

Operator Algebras Problem Set 7

To be submitted by Wednesday, **June 11**, 4 pm.

A state ϕ on a C^* -algebra is called *faithful* if $\phi(x^*x) = 0$ implies $x = 0$ for every element x .

Exercise 1

Let A be a C^* -algebra, let ϕ be a state on A and let $(H_\phi, \pi_\phi, \zeta_\phi)$ be the GNS-representation of A corresponding to ϕ . Prove the following statements:

- For every closed ideal I in A , we have $I \subset \ker(\pi_\phi)$ if and only if $I \subset \ker(\phi)$.
- If ϕ is a faithful state, then π_ϕ is a faithful representation.

Exercise 2

We consider the universal C^* -algebras

$$C^*(p, 1 \mid p \text{ is a projection, i.e. } p = p^2 = p^*)$$
$$C^*(s, 1 \mid s \text{ is a symmetry, i.e. } s^*s = ss^* = 1, s = s^*).$$

(Here, we consider 1 as a generator with the relations $1 = 1^* = 1^2$ and $1x = x1 = x$ for every other generator x .)

- Show that these C^* -algebras are isomorphic by writing down an explicit isomorphism. To do so, find a symmetry $s' \in C^*(p, 1)$ and a projection $p' \in C^*(s, 1)$ and use the universal property twice. (It might help to solve (a) and (b) at the same time.)
- How does the spectrum of a projection and the spectrum of a symmetry look like? Since the C^* -algebras above are commutative, they are isomorphic to the algebra of continuous functions on the spectrum, i.e.

$$C^*(p, 1) \cong C(\text{sp}(p)), \quad C^*(s, 1) \cong C(\text{sp}(s)).$$

What are images of $id_{\text{sp}(p)}$ and $id_{\text{sp}(s)}$ under the isomorphism between $C^*(p, 1)$ and $C^*(s, 1)$?

Exercise 3

Show that the following C^* -algebras are isomorphic.

- $C(\{1, \dots, n\})$
- $\mathbb{C}^n = \mathbb{C} \oplus \dots \oplus \mathbb{C}$
- $C^*(p_1, \dots, p_n, 1 \mid p_i \text{ projections, } \sum_{i=1}^n p_i = 1)$
- $C^*(u, 1 \mid u^*u = uu^* = 1, u^n = 1)$