3})^{\sigma_1} \\
&\overset{\text{Prop3.18(1)}}{=} ((b^{\sigma_2} \oplus a)^{\sigma_3} \oplus c)^{\sigma_1\sigma_4} \\
&\overset{\text{Prop3.18(1)}}{=} ((a \oplus b^{\sigma_2})^{\sigma_3\sigma_5} \oplus c)^{\sigma_1\sigma_4} \\
&= ((a \oplus b')^{\pi_1} \oplus c)^{\pi_2}
\end{aligned}$$

where $\sigma_1, \sigma_4 \in D_{k+l+m}, \sigma_2 \in D_l, \sigma_3\sigma_5 \in D_{k+l}$, and $\pi_1 = \sigma_3\sigma_5, \pi_2 = \sigma_1\sigma_4, b' = b^{\sigma_2}$. This case fulfils (ii).

Case 4: $(b \oplus c)^{\sigma} = (c_j, c_{j+1}, ..., c_{m-1}, b_1 + c_m, b_2, ..., b_{l-1}, b_l + c_1, c_2, ..., c_{j-1})$ for all $j \in \{2, ..., m-1\}$, then we have:

$$\begin{aligned}
& a \oplus (b \oplus c)^{\sigma} \\
&= (a_1, ..., a_k) \oplus (c_j, c_{j+1}, ..., c_{m-1}, b_1 + c_m, b_2, ..., b_{l-1}, b_l + c_1, c_2, ..., c_{j-1}) \\
&= (a_1 + c_{j-1}, a_2, ..., a_{k-1}, a_k + c_j, c_{j+1} ..., c_{m-1}, c_m + b_1, b_2, ..., b_{l-1}, b_l + c_1, c_2, ..., \\
& c_{j-2}) \\
&= (c_m + b_1, b_2, ..., b_{l-1}, c_1 + b_l, c_2, ..., c_{j-2}, a_1 + c_{j-1}, a_2, ..., a_{k-1}, a_k + c_j, c_{j+1} ..., \\
& c_{m-1})^{\sigma_1} \\
&= (b \oplus (c_1, c_2, ..., c_{j-2}, a_1 + c_{j-1}, a_2, ..., a_{k-1}, a_k + c_j, c_{j+1} ..., c_{m-1}, c_m))^{\sigma_1} \\
&= (b \oplus (a \oplus c^{\sigma_2})^{\sigma_3})^{\sigma_1} \\
&\overset{\text{Prop3.18(1)}}{=} ((a \oplus c^{\sigma_2})^{\sigma_3} \oplus b)^{\sigma_1\sigma_4} \\
&= ((a \oplus c')^{\pi_1} \oplus b)^{\pi_2}
\end{aligned}$$

where $\sigma_1, \sigma_4 \in D_{k+l+m}, \sigma_2 \in D_m, \sigma_3 \in D_{k+m}$, and $\pi_1 = \sigma_3, \pi_2 = \sigma_1\sigma_4, b' = b^{\sigma_2}$. This case fulfils (i).

□

3.3 Coxeter Frieze and Cuntz Frieze

The following proposition has been given by Cuntz and Holm in [11], but we give an alternative proof.

Proposition 3.21. (Equivalence between Coxeter frieze and Cuntz frieze) *Each tame Coxeter frieze is a Cuntz frieze, and vice versa.*

Proof. " $\Rightarrow$ " Let $E = (e_{i,j})_{i,j\in\mathbb{Z}}$ be a tame Coxeter frieze with width m. With Proposition 2.3 for any fixed $i \in \mathbb{Z}$ we obtain:

$$\prod_{k=i}^{i+m+1} \begin{pmatrix} e_{k,k} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e_{i,i+m+1} & -e_{i,i+m} \\ e_{i+1,i+m+1} & -e_{i+1,i+m} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -e_{i+1,i+m} \end{pmatrix}$$

Since $e_{i+1,i+m} \overset{\text{Prop1.5}}{=} e_{i+1,i+m+1}e_{i+m+2,i+m+2} - e_{i+1,i+m+2} = e_{i+m+2,i+m+2}$, we have:

$$\prod_{k=i}^{i+m+2} \begin{pmatrix} e_{k,k} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -e_{i+1,i+m} \end{pmatrix} \begin{pmatrix} e_{i+1,i+m} & -1 \\ 1 & 0 \end{pmatrix} = -\,\mathrm{id}$$

That means, E is a Cuntz frieze produced by $(e_{i,i}, ..., e_{i+m+2,i+m+2})$.

