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Operator Algebras Problem Set 6

To be submitted by Wednesday, **May 28**, 4 pm.

Exercise 1

(a) Let A be a commutative (not necessarily unital) C^*-algebra. Show that for every positive $h \in A$ and selfadjoint $x \in A$ with $h \geq x$, it holds that $h \geq x_+$.

(b) Find a counterexample to the above statement in $A = M_2(\mathbb{C})$.

Exercise 2

Let $N \in \mathbb{N}$ be fixed. We denote by tr the normalized trace on $M_N(\mathbb{C})$, i.e. the linear functional $M_N(\mathbb{C}) \to \mathbb{C}, T = (t_{i,j})_{i,j=1}^N \mapsto \frac{1}{N}\sum_{i=1}^N t_{ii}$. For a positive matrix $B \in M_N(\mathbb{C})$ we denote by τ_B the linear functional $M_N(\mathbb{C}) \to \mathbb{C}, T \mapsto \mathrm{tr}(BT)$.

(a) Show that tr is a faithful state on $M_N(\mathbb{C})$.

(b) Show that the mapping $B \mapsto \tau_B$ defines a bijection between the positive matrices in $M_N(\mathbb{C})$ and the positive functionals on $M_N(\mathbb{C})$.

(c) Determine the preimage of the set of states on $M_N(\mathbb{C})$ under the mapping from (b).

Talk

Consider the set of bounded $\mathbb{R}$-valued sequences ℓ^∞. A linear map $l : \ell^\infty \to \mathbb{R}$ is called *Banachlimes* if

i) $l(Lx) = l(x)$ for all $x \in \ell^\infty$, where L denotes the shiftoperator given by $L((x_n)_n) = (x_{n+1})_n$.

ii) If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $l((x_n)_n) \geq 0$.

iii) $l((1)_n) = 1$, where $(1)_n = (1, 1, 1, 1, \ldots)$.

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Show that there exists a Banachlimes, Furthermore if l is a Banachlimes, verify the following:

a) $l \in (\ell^\infty)'$ and $\|l\| = 1$.

b) $\liminf x_n \le l(x) \le \limsup x_n$ for all $x = (x_n)_n \in \ell^\infty$, in particular $l(x) = \lim_{n\to\infty} x_n$ for all converging sequences x.

c) l is not multiplicative, i.e. there are $x, y \in \ell^\infty$ such that $l(xy) \neq l(x)l(y)$.

Hints: For the existence of a Banachlimes, we recommend Exercise 3.6.5 in "Funktionalanalysis"" by Dirk Werner .
For a), you can start by showing that $\|x\|_\infty - x_n \ge 0$ and $\|x\|_\infty + x_n \ge 0$ for all $n \in \mathbb{N}$.
For b), fix $N \in \mathbb{N}$ and consider the sequence y defined by $y_n = x_n$ for $0 \le n \le N-1$ and $y_n = \inf_{k\ge N} x_k$ for $n \ge N$.
For c), consider $x = (1, 0, 1, 0, \dots)$ and $y = (0, 1, 0, 1, 0, \dots)$.