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Operator Algebras Problem Set 8

To be submitted by Wednesday, **June 18**, 4 pm.

Exercise 1

Let H be a complex Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, and let $\tilde{H}$ be a complex Hilbert space with orthonormal basis $(\tilde{e}_n)_{n\in\mathbb{Z}}$. For $\lambda \in S^1 \subset \mathbb{C}$, we define shift and diagonal operators via

$$\begin{aligned} S : H &\to H, & \tilde{S} : \tilde{H} &\to \tilde{H}, & d(\lambda) : H &\to H, & \tilde{d}(\lambda) : \tilde{H} &\to \tilde{H} \\ e_n &\mapsto e_{n+1} & \tilde{e}_n &\mapsto \tilde{e}_{n+1} & e_n &\mapsto \lambda^n e_n & \tilde{e}_n &\mapsto \lambda^n \tilde{e}_n \end{aligned}$$

Prove the following assertions:

(a) S is an isometry such that $1 - SS^*$ is the projection onto the one-dimensional subspace $\mathbb{C}e_1 \subset H$, while $\tilde{S}, d(\lambda), \tilde{d}(\lambda)$ are unitaries with

$$d(\lambda)^* = d(\bar{\lambda}), \quad d(\lambda)d(\lambda') = d(\lambda\lambda'), \quad \tilde{d}(\lambda)^* = \tilde{d}(\bar{\lambda}), \quad \tilde{d}(\lambda)\tilde{d}(\lambda') = \tilde{d}(\lambda\lambda').$$

(b) It holds that $d(\lambda)S = \lambda S d(\lambda)$ and $\tilde{d}(\lambda)\tilde{S} = \lambda\tilde{S}d(\lambda)$, and more generally

$$\tilde{d}(\lambda)^k \tilde{S}^l = \lambda^{kl}\tilde{S}^l\tilde{d}(\lambda)^k, \;\; k, l \in \mathbb{Z}.$$

Conclude that the set $\mathcal{S}$ of finite linear combinations of $\tilde{d}(\lambda)^k\tilde{S}^l$ is a dense $*$-subalgebra of $C^*(\tilde{S}, \tilde{d}(\lambda)) \subset \mathcal{B}(\tilde{H})$.

(c) The maps

$$\begin{aligned} \beta_\lambda : \mathcal{B}(H) &\to \mathcal{B}(H), & \tilde{\beta}_\lambda : \mathcal{B}(\tilde{H}) &\to \mathcal{B}(\tilde{H}) \\ T &\mapsto d(\lambda)Td(\lambda)^*, & T &\mapsto \tilde{d}(\lambda)T\tilde{d}(\lambda)^* \end{aligned}$$

are $*$-isomorphisms with $\beta_\lambda(C^*(S)) = C^*(S)$ and $\beta_\lambda(C^*(\tilde{S})) = C^*(\tilde{S})$.

(d) Use (c) to show that $\mathrm{sp}(\tilde{S}) = S^1$ and $\mathrm{sp}(\sigma(S)) = S^1$, where $\sigma : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ is the quotient map.

Exercise 2

Prove the " Five Lemma ", i.e. Lemma 6.27:
Assume we have the following commutative diagram of two short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & I_1 \arrow[r, "\iota_1"] \arrow[d, "\alpha"'] & A_1 \arrow[r, "\pi_1"] \arrow[d, "\varphi"'] & B_1 \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & I_2 \arrow[r, "\iota_2"'] & A_2 \arrow[r, "\pi_2"'] & B_2 \arrow[r] & 0 \end{tikzcd}$$

If α and β are $*$-isomorphisms, then also φ is a $*$-isomorphism.

Talk

The irrational rotation algebra A_ϑ was introduced in section 7.1. This talk should cover the results from section 7.2:

(a) Give the definition of a *conditional expectation* and faithful positive linear map.

(b) Construct the maps φ_1, φ_2 and outline why these are faithful conditional expectations.

(c) Use Lemma 7.8 to show one of the differences between $\vartheta \notin \mathbb{Q}$ and $\vartheta \in \mathbb{Q}$.