" $\Leftarrow$ " Let $F = (f_{i,j})_{i,j\in\mathbb{Z}}$ be a Cuntz frieze produced by a quiddity cycle c. With Proposition 3.6 we know that for $i \le j \le i+n-2$:

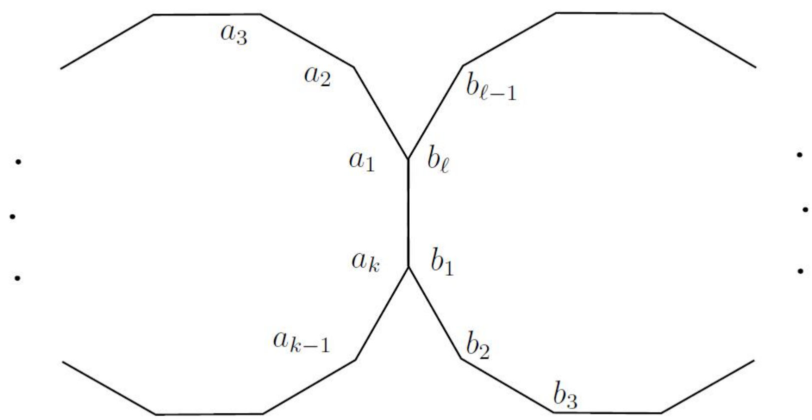
$$1 = \left| \prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \right| = \begin{vmatrix} f_{i,j} & -f_{i,j-1} \\ f_{i+1,j} & -f_{i+1,j-1} \end{vmatrix} = f_{i,j-1}f_{i+1,j} - f_{i,j}f_{i+1,j-1}$$

That means, the determinant of each 2×2 adjacent entries equals 1. F is a Coxeter frieze. With Proposition 3.5 we know $f_{i,j}, f_{i,j-1}$ and $f_{i,j-2}$ are linear dependent, so the determinant of each 3×3 adjacent entries equals 0. F is tame. $\square$

3.4 Combinatorial Models for integer-valued Frieze

In section 2 we have already introduced the triangulation for positive-integer-valued Coxeter Frieze. If we just consider Coxeter friezes in the positive-integer-valued case, we have the process to convert one quiddity cycle to one triangulation and verse vice. For the integer-valued case, we also have a similar system for transformation between quiddity cycles and triangulations. The details can be consulted in [11]. Here we don't give a deep explanation about it.

Proposition 3.22. [10] *Let R be a subset of a commutative ring, $a = (a_1, ..., a_k) \in R^k$ be a λ-quiddity cycle and $b = (b_1, ..., b_l) \in R^l$ be a λ'-quiddity cycle. If we denote a and b by using polygons (not necessary by triangulation), then $a \oplus b$ can be represent as following:*



$$a \oplus b = (a_1 + b_l, a_2, ..., a_{k-1}, a_k + b_1, b_2, ..., b_{l-1})$$

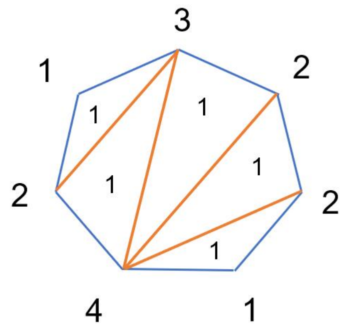
Definition 3.23. (Triangulation for integer-numbered Frieze)[11] For $m \in \mathbb{N}_{\geq 2}$, let T be a triangulation of a regular m-polygon. A *labelling* of T is an assignment of integers a_t, called labels, to the triangles t of T. Let d be the sum of the number of negaive labels and half the number of labels 0. We call $(-1)^d$ the *sign* of he labelling if d is an integer. A labelling is called admissible if the following conditions are satisfied:

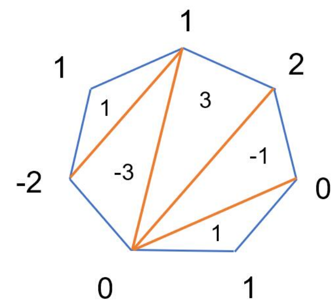
(1) The set of triangles t with $a_t \in \{1, -1\}$ can be written as a disjoint union of two-element subsets $\{t_1, t_2\}$ (called squares) such that t_1, t_2 have a common edge (i.e. are neighbouring triangles) and $a_{t_1} = -a_{t_2}$.

