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Operator Algebras Problem Set 4

To be submitted by Wednesday, **May 14**, 4 pm.

Exercise 1

Let A be a unital C^*-algebra and let $x, y \in A$. Show that

$$\operatorname{sp}(xy) \cup \{0\} = \operatorname{sp}(yx) \cup \{0\}$$

and provide an example where $\operatorname{sp}(xy)$ and $\operatorname{sp}(yx)$ do not coincide.

Exercise 2

Let H be a complex Hilbert space and $T \in \mathcal{B}(H)$.

(a) Show that there is a unique decomposition $T = VP$ where $V \in \mathcal{B}(H)$ is a partial isometry and $P \in \mathcal{B}(H)$ is positive such that $\ker(V) = \ker(P) = \ker(T)$.
Hint: Take $P = |T| := \sqrt{T^*T}$ and $V = V_0 \oplus 0$ where

$$V_0 : \overline{\operatorname{ran}(|T|)} \to \overline{\operatorname{ran}(|T|)}, Px \mapsto Tx.$$

In particular, justify that these expressions are well-defined and make sense. For the uniqueness statement, first prove that P is unique.

(b) Show that V is unitary whenever T is invertible.

(c) How does the decomposition $T = VP$ look like in the case $H = \mathbb{C}$?

Exercise 3

Let H be a complex Hilbert space and $T \in \mathcal{B}(H)$. Prove the following statements:

(a) $T = 0$ if and only if $\langle Tx, x\rangle = 0$ for all $x \in H$.

(b) $T = T^*$ if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.

(c) $T \geq 0$ if and only if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.
Hint: Show that for every $\lambda < 0$, the operator $\lambda 1 - T$ is bounded from below, that is, there exists a constant $c > 0$ such that $\|(\lambda 1 - T)x\| \geq c\|x\|$ for all $x \in H$.

The following exercises are **not graded** and will be discussed on Monday, May 12.

Exercise 4

Let A be a C^*-algebra, $x \in A$ normal and $f, g \in C(\mathrm{sp}(x))$. Show:

(a) Proposition 3.29 b), that is, $\mathrm{sp}(f(x)) = f(\mathrm{sp}(x))$.

(b) Recall Proposition 3.29 e) and Lemma 3.8 a) and verify the following arguments:

i) If x is selfadjoint and $f(t) = 2t^2$, then $f(x)$ is positive.

ii) If x is positive and $f(t) = t^3$, then $f(x)$ is positive.

iii) If x is positive and $x^n = 0$ for some $n \in \mathbb{N}$, then $x = 0$.

iv) If x is positive and $\lambda \in (0, \infty)$, then λx is also positive.

v) Recall the proof of Lemma 4.2.

vi) If x is normal and $y \in C^*(1, x)$ is positive, then there exists a positive function $\tilde{g} \in C(\mathrm{sp}(x))$ such that $\tilde{g}(x) = y$.

Exercise 5

Find the unique positive square root of the matrix $S = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$. The following steps may help you:

(a) Verify that S is selfadjoint and compute its eigenvalues. Make sure that all eigenvalues are positive.

(b) Diagonalize S, that is, find a unitary matrix U such that $S = U^*DU$, where D is diagonal matrix consisting of the eigenvalues of S.

(c) Verify that the square root of S is given by $U^*D'U$, where D' is the diagonal matrix consisting of the square roots of the eigenvalues of S.