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Operator Algebras Problem Set 5

To be submitted by Wednesday, **May 21**, 4 pm.

Exercise 1

Let A be a unital C^*-algebra and let $x \in A$ be selfadjoint. Show that:

(a) Recall that x is invertible if and only if $0 \notin \mathrm{sp}(x)$.

(b) Let x be invertible. Show that x^{-1} is selfadjoint.

(c) Let $\mathrm{sp}(x) \subset (0, \infty)$. Use the functional calculus to show $\mathrm{sp}(x^{-1}) \subset (0, \infty)$.

(d) Show that if $f, g \in C(\mathrm{sp}(x))$, then $f(x)$ and $g(x)$ commute; in particular, $f(x)$ and x commute.

(e) Show that $\mathrm{sp}(x - 1) \subset [0, \infty)$ if and only if $\mathrm{sp}(x) \subset [1, \infty)$.

(f) Show that if $\mathrm{sp}(x) \subset [1, \infty)$, then x is invertible and $\mathrm{sp}(1 - x^{-1}) \subset [0, \infty)$.

Exercise 2

Let I, A and B be C^*-algebras and let $\iota : I \to A$ and $\pi : A \to B$ be $*$-homomorphisms.

(a) Show that if the sequence

$$0 \to I \to A \to B \to 0$$

is exact, then ι is injective, π is surjective, $\iota(I) \lhd A$ is a closed ideal in A and $B \cong A/\iota(I)$.

(b) Conversely, if ι is injective and $\iota(I) \lhd$ is a closed ideal in A, show that

$$0 \to I \to A \to A/\iota(I) \to 0$$

is exact, where $A \to A/\iota(I)$ is the canonical quotient map.

Talk

In this talk we want to look at special C^*-subalgebras, namely *hereditary C^*-subalgebras*. The recommended literature is: “ C^*-algebras and Operator Theory “ by Gerald Murphy.

i) Read Section 3.2 and give the definition and an example for hereditary C^*-subalgebras. How are these linked to closed left ideals (Theorem 3.2.1)?

ii) In the separable case, what do hereditary C^*-subalgebras look like (Theorem 3.2.5 and Theorem 3.2.6)? How are closed ideals linked to hereditary C^*-subalgebras (Theorem 3.2.7)?

iii) Show that Theorem 3.2.5 does in general not hold for non-separable hereditary C^*-subalgebras.