(2) The sign is 1, i.e. the sum of the number of negative labels and half the number of labels 0 is even.

Example 3.24. (1) The following left figure represents the triangulation for the frieze in Example 1.13 (3).

(2) The following right figure shows the triangulation of quiddity cycle $(1, 1, 2, 0, 1, 0, -2)$.



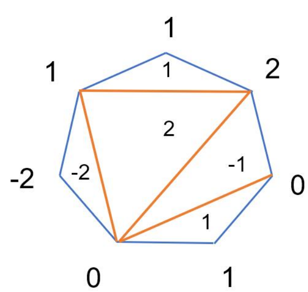


Theorem 3.25. [11]

(*a*) *Let T be a triangulation of a regular m-gon with vertices denoted (in counterclockwise order) $1, 2, ..., m$, and assume that we have an admissible labelling of T. For each vertex i let c_i be the sum of the labels of the triangles attached at the vertex i. Then $(c_1, ..., c_m)$ is a quiddity cycle over $\mathbb{Z}$.*

(*b*) *Every quiddity cycle over $\mathbb{Z}$ can be obtained as in* (*a*) *from an admissible labelling.*

Remark 3.26. [11] With 3.25 (b) we obtain that the mapping from triangulation to quiddity over $\mathbb{Z}$ is surjective. But it is not injective. For example, the following triangulation maps to the same quiddity as Example 3.24 (1).



4 Factorization of Reducibility

4.1 Factorization

This section is the main part of my bachelor thesis to present my own results. With section 3 we know that the "$\oplus$" operator fulfils neither commutativity nor associativity. So, it is difficult to say, which form is the decomposition of a $\{\pm 1\}$-quiddity cycle. In this section we define a rule of decomposition (Section 4.1), if we ignore the importance of this rule, we can find out that the decomposition is not unique. Furthermore, we introduce one more interesting property, which is called "Simple" (Section 4.2).

Definition 4.1. (Factorization) Let R be a subset of a commutative ring, $c = (c_1, c_2, ..., c_m) \in R^m$ be a reducible λ-quiddity cycle with $m > 2$. We denote by l_c the length of c. If there exits $n \in \mathbb{N}_+, n > 1$, $a_1, ..., a_n$ and $\sigma_1, ..., \sigma_{n-1}$ such that for all $i \in \{1, ..., n\}$ we have:

1. $a_i \in R^{l_{a_i}}$ is a $\lambda^{(i)}$-quiddity cycle, where $\lambda^{(i)} \in \{\pm 1\}$.
2. $l_{a_i} > 2$.
3. a_i is irreducible.
4. $\sigma_i \in D_{\sum_{k=1}^{i+1} l_{a_k}}$.
5. $c = (((((a_1 \oplus a_2)^{\sigma_1}) \oplus a_3)^{\sigma_2} \oplus ...)^{\sigma_{n-2}} \oplus a_n)^{\sigma_{n-1}} \qquad (**)$

Then we call $(**)$ the *factorization* of c. And $a_1, ..., a_n$ are called *factors* of this factorization.

Proposition 4.2. *Let R be a subset of a commutative ring, $c = (c_1, c_2, ..., c_m) \in R^m$ be a λ-quiddity cycle, and $a_1, ..., a_n$ are $\{\pm 1\}$-quiddity cycles. Then $a_1, ..., a_n$ are factors of c if and only if there exists $\pi_1, ..., \pi_{n-1}$ with $\pi_i \in D_{\sum_{k=i}^{n} l_{a_k}}, \forall i \in \{1, ..., n-1\}$ and $c = (a_n \oplus (a_{n-1} \oplus (... \oplus (a_2 \oplus a_1)^{\pi_1} ...)^{\pi_{n-3}})^{\pi_{n-2}})^{\pi_{n-1}}$.*

Proof. Notice that $a_1, ..., a_n$ are factors of c if and only if there exists $\sigma_1, ..., \sigma_{n-1}$ with $\sigma_i \in D_{\sum_{k=1}^{i+1} l_{a_k}}, \forall i \in \{1, ..., n-1\}$ and $c = (((((a_1 \oplus a_2)^{\sigma_1}) \oplus a_3)^{\sigma_2} \oplus ...)^{\sigma_{n-2}} \oplus a_n)^{\sigma_{n-1}}$. So we show this proposition by using induction over n.

