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Operator Algebras Problem Set 9

To be submitted by Wednesday, **June 25**, 4 pm.

Exercise 1

Let H be a complex Hilbert space.

(a) Show that the involution $* : \mathcal{B}(H) \to \mathcal{B}(H)$ is continuous in the norm topology and the weak operator topology, but not in the strong operator topology.

(b) Let $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}$ be two nets in $\mathcal{B}(H)$ (over the same index set Λ) with $x_\lambda \to x$ and $y_\lambda \to y$ in the strong operator topology. Show that if $(x_\lambda)_{\lambda\in\Lambda}$ is bounded, then $(x_\lambda y_\lambda)_{\lambda\in\Lambda}$ converges strongly to xy. (This fails to be true in general if $(x_\lambda)_{\lambda\in\Lambda}$ is not assumed to be bounded. A counterexample may be found in the book *A Hilbert Space Problem Book* by Paul Halmos.)

Exercise 2

Let H be a complex Hilbert space and let $S, T \subset \mathcal{B}(H)$ be subsets with $S \subset T$.

(a) Show that the commutant $S' \subset \mathcal{B}(H)$ of S is a weakly closed unital subalgebra.

(b) Show that if $S = S^*$, then the commutant $S' \subset \mathcal{B}(H)$ of S is a strongly closed unital $*$-subalgebra (and hence a von Neumann algebra).

(c) Show that $S \subset S''$ and $T' \subset S'$. Deduce that $S''' = S'$.

Talk

Show that the rational rotation algebra is not simple, while the irrational rotation algebra is simple. For the rational rotation algebra A_ϑ you can proceed as follows:

(a) Find a representation $\pi : A_\vartheta \to M_q(\mathbb{C})$.

(b) Find unital C^*-algebras B and D as well as unital $*$-homomorphisms $\varphi : A_\vartheta \to B$ and $\psi : A_\vartheta \to D$ such that $\varphi(v^q) = 1$ and $\psi(v^q) \neq 1$.

(c) Conclude that A_ϑ is not simple.

Use section 7.3 for the irrational rotation algebra A_ϑ:

(i) Define a *(normalized) trace*.

(ii) Briefly recall the defition of φ_1, φ_2.

(iii) Show that $\tau := \varphi_1 \circ \varphi_2$ is a unital faithful trace (Proposition 7.10).

(iv) Conclude that A_ϑ is simple (Theorem 7.11).