**UNIVERSITÄT DES SAARLANDES**
**FACHRICHTUNG MATHEMATIK**
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**Exercises for the Lecture Functional Analysis I**
Winter term 2026/2027

**Exercise Sheet 5**
**Submission:** Wednesday, 25 November 2026, 12:30 pm
in the post box of Benjamin Klasen in E2 5

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**Exercise 1** (10 points)**.** Show that there exists a function $g \in \mathcal{C}([0,1])$ that is differentiable at no point of the interval $[0, \frac{1}{2}]$. To this end, show that the sets

$$F_n := \left\{ f \in \mathcal{C}([0,1]) \;\middle|\; \exists x \in [0, \tfrac{1}{2}] : \sup_{0<h<\frac{1}{2}} \frac{|f(x+h) - f(x)|}{h} \leq n \right\}$$

are closed and contain no nonempty open balls for every $n \in \mathbb{N}$. Then use Baire's theorem.

**Exercise 2** (10 points)**.** Prove Corollary 4.3 of the lecture notes using Baire's theorem: there is no real or complex Banach space of countably infinite (vector space) dimension.

**Exercise 3** (10 points)**.** Prove Corollary 4.5 of the lecture notes using a principle of uniform boundedness: if $E$ is a normed space and $M \subseteq E$ is a subset such that $f(M)$ is bounded for every $f \in E'$, then $M$ is bounded.

**Exercise 4** (10 points)**.** Consider the space $f$ of finitely supported sequences (see Sheet 2) as a subspace of $\ell^\infty$. Let $T\colon f \to f$ be given by

$$T(a_1, a_2, a_3, \ldots, a_n, \ldots) := (a_1, 2a_2, 3a_3, \ldots, na_n, \ldots).$$

(a) Show that $f \subseteq \ell^\infty$ is not closed (and hence is not complete).

(b) Show that $T$ is discontinuous (but linear).

(c) Show that $T$ is the pointwise limit of continuous linear maps $T_n\colon f \to f$.

Thus the Banach–Steinhaus theorem (Corollary 4.6 of the lecture notes) can fail when $X$ is not a Banach space.