Induction base: $n = 2$, with Proposition 3.18(1) we have $\exists \vartheta_1 \in D_{l_{a_1}+l_{a_2}}, \pi_1 = \vartheta_1 \sigma_1$, such that $(a_1 \oplus a_2)^{\sigma_1} = ((a_2 \oplus a_1)^{\vartheta_1})^{\sigma_1} = (a_2 \oplus a_1)^{\pi_1}$.

Induction hypothesis: for n we have:

$$d := (((((a_1 \oplus a_2)^{\sigma_1}) \oplus a_3)^{\sigma_2} \oplus ...)^{\sigma_{n-2}} \oplus a_n)^{\sigma_{n-1}} = (a_n \oplus (a_{n-1} \oplus (... \oplus (a_2 \oplus a_1)^{\pi_1} ...)^{\pi_{n-3}})^{\pi_{n-2}})^{\pi_{n-1}}$$

Induction step: for $n+1$ we have $\exists \vartheta_{n+1} \in D_{l_d + l_{a_2}}, \pi_n = \vartheta_n \sigma_n$ such that:

$$
\begin{aligned}
&(((((a_1 \oplus a_2)^{\sigma_1}) \oplus a_3)^{\sigma_2} \oplus ...)^{\sigma_{n-2}} \oplus a_n)^{\sigma_{n-1}} \oplus a_{n+1})^{\sigma_n} \\
&= (d \oplus a_{n+1})^{\sigma_n} \\
&= ((a_{n+1} \oplus d)^{\vartheta_n})^{\sigma_n} \\
&= (a_{n+1} \oplus d)^{\pi_n} \\
&= (a_{n+1} \oplus (a_n \oplus (a_{n-1} \oplus (... \oplus (a_2 \oplus a_1)^{\pi_1} ...)^{\pi_{n-3}})^{\pi_{n-2}})^{\pi_{n-1}})^{\pi_n}
\end{aligned}
$$

□

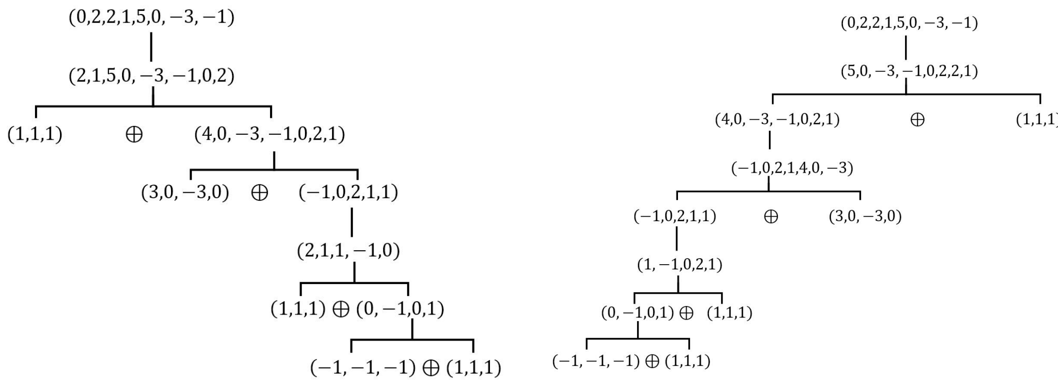
Corollary 4.3. (1) *Let R be a subset of a commutative ring, $c = (c_1, ..., c_m) \in R^m$ be a λ-quiddity cycle. If $a = (a_1, ..., a_k) \in R^k$ is a irreducible λ'-quiddity cycle, $b = (b_1, ..., b_l) \in R^l$ is a $(-\lambda\lambda')$-quiddity cycle and $\sigma \in D_m$, such that $c = (a \oplus b)^\sigma$ or $c = (b \oplus a)^\sigma$. Then with Proposition 4.2 a is always a factor of c.*

(2) *Let R be a subset of commutative ring, $e = (e_1, ..., e_m) \in R^m$ be a reducible λ-quiddity cycle. If there exists $a \in R^p, b \in R^q, c \in R^r, d \in R^s$ four $\{\pm 1\}$-quiddity cycles, such that $e = (a \oplus b) \oplus (c \oplus d)$, then c may not be a factor of e.*

