



Exercises for the Lecture Functional Analysis I
Winter term 2026/2027

Exercise Sheet 1

Submission: Wednesday, 28 October 2026, 12:30 pm
in the post box of Benjamin Klasen in E2 5

Exercise 1 (10 points). Let $f: X \rightarrow Y$ be a mapping between topological spaces. Show that f is continuous (in the sense of Definition 1.3 of the lecture) if and only if, for every net $(x_\lambda)_{\lambda \in \Lambda}$ in X with $x_\lambda \rightarrow x$ we have $f(x_\lambda) \rightarrow f(x)$. This proves Remark 1.9 (d) of the lecture.

Product topology. If X and Y are topological spaces, then the *product topology* on $X \times Y$ is generated by the family of subsets $\{U \times V \mid U \subseteq X, V \subseteq Y \text{ both open}\}$. Hence open sets in $X \times Y$ are arbitrary unions of such sets $U \times V$. Accordingly, $N \subseteq X \times Y$ is a neighbourhood of $(x, y) \in X \times Y$ if and only if there are open sets $U \subseteq X, V \subseteq Y$ with $x \in U, y \in V$ and $U \times V \subseteq N$. So a net $(x_\lambda, y_\lambda)_{\lambda \in \Lambda}$ in $X \times Y$ converges to a point (x, y) if and only if $x_\lambda \rightarrow x$ and $y_\lambda \rightarrow y$. It follows from Exercise 1 that $f: X \times Y \rightarrow Z$ is continuous if and only if for all nets $x_\lambda \rightarrow x$ and $y_\lambda \rightarrow y$ we have $f(x_\lambda, y_\lambda) \rightarrow f(x, y)$.

Exercise 2 (10 points). Let X be a topological space. Show that the following are equivalent.

- (a) Every two points in X have disjoint neighbourhoods.
- (b) Every point in X is the intersection of its closed neighbourhoods.
- (c) The diagonal $\{(x, x) \mid x \in X\} \subseteq X \times X$ is closed.
- (d) No net in X converges to two different points.

If one (and thus all) of the above statements are fulfilled, the space X is called a *Hausdorff space*. (Remark: All metric spaces are Hausdorff spaces.)

Exercise 3 (10 points). We extend the usual topology of $\mathbb{R}^2$ to a topology on $\mathcal{R}^2 := \mathbb{R}^2 \cup \{\infty\}$ by defining the neighbourhoods of ∞ to be the sets

$$U_\alpha := \{(x, y) \in \mathbb{R}^2 \mid x > c_\alpha, y > f_\alpha(x)\} \cup \{\infty\}$$

where

$$\alpha \in A := \{(c_\alpha, f_\alpha) \mid c_\alpha \in \mathbb{R}, f_\alpha: \mathbb{R} \rightarrow \mathbb{R} \text{ continuous, strictly increasing, unbounded}\}.$$

- (a) Prove that there is no sequence in $\mathbb{R}^2$ converging to ∞ .

(b) Construct a net in $\mathbb{R}^2$ which converges to ∞ .

Exercise 4 (10 points). Let Y be a complete metric space and $X \subseteq Y$ a subset. Prove that the completion $\hat{X}$ is isometrically isomorphic to the closure $\bar{X}$ of X in Y . (This is part of Theorem 1.12.)