UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Exercises for the Lecture Functional Analysis I
Winter term 2026/2027

Exercise Sheet 3
Submission: Wednesday, 11 November 2026, 12:30 pm
in the post box of Benjamin Klasen in E2 5

Exercise 1 (10 points)**.** A metric space (X, d) is called *separable* if there is a countable subset $A \subseteq X$ that is dense in X. Convince yourself that $\mathbb{R}$ and $\mathbb{C}$ are separable. Show that $(c_0, \|\cdot\|_\infty)$ is separable, whereas $(\ell^\infty, \|\cdot\|_\infty)$ is not.

Exercise 2 (10 points)**.** Show that every infinite-dimensional normed space admits discontinuous linear functionals.

Exercise 3 (10 points)**.** Let E be a normed space and E' its dual space. Show that the norm on E is given by

$$\|x\| = \sup\left\{|f(x)| \;\middle|\; f \in E',\ \|f\| \leq 1\right\}.$$

Moreover, the supremum is attained: for each $x \in E$, there exists $f \in E'$ with $\|f\| = 1$ and $f(x) = \|x\|$.

Exercise 4 (10 points)**.** Consider the Banach space $\ell^\infty_{\mathbb{R}}$ of bounded real-valued sequences, equipped with the supremum norm. Show that $q : \ell^\infty_{\mathbb{R}} \to \mathbb{R}$, defined by the limit superior $q((a_n)) = \limsup_{n\to\infty} a_n$, is a sublinear functional. Deduce that there exists a bounded linear functional $L : \ell^\infty_{\mathbb{R}} \to \mathbb{R}$ such that

$$\liminf_{n\to\infty} a_n \leq L((a_n)) \leq \limsup_{n\to\infty} a_n.$$

Show also that $L(1) = 1$, where $1 \in \ell^\infty_{\mathbb{R}}$ denotes the constant sequence $(1)_{n\in\mathbb{N}}$; that $L((a_n)) \geq 0$ if all $a_n \geq 0$; that $\|L\| = 1$; and that, if $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, then $L((a_n)) = \lim_{n\to\infty} a_n$.
In this way, one can in a certain sense define "limits" of bounded sequences (restricting L to convergent sequences recovers the usual limit of a sequence).