Example 4.4. (1) $(0, 2, 2, 1, 5, 0, -3, -1) \in R^8$ is a quiddity cycle. Notice that

$$
\begin{aligned}
&(0, 2, 2, 1, 5, 0, -3, -1) \\
&= (1, 1, 1) \oplus ((3, 0, -3, 0) \oplus ((1, 1, 1) \oplus ((-1, -1, -1) \oplus (1, 1, 1))^{\sigma_1}) \\
&= ((((-1, -1, -1) \oplus (1, 1, 1)) \oplus (1, 1, 1))^{\sigma_2} \oplus (3, 0, -3, 0))^{\sigma_3} \oplus (1, 1, 1)
\end{aligned}
$$

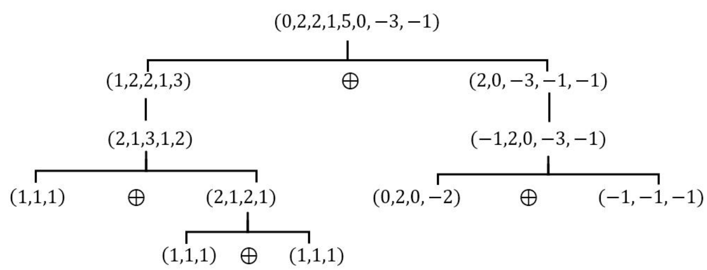
with $\sigma_1, \sigma_2 \in D_5, \sigma_3 \in D_7$, we know that $(1, 1, 1), (3, 0, -3, 0), (-1, -1, -1)$ are factors of the quiddity cycle $(0, 2, 2, 1, 5, 0, -3, -1)$ (No matter whether the form of decomposition in Definition 4.1 or in Proposition 4.2 is).



(2) If we decompose (0,2,2,1,5,0,-3,-1) in other forms, then we may have:

$$(0,2,2,1,5,0,-3,-1)$$
$$= (((1,1,1) \oplus ((1,1,1) \oplus (1,1,1)))^{\sigma_1} \oplus ((0,2,0,-2) \oplus (-1,-1,-1))^{\sigma_2})$$

with $\sigma_1, \sigma_2 \in D_5$, but $(0,2,0,-2)$ is not a factor of $(0,2,2,1,5,0,-3,-1)$.



Proposition 4.5. *Let $R = \mathbb{C}$ and $c = (c_1, ..., c_m) \in R^m$ be a $\{\pm 1\}$-quiddity cycle with $m > 2$, then $\exists n \in \mathbb{N}$, $\exists a_1, ..., a_n$ irreducible $\{\pm 1\}$-quiddity cycles over R and $\exists \sigma_1 \in D_{\sum_{k=1}^{2} l_{a_k}}, ..., \sigma_{n-1} \in D_{\sum_{k=1}^{n} l_{a_k}}$, such that $c = (((((a_1 \oplus a_2)^{\sigma_1}) \oplus a_3)^{\sigma_2} \oplus ...)^{\sigma_{n-2}} \oplus a_n)^{\sigma_{n-1}}$.*

Proof. This proposition will be shown by using induction over m.

Induction basis: $m = 3$, assume $c = (c_1, c_2, c_3)$, such that c is a $\{\pm 1\}$-quiddity cycle. With example 3.3 and Example 3.13 we know that $c = (1,1,1)$ or $c = (-1,-1,-1)$, therefore c is always irreducible.

Induction hypothesis: for m, $\exists n \in \mathbb{N}$, $\exists a_1, ..., a_n$ are irreducible $\{\pm 1\}$-quiddity cycles over R and $\exists \sigma_1 \in D_{\sum_{k=1}^{2} l_{a_k}}, ..., \sigma_{n-1} \in D_{\sum_{k=1}^{n} l_{a_k}}$, such that $c = (((((a_1 \oplus a_2)^{\sigma_1}) \oplus a_3)^{\sigma_2} \oplus ...)^{\sigma_{n-2}} \oplus a_n)^{\sigma_{n-1}}$.

Induction step: for $m + 1$, we consider the following cases.

Case 1: If c is irreducible, then this situation is trivial.

