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Operator Algebras Problem Set 7

To be submitted by Wednesday, **June 11**, 4 pm.

A state ϕ on a C^*-algebra is called *faithful* if $\phi(x^*x) = 0$ implies $x = 0$ for every element x.

Exercise 1

Let A be a C^*-algebra, let ϕ be a state on A and let $(H_\phi, \pi_\phi, \zeta_\phi)$ be the GNS-representation of A corresponding to ϕ. Prove the following statements:

(a) For every closed ideal I in A, we have $I \subset \ker(\pi_\phi)$ if and only if $I \subset \ker(\phi)$.

(b) If ϕ is a faithful state, then π_ϕ is a faithful representation.

Exercise 2

We consider the universal C^*-algebras

$$C^*(p, 1 \mid p \text{ is a projection, i.e. } p = p^2 = p^*)$$
$$C^*(s, 1 \mid s \text{ is a symmetry, i.e. } s^*s = ss^*, s = s^*).$$

(Here, we consider 1 as a generator with the relations $1 = 1^* = 1^2$ and $1x = x1 = x$ for every other generator x.)

(a) Show that these C^*-algebras are isomorphic by writing down an explicit isomorphism. To do so, find a symmetry $s' \in C^*(p, 1)$ and a projection $p' \in C^*(s, 1)$ and use the universal property twice. (It might help to solve (a) and (b) at the same time.)

(b) How does the spectrum of a projection and the spectrum of a symmetry look like? Since the C^*-algebras above are commutative, they are isomorphic to the algebra of continuous functions on the spectrum, i.e.

$$C^*(p, 1) \cong C(\mathrm{sp}(p)), \quad C^*(s, 1) \cong C(\mathrm{sp}(s)).$$

What are images of $id_{\mathrm{sp}(p)}$ and $id_{\mathrm{sp}(s)}$ under the isomorphism between $C^*(p, 1)$ and $C^*(s, 1)$?

Exercise 3

Show that the following C^*-algebras are isomorphic.

- $C(\{1, \dots, n\})$
- $\mathbb{C}^n = \mathbb{C} \oplus \dots \oplus \mathbb{C}$
- $C^*(p_1, \dots, p_n, 1 \mid p_i \text{ projections}, \sum_{i=1}^n p_i = 1)$
- $C^*(u, 1 \mid u^*u = uu^* = 1, u^n = 1)$