ONBs of Hilbert Modules, Cuntz Algebras, and Other Algebras Generated by Partial Isometries

(Joint with Malte Gerhold)

Abstract

We examine when the Hilbert modules $\mathcal{B}^m$ and $\mathcal{B}^n$ over a unital C^*–algebra $\mathcal{B}$ can be isomorphic.

In the case $m = 1$, this leads us naturally to the Cuntz algebras $\mathcal{O}_n$. For general m, we are lead to the universal C^*–algebra $\mathcal{O}_{n,m}$ generated by a biunital $n \times m$–matrix unit, generalizing $\mathcal{O}_n = \mathcal{O}_{n,1}$.

Since isomorphisms between $\mathcal{B}^m$ and $\mathcal{B}^n$ are, obviously, given by a unitary $n \times m$–matrix, it is evident that $\mathcal{O}_{n,m}$ ($\cong \mathcal{O}_{m,n}$) is intimately related to the universal C^*–algebra generated by a unitary $n \times m$–matrix, the rectangular Brown algebra $\mathcal{U}_{n,m}$ ($\cong \mathcal{U}_{m,n}$). In fact, we will see that $\mathcal{O}_{n,m} \cong M_m(\mathcal{U}_{n,m})$ ($\cong M_n(\mathcal{U}_{n,m})$), that it is a quotient of $\mathcal{U}_{n,m}$, and that it embeds naturally into the unital free product $\mathcal{O}_n \circledast_{\mathbf{1}} \mathcal{O}_m$.

It appears that the (adjoints of the) elements of the biunital matrix unit forming a unitary matrix in $M_{n,m}(\mathcal{O}_{n,m})$ (whose entries are partially isometric and generate $\mathcal{O}_{n,m}$) may be considered as a sort of a dilation of the unitary matrix in $M_{n,m}(\mathcal{U}_{n,m})$ (whose entries generate $\mathcal{U}_{n,m}$) through the natural conditional expectation $\mathcal{O}_{n,m} \cong M_m(\mathcal{U}_{n,m}) \to \mathcal{U}_{n,m}$.