Case 2: If c is reducible. That means, $\exists d, e$ two $\{\pm 1\}$ quiddity cycles over R, and $c = (d \oplus e)^\tau$ with $l_d > 2, l_e > 2, \tau \in D_{t+1}$. With Remark 3.15 $l_d + l_e - 2 = m + 1$, therefore $l_d = m + 1 + 2 - l_e < m + 1$, which implies $l_d \leq m$ and similarly $l_e \leq m$. With induction hypothesis, we know that, $\exists n_1, n_2 \in \mathbb{N}$, $\exists a_1, ..., a_{n_1}, b_1, ..., b_{n_2}$ are irreducible $\{\pm 1\}$-quiddity cycles over R and $\exists \sigma_1 \in D_{\sum_{k=1}^{2} l_{a_k}}, ..., \sigma_{n-1} \in D_{\sum_{k=1}^{n_1} l_{a_k}}, \pi_1 \in D_{\sum_{k=1}^{2} l_{b_k}}, ..., \pi_{n_2 - 1} \in D_{\sum_{k=1}^{n_2} l_{b_k}}$, such that

$$d = (((((a_1 \oplus a_2)^{\sigma_1}) \oplus a_3)^{\sigma_2} \oplus ...)^{\sigma_{n_1-2}} \oplus a_{n_1})^{\sigma_{n_1-1}}$$

$$e = (((((b_1 \oplus b_2)^{\pi_1}) \oplus b_3)^{\pi_2} \oplus ...)^{\pi_{n_2-2}} \oplus b_{n_2})^{\pi_{n_2-1}}$$

If we define $e_1 = (((((b_1 \oplus b_2)^{\pi_1}) \oplus b_3)^{\pi_2} \oplus ...)^{\pi_{n_2-3}} \oplus b_{n_2-1})^{\pi_{n_2-2}}$, then $e = (e_1 \oplus b_{n_2})^{\pi_{n_2-1}}$ and therefore $c = (d \oplus e)^\tau = (d \oplus (e_1 \oplus b_{n_2})^{\pi_{n_2-1}})^\tau$. With Proposition 3.20 we obtain one of the following cases:

(i) There exists $b'_{n_2} \in R^{l_{b_{n_2}}}, \tau_1 \in D_{l_d + l_{b_{n_2}}}, \tau_2 \in D_{t+1}$, such that $b'_{n_2} \sim b_{n_2}$ and $c = ((d \oplus b'_{n_2})^{\tau_1} \oplus e_1)^{\tau_2 \tau}$

(ii) There exists $e'_1 \in R^{\sum_{k=1}^{k=n_2-1} l_{b_k}}, \tau_1 \in D_{l_d + \sum_{k=1}^{k=n_2-1} l_{b_k}}, \tau_2 \in D_{t+1}$, such that $e'_1 \sim e_1$ and $c = ((d \oplus e'_1)^{\tau_1} \oplus b_{n_2})^{\tau_2 \tau}$

Notice that no matter in which case, the irreducible $\{\pm 1\}$-quiddity cycle b_{n_2} has been moved out of e. Since we need to move $b_2, ..., b_{n_2}$ out of e. So, we need to repeat Proposition 3.20 totally $n_2 - 1$ times, then this proposition will hold.

□

Remark 4.6. Proposition 4.5 is equivalent to that each λ-quiddity over $\mathbb{C}$ has a factorization. But this doesn't means, this factorization is unique.
Actually, with Remark 3.19(3) we know that if R is a subset of a commutative ring, $c \in R^n$ is a quiddity cycle and a,b are two (irreducible) $\{\pm 1\}$-quiddity cycles, such that $c = (a \oplus b)^\sigma$, then b is not unique. That implies, no matter in which form, if $c \in R^n$ is reducible, then its decomposition can always be not unique.

