UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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Exercises for the Lecture Functional Analysis I
Winter term 2026/2027

Exercise Sheet 4
Submission: Wednesday, 18 November 2026, 12:30 pm
in the post box of Benjamin Klasen in E2 5

Exercise 1 (10 points)**.** Give an example showing that the convex hull of a compact subset of a normed space need not be closed. For instance, consider a sequence in c_0 that converges to the zero sequence.

Exercise 2 (10 points)**.** Show that the closed unit ball of the Banach space $(c_0, \|\cdot\|_\infty)$ has no extreme points.

Exercise 3 (15 points)**.**

(i) Let $T\colon X \to Y$ be a linear map between vector spaces. Show that the preimage $T^{-1}(A)$ of a convex subset $A \subseteq Y$ is convex.

For $0 < p < 1$, consider the vector space

$$L^p = \left\{ f\colon [0,1] \to \mathbb{C} \;\middle|\; f \text{ is Borel measurable and } \int_{[0,1]} |f(x)|^p \, d\lambda(x) < \infty \right\},$$

where λ is Lebesgue measure on the Borel subsets of $[0,1]$ and functions that agree almost everywhere are identified.

(ii) Show that $d(f,g) := \int_{[0,1]} |f(x) - g(x)|^p \, d\lambda(x)$ defines a metric on L^p.

(iii) Let $f \in L^p$. Given a partition $0 = x_0 < x_1 < \cdots < x_n = 1$, write f as the convex combination

$$f = \frac{1}{n} \sum_{i=1}^{n} g_i, \qquad g_i = n f \, \mathbf{1}_{[x_{i-1}, x_i]}.$$

Show that, for every $\varepsilon > 0$, one can choose the partition so that $d(0, g_i) < \varepsilon$ for every i.

(iv) Deduce that L^p is the only open convex subset of (L^p, d) containing 0. In particular, (L^p, d) is not locally convex.

(v) Show that $(L^p, d)' = \{0\}$, i.e. every continuous linear functional on (L^p, d) is zero.

(vi) Show that the Hahn–Banach extension theorem (Theorem 2.6 in the lecture notes) can fail for metric vector spaces. Find a continuous linear functional on a closed subspace of (L^p, d) that has no continuous linear extension to L^p.