Proposition 4.7. (Positive-integer frieze and factorization) *Let $R = \mathbb{N}_+$, $c = (c_1, c_2, ..., c_n) \in R^n$ be a quiddity cycle with $n > 2$, If c produces a positive-integer-valued Cuntz frieze (expect the bound 0's at top and bottom), then one of the following cases will be satisfied:*

(i) $c = (1,1,1)$

(ii) $\exists d \in R^{n-1}$ *is a quiddity cycle and* $\exists \sigma \in D_n$*, such that* $c = (d \oplus (1,1,1))^{\sigma}$ *and* d *produces a positive-integer-valued Cuntz frieze.*

Proof. Since the by c produced Cuntz Frieze F consists of positive integers, c consists of positive integers as well. If $n = 3$, then $c = (1,1,1)$ is irreducible, which fulfils (i). If $n = 4$, then with Example 3.3 we have $c = (2,1,2,1) = (1,1,1) \oplus (1,1,1)$ or $c = (1,2,1,2) = ((1,1,1) \oplus (1,1,1))^{\pi_1}$ for a $\pi_1 \in D_4$.
For $n \geq 5$, with Proposition 3.11 and Proposition 1.10 $\exists t \in \{1,...,n\}, c_t = 1$ and $c_{t+1} \neq 1, c_{t-1} \neq 1$. Then $\exists \pi_{n-1} \in D_n$ such that:

$$c = ((c_{t+1} - 1, c_{t+2}, ..., c_n, c_1, ..., c_{t-2}, c_{t-1} - 1) \oplus (1,1,1))^{\pi_{n-1}}$$

We denote by $d = (c_{t+1} - 1, c_{t+2}, ..., c_n, c_1, ..., c_{t-2}, c_{t-1} - 1)$ a finite sequence of length $n-1$. With Proposition 3.16 we obtain that d is a quiddity cycle as well, and clearly all the entries in d are positive integer.
Furthermore, let $\widetilde{F}$ be the Cuntz Frieze produced by d. Then with Proposition 3.5 we obtain $\widetilde{F}$ satisfies linear recurrence:

$$\widetilde{f}_{i,j} = \widetilde{f}_{i,j-1}\widetilde{f}_{j,j} - \widetilde{f}_{i,j-2}, \forall i,j \in \mathbb{Z}, i-2 \leq j \leq i+n-2$$

Since all the entries in d are positive integers, we have $\widetilde{f}_{i,i} > 0$, for all $i \in \mathbb{Z}$. Since F is a positive-integer-valued Cuntz frieze, with Remark 3.7 we get:

$$f_{i,j} = \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} > 0, \forall i,j \in \mathbb{Z}, i \leq j \leq i+n-4$$

In particular, for the fixed $t+2$ we have:

$$\widetilde{f}_{t+2,j} = \begin{pmatrix} 1 & 0 \end{pmatrix} \prod_{k=t+2}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} > 0, \forall j \in \mathbb{Z}, t+2 \leq j \leq t+n-2$$

where $c_{i+n} = c_i$ for all $i \in \mathbb{Z}$. With Proposition 1.9, $\widetilde{F}$ consists of positive integers. □

Remark 4.8. (1) All λ-quiddity cycles having the form $(1,...,1)$ are irreducible over $R = \mathbb{N}_+$.

Proof. Suppose $c = (c_1,...,c_m) = (1,...,1) \in R^m$ is reducible as $c = (a \oplus b)^{\sigma}$ with $a \in R^k, b \in R^l, \sigma \in D_m$. Then we obtain $m = k + l - 2$ and $m = \sum_{t=1}^{m} c_t = \sum_{t=1}^{k} a_t + \sum_{t=1}^{l} b_t \geq k + l = m + 2$, which is a contradiction. □

(2) The reducibility of a λ-quiddity cycle is dependent on the commutative ring R. For example, $(1,1,1,1,1,1,1,1,1)$ is irreducible over $R = \mathbb{N}_+$ since (1).
But $(1,1,1,1,1,1,1,1,1) = (1,1,1) \oplus (0,1,1,1,1,1,1,0)$ is reducible over $R = \mathbb{N}$.

Theorem 4.9. *Let $R = \mathbb{N}_+$, $c = (c_1, ..., c_n) \in R^n$ be a quiddity cycle of a positive-integer-valued Cuntz frieze. Then c has a factorization such that $c = ((((1,1,1) \oplus (1,1,1))^{\sigma_{n-3}} ...)^{\sigma_2} \oplus (1,1,1))^{\sigma_1}$, where $\sigma_1 \in D_n, ..., \sigma_{n-3} \in D_4$.*

Proof. If $m = 3$, then $c = (1,1,1)$ trivial. If $m > 3$, then since $c = (c_1, ..., c_n) \in R^n$ is a quiddity cycle of a positive-integer-valued Cuntz frieze, with Proposition 4.7 we obtain that $\exists d_1 \in R^{n-1}$ quiddity cycle and $\exists \sigma \in D_n$, such that $c = (d_1 \oplus (1,1,1))^{\sigma}$. Recursively using Proposition 4.7 we obtain $c = (((d_{n-3} \oplus (1,1,1))^{\sigma_{n-3}} ...)^{\sigma_2} \oplus (1,1,1))^{\sigma_1}$, where $d_{n-3} \in R^3, \sigma_i \in D_{n-i+1}, \forall i \in \{1, 2, ..., n-3\}$. Since $d_{n-3} \in R^3$ is a quiddity cycle, we have $d_{n-3} = (1,1,1)$.

□

4.2 Simple Frieze Patterns

Definition 4.10. (Simple quiddity cycle) Let R be a subset of a commutative ring, $c = (c_1, c_2, ..., c_n) \in R^n$ be a λ-quiddity cycle with $n > 2$. We denote $c_{tn+i} = c_i$ for $\forall t \in \mathbb{Z}$ and $i \in \{1, ..., n\}$. If $(c_i, c_{i+1}, ..., c_j)$ is not a $\{\pm 1\}$-quiddity cycle for all $i, j \in \mathbb{Z}$ with $|i - j| < n - 1$, then we call c simple.

Example 4.11. (1) $(1,1,1)$ and $(-1,-1,-1)$ are simple over $\mathbb{Z}$.

(2) The quiddity cycle $(1,1,1,1,1,1)$ is not simple over $\mathbb{Z}$, since $(1,1,1)$ is also a quiddity cycle.

Proposition 4.12. *Let R be a subset of a commutative ring and $c = \{c_1, ..., c_n\} \in R^n$ be a quiddity cycle.*

(1) *If c is not simple and we denote $c_{t+n} = c_t$ for all $t \in \mathbb{Z}$, then $\exists i, j \in \{1, ..., 2n\}, i < j$, such that $c' = (c_i, ..., c_j)$ is a quiddity cycle.*

(2) *If $0 \in R$ and c is irreducible, then c is simple.*

(3) *If $0 \notin R$ and c is irreducible, then c may be not simple.*

Proof. (1) If c is not simple, then $\exists i, j \in \{1, ..., n\}$ such that $a = (c_i, ..., c_j)$ is a λ-quiddity cycle and $b = (c_{j+1}, ..., c_n, c_1, ..., c_{i-1}) = (c_{j+1}, ..., c_n, c_{n+1}, ..., c_{n+i-1})$ is a λ'-quiddity cycle. With

$$\begin{aligned}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} &= \prod_{k=1}^{n} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} = \prod_{k=i}^{j} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \prod_{k=j+1}^{n+i-1} \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda' & 0 \\ 0 & \lambda' \end{pmatrix} = \begin{pmatrix} \lambda\lambda' & 0 \\ 0 & \lambda\lambda' \end{pmatrix}\end{aligned}$$

we obtain $\lambda\lambda' = -1 \Rightarrow \lambda = -1$ or $\lambda' = -1$.

(2) Suppose c not simple. That means, with (1) $\exists i, j \in \{1, ..., n, n+1, ..., 2n\}, i \leq j$, such that $c' = (c_i, ..., c_j)$ is a quiddity cycle, which is a contradiction to c irreducible.

(3) For example, $R = \mathbb{N}_+$, $c = (1, 1, 1, 1, 1, 1, 1, 1, 1)$ is irreducible but not simple.

□

5 Open Questions

1. Can we show that, if c is a simple λ-quiddity cycle over $\mathbb{N}_+$, then $\lambda = -1$. In other words, there is no 1-quiddity cycle over $\mathbb{N}_+$.

2. Can we show that, if $c = (c_1, ..., c_n)$ is a simple quiddity cycle over $\mathbb{C}$, then for $\forall i \in \{1, ..., n\}, \exists j \in \{1, ..., n\}$, such that $c_j = \overline{c_i}$

3. Can we show that, if R is a subset of a commutative ring and c is a simple quiddity cycle over R, then the Cuntz Frieze pattern $(f_{i,j})_{i,j\in\mathbb{Z}}$, which is produced by c, satisfies that $f_{i,j} \in R, \forall i, j \in \mathbb{Z}, i \leq j \leq i + n - 4$.

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